# TRANSVERSAL MAPPINGS AND PROJECTIONS OF INVARIANT MEASURES ON MANIFOLDS

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Academic dissertation

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I dedicate this thesis to my wife and our beloved daughter Aino.

Helsinki, September 2012 Risto Hovila

## LIST OF INCLUDED ARTICLES

This thesis consists of an introductory part and the following four articles:

[A] R. HOVILA, *The dimension spectrum of projected measures on Riemann manifolds*, Ann. Acad. Sci. Fenn. Math. **35** (2010), 595–608.

[B] R. HOVILA, E. JÄRVENPÄÄ, M. JÄRVENPÄÄ AND F. LEDRAPPIER, *Singularity of projections of 2-dimensional measures invariant under the geodesic flow*, Comm. Math. Phys. **312** (2012), 127–136.

[C] R. HOVILA, E. JÄRVENPÄÄ, M. JÄRVENPÄÄ AND F. LEDRAPPIER, *Besicovitch-Federer projection theorem and geodesic flows on Riemann surfaces*, to appear in Geom. Dedicata.

[D] R. HOVILA, *Transversality of isotropic projections, unrectifiability and Heisenberg groups*, submitted.

The author of this dissertation has actively taken part in the research of the joint papers [B] and [C].

## 1. Overview

The dimensional properties of projections of sets and measures have been studied for decades. One of the most fundamental results in this area is the projection theorem of J. M. Marstrand. In his paper [Mar] from 1954 he proved that if a planar set has Hausdorff dimension at most one, then its dimension is preserved under typical projections onto lines. If the set has Hausdorff dimension greater than one, then typical projections have positive measure. The influence of this result to geometric measure theory cannot be overestimated and also the majority of the research done in this thesis has its roots in Marstrand's theorem. However, Marstrand was not the first one to study orthogonal projections from a measure theoretic point of view. In 1939 A. S. Besicovitch [Be] studied projectional properties of unrectifiable sets and discovered that a set in the plane is purely unrectifiable, if and only if almost all of its projections onto lines have zero measure. This result has also had a great impact on the study of geometric measure theory and is perhaps the second most influential theorem for the research done in this thesis. The paper [Be] of Besicovitch is one of the three papers in which he studied properties of one dimensional sets in the plane in great detail. These papers may be regarded as the foundation of geometric measure theory. H. Federer [Fe1] generalized Besicovitch's theorem to higher dimensional setting. In this generality the theorem states that a set in $\mathbb{R}^n$ is purely $m$-unrectifiable, if and only if almost all of its projections onto $m$-planes have zero Lebesgue measure. Nowadays this result is often called the Besicovitch-Federer projection theorem.

There are numerous extensions and generalizations of Marstrand's theorem. First the development of the field was quite slow, as one could imagine from the fact that the theorem of Besicovitch concerns only one dimensional sets, and it took 15 years before Marstrand proved his theorem for sets with dimension other than one. The reason for this is that back then the Hausdorff dimension did not have the importance that it nowadays has in many fields of mathematics.

After [Mar], the development still was not very rapid. The original proof of Marstrand was geometric, and in 1968 R. Kaufman [Ka] reproved the theorem using potential theoretic methods. He also provided a dimension estimate for the set of exceptional directions. P. Mattila [Mat1] generalized Marstrand's theorem and Kaufman's exceptional set estimate to higher dimensions.

To state Marstrand's theorem, we need to fix some notation. We denote by $G(n,m)$ the Grassmann manifold of all $m$-dimensional linear subspaces of $\mathbb{R}^n$, by $\gamma_{n,m}$ the natural orthogonally invariant probability measure on $G(n,m)$, and by $P_V : \mathbb{R}^n \to V$ the orthogonal projection onto an $m$-plane $V \in G(n,m)$. For any $m$-plane $V$, we denote by $\mathcal{L}^m$ the Lebesgue measure on $V$. The $s$-dimensional Hausdorff measure will be denoted by $\mathcal{H}^s$ and the Hausdorff dimension by $\dim_{\mathrm{H}}$.

With this notation, Mattila's generalization of Marstrand's theorem may be formulated as follows:

**Theorem 1.1.** *Let $E \subset \mathbb{R}^n$ be a Borel set with $\dim_{\mathrm{H}} E = s$.*

*(i) If $s \leq m$, then $\dim_{\mathrm{H}} P_V(E) = s$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$.*
*(ii) If $s > m$, then $\mathcal{L}^m(P_V(E)) > 0$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$.*

There exists an analogous dimension preservation principle for measures in $\mathbb{R}^n$, as discovered by Kaufman [Ka], Mattila [Mat2], X. Hu and J. Taylor [HT] and K. Falconer and Mattila [FM]. It may be stated as follows: Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$. Then

$$\dim_{\mathrm{H}} P_{V*}\mu = \min\{\dim_{\mathrm{H}} \mu, m\}$$

for $\gamma_{n,m}$-almost all $V \in G(n,m)$, where $P_{V*}\mu$ denotes the push-forward of the measure $\mu$ under the mapping $P_V$, defined by $P_{V*}\mu(A) = \mu(P_V^{-1}(A))$ for $A \subset V$. Moreover, if $\dim_{\mathrm{H}} \mu > m$, then for $\gamma_{n,m}$-almost all $V \in G(n,m)$ the projected measure $P_{V*}\mu$ is absolutely continuous with respect to the Lebesgue measure, $P_{V*}\mu \ll \mathcal{L}^m$.

