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Solutions and Phase Portraits of Endogenous Growth Models with Optimal Saving

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Abstract

This paper explores the optimal saving decisions within the context of endogenous economic growth modeled as a Ramsey model. In contrast to the common wisdom, the Ramsey model is capable of generating endogenous growth. The conditions sufficient for persistent growth in an otherwise standard Ramsey model are given. In the phase diagram of the model, the optimal path shows up as a separator. Below the separator, over-saving diminishes consumption, ultimately leading to a sub-optimal situation where all incomes are saved. On the other hand, above the separator under-saving suddenly collapses the economy as its productive capital vanishes to zero. The paper also gives numerical examples for CRRA preferences and CES and Cobb Douglas technologies together with a sensitivity analysis.

JEL Classification: C68, O40, O41

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1 Introduction

One of the most important decisions for individuals is the choice about the rate of savings, i.e. about the share of income to be devoted to investment and hence not available for consumption. In this paper, we study the optimal saving decisions within the context of endogenous economic growth modeled as a Ramsey model.

In contrast to the common wisdom, the Ramsey model is capable of generating endogenous growth. In this paper, we explore the conditions sufficient for persistent growth in an otherwise standard Ramsey model. We also show that, in the case of persistent growth, much of the ingenious phase diagram can be preserved, even though the saddle path and steady state structure disappears because the optimal path shows up as a separator in the phase portrait. Beneath this separator, over-saving diminishes consumption, ultimately leading to a sub-optimal situation where all incomes are saved. On the other hand, above the separator under-saving suddenly collapses the economy as its productive capital vanishes to zero.

The paper is organized as follows: Section 2 introduces the standard optimal saving model and the sufficient condition for persistent growth is given in Section 3. In Section 4, numerical examples for CRRA preferences and CES and Cobb Douglas technologies show both the phase portrait and the time paths of the endogenous variables. A sensitivity analysis is also provided. Section 5 discusses the results. Technical details are placed in the Appendix.

2 Standard Optimal Saving Model

Consider infinitely lived consumers maximizing total *lifetime* utility. The representative consumer has a time additive intertemporal utility function

$$U = \int_0^\infty u[c(t)]e^{-\rho t}dt, \quad c = C/L \tag{1}$$

where the decision variable, c, is per capita consumption and ρ is the constant subjective rate of time preference (discount rate). The *contemporaneous* utility (felicity) function, $u(c)$, is *concave, increasing*, and satisfies the Inada conditions

$$\lim_{c\to 0} u'(c) = \infty, \qquad \lim_{c\to\infty} u'(c) = 0. \tag{2}$$

In an economy with exogenous population (labor) growth, $L(t) = L_0 e^{nt}$, a social planner may maximize the objective (welfare) function,

$$V = \int_0^\infty u[c(t)]L(t)e^{-\rho t}dt = L_0 \int_0^\infty u[c(t)]e^{-(\rho-n)t}dt \tag{3}$$

With $\rho > 0$ and $\rho - n > 0$, total utility, U and V, are bounded, if $u(c)$ with (2) is also bounded over time. In principle, there is no difference between the problem of an individual, (1), or a society (planner), (3). Hence we stick with (3) as in Barro & Sala-i-Martin (2004, p.87). But an *upper bound* imposed on $u(c)$ is a *critical assumption* that we shall dispense with and elaborate on below.

The *technology*, $F(L,K)$, is described by a smooth concave homogeneous production function with *constant returns* to scale in labor and capital,

$$Y = F(L,K) = Lf(k) \equiv Ly, L \neq 0; \;\; F(0,0) = 0 \tag{4}$$

where the function $f(k)$, is strictly concave and monotonically increasing in the capital-labor ratio $k \in [0, \infty[$, i.e., $f(k)$ has the properties

$$\forall k > 0\colon \;\; f'(k) > 0, \quad f''(k) < 0 \tag{5}$$

$$\lim_{k \to 0} f'(k) \equiv \bar{b} \; \leq \infty, \; \lim_{k \to \infty} f'(k) \equiv \underline{b} \; \geq 0, \; f'(k) \in \left[\underline{b}, \bar{b}\right] \tag{6}$$

For the macro (one-sector) model, factor accumulation is given by, cf. (1), (4)

$$\dot{K} = Y - C - \delta K = L\left[f(k) - c - \delta\,\right], \qquad \dot{L} = nL \tag{7}$$

Hence

$$\dot{k} \; = \; f(k) - c - (n+\delta)k \; = \; h(k,c). \tag{8}$$

Thus the Ramsey optimization problem is (with $L_0 = 1$),

$$\max V = \max_{c(t)} \int_0^\infty u[c(t)] e^{-(\rho - n)t} dt \tag{9}$$

$$\text{s.t.} \quad \dot{k} = f(k) - c - (n+\delta)k = h(k,c), \quad c \geq 0, \tag{10}$$

which is equivalent to maximizing the current value Hamiltonian function

$$\mathcal{H} = u[c(t)] + \lambda(t)\left[f(k) - c - (n+\delta)k\,\right], \tag{11}$$

with the costate variable, $\lambda(t)$, and the transversality conditions:

$$k(0) = k_0, \quad \lim_{t \to \infty} \lambda(t)k(t)e^{-(\rho - n)t} = 0. \tag{12}$$

Besides, $\partial\mathcal{H}/\partial\lambda = h(k,c)$, cf. (11), (8), the first order (necessary) condition is,

$$\frac{\partial \mathcal{H}}{\partial c} = u'(c) - \lambda(t) = 0, \tag{13}$$

The maximum principle also gives a necessary costate equation ("Euler") as,

$$\dot{\lambda}(t) \quad = \quad -\frac{\partial \mathcal{H}}{\partial k} + (\rho - n)\lambda(t). \tag{14}$$

In addition to necessity, the sufficiency of (9)–(14) can be proved simply. It is observed that the objective function $u\,[c(t)]\,e^{-(\rho-n)t}$ is a *concave* function in (k,c)-space, cf. (2). Furthermore, it can be shown, by using (13) and (2), that $\lambda(t) = u'(c) > 0$, and that the function, $h(k,c)$, in (10) is also *concave* in c, k. Therefore the necessary conditions provided by the *maximum* principle are also sufficient for obtaining the optimal solutions of $[k(t), \lambda(t)\,; c(t)]$.

