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EFFECTIVE ELASTIC MODULI OF INHOMOGENEOUS SOLIDS BY EMBEDDED CELL MODEL*

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ABSTRACT: An embedded cell model is presented to obtain the effective elastic moduli for three-dimensional two-phase composites which is an exact analytic formula without any simplified approximation and can be expressed in an explicit form. For the different cells such as spherical inclusions and cracks surrounded by sphere and oblate ellipsoidal matrix, the effective elastic moduli are evaluated and the results are compared with those from various micromechanics models. These results show that the present model is direct, simple and efficient to deal with three-dimensional two-phase composites.

KEY WORDS: embedded cell model, effective elastic moduli, crack, inclusion

1 INTRODUCTION

One of the most basic problems in composite-materials theory is the prediction of effective elastic properties of composite materials. Equivalent-inclusion principle was presented by Eshelby[1] in 1957. Hill[2] and Budiansky[3] used the self-consistent method to evaluate the effective elastic moduli of composite materials, Budiansky and O'Connell[4] have also applied this method to cracked materials. A generalized self-consistent method was first investigated by Christensen and Lo[5] for a single-phase reinforced composite (whose matrix containing one phase of cylindrical or spherical reinforcement). This has been further developed by Huang et al.[6] to microcracked solids. Mclaughlin[7] studied the effective elastic properties by the differential scheme. Hashin[8] has used this scheme to cracked materials. The Mori-Tanaka method (Mori and Tanaka [9]) is an effective field scheme that was applied to composites by Weng[10] and a 2-D cracked solid by Benveniste[11]. The analysis is reduced to the consideration of one isolated crack placed into the undamaged matrix but subjected to a certain effective field, which, a priori, does not necessarily coincide with the remotely applied one. In addition, the bound of effective elastic moduli of composite materials is one of the classical problems in micromechanics, as has been studied by Voigt and Reuss[12]. The Voigt approximation gives the upper bounds and the Reuss approximation gives the lower bounds of the average elastic moduli. Hashin and Shtrikman[13] have proposed a variational principle for finding the upper and lower bounds on the average elastic moduli of a composite material. Recently, based on a generalized three-phase sphere model, new improved bounds

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on the effective moduli for both two-phase and three-phase hybrid composites were obtained by Dai et al.[14]. Liang et al.[15] studied the elastic constants of composites by Eshelby and Mori-Tanaka theories combined with Stiffness Averaging Method. The closed-form interacting solutions for the overall elastic moduli of an isotropic matrix with various multi-phase and multi-shape isotropic inclusions are derived by Zheng and Du[16].

Based on Eshelby's equivalence principle and Tanaka-Mori's theorem, embedded cell model is presented to obtain the effective elastic moduli which can be expressed in a simple explicit form. The effective elastic moduli are evaluated for different cells of spherical inclusions and cracks in spheres and oblate spheroids. The results are compared with those from various micromechanics models. For brevity, the symbolic notation will be used in the general theory. Bold-face, Greek letters denote the 2nd-rank tensors, and ordinary capital letters denote the 4th-rank ones. The inner product of two tensors is written such that $\boldsymbol{\sigma\varepsilon} = \sigma_{ij}\varepsilon_{ij}$, $\boldsymbol{L\varepsilon} = L_{ijkl}\varepsilon_{kl}$, and $\boldsymbol{LS} = L_{ijkl}S_{klmn}$, in terms of the indicial components.

