Helsinki
Center
of
Economic
Research

Discussion Papers

Integration of Micro and Macro Explanations in Economics: Aggregation of the Marginal Propensity to Consume

Yrjö Vartia
University of Helsinki and HECER

Discussion Paper No. 353
October 2012

ISSN 1795-0562

HECER – Helsinki Center of Economic Research, P.O. Box 17 (Arkadiankatu 7), FI-00014 University of Helsinki, FINLAND, Tel +358-9-191-28780, Fax +358-9-191-28781, E-mail info-hecer@helsinki.fi, Internet www.hecer.fi

HECER
Discussion Paper No. 353

Integration of Micro and Macro Explanations in Economics: Aggregation of the Marginal Propensity to Consume*

Abstract

Do the parts determine the whole or is the whole more than the sum of its parts? We consider the possibility of unified science in economics, where the macro behaviour is determined from its micro foundations. Economists think that this is impossible except in unrealistic special cases, e.g. when all the micro coefficients are the same. When behavioural equations are expressed in changes instead of levels, the connection between micro and macro formulations becomes a real possibility. Results are applied to the consumption function where the marginal propensity to consume is aggregated. The effective macro level marginal propensity to consume MC is found to be an affine combination of its components MC^{UA} and MC^{UP} related to uniform absolute and uniform proportional changes of income. Their weights depend on the way and how much the income distribution is changing. This implies, that time series estimations of the consumption function are not altogether in error, but inaccurate and unreliable because the estimated coefficients are not constant and the number of macro observations is necessarily small.

JEL Classification: B41, C02, C43, C81, C82, E01

Keywords: Aggregation, micro foundations, methodology of economics, organizing micro- and macroeconomic data.

Yrjö Vartia

Professor of Econometrics,
Department of Political and Economic Studies,
P.O. Box 17 (Arkadiankatu 7)
University of Helsinki
FI-00014
FINLAND

E-mail: yrjo.vartia@helsinki.fi

* I thank M.Sc. Matti Jokinen, prof. Ulla Lehmijoki, ass. Olli Ropponen, licentiate Pekka Tavaila, prof. Otto Toivanen, planning officer Antti Suoperä, chief economist Juhani Turkkila and prof. Pentti Vartia for useful comments on previous versions of the paper.

Integration of Micro and Macro Explanations in Economics: Aggregation of the Marginal Propensity to Consume[1]

1. Introduction

Do the parts determine the whole or is the whole more than the sum of its parts? Are the methods of the science effective also in the Humanities or must they differ from each other as the vitalists and humanists believe? **Integration** in our title refers to mathematics and **explanations** to the philosophy of science. The Grand Unified Theory *GUT* in physics and Aggregation in economics are designed to unify the micro based and holistic views.

The views of parts and the whole, of analysis and synthesis, have divided philosophy and science more than two thousand years. Classical separations are materialistic/ idealistic, descriptive/explanatory, mechanical/intentional etc. Special terminology includes emergence, vitalism/reductionism, interaction, feedback and holism. Of the general philosophy see e.g. Wilson (1998), Enqvist (2007) and Monod (1971). **The schemes of unified science** (three m's, both Latin and Greek ones) generally, in chemistry C, and in economics E are

$$\begin{array}{ccccccc} & M & & & M(C) & & & M(E) \\ \mu & \uparrow & & \mu(C) & \uparrow & & \mu(E) & \uparrow \\ & m & & & m(C) \subset M(PF) & & & m(E) \end{array}$$

The actual topic of a research area (its macro level) is on the top, some chosen meaningful micro level at the bottom and μ denotes mathematical-logical methods in their integration. Mathematical reasoning in chemistry $\mu(C)$ differs from that of econometrics $\mu(E)$. The micro $m(C)$ of chemistry is, of course, also part of particle physics $M(PF)$. While $M(C)$ is based mostly on observation and experiments, its mathematics $\mu(C)$ is based on statistical physics and other mathematical models of particle physics, see Vartia (2009, p. 1-2). For instance, deriving the well-known properties of H_2O from those of hydrogen and oxygen requires ultimately quantum theory. Apparent but extremely important experimental facts such as transparency of water or that its maximum density occurs at the temperature 3.98 ^oC are not easily reduced to its atomic properties. Chemical compounds and atomic bonds serve as a model for macro effects of economic variables such as sex, occupation, and organization structure or production sector.

Math-logical methods μ cannot be replaced here by semantics or philosophy. Hegel, Kant, Marx or Derrida "do not work", because micro and macro cannot be integrated in one´s head easier in sociology than in economics or chemistry. In our interpretation, **integration of parts and the whole is basically a mathematical problem**. Philosophical analysis and mere semantics are not powerful enough to deal with it, although humanists may think otherwise.