Economic intuition about the implications of the necessary conditions are well-known and easily provided. By time derivation of (13), we get

$$\dot{\lambda}(t) = u''(c)\,\dot{c}(t)\,; \quad \dot{\lambda}/\lambda = [u''(c)/u'(c)]\,(\dot{c}/c)\,; \quad u''(c) < 0\,,\; u''(c) < 0 \tag{15}$$

Thus if consumption along the optimal solution is growing ($\dot{c}(t) > 0$), the concavity of $u(c)$ ensures that marginal utility of consumption ["shadow price," λ, "opportunity costs" in utils of postponing consumption (saving, accumulating capital)] is falling. The necessary condition (14) provides additional information about the optimal time paths of this "shadow price" of consumption,

$$\begin{aligned} \dot{\lambda}(t) &= -\frac{\partial \mathcal{H}}{\partial k} + (\rho - n)\lambda(t)\left[-f'(k) + (n+\delta) + \rho - n\right] \\ &= -\lambda(t)\left[f'(k) - \delta - \rho\right]. \end{aligned} \tag{16}$$

Hence along the optimal consumption solution (path), the relative change (fall /increase) in its "shadow price", $\dot{\lambda}/\lambda$, must always be equal to the net return (positive/negative) on capital. This necessary optimality condition (16) is "Ramsey's rule", cf. Ramsey (1928, p. 548, eq.(3), $\delta = 0, \rho = 0$; p. 554, eq.(9), $\delta = 0, \rho > 0$). A related "Keynes-Ramsey" rule relates changes in the product $[\lambda(t)\,\dot{k}]$ to attained levels of utility, but the stated rule relies on an upper bounded u(c) function (bliss point), see, Ramsey (1928,p. 527), Wan, Jr. (1971, p. 315), Newbery (2008). We return to the issues with unbounded u(c).

The alternative characterizations of optimal changes in the shadow price, $\dot{\lambda}(t)$, in (15), (16), can be combined to obtain the differential equation for the optimal changes in per capita consumption, c, as

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - (\delta + \rho)\right] \equiv \eta(c)\, c\left[f'(k) - (\delta + \rho)\right] \tag{17}$$

or

$$\dot{c}/c \;=\; \widehat{c} \;=\; \eta(c)\;\left[f'(k) - (\delta + \rho)\right] \tag{18}$$

where, $\eta(c) = -u'(c)/[u''(c)c]$, is the *intertemporal substitution elasticity.* As c(t) is the "decision (control) variable" in (9)–(10), it is more convenient to have (17) directly instead of the costate variable $\lambda(t)$ needed in (11).

Thus combined the differential equations of optimal capital accumulation and consumption (8), (17), define a *dynamic system* in the state variables k and c with the governing function, $h(k, c)$, and $g(k, c)$ on the *closed* set (first quadrant), $\overline{\Re}^2_+$:

$$\begin{aligned} \dot{k} &= h(k, c) = f(k) - (n + \delta)k - c, &(19)\\ \dot{c} &= g(k, c) = \eta(c)\, c\left[f'(k) - (\delta + \rho)\right], &(20) \end{aligned}$$

By (13) the transversality condition (12) becomes

$$u'(c)k(t)e^{-(\rho-n)t} \to 0 \text{ as } t \to \infty. \tag{21}$$

With the solutions, k(t), c(t), of (19-20), we can form relevant (diagnostic, performance) auxiliary time paths, e.g., saving rate s(t) as given by, cf. (4), (7),

$$s(t) \;=\; 1 \;-\; C(t)/\,Y(t) \;=1 \;-\; c(t)/\,y(t) \;=\; 1 \;-\; c(t)/\,f(k(t)) \tag{22}$$

A realistic behavior of s(t) is often a test and hallmark for evaluating the concrete specifications of the optimal saving dynamics (19-20).

Steady States (Saddle Points) within the Standard Optimal Saving Model

If they exist, *steady-state* values of capital-labor ratios and per capita consumptions in optimal one-sector growth models are *critical points* of (19-20):

$$[\,\dot{c} = 0 \Leftrightarrow k(t) = \kappa\,] \;\Leftrightarrow\; [\,f'(\kappa) = \rho + \delta\,]\;;\quad MP_K(\kappa) = \rho + \delta \tag{23}$$

$$[\,\dot{k} = 0 \Leftrightarrow c(t) = c(\kappa)\,] \;\Leftrightarrow\; [\,c(\kappa) \;=\; f'(\kappa) - (n+\delta)\kappa\,] \tag{24}$$

For the utility functions, $u(c)$, a common practice is to use the class of *isoelastic* CRRA utility function. Such convenient CRRA parametric form of $u(c)$ is, see Barro and Sala-i-Martin (1995, p. 141), Solow (2000, p. 114) :

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta},\ \theta > 0;\quad R_r(c) = \theta\ ;\ \eta(c) = 1/\theta \tag{25}$$

Consider then the CES production function $Y = F(L,K)$ and the associated marginal and average products of capital are (Arrow et al. 1961, p. 230, La Grandeville 2009, p. 90):

$$\begin{aligned} Y \;&=\; F(L,K) = \gamma\left[(1-a)L^{\frac{\sigma-1}{\sigma}} + aK^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\ ;\ 0 < a < 1,\ \sigma > 0 \\ AP_L(k) \;&=\; Y/L \;=\; y = f(k) = \gamma\left[(1-a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{(\sigma-1)}} \end{aligned} \tag{26}$$

in which γ is the efficiency parameter, a is the distribution parameter and σ is the substitution parameter, respectively.

In the CES case (26), the dynamic system (19-20) becomes

$$\dot{k} \;=\; \gamma\left[(1-a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{(\sigma-1)}} - c - (n+\delta)k \tag{27}$$

$$\dot{c} \;=\; (c/\theta)[\gamma a\left[a + (1-a)k^{\frac{-(\sigma-1)}{\sigma}}\right]^{\frac{1}{(\sigma-1)}} - \delta - \rho] \tag{28}$$

The steady-state values, (23), (24), for an extensive set of **CD** and **CES** parameter cases are exhibited in **Table 1**. The steady state values of Table 1 are as is well-known the *saddle points* depicted below in Figure 1.1

Transitional Dynamics within the Standard Optimal Saving Model

Table 1. Parametrizations for optimal one-sector growth models: CD and CES with CRRA