2 EMBEDDED CELL MODEL AND BASIC FORMULAE

An ellipsoidal inclusion Ω is considered in a homogeneous infinite body. The traction at infinity is prescribed corresponding to a uniform stresses. An embedded cell model consists of an ellipsoidal inclusion Ω surrounded by a finite ellipsoid V and is embedded in an infinite matrix (Fig.1). The embedded cell model is different from the classical three-phase and two-phase models, because the embedded cell is only concerned with the matrix phase and the inclusion phase and it is embedded in an infinite matrix that the traction and the displacements of the cell's boundary are determined by uniform remote loads. The elasticity tensor of the matrix and the ellipsoidal inclusion Ω are $\boldsymbol{L}_0$ and $\boldsymbol{L}_1$, respectively. (see Fig.1)

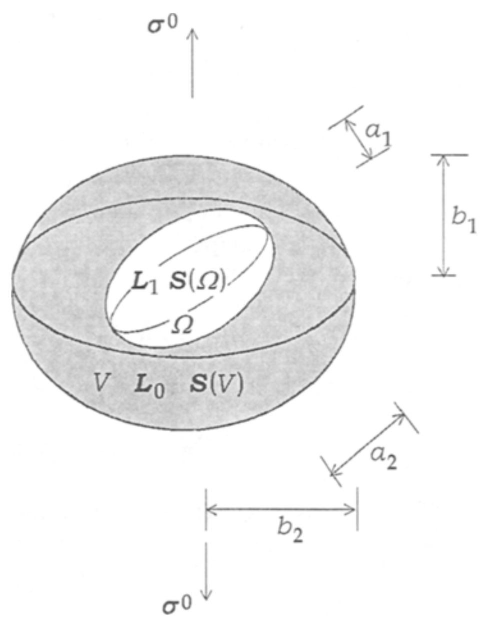


Fig.1 An embedded cell model

Suppose that the infinite homogeneous body is subjected to far field stress $\boldsymbol{\sigma}^0$, with accompanying strain $\boldsymbol{\varepsilon}^0$. The strain and stress fields in the matrix naturally differ from $\boldsymbol{\sigma}^0$ and $\boldsymbol{\varepsilon}^0$ due to the presence of the inclusion. Denoting these disturbances by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ respectively, the total stress and strain are given by

$$\boldsymbol{\sigma}_{\text{total}} = \boldsymbol{\sigma}^0 + \boldsymbol{\sigma} \tag{1}$$

$$\boldsymbol{\varepsilon}_{\text{total}} = \boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon} \tag{2}$$

The average stress and strain are

$$\overline{\boldsymbol{\sigma}} = \frac{1}{V}\int_V \boldsymbol{\sigma}_{\text{total}}\mathrm{d}V \tag{3}$$

$$\bar{\varepsilon} = \frac{1}{V}\int_V \varepsilon_{\mathrm{total}} \mathrm{d}V \tag{4}$$

By means of Eshelby's solutions and the equivalent eigenstrain principle [12,17], the stress and strain in the inclusion are uniform, given by

$$\int_\Omega \sigma \mathrm{d}V = \Omega\sigma = \Omega \boldsymbol{L}_0(\varepsilon - \varepsilon^*) = \Omega \boldsymbol{L}_0(\boldsymbol{S}(\Omega) - \boldsymbol{I})\varepsilon^* \tag{5}$$

$$\int_\Omega \varepsilon \mathrm{d}V = \Omega\varepsilon = \Omega \boldsymbol{S}(\Omega)\varepsilon^* \tag{6}$$

where ε^* is eigenstrain, $\boldsymbol{S}(\Omega)$ is Eshelby's tensor for the ellipsoid Ω. The components of the $\boldsymbol{S}$-tensor for a general isotropic spheroid are given in Appendix, assuming direction 1 to be the axis of symmetry.

In terms of Tanaka-Mori's theorem [17,18], we have

$$\int_{V-\Omega} \sigma \mathrm{d}V = \Omega \boldsymbol{L}_0(\boldsymbol{S}(V) - \boldsymbol{S}(\Omega))\varepsilon^* \tag{7}$$

$$\int_{V-\Omega} \varepsilon \mathrm{d}V = \Omega(\boldsymbol{S}(V) - \boldsymbol{S}(\Omega))\varepsilon^* \tag{8}$$

where $\boldsymbol{S}(V)$ is Eshelby's tensor for the ellipsoid V. Equations (7) and (8) show the volume integrals of the stress $\boldsymbol{\sigma}$ and the strain ε over $V - \Omega$ are independent of the absolute position of V and Ω; they depend only on the Eshelby's tensors of V and Ω; they vanish when V and Ω are similar in shape and have the same orientation.