All the dimension results above concern only the Hausdorff dimension. However, there are also other notions of dimension for which there exist Marstrand-type theorems. The geometry of the packing dimension, $\dim_{\mathrm{p}}$, turns out to be less regular than that of the Hausdorff dimension. The packing dimension of projections of sets and measures is studied for example by M. Järvenpää [Jä], Falconer and Mattila [FM] and Falconer and J. D. Howroyd [FH1], [FH2]. The following result for measures is from [FH2]: Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$. Then

$$\dim_{\mathrm{p}} P_{V_*}\mu = \dim_m \mu$$

for $\gamma_{n,m}$-almost all $V \in G(n,m)$, where $\dim_m$ is a packing-type dimension defined by using a certain $m$-dimensional kernel. Even in the case $\dim_{\mathrm{p}} \mu \leq m$, it can happen that $\dim_m \mu < \dim_{\mathrm{p}} \mu$, which means that the dimension can decrease under typical projections, but still the packing dimension of the projected measure is the same for almost all projections.

The behaviour of the $q$-dimension of projections of measures has been studied by B. R. Hunt and V. Y. Kaloshin [HK], Falconer and T. C. O'Neil [FO] and E. and M. Järvenpää [JJ]. For $1 < q \leq 2$, the lower $q$-dimension behaves similarly to the Hausdorff dimension under projections, while the behaviour of the upper $q$-dimension is similar to that of the packing dimension. These facts will be discussed in greater detail in Section 3. The $q$-dimension of images of measures under projection-like mappings is studied in article [A] and this research will also be discussed in Section 3.

Orthogonal projections are not the only parametrized families of mappings for which there exist Marstrand-type theorems. The generalized projection formalism of Y. Peres and W. Schlag has turned out to be extremely useful in many situations. In [PS] Peres and Schlag study parametrized families of transversal mappings and prove dimension estimates for exceptional sets. In particular, Marstrand's projection theorem follows from their results. Transversal projection families will be discussed in Section 2. In this thesis transversality is used in articles [A], [C] and [D]. Transversal families also satisfy the Besicovitch-Federer projection theorem, which is shown in article [C].

One thing that is common for all theorems mentioned above is that they are "almost all" -type results, which give no information about any specific projection. However, in 2003 F. Ledrappier and E. Lindenstrauss discovered that similar methods used to prove theorems for typical projections can be used to study one specific projection. In [LL] they consider measures on the unit tangent bundle $T^1M$ of a Riemann surface $M$, and the natural projection $\Pi : T^1M \to M$. They show that if $\mu$ is a Radon probability measure on $T^1M$, which is invariant under the geodesic flow, then its projection $\Pi_*\mu$ satisfies a Marstrand-type theorem. Although there is only one projection, E. and M. Järvenpää and M. Leikas [JJLe] have showed that the situation may still be interpreted as a projection problem for a transversal family of mappings, which makes the existence of such a result little less surprising. Projections of measures invariant under the geodesic flow are discussed in Sections 2–5 and in articles [A], [B] and [C] of this thesis.

As mentioned above, Marstrand's theorem and its generalizations give information about the typical behaviour of dimensions of sets and measures under projections. When considering some lower dimensional subspace of the parameter space and the projections corresponding to these parameters, the dimension may drop. E. Järvenpää, M. Järvenpää and T. Keleti [JJK] and D. Oberlin [Ob] have studied subfamilies of orthogonal projections and obtained dimension estimates for the projections in this case. However, in some cases the dimension can still be preserved. In article [D] we consider one particular subfamily of orthogonal projections. We show that the family of isotropic projections is transversal, which in particular implies that it satisfies Marstrand's projection theorem. This family also relates to horizontal projections of the Heisenberg group, so the result may be used to obtain information on these projections.

For more detailed information on topics related to the behaviour of Hausdorff dimension under projection-type mappings, see the survey of Mattila [Mat4] and references therein.

## 2. Transversal families of mappings and projections on manifolds

The first thing that might come to mind when talking about projections are the orthogonal ones. However, beginning from the 90's there has been much research showing that many of the projection theorems for orthogonal projections hold also for more general families of mappings. In the paper [PS] from 2000, Peres and Schlag study extensively the properties of transversal families of mappings and obtain dimension preservation results analogous to Marstrand's projection theorem and also estimates on the dimensions of exceptional sets.

2.1. **Transversality.** First we introduce the definition:

**Definition 2.1.** Let $\Lambda \subset \mathbb{R}^l$ be open. A family of maps $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ is *transversal*, if it satisfies the following conditions for each compact set $K \subset \mathbb{R}^n$:

(1) The mapping $P : \Lambda \times K \to \mathbb{R}^m$, $(\lambda, x) \mapsto P_\lambda(x)$, is continuous with respect to $x$ and twice differentiable with respect to $\lambda$.

(2) For $j = 1, 2$ there exist constants $C_j$ such that the derivatives with respect to $\lambda$ satisfy

$$\|D^j_\lambda P(\lambda, x)\| \leq C_j \text{ for all } (\lambda, x) \in \Lambda \times K.$$

(3) For all $\lambda \in \Lambda$ and $x$, $y \in K$ with $x \neq y$, define

$$T_{x,y}(\lambda) := \frac{P_\lambda(x) - P_\lambda(y)}{|x - y|}.$$

Then there exists a constant $C_T > 0$ such that the property

$$|T_{x,y}(\lambda)| \leq C_T$$

implies that

$$\det\left(D_\lambda T_{x,y}(\lambda)\,(D_\lambda T_{x,y}(\lambda))^T\right) \geq C_T^2.$$

(4) There exists a constant $C_L$ such that

$$\|D^2_\lambda T_{x,y}(\lambda)\| \leq C_L$$

for all $\lambda \in \Lambda$ and $x, y \in K$ with $x \neq y$.

In articles [C] and [D] we also require $P$ to be continuously differentiable as a mapping $\Lambda \times K \to \mathbb{R}^m$.