Parameter values - steady state models							Model characteristics									
case	ρ	n	δ	γ	a	σ	κ	c(κ)	f(κ)	MP$_L$	f(κ)/κ	f '(κ)	ε_κ	K/Y	s(κ)	$\rho+\delta$
1	0.050	0.02	0.05	1.0	0.20	1.0	2.378	1.023	1.189	0.951	0.500	0.100	0.200	2.000	0.140	0.100
2	0.050	0.02	0.05	1.0	0.25	1.0	3.393	1.119	1.357	1.018	0.400	0.100	0.250	2.500	0.175	0.100
3	0.050	0.02	0.05	1.0	0.40	1.0	10.079	1.814	2.520	1.512	0.250	0.100	0.400	4.000	0.280	0.100
4	0.050	0.02	0.05	3.0	0.40	1.0	62.898	11.321	15.724	9.435	0.250	0.100	0.400	4.000	0.280	0.100
5	0.070	0.02	0.05	1.0	0.40	1.0	7.438	1.710	2.231	1.339	0.300	0.120	0.400	3.333	0.233	0.120
6	0.075	0.02	0.08	1.0	0.40	1.0	4.855	1.396	1.881	1.129	0.388	0.155	0.400	2.581	0.258	0.155
7	0.100	0.02	0.08	1.0	0.40	1.0	3.784	1.325	1.703	1.022	0.450	0.180	0.400	2.222	0.222	0.180
8	0.120	0.02	0.08	1.0	0.40	1.0	3.175	1.270	1.587	0.952	0.500	0.200	0.400	2.000	0.200	0.200
9	0.100	0.02	0.08	2.0	0.40	1.0	12.014	4.205	5.406	3.244	0.450	0.180	0.400	2.222	0.222	0.180
10	0.050	0.02	0.08	0.3	0.40	1.0	0.875	0.197	0.284	0.171	0.325	0.130	0.400	3.077	0.308	0.130
11	0.050	0.02	0.08	1.0	0.60	1.0	45.764	5.339	9.915	3.966	0.217	0.130	0.600	4.615	0.461	0.130
12	0.050	0.02	0.08	1.0	0.40	1.0	6.509	1.464	2.115	1.269	0.325	0.130	0.400	3.077	0.308	0.130
13	0.150	0.02	0.05	1.0	0.60	1.0	15.588	4.105	5.196	2.078	0.333	0.200	0.600	3.000	0.210	0.200
1	0.050	0.02	0.05	1.0	0.25	0.5	1.775	0.999	1.123	0.945	0.632	0.100	0.158	1.581	0.111	0.100
2	0.050	0.02	0.05	1.0	0.40	0.5	2.667	1.146	1.333	1.067	0.500	0.100	0.200	2.000	0.140	0.100
3	0.050	0.02	0.05	1.0	0.60	0.5	4.624	1.564	1.888	1.425	0.408	0.100	0.245	2.449	0.172	0.100
4	0.050	0.02	0.05	3.0	0.60	0.5	9.107	5.802	6.439	5.529	0.707	0.100	0.141	1.414	0.099	0.100
5	0.075	0.02	0.05	1.0	0.40	1.5	53.718	5.624	9.384	2.669	0.175	0.125	0.716	5.724	0.400	0.125
6	0.075	0.02	0.05	1.0	0.60	1.2	820.885	67.504	124.966	22.356	0.152	0.125	0.821	6.569	0.461	0.125
7	0.100	0.02	0.08	0.3	0.40	1.5	0.385	0.174	0.212	0.143	0.551	0.180	0.327	1.814	0.181	0.180
8	0.100	0.02	0.08	0.2	0.40	1.5	0.162	0.094	0.110	0.080	0.675	0.180	0.267	1.481	0.148	0.180
9	0.060	0.02	0.05	1.0	0.40	1.7	108201.257	4478.988	12053.076	150.937	0.111	0.110	0.987	8.977	0.631	0.110
Parameters - persistent growth models							Limits for $k \to \infty$									
1	0.100	0.02	0.05	1.0	0.60	1.5	∞	∞	∞	∞	0.216	0.216	1.000	4.630	0.324	0.150
2	0.100	0.02	0.05	1.0	0.40	2.0	∞	∞	∞	∞	0.160	0.160	1.000	6.250	0.438	0.150
3	0.060	0.02	0.08	1.0	0.40	3.0	∞	∞	∞	∞	0.253	0.253	1.000	3.952	0.395	0.140
4	0.070	0.02	0.08	1.0	0.40	7.0	∞	∞	∞	∞	0.343	0.343	1.000	2.915	0.292	0.150
5	0.080	0.02	0.05	1.0	0.40	3.0	∞	∞	∞	∞	0.253	0.253	1.000	3.952	0.395	0.130

3 Persistent growth: Solutions - Phase Portrait

With no critical points of (19-20), persistent per capita growth will prevail.

Assumption 1. *Technology : The per capita function $f(k)$, (5), has the continuity and differentiability properties as follows,*

$$(i) \quad f(k) \in C^0([0,\infty[) \cap C^1(]0,\infty[), \qquad (ii) \quad f(0) \geq 0. \tag{29}$$

It is further assumed that

$$(iii) \quad \forall k > 0 : \quad f'(k) > \delta + \rho. \tag{30}$$

For a concave function with $f(k) \to \infty$ as $k \to \infty$, (30) becomes, cf. (6),

$$(iv) \quad \lim_{k\to\infty} f'(k) \equiv \underline{b} > \delta + \rho. \tag{31}$$

It follows from (29)–(30) or (29) and (31) that the dynamic system (19)–(20) has *no stationary solutions* in *closed* first quadrant, $\overline{\Re}^2_+$, [except possibly for $(0,0)$], and that the *positive k-axis* ($c = 0$) is a *trajectory* (orbit).

With regard to Ramsey (optimal) saving, it has been incumbent on us to obtain *sufficient conditions* – applicable to a general GNP-*function*, $f(k)$ and a *general utility* function $u(c)$ – that ensure *persistent* per *capita* growth.