According to the equivalent inclusion method [12,17], eigenstrain ε^* is given by

$$\varepsilon^* = -[\boldsymbol{L}_0 + (\boldsymbol{L}_1 - \boldsymbol{L}_0)\boldsymbol{S}(\Omega)]^{-1}[\boldsymbol{L}_1 - \boldsymbol{L}_0]\varepsilon^0 = -\boldsymbol{A}\varepsilon^0 \tag{9}$$

where $\boldsymbol{A}$ is the concentration factor tensor,

$$\boldsymbol{A} = [\boldsymbol{L}_0 + (\boldsymbol{L}_1 - \boldsymbol{L}_0)\boldsymbol{S}(\Omega)]^{-1}[\boldsymbol{L}_1 - \boldsymbol{L}_0] \tag{10}$$

Then, the average stress and strain of the embedded cell are

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{L}_0[\boldsymbol{I} + \rho(\boldsymbol{I} - \boldsymbol{S}(V))\boldsymbol{A}]\varepsilon^0 \tag{11}$$

$$\bar{\varepsilon} = [\boldsymbol{I} - \rho\boldsymbol{S}(V)\boldsymbol{A}]\varepsilon^0 \tag{12}$$

where ρ is the volume fraction of the inclusion, $\rho = \Omega/V$.

The effective elastic moduli of the cell, $\boldsymbol{L}$, then follow from $\bar{\boldsymbol{\sigma}} = \boldsymbol{L}\bar{\varepsilon}$, as

$$\boldsymbol{L} = \boldsymbol{L}_0\{\boldsymbol{I} + \rho\boldsymbol{A}[\boldsymbol{I} - \rho\boldsymbol{S}(V)\boldsymbol{A}]^{-1}\} \tag{13}$$

Expression (13) is an exact analytic formula for the effective elastic moduli of the embedded cell model. In Eq.(13), $\boldsymbol{S}(V)$ reflects the effect of the shape of the ellipsoid V, and $\boldsymbol{S}(\Omega)$ is relative to the shape of the inclusion Ω. So, the explicit expression (13) is simple, but quite meaningful.

When the ellipsoid V and the ellipsoidal inclusion Ω are similar in shape and have the same orientation, we have

$$\boldsymbol{S}(V) = \boldsymbol{S}(\Omega) = \boldsymbol{S} \tag{14}$$

Then the effective elastic moduli of the cell are given by

$$\boldsymbol{L} = \boldsymbol{L}_0\{\boldsymbol{I} + \rho\boldsymbol{A}[\boldsymbol{I} - \rho\boldsymbol{S}\boldsymbol{A}]^{-1}\} \tag{15}$$

where

$$\boldsymbol{A} = [\boldsymbol{L}_0 + (\boldsymbol{L}_1 - \boldsymbol{L}_0)\boldsymbol{S}]^{-1}[\boldsymbol{L}_1 - \boldsymbol{L}_0] \tag{16}$$

From (15) it is seen that the effective elastic moduli are the same as those obtained by Mori-Tanaka method[19].

For voids, by setting $\boldsymbol{L}_1 = \boldsymbol{0}$, the effective moduli are

$$\boldsymbol{L} = \boldsymbol{L}_0\{\boldsymbol{I} + \rho\boldsymbol{A}[\boldsymbol{I} - \rho\boldsymbol{S}(V)\boldsymbol{A}]^{-1}\} \tag{17}$$

where

$$\boldsymbol{A} = -[\boldsymbol{I} - \boldsymbol{S}(\Omega)]^{-1}$$

3 EXAMPLES AND RESULTS

3.1 Spherical Inclusion

When the inclusion is spherical, two aspect ratios of the ellipsoid V are considered in order to investigate the effect of the shape of the cell on the effective elastic moduli. When V is spherical (the aspect ratio is 1), $\boldsymbol{S}(V)(=\boldsymbol{S}(\Omega))$ is given in Appendix (see A3). Equation (15) readily provides the effective bulk and shear moduli of the cell