Can one write anything new and sensible of a more or less chaotic topic like this? We have interpreted the problem in a new and mathematically accurate way; see Lintunen et al (2009) and Vartia (2008a-b, 2009).

[1] This paper is an extended version of Vartia (2008a). We have formulated the aggregation problem in terms of consumption function and its marginal propensity to consume because it states the problem in most familiar and classic terms. Mathematical results and formulas generalize to any such dependence.

We illustrate the problem of aggregation by the Keynesian consumption function. We allow the individual non-linear consumption functions $c(a) = f^a(x(a))$ to differ from each other in arbitrary ways, even when there are millions of them. Marginal propensities to consume (MPC) $\overline{m}_1(a)$ and the changes in income $\Delta x_1(a)$ for all households *a* map the micro level information to the macro level. As shown, MPC can be directly and naturally aggregated. The main question is how to express the inner product $\langle \overline{m}_1 \Delta x_1 \rangle$ in terms of meaningful macro characteristics. This can be done for MPC ´s using the Basic Lemma of Aggregation (see page 11), which decomposes the average of a product into the product of averages plus a covariance. A proper weighting is determined to make the covariance a small random like term.

In chapter 3.1 uniform absolute changes of income $\Delta x^{UA}(a)$ are considered. (We have dropped the subindex 1 from $\Delta x_1^{UA}(a)$.) It is shown, that their effective marginal coefficient MC^{UA} is an evenly weighted average of the micro mpc´s. In chapter 3.2 uniform proportional changes of income $\Delta x^{UP}(a)$ are investigated. Their effective marginal coefficient MC^{UP} is an income weighted average of the micro mpc´s. Proposition 4 seems to be a novelty.

Arbitrary income changes are decomposed in chapter 3.3 into their uniform marginal components in Proposition 6. Effective macro marginal coefficient *MC* can be expressed as an affine combination of the uniform marginal components MC^{UA} and MC^{UP}. This affine combination captures the effects of the relative income inequality *RII* on the macro consumption. Propositions 6-8 in the chapter 3.3 seem to be new contributions.

Illustrations in chapter 4 deal with three cases: relative income inequality *RII* stays almost constant, increases and decreases. General conclusions 5 sketch the implications of the paper to the macro time series methodology of economics.

2. Analysis and synthesis in economics

Let us elaborate the economic background more carefully. How do incomes and consumption functions, say, of two million households $a \in \Omega$ determine total consumption? For a single agent *a*, consumer theory specifies its consumption function which are treated for simplicity as a function of incomes $x_1(a)$ only mostly in the paper. (Or rather, consider changes caused by income only without assuming that it is the only explanatory variable. Think prices and other socio-economic input variables as fixed.) We allow the micro consumption functions $c(a) = f^a(x(a)) + \varepsilon(a) = f^a(x_1(a),..., x_K(a)) + \varepsilon(a)$ to be non-linear, stochastic and agent-specific. The vector of explanatory variables may contain lags, which makes the specification dynamic. Also common macro inputs, say economic expectations and published results of barometers are allowed. It does not matter in aggregation what kind of other explanatory variables in addition to income are included although they change the numerical magnitude of effect of income, because its partial contribution has all the time the same functional form of an inner product, see (2b). Additive error terms are supposed to have vanishing conditional expectations $E(\varepsilon(a)|x(a)) = 0$, finite variances and small correlations across households. Their mean is practically zero by the Law of Large Numbers. In the notation of Vartia (2009) and Suoperä and Vartia (2011) all agent behaviours are denoted by $c = A(f, x) + \varepsilon$, where *f* and *x* are vectors of behaviours and inputs and *A* is the analysis operator. The incomes $x_1(a)$ are treated as arbitrary positive real numbers. At the macro level the set of all the households $\Omega = \{a_1,..., a_n\}$ form the aggregate (collection, set) under consideration.

In the System of National Accounts *SNA*, the *total* consumption $\tilde{c} = \sum_{\Omega} c(a) = \sum_{i=1}^{n} c(a_i)$ (as a current value or in constant prices) over households $\Omega = \{a_1,..., a_n\}$ is chosen as the macro level output. In this paper it is more natural to consider $\langle c \rangle$ = the average consumption per household $= \frac{\tilde{c}}{n} = \frac{1}{n}\sum_{\Omega} c(a)$ as the macro output. Averages or per capita outputs are also easier to consider in aggregation than the totals. As the number of households n is close to constant (say in time) or slowly changing, the variables $\langle c \rangle$ and $\tilde{c}$ are practically constant multiples of each other. Therefore, aggregation results of average variables can be accurately applied also for totals of the system of national accounts *SNA*.