The definition above is not in the same generality as the definition used by Peres and Schlag in [PS]. They use a more general notion of $\beta$-transversality. The essential difference is that they have an additional parameter $\beta \in [0, 1)$, and the condition (3) in the definition is replaced by

(3′) For all $\lambda \in \Lambda$ and $x$, $y \in K$ with $x \neq y$, there exists a constant $C_\beta > 0$ such that the property

$$|T_{x,y}(\lambda)| \leq C_\beta |x - y|^\beta$$

implies that

$$\det\left(D_\lambda T_{x,y}(\lambda)\,(D_\lambda T_{x,y}(\lambda))^T\right) \geq C_\beta^2 |x - y|^{2\beta}.$$

Our notion of transversality then corresponds to the case $\beta = 0$. Moreover, in their definition the space $\mathbb{R}^n$ is replaced by a general compact metric space, but since the results of this thesis do not concern families of this generality, from now on by saying that a family is transversal, we refer to families satisfying Definition 2.1.

2.2. **Projections of measures invariant under the geodesic flow.** The results concerning the behaviour of dimension under projections are very often such that they do not give information on any specific projection, instead they state that something is true for almost all of them. However, in 2003 Ledrappier and Lindenstrauss [LL] studied measures on the unit tangent bundle of a Riemann surface and discovered that methods similar to those used in the classical projection theorems can also be applied to study the natural projection from the unit tangent bundle to the base manifold. They proved the following theorem:

**Theorem 2.2.** *Let $T^1M$ be the the unit tangent bundle of a compact Riemann surface $M$. If $\mu$ is a Radon probability measure on $T^1M$ which is invariant under the geodesic flow, then*

$$\dim_{\mathrm{H}} \Pi_*\mu = \dim_{\mathrm{H}} \mu, \textit{ if } \dim_{\mathrm{H}} \mu \leq 2 \quad \textit{and}$$

$$\Pi_*\mu \ll \mathcal{H}^2|_M, \textit{ if } \dim_{\mathrm{H}} \mu > 2,$$

*where $\Pi : T^1M \to M$, $\Pi(x,v) = x$, is the natural projection and $\mathcal{H}^2|_M$ denotes the 2-dimensional Hausdorff measure on $M$.*

At first sight the theorem might seem a bit surprising, due to the fact that instead of a family of projections, Ledrappier and Lindenstrauss obtained a dimension preservation result for one particular projection. However, it turns out that the situation can be interpreted as a projection problem for a family of mappings.

In [JJLe] E. and M. Järvenpää together with Leikas reproved Theorem 2.2 using the generalized projection formalism of Peres and Schlag. They showed that each point $p \in T^1M$ has a neighbourhood $U \subset T^1M$ such that there are bi-Lipschitz mappings $\psi_1 : U \to I^3$ and $\psi_2 : I^2 \to \Pi(U)$ and a smooth mapping $P : I^3 \to I^2$ so that

$$\Pi|_U = \psi_2 \circ P \circ \psi_1,$$

where $I \subset \mathbb{R}$ is the open unit interval. The mapping $P$ is defined by $P(y_1, y_2, t) = (P_t(y_1, y_2), t)$, where $\{P_t : I^2 \to I\}_{t \in I}$ is a transversal family of smooth mappings. With their new approach to the problem, they also showed that there is no corresponding dimension preservation result in higher dimensional manifolds and answered a question of Ledrappier and Lindenstrauss concerning the fractional derivatives of the projected measure.

In [FH2], Falconer and Howroyd showed that although the packing dimension of a measure is not necessarily preserved under orthogonal projections, the dimension of the projected measure is constant under almost all projections. Combining the techniques of [FH2] and [JJLe], Leikas [Le] showed that the result of Falconer and Howroyd holds for transversal families of mappings between manifolds and measures on them and computed the packing dimension of the natural projection of a probability measure which is invariant under the geodesic flow on the unit tangent bundle of a two-dimensional Riemann manifold.

## 3. Dimension spectrum of projected measures

3.1. **Background.** *The dimension spectrum* of a measure is a parametrized family of dimensions, called the $q$-dimensions of the measure. We begin with the definition:

Let $\mu$ be a Borel probability measure with compact support on a metric space $(X,d)$. For every $q \neq 1$ the lower and upper *$q$-dimensions* are defined by

$$\underline{D}_q(\mu) = \liminf_{r\to 0} \frac{\log \int \mu(B(x,r))^{q-1}\, d\mu(x)}{(q-1)\log r}$$

and

$$\overline{D}_q(\mu) = \limsup_{r\to 0} \frac{\log \int \mu(B(x,r))^{q-1}\, d\mu(x)}{(q-1)\log r},$$

where $B(x,r)$ denotes the open ball with centre at $x \in X$ and radius $r > 0$.

There exists a Marstrand-type projection theorem also for these notions of dimension. Using potential theoretic methods Hunt and Kaloshin [HK] showed that the lower $q$-dimension behaves similarly to the Hausdorff dimension under projections, provided that $1 < q \leq 2$. Falconer and O'Neil [FO] reproved this result by studying certain appropriately defined convolution kernels. By these methods they also showed that the behaviour of the upper $q$-dimension under projections is similar to that of the packing dimension, that is, the dimension may decrease, but it is constant under almost all projections. In [JJ] E. and M. Järvenpää also considered the upper $q$-dimension using potential theoretic methods and they presented an alternative proof for the behaviour of the upper $q$-dimension.