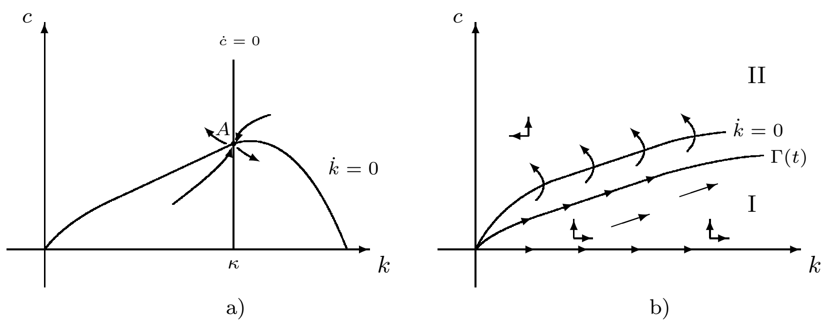

Figure 1: Dynamics with optimal saving

To characterize such situation, we give

Theorem 1. *With optimal (Ramsey) saving, persistent (endogenous) per capita growth is obtained if the concave per capita function $f(k)$ and the intertemporal substitution elasticity $\eta(c)$ of $u(c)$ or the rate of time preference ρ satisfies, respectively*

$$\lim_{k\to\infty} f'(k) \equiv \underline{b} > \rho + \delta, \tag{32}$$

$$\overline{\eta} = \sup_{c>0} \eta(c) < \frac{\underline{b} - (n+\delta)}{\underline{b} - (\rho+\delta)} \Leftrightarrow \rho > (\underline{b} - \delta)\left[\frac{\overline{\eta} - 1}{\overline{\eta}}\right] + \frac{n}{\overline{\eta}} \tag{33}$$

The two sufficient conditions, (32), (33), ensure the existence – below the isocline $\dot{k} = h(k,c) = 0$ – of a separator, $\Gamma(t)$, the particular orbit, depicted in Figure 1.2:

$$\Gamma(t) \equiv [k^*(t),\, c^*(t)], \tag{34}$$

This existence of the separator, (34) depicted in Figure 1.2, is required (necessary) for persistent (endogenous) per capita growth.
With nondecreasing utility functions, $u(c)$, satisfying (33), this separator *(34) in Figure 1.2 is the* unique optimal *solution satisfying (9)–(12) or (19) - (21) .*

Proof.

Lemma 1. *Separator Existence. If there exists a number $d > 0$ and a $k_0 > 0$ such that $\forall k \geq k_0$, $\forall c > 0$:*

$$f(k)/k - (n+\delta) - \eta(c)\,[f'(k) - (\delta+\rho)] \geq d, \tag{35}$$

then there exists to the system (19)–(20) an orbit – $\Gamma(t) \equiv [k^*(t), c^*(t)]$, $t \in \Re$, such that $k^*(t) \to \infty$, $c^*(t) \to \infty$, as $t \to \infty$ – which separates the first quadrant into two regions I and II in Figure 1.2.
A solution (orbit), $[k(t), c(t)]$, $t \in \Re$, starting in the lower region I has the same behavior as $\Gamma(t)$ for $t \to \infty$ - whereas an orbit, $[k(t), c(t)]$, $t \in \Re$, starting in the upper region II eventually meets the c-axis, $k = 0$.

Proof. Consider the region $W_\alpha = \{(k, c) \mid 0 \leq c \leq \alpha k \wedge k \geq k_0\}$, where α is a positive constant chosen such that W_α becomes *positively invariant*, cf. Figure 2.

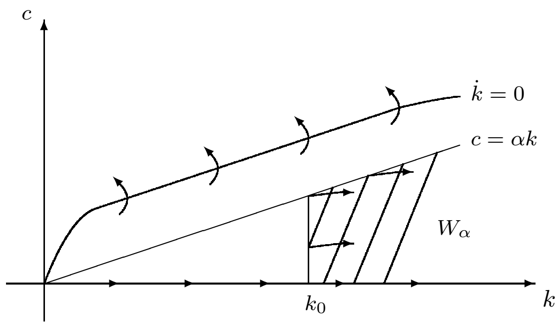

Figure 2: The positive invariant region, $\mathbf{W}_\alpha$, with endogenous (persistent) per capita growth

Since the *vector field* (19)–(20) is directed inward on the line $k = k_0$, and since the positive k-axis is a trajectory, the region W_α is positive *invariant* iff the vector field points inward on the line $c = \alpha k$, $(k > k_0)$. Since the inward pointing normal to the line $c = \alpha k$ is $(\alpha, -1)$, we require

$$\alpha h(k, \alpha k) - g(k, \alpha k) > 0, \quad \text{for } k \geq k_0. \tag{36}$$

Inserting the expressions for h, (19), and g, (20), into (36), and simplifying, we find the *requirement* :

$$\alpha < f(k)/k - (n+\delta) - \eta(\alpha k)\,[f'(k) - (\delta+\rho)] \equiv R(k) \quad \text{for } k \geq k_0. \tag{37}$$

A positively invariant region W_α (with some $\alpha > 0$) exists iff $R(k)$, (37), is bounded from below by a positive constant. By (35), we have for $k \geq k_0$

$$R(k) \;\geq\; d \;>\; 0. \tag{38}$$

Choose α to be any *positive* constant *less* than d. Then W_α is positively invariant. For any orbit in the *open* first quadrant, $\Re^2_+$, we have by (30) that $\dot{c} > 0$. Accordingly, it follows that any orbit starting in W_α must satisfy $k(t) \to \infty$, $c(t) \to \infty$, as $t \to \infty$. Any orbit in $\Re^2_+$ must either behave as just characterized (class I) or cross the $\dot{k} = 0$ nullcline (class II). In the latter case, the orbit will meet the c-axis eventually, since otherwise $c(t) \to \infty$ and $k(t) \to k_\varepsilon$ as $t \to \infty$, for some $k_\varepsilon \geq 0$.

For t sufficiently large, i.e., k sufficiently small, say $0 < k \leq k^*$, we have from the sytem (19)–(20)

$$-\frac{dc}{dk} = -\frac{\dot{c}}{\dot{k}} = \frac{\eta(c)[f'(k) - \rho - \delta]}{-(1/c)[f(k) - (n+\delta)k] + 1} \leq \frac{\bar{\eta} f'(k)}{1/2} \equiv a f'(k), \tag{39}$$

where $\bar{\eta}$ is an upper bound for $\eta(c)$. This immediately rules out $k_\varepsilon > 0$ since then $-\frac{dc}{dk}$ would be bounded above by a constant. If $k_\varepsilon = 0$, we find by integrating from k to k^* for $0 < k < k^*$

$$c(k) \leq c(k^*) + a f(k^*), \quad 0 < k \leq k^*, \tag{40}$$

contradicting $c(k) \to \infty$ as $k \to 0^+$.