3. Integration of micro and macro information

Macro output in terms of micro inputs can be written

(1) $\langle c \rangle = \frac{1}{n}\sum_{\Omega} f^{a}(x(a)) = \frac{1}{n}\sum_{\Omega} f^{a}(x_1(a), x_2(a),..., x_K(a))$.

The average of the additive error terms $\langle \varepsilon \rangle = \frac{1}{n}\sum_{\Omega} \varepsilon(a)$ is left out from the equation because it is practically zero for millions of households. Micro level disturbances do not affect the macro behaviour. In Vartia (2008a) we considered for simplicity he case $K = 1$, but the main results are formally the same as here. In our previous aggregation papers Vartia (2008b) and Suoperä and Vartia (2011) we considered levels elaborating the level equation $\langle c \rangle = \frac{1}{n}\sum_{\Omega} f^{a}(x(a)) = S(f, x)$, where $S(f, x)$ is the synthesis operator. The references derive its fundamental decomposition $S(f, x) = \bar{f}\langle x \rangle + NLE(x) + HE(x)$, where $\bar{f}\langle x \rangle$, $NLE(x)$ and $HE(x)$ are average behaviours at mean inputs, non-linearity effects and heterogeneity effects, respectively. Average consumption $\langle c \rangle = S(f, x)$ can be described as a MicMac-function, because it maps the micro information (behaviours *f* and inputs *x*) to the macro level. It depends in a very complicated way on the micro information. Despite its short representation, this is a very complicated function of the input vector $(x(a_1),..., x(a_n))$ when n is large and $f^{a} : R^{K} \to R$ are non-linear agent specific functions of *K* arguments. We do not need to specify the model or theory from which the behavioural equations are derived, because it does not matter in the problem of aggregation. Taxation and its deductions are simple examples of such dependences, where in addition functions $f^{a} : R^{K} \to R$ are the same for all agents, given by tax legislation, see Edgren, C., Turkkila, J. and Vartia, Y. (1978) and Turkkila, J. (2011). This scheme is general enough to cover any micro-theoretic situation.

Potential explanatory variables in (1) may include income, wealth, socio-economic factors (family composition, employment, sickness, age, …) and also macro conditions (consumption expectations, asset prices, barometer results, time) and their lagged values. Some of the potential explanatory variables (family composition, gender, sickness) do not normally change systematically in the macro level and thus drop away as actual variables in the macro equations. Macro variables are particularly convenient in aggregation, because their absolute changes are uniform across agents. Nonlinearities of behaviours allow levels of macro variables (say of capacity utilization) affect reaction parameters. Though qualitatively important on the macro level, these kinds of nonlinearity effects are not managed well quantitatively and more research is needed in their implementation. After some motivating comments we concentrate mostly on one input variable - the income - only. We are asking, whether it is possible to write this relationship (1) **in**

terms of some familiar macro variables, such as average income, income inequality and average marginal propensities to consume.

Similar integration problems have been considered in statistical physics but without the heterogeneity of economic behavioural functions. Klein (1946, 1956, pp. 192-6), Leontief (1947), Nataf (1948), Gorman (1959), Theil (1954), Arrow (1963), Green (1964) and other almost Nobel-level classic experts of aggregation in economics have shown, that their desired solutions are impossible (or at least naturally implemented) except under restrictive special cases. In our opinion, classic experts pose a wrong question. They seek for conditions which allow a given aggregation procedure. In the case of Klein-Nataf approach the horizontal and vertical aggregations are requested to give consistent results, which leads to additively separable functional forms, see van Dahl and Merkies (1984).

We suggest that the right question is what happens in the macro level or how the output aggregate behaves *under realistic conditions*. The fault in classical aggregation literature is the requirement, that the micro and macro relations must have the same functional form and satisfy complicated separability conditions in their input variables. These requirements are neither necessary nor natural and we do not make them.

The problem of aggregation has been almost bypassed during the last 40 years, although according to the classic experts e.g. the **macro economics of text-books is contradictory with micro behaviours** without unrealistic homogeneity assumptions, see Allen (1964, last chapter) or van Daal and Merkies (1984, chapter 1). The general attitude of economists is that (consistent or perfect) aggregation is practically impossible. It is possible that this is in fact exaggeration. These impossibility results consider behaviours expressed in levels and they require that micro and macro relations have the same functional form. Comparing these with relations between changes, the latter are much simpler and more uniform than behavioural functions expressed as level relations. This is due to the power of differential calculus and it allows rather surprisingly natural aggregation of change equations as is shown below. The art of successfully constructing price and quantity index numbers is also based to the fact that they are relative changes (actually price or quantity ratios), see Vartia (1976).