Defining the *modified $q$-dimensions* by

$$\underline{D}_q^k(\mu) = \liminf_{r\to 0} \frac{\log \int F_\mu^k(x,r)^{q-1}\, d\mu(x)}{(q-1)\log r}$$

and

$$\overline{D}_q^k(\mu) = \limsup_{r\to 0} \frac{\log \int F_\mu^k(x,r)^{q-1}\, d\mu(x)}{(q-1)\log r},$$

where $F_\mu^k(x,r) = \int_X \min\{1, r^k d(x,y)^{-k}\}\, d\mu(y)$, the above results may be summarized as follows:

**Theorem 3.1.** *Let $\mu$ be a compactly supported Borel regular probability measure on $\mathbb{R}^n$ and let $1 \leq m \leq n$. Then for all $q > 1$ and all $V \in G(n,m)$*

$$\underline{D}_q(\mu_V) \leq \underline{D}_q^m(\mu) = \min\left\{m, \underline{D}_q(\mu)\right\} \quad \textit{and} \quad \overline{D}_q(\mu_V) \leq \overline{D}_q^m(\mu).$$

*Moreover, for all $1 < q \leq 2$ and for $\gamma_{n,m}$-almost all $V \in G(n,m)$*

$$\underline{D}_q(\mu_V) = \underline{D}_q^m(\mu) = \min\left\{m, \underline{D}_q(\mu)\right\} \quad \textit{and} \quad \overline{D}_q(\mu_V) = \overline{D}_q^m(\mu).$$

*Here $\mu_V := P_{V*}\mu$.*

In view of the generalization of Marstrand's theorem to transversal mappings in [PS] and the corresponding result for the packing dimension in [Le], the natural question would be whether such a generalization exists for Theorem 3.1. Furthermore, from the results in [LL], [JJLe] and [Le] concerning the behaviour of Hausdorff and packing dimensions of the natural projection of a measure invariant under the geodesic flow, arises the question of how does the dimension spectrum behave in this projection. These questions are answered in the first article of this thesis.

3.2. **Article [A].** In article [A] we generalize Theorem 3.1 to parametrized families of transversal mappings between smooth manifolds and measures on them. Before stating the main result we fix some notation.

Let $L, N$ and $M$ be smooth Riemann manifolds with dimensions $l, n$ and $m$, respectively, such that $l, n \geq m$. Suppose that $P : L \times N \to M, P(\lambda, x) = P_\lambda(x)$ satisfies Definition 2.1 locally, that is, for all points $\lambda \in L$ and $x \in M$ there are coordinate neighbourhoods of $\lambda$ and $x$ on which the transversality conditions hold.

**Theorem 3.2.** *Let $L$, $N$, $M$ and $P : L \times N \to M$ be as above, and let $\mu$ be a compactly supported Borel regular probability measure on $N$.*

1) *For all $q > 1$ and all $\lambda \in L$*
$$\underline{D}_q(\mu_\lambda) \leq \underline{D}_q^m(\mu) = \min\{m, \underline{D}_q(\mu)\} \quad \textit{and} \quad \overline{D}_q(\mu_\lambda) \leq \overline{D}_q^m(\mu).$$
2) *For all $1 < q \leq 2$ and $\mathcal{H}^l$-almost all $\lambda \in L$*
$$\underline{D}_q(\mu_\lambda) = \underline{D}_q^m(\mu) = \min\{m, \underline{D}_q(\mu)\} \quad \textit{and} \quad \overline{D}_q(\mu_\lambda) = \overline{D}_q^m(\mu).$$

*Here $\mu_\lambda = P_{\lambda *}\mu$.*

The proof uses similar methods to those of Falconer and O'Neil in [FO]. Generalizing every step in their proof and circumventing some technical problems eventually leads to a proof for Theorem 3.2.

3.3. **Applications.** Using Theorem 3.2, we can prove the following theorem, already presented in [A], concerning the dimension spectrum of the natural projection of a measure which is invariant under the geodesic flow.

**Theorem 3.3.** *Let $M$ be a smooth, compact Riemann surface, let $\mu$ be a Borel regular probability measure on the unit tangent bundle $T^1M$, and let $\Pi : T^1M \to M$ be the natural projection. If $\mu$ is invariant under the geodesic flow, then for $1 < q \leq 2$,*
$$\underline{D}_q(\Pi_*\mu) = \underline{D}_q^2(\mu) \quad \textit{and} \quad \overline{D}_q(\Pi_*\mu) = \overline{D}_q^2(\mu).$$

As one can see, the result for the $q$-dimension is similar to those concerning the Hausdorff dimension and the packing dimension and the proof uses many ideas from [JJLe] and [Le]. However, we need to use slightly different methods for the proof, because the proofs in [JJLe] and [Le] use certain results concerning the dimension of slices of a measure. While such results hold for the Hausdorff and the packing dimension, the $q$-dimensions do not have the corresponding properties.

## 4. Invariant measure with singular projection

4.1. **Background.** Let $M$ be a compact surface with possibly variable negative curvature and let $\varphi = \varphi_t$, $t \in \mathbb{R}$, be the geodesic flow on the unit tangent bundle $T^1M$. In Theorem 2.2 it is shown that the natural projection of a $\varphi$-invariant measure of dimension greater than 2 is absolutely continuous with respect to the Lebesgue measure. When the dimension of the original measure is exactly 2, the projected measure is also known to have dimension 2, but it remains unclear whether in this case the projected measure is always absolutely continuous or not. In article [B] we study this question. It turns out that in this case the projected measure can be singular with respect to the Lebesgue measure.

4.2. **Quantum Unique Ergodicity.** Motivation for the research in the article [B] comes from Quantum Unique Ergodicity (QUE) conjecture, which can be described as follows. Let $\psi_n$ be a sequence of orthonormal eigenfunctions of the Laplacian on $M$. The associated eigenvalues converge to infinity and the problem of QUE is to describe the possible weak* limits of the probability measures with density $|\psi_n|^2$ as $n$ tends to infinity. The conjecture is that the only limit is the normalized Lebesgue measure. This was verified by Lindenstrauss [Lin] for arithmetic hyperbolic surfaces in the case when $\psi_n$'s are also eigenfunctions of the Hecke operators. In the case of a general hyperbolic surface, or of more general eigenfunctions, if any, on an arithmetic surface, N. Anantharaman and S. Nonnenmacher [AN] proved that any limit is the projection to $M$ of a $\varphi$-invariant measure with dimension at least 2. G. Rivière [Ri] showed that this property is still true on surfaces with variable negative curvature. Our result shows that one still cannot conclude from the results in [AN] and [Ri] that any weak* limit is nonsingular, which is a weak form of the QUE conjecture.