To get the *separating orbit*, Γ, consider a curve, C, connecting $(k, c) = (1, 0)$ with $(k, c) = (0, 1)$ and intersecting the nullcline $\dot{k} = 0$ once (think of a circle). We can write $C = C_I \cup C_{II} \cup \{(1, 0), (0, 1)\}$ where C_I and C_{II} consists of the points through which pass orbits of class I and II, respectively. C_{II} must be an *open* and *connected* part of C. Since C_I and C_{II} are both non-empty, $C_I \cup \{(1, 0)\}$ must be *closed*. The separating orbit Γ goes through the end point of C_I.$\square$

A powerful and useful extension of Lemma 1A is the *simpler separator condition* stated in:

Corollary 1. *Separator, Sufficient Condition. With the assumptions of (29) and (31), the sufficient conditions for existence of the separating orbit,* $\Gamma(t)$, *cf. Lemma 1, is given by the restriction*

$$\overline{\eta} \; = \; \sup_{c>0} \eta(c) \; < \; \frac{\underline{b} - (n+\delta)}{\underline{b} - (\rho+\delta)}, \tag{41}$$

where $\overline{\eta}$ *is the upper bound of the intertemporal substitution elasticity* $\eta(c)$ *of* $u(c)$ *and where* $\underline{b}$ *is given in (31).*

Proof. Since $f(k) \to \infty$ as $k \to \infty$, it follows from (31) and l'Hospital that $y(k)/k \to \underline{b}$ as $k \to \infty$. Thus for any number $\varepsilon > 0$, there exists a number, k_ε, such that for $k > k_\varepsilon$, we have, cf. (35)

$$\begin{aligned} & f(k)/k - (n+\delta) - \eta(c)\,[f'(k) - (\delta+\rho)] \\ \geq \;& \underline{b} - \varepsilon - (n+\delta) - \sup_{c>0} \eta(c)\,[\underline{b} + \varepsilon - (\delta+\rho)] \\ = \;& \underline{b} - (n+\delta) - \overline{\eta}\,[\underline{b} - (\delta+\rho)] - \varepsilon(1+\overline{\eta}) \equiv d. \end{aligned} \tag{42}$$

By assumption (31) and (41), the sum of the first three terms of d is positive. Thus, by choosing $\varepsilon > 0$ sufficiently small, also d is positive. Thus, the requirements of **Lemma 1** are satisfied, and hence the separating orbit $\Gamma(t)$ exists.$\square$

Lemma 2. *Convergence. If any chosen utility function* $u(c)$ *is assumed to satisfy*

$$\forall c \geq c_0 \geq 0: \quad 0 \leq u(c) \leq Ac, \tag{43}$$

where A *is any positive constant, then the convergence of the integral, (3),* $V = \int_0^\infty u[c(t)]e^{-(\rho-n)t}dt$ *for a solution* $[k(t), c(t)]$ *to the system (19)–(20) is assured, if*

$$\overline{\eta} = \sup_{c>0} \eta(c) < \frac{\rho - n}{\underline{b} - (\delta + \rho)}, \tag{44}$$

where $\underline{b}$ *is given by (31).*

Proof. Let $\varepsilon > 0$. Choose k_ε such that $f'(k) < \underline{b} + \varepsilon$ for $k \geq k_\varepsilon$. Choose t_ε such that $k(t) > k_\varepsilon$ for $t > t_\varepsilon$. Then from (20), we find

$$\forall t \geq t_\varepsilon: \quad \dot{c} < c \sup_{c>0} \eta(c)[(\underline{b} + \varepsilon) - (\delta + \rho)] \equiv \alpha c. \tag{45}$$

It follows from (45) that $c(t) \leq c(t_\varepsilon)e^{\alpha(t-t_\varepsilon)}$, and hence, cf. (43)

$$u[c(t)]e^{-(\rho-n)t} \leq Ac(t)e^{-(\rho-n)t} \leq Ac(t_\varepsilon)e^{-\alpha t_\varepsilon}e^{-(\rho-n)-\alpha)t}. \tag{46}$$

Thus the convergence of the integral U is assured, if $\alpha < \rho - n$, which by (45) says

$$\overline{\eta} = \sup_{c>0} \eta(c) < \frac{\rho - n}{\underline{b} + \varepsilon - (\delta + \rho)}. \tag{47}$$

With $\varepsilon > 0$ chosen sufficiently small, the requirement (47) can be satisfied by the condition (44). □

Remark 1. The condition (44) is stronger that (41) of Corollary 1, since by assumption (31), we have

$$\frac{\rho - n}{\underline{b} - (\rho + \delta)} < \frac{\underline{b} - (n + \delta)}{\underline{b} - (\rho + \delta)}. \tag{48}$$

In short, the *existence* of *separating orbit* Γ is *assured* by $\overline{\eta} < 1$, but $\overline{\eta} < 1$ does not itself ensure convergence of U. However, for *isoelastic* $u(c)$ with $\forall c$, $\eta(c) = \eta$ (constant), it can be verified that the *convergence* of U is in fact also ensured by the *existence* condition of the separating orbit, (41).

Indeed, with *constant* intertemporal elasticity of substitution, the *separating orbit* in Figure 1.2 is the *optimal solution* $[k^*(t), c^*(t)]$, satisfying the transversality condition; see hereto Gandolfo (1996, p. 390).
It remains to be seen how (44) may be relaxed for general *non-isoelastic* $u(c)$ in Ramsey problems. □

Condition (32) is analogous to with a low ρ - taking over the role of a large s. But (32) is not always enough to ensure persistent growth, as (33) is also needed. However, if $u(c)$ always has $\eta(c) \leq 1, \forall c > 0$, then (33) is automatically satisfied (Hall 1988) estimated that η is much below unity, $0.1 < \eta < 0.4.$, irrespective of the size of $\rho > 0$. If $\eta(c) > 1$, then ρ must be large enough to satisfy (33).

4 Numerical examples - CES and Cob Douglas technologies and CRRA preferences

In this section, we provide some numerical examples to illustrate the optimal system dynamics (19-20) and Theorem 1. These examples are provided for two types of production functions, namely for CES and extended Cobb-Douglas.