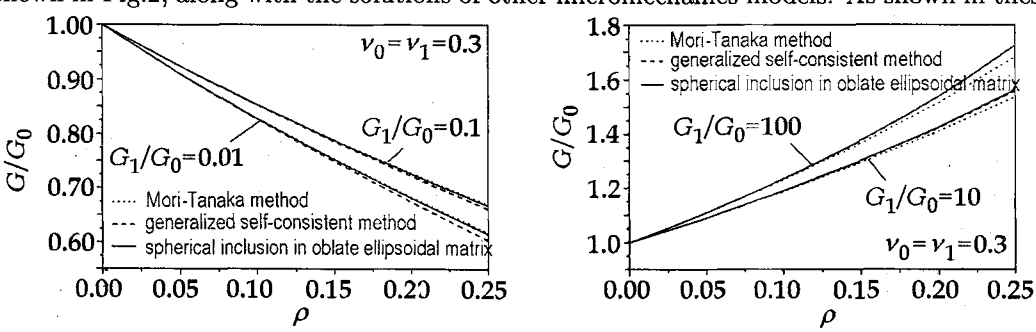
$$\frac{K}{K_0} = 1 + \frac{\rho(K_1 - K_0)}{K_0 + \alpha'(1-\rho)(K_1 - K_0)} \qquad \frac{G}{G_0} = 1 + \frac{\rho(G_1 - G_0)}{G_0 + \beta'(1-\rho)(G_1 - G_0)} \tag{18}$$

where

$$\alpha' = \frac{1}{3}\frac{1+\nu_0}{1-\nu_0} \qquad \beta' = \frac{2}{15}\frac{4-5\nu_0}{1-\nu_0}$$

where K_1 and G_1 are the bulk and shear moduli of the inclusion, respectively. K_0 and G_0,ν_0 are the bulk and shear moduli, the Poisson ratio of the matrix, respectively. They are the same as those in the Mori-Tanaka method [10].

When V is the rotative oblate spheroid, $\boldsymbol{S}(V)$ is given (in Appendix A1) for the aspect ratio $\alpha_0 = b_1/b_2 = 1/2$. The inclusion Ω is spherical, $\boldsymbol{S}(\Omega)$ can be obtained from Appendix A3. Substituting $\boldsymbol{S}(V)$ and $\boldsymbol{S}(\Omega)$ into Eq.(13), the effective moduli are obtained, as shown in Fig.2, along with the solutions of other micromechanics models. As shown in these



(a) Normalized effective shear moduli vs volume fraction ($G_1/G_0 < 1$)

(b) Normalized effective shear moduli vs volume fraction ($G_1/G_0 > 1$)

Fig.2

figures, the moduli are close to the solutions from the generalized self-consistent model and the Mori-Tanaka method, indicating that the effect of the shape of the cell on the effective elastic moduli is slight.

3.2 Penny-Shaped Crack

3.2.1 Unidirectionally Aligned Penny-Shaped Crack

The normal to the crack surface is taken as axis 1, and therefore the 2-3 plane is the isotropic plane.

The effective elastic moduli are evaluated for the cell that consists of a penny-shaped crack surrounded by an oblate spheroid matrix. We consider three aspect ratio of the oblate spheroid V, which are α_0 =1, 1/2 and 1/3, respectively. In order to calculate the effective moduli of the cell with crack, the effective moduli are firstly evaluated for the cell including the penny-shaped void ($\alpha_1 = a_1/a_2 \ll 1$). When a penny-shape void is in a sphere, $\boldsymbol{S}(V)$ and $\boldsymbol{S}(\Omega)$ are given in Appendix (see A3 and A4, respectively). Substituting $\boldsymbol{S}(V)$ and $\boldsymbol{S}(\Omega)$ into Eq.(17), the effective moduli are obtained as