4.3. **Article [B].** The main theorem of article [B] reads as follows.

**Theorem 4.1.** *For any compact surface $M$ whose curvature is everywhere negative, there exists an ergodic $\varphi$-invariant measure $m$ on $T^1M$ such that $\Pi_* m$ has Hausdorff dimension equal to 2 and is singular with respect to the Lebesgue measure on $M$.*

In order to prove Theorem 4.1, it suffices to construct an ergodic $\varphi$-invariant measure $m$ with dimension 2 and a measurable set $A \subset T^1M$ with the properties that $m(A^c) = 0$ and $\mathcal{H}^2(A) = 0$, where $A^c = T^1M \setminus A$. Since the natural projection is 1-Lipschitz, it cannot increase the Hausdorff measure, and so the projected measure $\Pi_* m$ is singular with respect to the Lebesgue measure $\mathcal{L}^2$ on $M$. But, since the Hausdorff dimension of a $\varphi$-invariant measure is preserved under the projection by Theorem 2.2, the Hausdorff dimension of $\Pi_* m$ is 2, as claimed.

To obtain such a measure, we use the symbolic coding of the geodesic flow. In this symbolic coding we may construct Markov measures, each of which corresponds to a $\varphi$-invariant measure on $T^1M$. First we prove that there are many Markov measures for which the resulting invariant measure has Hausdorff dimension 2. After this we show that among these Markov measures there exists a measure $\mu$ such that for the corresponding $\varphi$-invariant measure $m$,

$$\limsup_{\epsilon\to 0} \frac{m\left(B((x,v),\epsilon)\right)}{\epsilon^2} = +\infty \tag{4.1}$$

for $m$-almost all $(x,v) \in T^1M$, implying that $m$ is singular with respect to $\mathcal{H}^2$. To show that the fluctuations of the measures of the balls $B((x,v),\epsilon)$ are large enough, we use a vector valued almost sure invariance principle for hyperbolic dynamical systems proved by I. Melbourne and M. Nicol in [MN]. We define observables $(X_n^u, Y_n^u)$ and $(X_n^s, Y_n^s)$ so that $X_n$ describes the mass and $Y_n$ describes the radius. Here $u$ stands for the unstable foliation and $s$ for the stable one. The almost sure invariance principle then implies that these observables can be approximated by 2-dimensional Brownian motions. The main technical difficulty is to show that for an appropriate choice of the Markov measure $\mu$ the covariance matrix is nonsingular. Using properties of the Brownian motion we show that the fluctuations of the observables are large enough, which implies the equation (4.1).

## 5. Absolutely continuous measure with singular projection

5.1. **Background.** The measures constructed in article [B] are supported on the whole unit tangent bundle $T^1M$ and they are singular with respect to $\mathcal{H}^2$ on $T^1M$. When constructing measures with the property that their projections are singular, in a way it would be more satisfactory to obtain measures such that the singularity of the projections is actually due to the projection. In the setting of [B] this is not the case: the singularity is actually preserved by the projection, not generated by it. Therefore from the proof of Theorem 4.1 arises naturally the question of the existence of $\varphi$-invariant measure $m$ on the unit tangent bundle $T^1M$, which has 2-dimensional support and is absolutely continuous with respect to $\mathcal{H}^2$ on $T^1M$, but whose projection $\Pi_* m$ is singular with respect to the Lebesgue measure on $M$. In article [C] we consider this question and show that on certain classes of Riemann surfaces with constant negative curvature and with boundary there are such measures.

5.2. **Article [C].** We consider measures on the unit tangent bundle of particular Riemann surfaces called the *pair of pants*. A pair of pants $M$ is a 2-sphere minus three points endowed with a metric of constant curvature $-1$ in such a way that the boundary consists of three closed geodesics of length $a$, $b$ and $c$, called the *cuffs*, see Figure 1.

For each point $x$ in $M$, let $\Omega_x$ be the set of unit tangent vectors $v \in T_x^1M$ for which the geodesic ray $\gamma_v(t), t \geq 0$, with initial condition $(x,v)$, never meets the boundary of $M$. The set $\Omega_x$ is a Cantor set of Hausdorff dimension $\delta = \delta(a,b,c)$. The number $\delta$ is an important geometric invariant of the pair of pants $M$: it is the critical exponent of the Poincaré series of $\pi_1(M)$ and the topological entropy of the geodesic flow on $T^1M$ (cf. [Su2]).

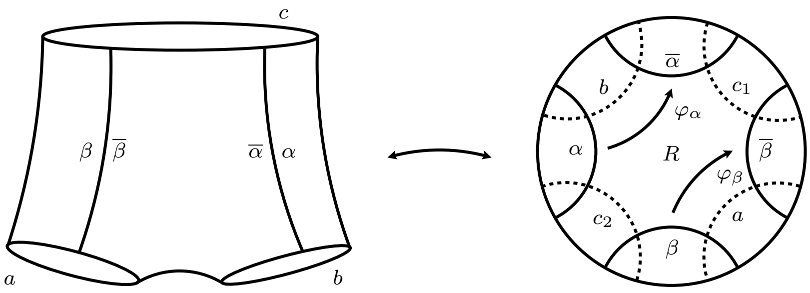


FIGURE 1. Pair of pants and its representation in the unit disc.

We are interested in the set

$$C(M) := \{x \in M \mid \text{ there exists } v \in T^1_x M \text{ such that } v \in \Omega_x \text{ and } -v \in \Omega_x\}. \tag{5.1}$$

In other words, $C(M)$ is the set of points in complete geodesics in $M$. Let

$$D(M) := \{(x, v) \in T^1 M \mid x \in C(M), v \in \Omega_x, -v \in \Omega_x\} \tag{5.2}$$

be the subset of $T^1 M$ where the geodesic flow is defined for all $t \in \mathbb{R}$. Clearly, $\Pi(D(M)) = C(M)$.