For CRRA (25), the two conditions, (32), (33), for the existence of both *endogenous* growth and *optimal solutions* [a finte integral of V, (9), and the separator, (34)] can be summarized as:

$$\underline{b} - \delta > \rho > (\underline{b} - \delta)(1-\theta) + n\theta \tag{49}$$

The long-run (asymptotic) saving rate (s*) is given by

$$s^* = 1 - x^*/(z^* = \underline{b}) = 1 - \frac{\rho - (\underline{b} - \delta)(1-\theta) - n\theta}{\underline{b}\,\theta} \tag{50}$$

In the CES case (26), the isocline $\dot{k} = 0$ of the dynamic system (19-20) with (49) becomes

$$\dot{k} = 0 \;\Leftrightarrow\; c = \gamma\left[(1-a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{(\sigma-1)}} - (n+\delta)k. \tag{51}$$

4.1 Baseline CES

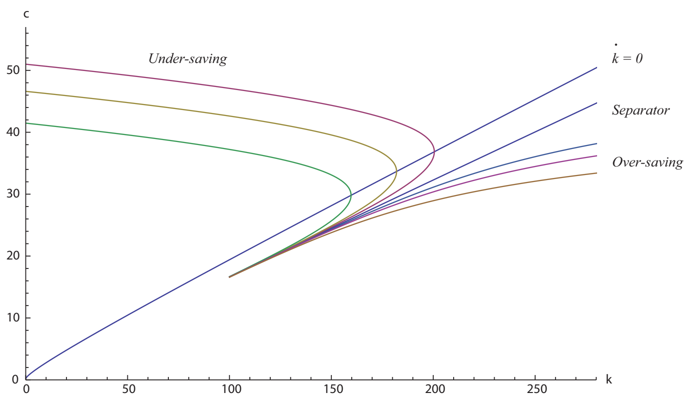

Figure 3: The phase portrait for the original space. CES-baseline.

We calculate the parametric time paths by the time elimination method suggested by Mulligan and Sala-i-Martin (1993). This method takes the (known) steady state as its fixed point and proceeds backwards in the time along the stable saddle path to discover the route from the initial and to the asymptotic state. In system (27-51), however, the asymptotic growth rate is positive indicating that the necessary fixed point is not available. Therefore, (27-51) is transformed into another system with a well-defined steady state by adopting new variables $z = f(k)/k$ and $x = c/k$ (see also Barro and Sala-i-Martin 2003, p. 230-231).[1]

$$MP_K(k) = f'(k) = \gamma a \left[a + (1-a)k^{\frac{-(\sigma-1)}{\sigma}}\right]^{\frac{1}{(\sigma-1)}} = \gamma^{\frac{\sigma-1}{\sigma}} a \, [AP_K(k)]^{\frac{1}{\sigma}} \quad (52)$$

The transformed system is:

$$\begin{aligned}
\dot{z} &= -z[1 - \gamma^{\frac{\sigma-1}{\sigma}} a z^{\frac{1-\sigma}{\sigma}}][z - x - (n+\delta)] & (53)\\
&= -z[1 - (z/A)^{-\Psi}][z - x - (n+\delta)] \\
\dot{x} &= x[x - z(1 - \frac{1}{\theta}\gamma^{\frac{\sigma-1}{\sigma}} a z^{\frac{1-\sigma}{\sigma}}) + n + \delta - \frac{\delta+\rho}{\theta}] & (54)\\
&= x[x - z(1 - \frac{1}{\theta}(z/A)^{-\Psi}) + n + \delta - \frac{\delta+\rho}{\theta}] & (55)
\end{aligned}$$

with the isoclines and the steady state

$$\begin{aligned}
\dot{z} &= 0 \Leftrightarrow x = \gamma a^{\frac{\sigma}{\sigma-1}} \quad or \; x = z - (n+\delta) & (56)\\
\dot{x} &= 0 \Leftrightarrow x = z(1 - \frac{1}{\theta}\gamma^{\frac{\sigma-1}{\sigma}} a z^{\frac{1-\sigma}{\sigma}}) - (n+\delta) + \frac{\delta+\rho}{\theta} & (57)\\
z^* &= \gamma a^{\frac{\sigma}{\sigma-1}} & (58)\\
x^* &= z^*(1 - \frac{1}{\theta}) - (n+\delta) + \frac{\delta+\rho}{\theta} & (59)\\
&= (z^* - \delta)(1 - \frac{1}{\theta}) + \frac{\rho}{\theta} - n
\end{aligned}$$

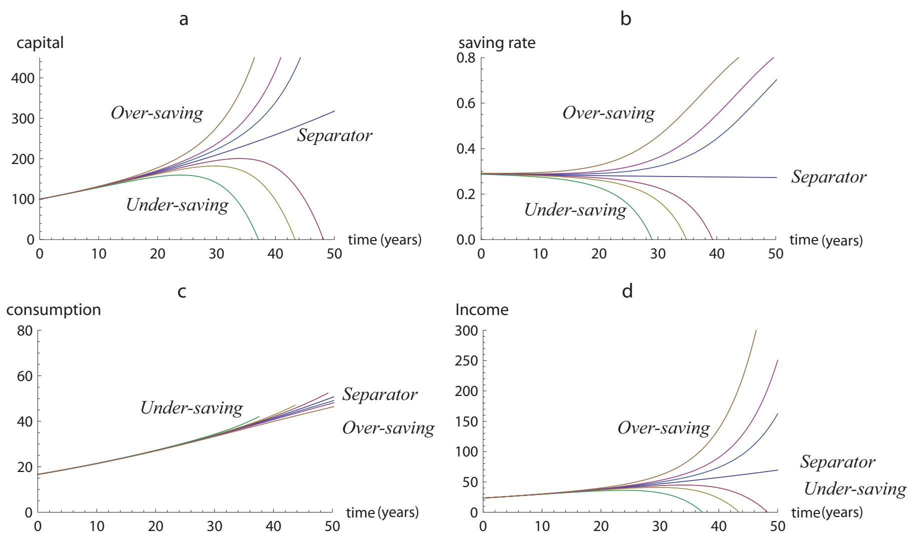

Figure 4: The time paths for saving and capital. CES-baseline.

Consider now parameters of values $\gamma = 1.6$, $a = 0.5$, $\sigma = 1.5$, $\delta = 0.03$, $\rho = 0.12$, $n = 0.01$, and $\theta = 5$. For this CES-baseline parameter set we have

[1] But compare 55 with (4.66) in Barro and Sala-i-Martin (2003).

$x^* = 0.15$, $z^* = 0.20$, and $s^* = 0.25$. Figures 3 and below illustrate the CES-baseline set, while the phase portrait for the the transformed z, x-space is given in the appendix. Figure 3 shows the system (27-51) in the k, c-space.[2] The separator appears in the latter as the counterpart of the stable saddle path in the former.