$$\begin{aligned}\frac{E_{11}}{E_0} &= \left[45\pi^2{\alpha_1}^2(1+\nu_0) - 30\pi\alpha_1(3+\rho+4\nu_0\rho) + 16\,\rho(12+(-7+5\nu_0)\rho)\right]/\\ &\quad \left[45\pi^2{\alpha_1}^2(1+\nu_0) - 8\rho(21+15{\nu_0}^2(-3+\rho) - 13\rho + 2\nu_0\rho) - \right.\\ &\quad \left. 15\pi\alpha_1(6+(-7+5\nu_0+12{\nu_0}^2)\rho)\right]\end{aligned} \tag{19}$$

When α_1 is taken to approach zero, the void has turn into the crack. The volume fraction ρ is no longer an adequate parameter, a more measurable one is perhaps the crack density parameter introduced by Budiansky and O'Connell[4]. For the cell, it is defined as

$$\eta = a_2/V \tag{20}$$

then the volume fraction of the crack is

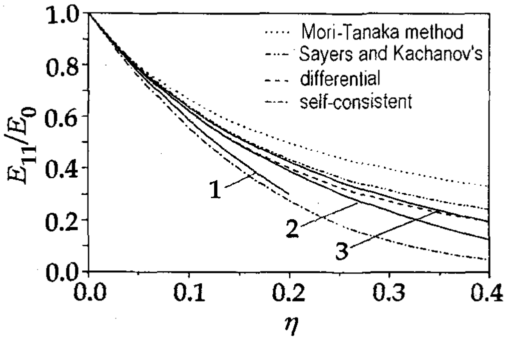
$$\rho = \frac{4}{3}\pi a_2 a_1/V = \frac{4}{3}\pi\alpha_1\eta \tag{21}$$

Substituting Eq. (21) into Eq. (19), the effective moduli of the cell including the crack are obtained by letting α_1 approach zero

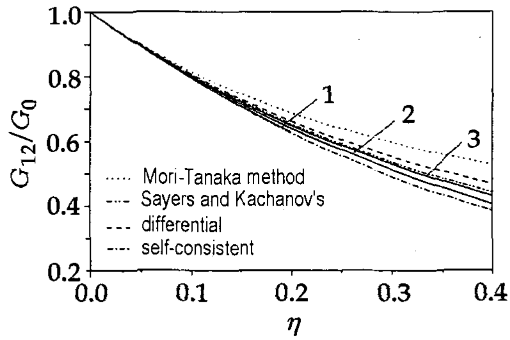
$$\frac{E_{11}}{E_0} = 1 + \frac{240\eta(1-{\nu_0}^2)}{-45+16\eta(-7+15{\nu_0}^2)} \tag{22}$$

As is stated above, the effective moduli of the cell having a crack in an oblate spheroid V are obtained in terms of the crack-density parameter when the aspect ratio of V is 1/2 and 1/3, respectively. The results are shown in Fig.3, along with the solutions of other micromechanics models.

As shown in Fig.3(a), the effect of the shape of the cell on the effective moduli is quite significant. The effective moduli are scattered for different aspect ratios of the oblate ellipsoidal matrix α_0. When α_0 approaches zero, the effective Young's moduli by the embedded cell model are the same as those of the Mori-Tanaka method. The solutions obtained by the differential scheme and Sayers and Kachanov[20] are in the scattered zone of our results. It is observed that the effect of the shapes of the cells appears not significant from Fig.3(b).