The following theorem is the main result of the article [C].

**Theorem 5.1.** *With the above notation,*

*a)* $\mathcal{L}^2(C(M)) > 0$, *provided that* $\delta > 1/2$ *and*

*b)* $\dim_{\mathrm{H}} C(M) = 1 + 2\delta$ *and* $\mathcal{L}^2(C(M)) = 0$, *provided that* $\delta \leq 1/2$.

The Hausdorff dimension of the set $D(M)$ is $1 + 2\delta$, so most of the above theorem follows already from the projection theorem 2.2. The new part of the result is when $\delta$ is exactly $1/2$. In that case it is known that $\dim_{\mathrm{H}} C(M) = 2$, and we sharpen this by proving that $C(M)$ is Lebesgue negligible.

The main ingredient in the proof is the Besicovitch-Federer projection theorem for transversal families of mappings.

**Theorem 5.2.** *Let* $E \subset \mathbb{R}^n$ *be* $\mathcal{H}^m$*-measurable with* $\mathcal{H}^m(E) < \infty$. *Assume that* $\Lambda \subset \mathbb{R}^l$ *is open and* $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ *is a transversal family of maps. Then* $E$ *is purely* $m$*-unrectifiable, if and only if* $\mathcal{H}^m(P_\lambda(E)) = 0$ *for* $\mathcal{L}^l$*-almost all* $\lambda \in \Lambda$.

The proof of Theorem 5.2 is similar to the classical proof found for example in [Fe2] and [Mat3], but naturally some modifications are needed to prove this more general version.

As mentioned in Section 2.2, E. and M. Järvenpää together with Leikas have showed that there exists a transversal family of mappings so that the properties of the natural projection can be studied using this family. Representing the surface $M$ in the unit disc (see e.g. [Se]), it may be shown that when $\delta = 1/2$, the set

$D(M)$ is locally of the form $E \times I$, where $E \subset I^2$ is a purely 1-unrectifiable set and $I$ is the unit interval. This fact together with the discussion in Section 2.2 and Theorem 5.2 implies that the set $C(M) = \Pi(D(M))$ is Lebesgue null.

As a corollary of Theorem 5.1, one can show that there exists a $\varphi$-invariant measure $m$ on the unit tangent bundle $T^1M$ with the following properties: both $m$ and its support $\operatorname{spt} m = D(M)$ have Hausdorff dimension 2 and $m$ is absolutely continuous with respect to $\mathcal{H}^2$ on $T^1M$, but $\mathcal{L}^2(\operatorname{spt}(\Pi_* m)) = 0$. Indeed, the invariant measure constructed in [Su1, Section 4] (called the Bowen-Margulis-Patterson-Sullivan measure) has these properties.

5.3. **Applications.** Theorem 5.2 seems to have applications also other than the study of $\varphi$-invariant measures. The concept of rectifiability comes up in many connections in geometric measure theory and the theorem gives new tools to study this notion.

One application, studied in article [D], is the behaviour of unrectifiable sets under isotropic projections. These projections also relate to projections in Heisenberg groups, so Theorem 5.2 can also be used in this context.

## 6. Isotropic projections

6.1. **Background.** In 2011 the study of projections took yet another turn when Z. M. Balogh, E. Durand Cartagena, K. Fässler, Mattila and J. T. Tyson initiated the study of projections in Heisenberg groups. In [BCFMT] they prove Marstrand-type projection theorems for projections in the first Heisenberg group and in [BFMT] Balogh, Fässler, Mattila and Tyson prove analogues of these results in general Heisenberg groups. There are two kinds of projections in Heisenberg groups, horizontal and vertical ones. The Horizontal projections correspond to certain Euclidean projections, while the vertical ones are much more complicated to handle. This is why only partial results for the vertical projections are obtained in the aforementioned papers. The correspondence between the horizontal and Euclidean projections involves the study of symplectic geometry, which is discussed in the next subsection.

6.2. **Symplectic geometry.** Let $M$ be a manifold of dimension $2n$. A *symplectic form* on $M$ is a closed non-degenerate 2-form on $M$. The *standard form* $\omega$ on $\mathbb{R}^{2n}$ is defined by

$$\omega(x, y) = \sum_{i=1}^{n} x_{i+n} y_i - x_i y_{i+n} = (Jx \mid y),$$

where $(\cdot \mid \cdot)$ is the Euclidean inner product on $\mathbb{R}^{2n}$ and $J$ is the $2n \times 2n$-matrix

$$J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}.$$

Here $I_{n\times n}$ denotes the identity matrix of size $n \times n$. By a well-known theorem of Darboux, every symplectic form on $M$ is locally diffeomorphic to the standard form $\omega$ on $\mathbb{R}^{2n}$. Furthermore, every symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega)$. Below we work only on $\mathbb{R}^{2n}$ equipped with the standard form $\omega$.

For a linear subspace $V \subset \mathbb{R}^{2n}$, we define its *symplectic orthogonal* $V^\omega$ by

$$V^\omega = \{w \mid \omega(w, v) = 0 \text{ for all } v \in V\}.$$

A linear subspace $V$ is said to be *isotropic*, if $V \subset V^\omega$, and *Lagrangian*, if $V = V^\omega$. A subspace $V$ can be Lagrangian only when $\dim V = n$. For integers $0 < m \leq n$, we define the *isotropic Grassmannian* $G_h(n, m)$ by

$$G_h(n, m) = \{V \in G(2n, m) \mid V \text{ is an isotropic subspace of } \mathbb{R}^{2n}\}.$$

In the case $m = n$, $G_h(n, n)$ is called the *Lagrangian Grassmannian*. The isotropic Grassmannian $G_h(n, m)$ is a smooth manifold of dimension $2nm - \frac{m(3m-1)}{2}$, so for $m > 1$ it is a submanifold of $G(2n, m)$ with positive codimension. For $m = 1$, the manifolds are the same, $G_h(1, 1) = G(2, 1)$. The isotropic Grassmannian can be endowed with a natural measure $\mu_{n,m}$ in a similar way as the usual Grassmannian is endowed with the measure $\gamma_{n,m}$, using unitary matrices instead of orthogonal ones. See [BFMT, Section 2] for more details.