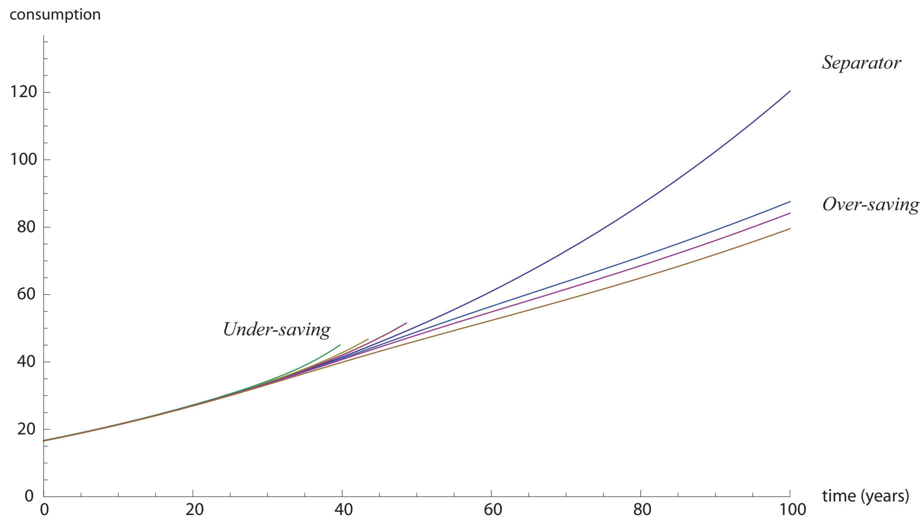

Figure 5: The time paths for consumption, short run and long run. CES-baseline.

Figure 4 illustrates the time paths for capital accumulation and saving, showing how over-saving soon wedges the capital stock above its optimal path (separator) and capital starts to accumulate very fast as almost all output is devoted to saving. On the other hand, under-saving may dilute the capital stock close to zero in some $35 - 50$ years. Figure 5 for consumption thus illustrates that under-saving collapses consumption but, on the other hand, over-saving causes minor deviations from the optimal path in the short run. In the long run, however, over-saving starts to bite; the rightmost panel of Figure 5 shows that such consumption which lies $0.001 - 0.005\%$ below its optimal level initially, causes it to fall some 50% below the optimum in hundred years. Nevertheless, the interesting conclusion is that there exist paths in the neighborhood of the separator which, in terms of short-run consumption, generate very similar outcomes than the optimal path.

4.2 Sensitivity analysis for CES

Consider now variations in the CES-baseline parameter set above. Several types of sensitivity analysis are possible, including variations of one parameter at a time as well as simultaneous variations of several parameters. In this paper, we provide two new parameter sets, each of which generates exactly the same asymptotic outcome $x^* = 0.15$, $z^* = 0.20$, and $s^* = 0.25$ as the CES-baseline. In both parameter sets, only two parameters are varied. In the first set, denoted

[2] See also Barro and Sala-i-Martin (2003, fig. 4.3) and Gandolfo (1997, fig. 22.3).

as $low - \theta - high - \rho$, we decrease the value of θ from 5 to 2 and increase the value of ρ from 0.12 to 0.15. In the second set ($low - \sigma - high - \gamma$), we decrease the value of σ from 1.5 to 1.2 and increase the value of γ from 1.60 to 12.80.

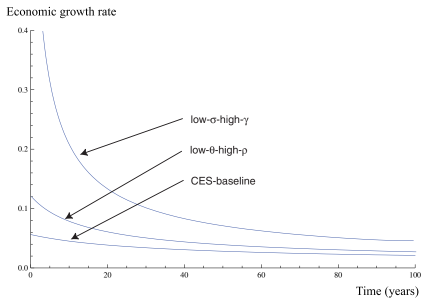

Figure 6: The time paths for the economic growth rates and saving rates.

Figure 6 (left panel) illustrating the optimal economic growth rates for each set, shows that considerable differences exist as the CES-baseline generates low and stable growth rates while $low - \sigma - high - \gamma$ generates excessive and unstable ones and those for $low - \theta - high - \rho$ lie between these two extremes. Figure 6 (right panel) also shows the saving rates with analogous differences: for $low - \sigma - high - \gamma$ the saving rates are high and decreasing but for CES-baseline low and increasing while $low - \theta - high - \rho$ again generates the intermediate values. The time paths of consumption and capital naturally respond to these findings. For $low - \sigma - high - \gamma$, for example, excessive capital accumulation takes place initially. Later, capital accumulation levels off but it exceeds that in the CES-baseline and $low - \theta - high - \rho$ even in the long run.

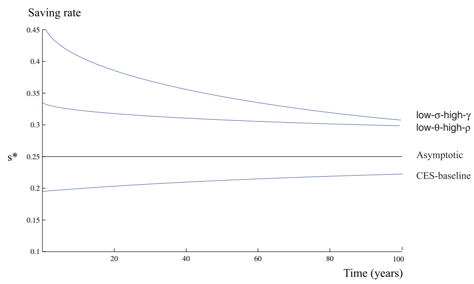

Figure 7: The time paths for the

To summarize, one can say that, in spite of asymptotic similarity, different parameter sets can produce very different temporal outcomes. Therefore, depending upon the values of the parameters, the endogenous growth model discussed here is able to explain several historical patterns of capital accumulation, saving and consumption.

4.3 Extended Cobb-Douglas

Consider the extended Cobb-Douglas production function $Y = F(L, K)$ and the associated average and marginal products of capital:

$$\begin{aligned} Y &= F(L,K) = AK + BK^{\alpha} \cdot L^{1-\alpha}\,, \\ AP_L(k) &= Y/L \;=\; AP_L(k) = f(k) = Ak + Bk^{\alpha}\,, \\ MP_K(k) &= f'(k) = A + \alpha Bk^{\alpha-1}\,, \end{aligned} \tag{60}$$

where $A > 0\,;\, B > 0\,;\, 0 < \alpha < 1$. The dynamics of the Ramsay system discussed here are

$$\dot{k} = f(k) - c - (n+\delta)k \tag{61}$$

$$\dot{c} = (c/\theta)[f'(k) - \sigma - \rho] \tag{62}$$

In the extended Cobb-Douglas case this implies

$$\dot{k} = Ak + Bk^{\alpha} - c - (n+\delta)k \tag{63}$$

$$\dot{c} = (c/\theta)[A + \alpha Bk^{\alpha-1} - \sigma - \rho] \tag{64}$$

The isocline for k is:

$$\dot{k} = 0 \Leftrightarrow c = Ak + Bk^{\alpha} - (n+\delta)k \tag{65}$$

Given the transformations $z = f(k)/k$ and $x = c/k$, the transformed system becomes:

$$\dot{z} = -(1-\alpha)(z-A)(z-x-n-\delta)\,, \tag{66}$$

$$\dot{x} = x[(x-\varphi) - \frac{\theta-\alpha}{\theta} \cdot (z-A)\,, \tag{67}$$

where $\varphi = (A-\delta) \cdot (\theta - 1)/\theta + \rho/\theta - n$. The isoclines and the steady state of the transformed system are:

$$\dot{z} = 0 \Leftrightarrow x = z - n - \delta \tag{68}$$

$$\dot{x} = 0 \Leftrightarrow x = \varphi + (z-A) - \frac{\alpha}{\theta} \cdot [\Big(\frac{z}{A}\Big)^{1-\varphi} - 1] \tag{69}$$

$$z^* = A \tag{70}$$

$$x^* = \varphi \tag{71}$$

To compare CES and Cobb-Douglas results, consider Cobb-Douglas parameters $A = 0.20$, $B = 1.5$, and $\alpha = 0.5$ and keep other parameters as in the CES-baseline ($\delta = 0.03$, $\rho = 0.12$, $n = 0.01$, and $\theta = 5$). This parameter set generates exactly the same asymptotic values $x^* = 0.15$, $z^* = 0.20$, and $s^* = 0.25$ as the CES-baseline. Furthermore, Figure 8 shows that the Cobb-Douglas phase portrait is practically identical to its CES counterpart in Figure 3, and the time paths for capital and consumption in Figure 9 closely resemble those in Figures 5 and , indicating that, for suitable parameter sets, one can generate identical results for these two production functions.

They were both as in natural sciences concerned with exact *parametric laws* of *description* and *motion*.

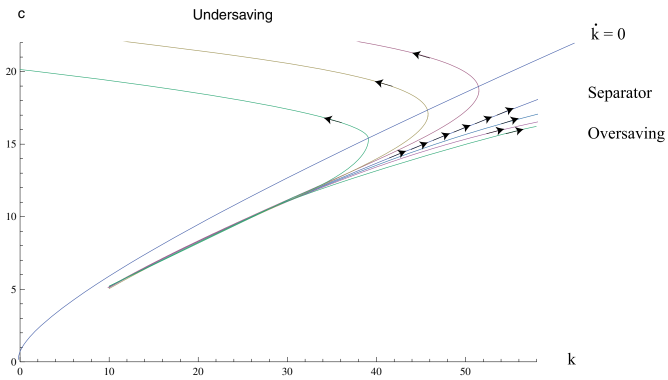

Figure 8: The phase portrait for the original space. Cobb-Douglas production function.

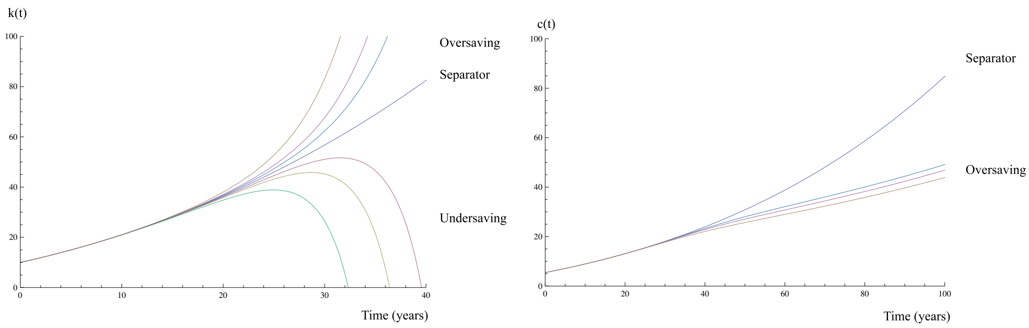

Figure 9: The time paths for capital accumulation and consumption. Cobb-Douglas production function.

5 Final Comments and Conclusion

In this paper, we explore the conditions sufficient for persistent growth in an otherwise standard Ramsey model. Newbery (2008) has stated: "Ramsey´s formulation of the problem served as a model for almost all subsequent studies of optimal economic growth, and, with the critical addition of a growing population, might have created neoclassical growth theory about 30 years before Solow´s (1956) contribution."

In contrast to the common wisdom, the Ramsey model is capable of generating endogenous growth. We also show that, in the case of persistent growth, much of the ingenious phase diagram can be preserved, even though the saddle path and steady state structure disappears because the optimal path shows up as a separator in the phase portrait. Beneath this separator, over-saving diminishes consumption, ultimately leading to a sub-optimal situation where all incomes are saved. On the other hand, above the separator under-saving suddenly collapses the economy as its productive capital vanishes to zero while over-saving causes minor deviations from the optimal path in the short run.

The numerical examples for CRRA preferences and CES and Cobb Douglas technologies show that under-saving seriously damages the economy, even in a markedly short time of $20 - 45$ years. On the other hand, over-saving seekps consumption at almost optimal level in the short run. In the long run, however, over-saving starts to bite; our results shows that such saving, which lies $0.001 - 0.005\%$ below its optimal level initially, causes consumption to fall some 50% below the optimum in a hundred years. Nevertheless, the interesting conclusion is that there exist paths in the neighborhood of the separator which, in terms of short-run consumption, generate very similar outcomes than the optimal path.

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Transformed space

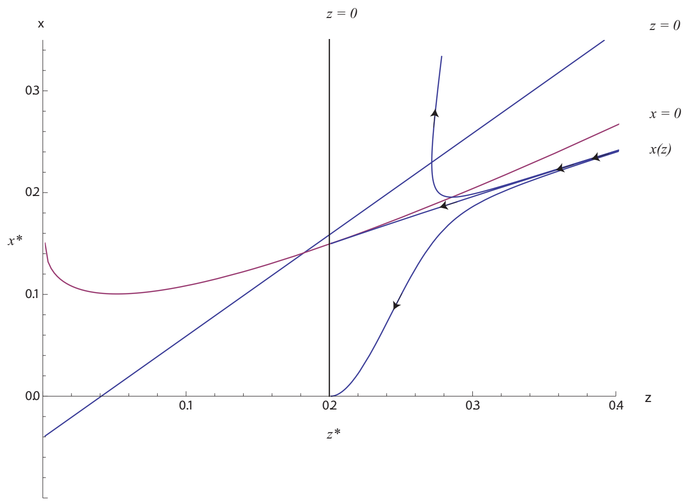

Figure A: The phase portrait for the transformed space. CES-baseline.