(a) Normalized effective Young's moduli vs crack density ($\nu_0 = 0.25$)
(Unidirectionally aligned penny-shaped crack)
present results: 1 crack in sphere, 2 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/2$), 3 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/3$)



(b) Normalized effective shear moduli vs crack density ($\nu_0 = 0.25$)
(Unidirectionally aligned penny-shaped crack)
present results: 1 crack in sphere, 2 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/2$), 3 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/3$)

Fig.3

3.2.2 3-D Randomly Oriented Penny-Shaped Crack

Consider the volume element, consisting of 3-D randomly oriented penny-shaped crack, see Fig.4. In a local coordinate system, the average stress and strain of a unit cell are given by

$$\left.\begin{aligned} \overline{\sigma}' &= \boldsymbol{L}_0\{\boldsymbol{I} + \rho[\boldsymbol{I} - \boldsymbol{S}(V)]\boldsymbol{A}\}\varepsilon^{0'} = \boldsymbol{L}_\sigma \varepsilon^{0'} \\ \overline{\varepsilon}' &= \{\boldsymbol{I} - \rho \boldsymbol{S}(V)\boldsymbol{A}\}\varepsilon^{0'} = \boldsymbol{L}_\varepsilon \varepsilon^{0'} \end{aligned}\right\} \tag{23}$$

where

$$\boldsymbol{A} = -[\boldsymbol{I} - \boldsymbol{S}(\Omega)]^{-1}$$

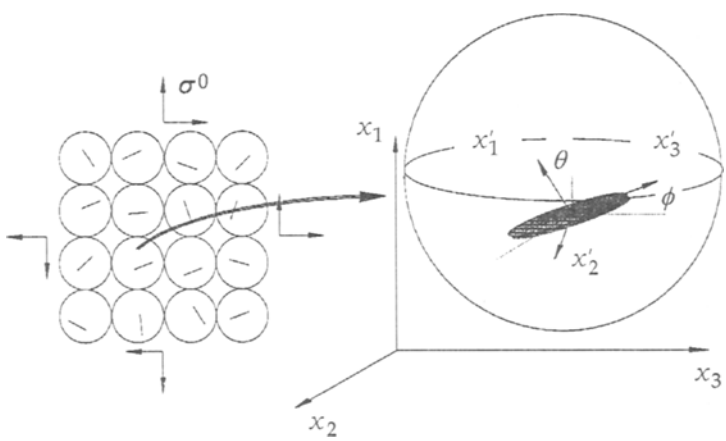


Fig.4 A volume element model for 3-D randomly oriented crack

In a global Cartesian coordinate system, according to the formulae of coordinate system transformation, the average stress and strain are

$$\overline{\sigma} = \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{L}_\sigma \boldsymbol{\beta} \varepsilon^0 \qquad \overline{\varepsilon} = \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{L}_\varepsilon \boldsymbol{\beta} \varepsilon^0 \tag{24}$$

where $\boldsymbol{\beta}$ is the matrix of coordinate system transformation

$$\boldsymbol{\beta} = \begin{bmatrix} \cos\theta & \sin\theta\cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & \cos\theta\cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}$$

Suppose that the crack are 3-D randomly oriented in the volume element, then the average stress and strain of the volume element can be written as

$$\Sigma = \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi} \overline{\sigma}\sin\theta \mathrm{d}\varphi \mathrm{d}\theta = \boldsymbol{A}_\sigma \boldsymbol{\varepsilon}^0 \tag{25a}$$

$$\Xi = \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi} \overline{\varepsilon}\sin\theta \mathrm{d}\varphi \mathrm{d}\theta = \boldsymbol{A}_\varepsilon \boldsymbol{\varepsilon}^0 \tag{25b}$$

where

$$\left.\begin{aligned} \boldsymbol{A}_\sigma &= \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi} \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{L}_\sigma\boldsymbol{\beta}\sin\theta \mathrm{d}\varphi \mathrm{d}\theta \\ \boldsymbol{A}_\varepsilon &= \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi} \boldsymbol{\beta}^{T}\boldsymbol{L}_\varepsilon\boldsymbol{\beta}\sin\theta \mathrm{d}\varphi \mathrm{d}\theta \end{aligned}\right\} \tag{26}$$

Then the effective elastic moduli $\boldsymbol{L}$ can be obtained by canceling $\boldsymbol{\varepsilon}^0$ in Eq. (25)