The orthogonal projections $P_V : \mathbb{R}^{2n} \to V$, where $V \in G_h(n, m)$ will be called $m$-dimensional isotropic projections. For $m > 1$ the family of $m$-dimensional isotropic projections has dimension less than the dimension of $G(2n, m)$, and therefore one cannot apply standard projection theorems (e.g. Marstrand's projection theorem or Besicovitch-Federer projection theorem) to obtain dimension results for these projections.

6.3. **Heisenberg groups.** The Heisenberg group $\mathbb{H}^n$ is the unique simply connected, connected nilpotent Lie group of step two and dimension $2n+1$ with one dimensional centre. As a manifold $\mathbb{H}^n$ may be identified with $\mathbb{R}^{2n+1}$. The relation between symplectic geometry and Heisenberg groups is the following: writing points $p \in \mathbb{H}^n$ in coordinates as

$$p = (z, t) = (z_1, \ldots, z_{2n}, t) \in \mathbb{R}^{2n} \times \mathbb{R},$$

the group operation is given by

$$p * p' = (z, t) * (z', t') = (z + z', t + t' + 2\omega(z, z')),$$

where $\omega$ is the standard symplectic form on $\mathbb{R}^{2n}$.

The Heisenberg metric $\mathrm{d_H}$ of $\mathbb{H}^n$ is defined by

$$\mathrm{d_H}(p, p') := \|p^{-1} * p'\|_{\mathrm{H}}, \quad \text{where } \|p\|_{\mathrm{H}} := (\|z\|^4 + t^2)^{1/4},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{2n}$. This metric is bi-Lipschitz equivalent to the usual Carnot-Caratheodory metric on $\mathbb{H}^n$. The metric $\mathrm{d_H}$ induces the Euclidean topology, but the properties of the metric space $(\mathbb{H}^n, \mathrm{d_H})$ differ significantly from those of the underlying Euclidean space. For example, the Hausdorff dimension of $(\mathbb{H}^n, \mathrm{d_H})$ is $2n+2$. Thus, when speaking of the metric properties of $\mathbb{H}^n$, one needs to specify which metric is being used.

A *homogeneous subgroup* $\mathbb{G}$ of $\mathbb{H}^n$ is a subgroup which is closed under the intrinsic dilatations $\delta_s(z,t) = (sz, s^2t), s > 0$. There are two kinds of homogeneous subgroups of $\mathbb{H}^n$. The *horizontal subgroups* are the ones which are contained in $\mathbb{R}^{2n} \times \{0\}$ and the *vertical subgroups* are the ones which contain the $t$-axis $\{0\} \times \mathbb{R}$. Horizontal subgroups can be identified with linear subspaces of $\mathbb{R}^{2n+1}$ which are contained in $\mathbb{R}^{2n} \times \{0\}$. However not every linear subspace of this form is a horizontal subgroup, only those corresponding to isotropic subspaces $V$ of $\mathbb{R}^{2n}$.

Let $\mathbb{V} = V \times \{0\}$ be a horizontal subgroup. Consider $\mathbb{V}^\perp = V^\perp \times \mathbb{R}$, where $V^\perp$ is the orthogonal complement of $V$ in $\mathbb{R}^{2n}$. Then $\mathbb{V}^\perp$ is a vertical subgroup of $\mathbb{H}^n$, called the vertical subgroup associated to $V$. Each point $p \in \mathbb{H}^n$ can be written uniquely as

$$p = P_{\mathbb{V}^\perp}(p) * P_{\mathbb{V}}(p),$$

with $P_{\mathbb{V}^\perp}(p) \in \mathbb{V}^\perp$ and $P_{\mathbb{V}}(p) \in \mathbb{V}$. This gives rise to a well-defined *horizontal projection*

$$P_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V},\ (z,t) \mapsto P_{\mathbb{V}}(z,t) = (P_V(z), 0),$$

and a *vertical projection*

$$P_{\mathbb{V}^\perp} : \mathbb{H}^n \to \mathbb{V}^\perp,\ (z,t) \mapsto P_{\mathbb{V}^\perp}(z,t) = (P_{V^\perp}(z), t - 2\omega(P_{V^\perp}(z), P_V(z))).$$

Since there is a one-to-one correspondence between isotropic subspaces of $\mathbb{R}^{2n}$ and horizontal subgroups of $\mathbb{H}^n$, these projections may be parametrized by the isotropic Grassmannian $G_h(n,m)$. For more information on the projections of the Heisenberg group, see [BCFMT] and [BFMT].

As can be seen from the definition, the horizontal projections correspond to orthogonal projections in the underlying Euclidean space. They are Lipschitz continuous with respect to the Heisenberg and the Euclidean metric and they are also group homomorphisms. The vertical projections do not have any of these properties, which makes it more difficult to study the behaviour of the Hausdorff dimension under these projections.

6.4. **Article [D].** In article [D] we show that the family of $m$-dimensional isotropic projections in $\mathbb{R}^{2n}$ is transversal.