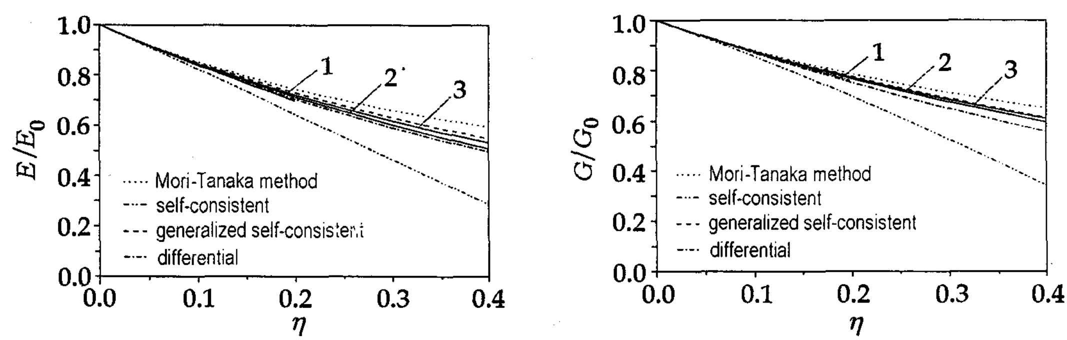
$$\Sigma = \boldsymbol{L}\Xi \tag{27}$$

where

$$\boldsymbol{L} = \boldsymbol{A}_\sigma \boldsymbol{A}_\varepsilon^{-1}$$

The effective elastic moduli are calculated and expressed in an explicit form for the spheroid V and the oblate ellipsoid V ($\alpha_0 = 1/2$ and $\alpha_0 = 1/3$, respectively) by using the MATHEMATICA Software.

The results are displayed in Fig.5, along with the solutions of other micromechanics



(a) Normalized effective Young's moduli vs crack density ($\nu_0 = 0.3$) (3-D randomly oriented penny-shaped crack) present results: 1 crack in sphere, 2 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/2$), 3 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/3$)

(b) Normalized effective shear moduli vs crack density ($\nu_0 = 0.3$) (3-D randomly oriented penny-shaped crack) present results: 1 crack in sphere, 2 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/2$), 3 crack in oblate ellipsoidal matrix ($b_1/b_2 = 1/3$)

Fig.5

models. As shown from Fig.5, the present results are close to those of the differential scheme and the generalized self-consistent method, and the effect of the shapes of the cells on the effective moduli is slight.

4 CONCLUSION

In summary, an embedded cell model is presented to obtain the effective elastic moduli for three-dimensional two-phase composites which is an exact analytic formula without any simplified approximation and can be expressed in an explicit form. For the different cells including spherical particle and crack in sphere and oblate spheroid matrix, the effective elastic moduli are evaluated and the results are compared with those from various micromechanics models. These results show that the present model is direct, simple and efficient for three-dimensional two-phase composites. The results show that the present results are close to those of the solutions from the generalized self-consistent model and the Mori-Tanaka method for composites with the spherical inclusion, indicating that the effect of the shape of the cell on the effective elastic moduli is slight. For unidirectionally aligned penny-shaped crack, the effect of the shape of the cell on the effective moduli is quite significant. When the crack is randomly oriented, the effective moduli agree well with the solutions of the differential scheme and the generalized self-consistent method, and the effect of the shapes of the cells on the effective moduli is slight.

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Appendix Components of Eshelby's S_{ijkl} Tensor

For a spheroidal inclusion with the symmetric axis identified as x_1, the components of Eshelby's tensor S_{ijkl} are

$$\left.\begin{aligned}
S_{1111} &= \frac{1}{2(1-\nu_0)}\left\{1-2\nu_0+\frac{3\lambda^2-1}{\lambda^2-1}-\left[1-2\nu_0+\frac{3\lambda^2}{\lambda^2-1}\right]g\right\}\\
S_{2222} = S_{3333} &= \frac{1}{8(1-\nu_0)}\frac{\lambda^2}{\lambda^2-1}+\frac{1}{4(1-\nu_0)}\left[1-2\nu_0-\frac{9}{4(\lambda^2-1)}\right]g\\
S_{2233} = S_{3322} &= \frac{1}{4(1-\nu_0)}\left\{\frac{\lambda^2}{2(\lambda^2-1)}-\left[1-2\nu_0+\frac{3}{4(\lambda^2-1)}\right]g\right\}\\
S_{2211} = S_{3311} &= -\frac{1}{2(1-\nu_0)}\frac{\lambda^2}{\lambda^2-1}+\frac{1}{4(1-\nu_0)}\left[\frac{3\lambda^2}{\lambda^2-1}-(1-2\nu_0)\right]g\\
S_{1122} = S_{1133} &= -\frac{1}{2(1-\nu_0)}\left[1-2\nu_0+\frac{1}{\lambda^2-1}\right]+\frac{1}{2(1-\nu_0)}\left[1-2\nu_0+\frac{3}{\lambda^2-1}\right]g\\
S_{2323} &= \frac{1}{4(1-\nu_0)}\left\{\frac{\lambda^2}{2(\lambda^2-1)}+\left[1-2\nu_0-\frac{3}{4(\lambda^2-1)}\right]g\right\}\\
S_{1212} = S_{1313} &= \frac{1}{4(1-\nu_0)}\left\{1-2\nu_0-\frac{\lambda^2+1}{\lambda^2-1}-\frac{1}{2}\left[1-2\nu_0-\frac{3(\lambda^2+1)}{\lambda^2-1}\right]g\right\}
\end{aligned}\right\} \tag{A1}$$

where ν_0 is Poisson's ratio of the matrix, λ is the aspect ratio of the inclusion and g is given by

$$g = \begin{cases} \dfrac{\lambda}{(\lambda^2-1)^{3/2}}\{\lambda(\lambda^2-1)^{1/2}-\arccos\lambda\} & \lambda > 1 \\ \dfrac{\lambda}{(1-\lambda^2)^{3/2}}\{\arccos\lambda-\lambda(1-\lambda^2)^{1/2}\} & \lambda < 1 \end{cases} \tag{A2}$$

For a spherical inclusion, they are simplified to

$$\left.\begin{aligned}
S_{1111} = S_{2222} = S_{3333} &= \frac{7-5\nu_0}{15(1-\nu_0)}\\
S_{1122} = S_{2233} = S_{3311} &= \frac{5\nu_0-1}{15(1-\nu_0)}\\
S_{1212} = S_{2323} = S_{3131} &= \frac{4-5\nu_0}{15(1-\nu_0)}
\end{aligned}\right\} \tag{A3}$$

For a penny-shaped disc with an aspect atio ($\lambda \ll 1$), they are

$$\left.\begin{aligned}
S_{1111} &= 1 - \frac{1-2\nu_0}{4(1-\nu_0)}\pi\lambda \\
S_{2222} &= S_{3333} = -\frac{13-8\nu_0}{32(1-\nu_0)}\pi\lambda \\
S_{2233} &= S_{3322} = \frac{8\nu_0-1}{8(1-\nu_0)}\pi\lambda \\
S_{2211} &= S_{3311} = \frac{2\nu_0-1}{8(1-\nu_0)}\pi\lambda \\
S_{1122} &= S_{1133} = \frac{\nu_0}{1-\nu_0}\left(1 - \frac{1+4\nu_0}{8\nu_0}\pi\lambda\right) \\
S_{2323} &= \frac{7-8\nu_0}{32(1-\nu_0)}\pi\lambda \\
S_{1212} &= S_{1313} = \frac{1}{2}\left[1 - \frac{2-\nu_0}{4(1-\nu_0)}\pi\lambda\right]
\end{aligned}\right\} \tag{A4}$$