**Theorem 6.1.** *Let $n, m$ be integers such that $0 < m \le n$, let $G_h(n,m)$ be the submanifold of the Grassmannian $G(2n,m)$ consisting of all isotropic subspaces of $\mathbb{R}^{2n}$ and denote by $P_V : \mathbb{R}^{2n} \to V$ the orthogonal projection onto the $m$-plane $V \in G_h(n,m)$. Then the projection family $\{P_V : \mathbb{R}^{2n} \to V\}_{V \in G_h(n,m)}$ is transversal.*

To prove Theorem 6.1, we introduce local coordinates on the isotropic Grassmannian $G_h(n,m)$. The local coordinates on the Grassmannian $G(2n,m)$ are defined by using the space of all $(2n-m) \times m$ matrices, and the coordinates on the isotropic Grassmannian are obtained by restricting to a certain $2nm - \frac{m(3m-1)}{2}$ -dimensional submanifold of these matrices.

It is easy to see that isotropic projections satisfy properties (1), (2) and (4) of Definition 2.1, so all the work in the proof is in showing that the transversality property (3) holds. The proof is a bit technical, but the idea is clear; we formulate the equations in the local coordinates and show that the property (3) holds at 0. Then the transversality property follows by a continuity argument.

6.5. **Applications.** Transversality yields many corollaries, which are already presented in [D]. First of all, from Theorem 5.2 we see that unrectifiability can be characterized by isotropic projections.

**Theorem 6.2.** *Let $E \subset \mathbb{R}^{2n}$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(E) < \infty$. Then $E$ is purely $m$-unrectifiable, if and only if $\mathcal{H}^m(P_V(E)) = 0$ for $\mu_{n,m}$-almost all $V \in G_h(n,m)$.*

Applying Theorem 7.3 of [PS], we obtain the following theorem on exceptional sets of isotropic projections.

**Theorem 6.3.** *Let $n, m$ be integers such that $0 < m \leq n$ and let $E \subset \mathbb{R}^{2n}$ be a Borel set with $\dim_{\mathrm{H}} E = s$.*

(1) *If $s \leq m$, then $\dim_{\mathrm{H}}\{V \in G_h(n,m) \mid \dim_{\mathrm{H}} P_V(E) < s\} \leq 2nm - \frac{m(3m+1)}{2} + s$.*
(2) *If $s > m$, then $\dim_{\mathrm{H}}\{V \in G_h(n,m) \mid \mathcal{H}^m(P_V(E)) = 0\} \leq 2nm - \frac{3m(m-1)}{2} - s$.*

Due to the correspondence between isotropic subspaces and horizontal subgroups of the Heisenberg group, the above theorems yield results on horizontal projections in $\mathbb{H}^n$.

Since the metric properties of the Heisenberg group $\mathbb{H}^n$ differ from those of $\mathbb{R}^n$, we need to specify the metric we are using. In the following corollaries we denote the Hausdorff dimension with respect to the Heisenberg metric by $\dim_{\mathrm{H}}^{\mathrm{H}}$ and with respect to the Euclidean metric by $\dim_{\mathrm{H}}^{\mathrm{E}}$. The $m$-dimensional Hausdorff measure with respect to the Euclidean metric is denoted by $\mathcal{H}^m_{\mathrm{E}}$. The restriction of the Heisenberg metric to a horizontal subgroup coincides with the Euclidean metric, and in that case we use the same notation as before, $\mathcal{H}^m$ for the Hausdorff measure and $\dim_{\mathrm{H}}$ for the Hausdorff dimension.

The first corollary is an immediate consequence of Theorem 6.2. We denote by $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}, \pi(z,t) = z$ the projection onto the first $2n$ coordinates.

**Corollary 6.4.** *Let $E \subset \mathbb{H}^n$ be a Borel set with $\mathcal{H}^m_{\mathrm{E}}(\pi(E)) < \infty$. Then $\mathcal{H}^m(P_{\mathbb{V}}(E)) = 0$ for $\mu_{n,m}$-almost all $V \in G_h(n,m)$, if and only if $E \subset A \times \mathbb{R}$, where $A \subset \mathbb{R}^{2n}$ is purely $m$-unrectifiable in the Euclidean sense.*

The second corollary follows from Theorem 6.3 together with the observation $P_{\mathbb{V}} = P_V \circ \pi$ and the fact that for $E \subset \mathbb{H}^n$, $\dim_{\mathrm{H}}^{\mathrm{E}} \pi(E) \geq \dim_{\mathrm{H}}^{\mathrm{H}} E - 2$ (see the proof of Theorem 1.1 in [BFMT]).

**Corollary 6.5.** *Let $n, m$ be integers such that $0 < m \leq n$ and let $E \subset \mathbb{H}^n$ be a Borel set with $\dim_{\mathrm{H}}^{\mathrm{H}} E = s$.*

(1) *If $s \leq m + 2$, then*

$$\dim_{\mathrm{H}}\{V \in G_h(n,m) \mid \dim_{\mathrm{H}} P_{\mathbb{V}}(E) < s - 2\} \leq 2nm - \frac{m(3m+1)}{2} + s - 2.$$

(2) *If $s > m + 2$, then*

$$\dim_{\mathrm{H}}\{V \in G_h(n,m) \mid \mathcal{H}^m(P_{\mathbb{V}}(E)) = 0\} \leq 2nm - \frac{3m(m-1)}{2} - s + 2.$$

6.6. **Further developments.** One could ask whether there exists a Besicovitch-Federer-type projection theorem for the projections of the Heisenberg group. Corollary 6.4 has this kind of flavour to it, but it is a bit unsatisfactory since everything is formulated with respect to the Euclidean metric. Furthermore, the corollary concerns only horizontal projections and one could ask what can be said about the vertical ones. The rectifiability in $\mathbb{H}^n$ differs from the Euclidean one, so one needs to consider rectifiability with respect to the Heisenberg structure. For more discussion on rectifiability in $\mathbb{H}^n$, see for example [MSSC] and references therein.

As mentioned above, the results in [BFMT] for vertical projections are only partial, so obtaining more complete results for these projections would be a good place for future research. It seems, however, that one cannot apply the projection formalism of Peres and Schlag in this case.

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