We denote vector z with components $z(a)$ by $z=(z(a))$. Also define functions of vectors component-wise like in $\log z=(\log z(a))$, $\Delta\log z=(\Delta\log z(a))$, $h(z)=(h(z(a)))$. This idea generalizes to operations between two vectors. Multiplication and division of two vectors x and y are defined by $xy=(x(a)y(a))$ and $\frac{x}{y}=(\frac{x(a)}{y(a)})$. More generally $h(x,y)=(h(x(a),y(a)))$.

The *inner product* (or scalar product) $\langle xy\rangle=\langle(x(a)y(a))\rangle=\frac{1}{n}\sum_{\Omega}x(a)y(a)$ of two n-vectors x and y has the defining properties (see Kolmogorov & Fomin 1970, p. 142)

1) $\langle xx\rangle\geq 0$ and $\langle xx\rangle=0$ if and only if $x=0$;
2) $\langle xy\rangle=\langle yx\rangle$;
3) $\langle(\lambda x)y\rangle=\langle x(\lambda y)\rangle=\lambda\langle xy\rangle$;
4) $\langle x(y+z)\rangle=\langle xy\rangle+\langle xz\rangle$

(valid for all $x,y,z\in R^n$ and all real λ).

Aggregation of quadratic and arbitrary level form micro functions are considered in Vartia (2008b, 2009). These papers show that the macro behaviours inherit properties of the micro functions only partially. Fortunately the functions expressed in change form complement these level representations and will give more generally applicable results.

For any individual (household) we can express the change in output $\Delta c(a)$ induced by input changes $\Delta x(a) = (\Delta x_1(a), \Delta x_2(a), ..., \Delta x_K(a))$ using the mean value theorem as follows $\Delta c(a) = \sum_{k=1}^{K} D_k f^a(\xi(a))\Delta x_k(a) = \sum_{k=1}^{K} \overline{m}_k(a)\Delta x_k(a)$. Here all input variables are supposed to be continuous and the function differentiable. An alternative way is to apply the differential $dc(a) = df^a(x^0(a), \Delta x(a)) = \sum_{k=1}^{K} D_k f^a(x^0(a))\Delta x_k(a) = \sum_{k=1}^{K} m_k^0(a)\Delta x_k(a)$. In terms of gradient function $\nabla f^a(x^0) = (D_1 f^a(x^0), ..., D_K f^a(x^0))$, this is expressed shortly $df^a(x^0(a), \Delta x(a)) = \nabla f^a(x^0(a)) \cdot \Delta x(a)$. The differential gives accurate estimations of the actual output change for small changes in the input variables, see Apostol (1963, p. 107). The first term of these sums gives the contribution of income to the change in consumption. The third possibility is to change only the first variable (income) and keep all other variables constant, which would reduce the problem into one variable case considered in Vartia (2008a). Here the mean value theorem reduces to $\Delta c(a) = (\Delta c(a)/\Delta x_1(a))\Delta x_1(a) = D_1 f^a(\xi(a))\Delta x_1(a) = \overline{m}_1^*(a)\Delta x_1(a)$, where all the other components than the first on in $\xi(a)$ are in their initial values. In these expressions the changes $\Delta x_k(a) = x_k^1(a) - x_k^0(a)$ are either planned exogenous parameters or realized values. For millions of households only their overall distribution can be supposed to be known. The knowledge of the *size* of derivatives $\overline{m}_k(a) = D_k f^a(\xi(a))$ at some mean input value $\xi(a)$ - or its other variants $m_k^0(a)$ and $\overline{m}_1^*(a)$ - is even more hypothetical. But for differentiable functions such derivatives or marginal coefficients exist between the old and new values for any input changes, see Apostol (1963, p. 117).

We have assumed that all variables are continuous and the function is differentiable. In period (say quarterly) analysis – which we are discussing – variables that are normally considered as discrete (say number of children) become actually continuous when dated accurately. If the number of children increases by one in the middle of the quarter, the effective increase during the quarter is one half. This is the effect of temporal aggregation. Therefore, discrete variables in instantaneous analysis can and should be treated as continuous in the period analysis.

The macro difference of the consumption satisfies

$$(2)\ \langle \Delta c \rangle = \tfrac{1}{n}\sum_{\Omega}\sum_{k=1}^{K} \overline{m}_k(a)\Delta x_k(a) = \sum_{k=1}^{K} \tfrac{1}{n}\sum_{\Omega} \overline{m}_k(a)\Delta x_k(a) = \sum_{k=1}^{K} \langle \overline{m}_k \Delta x_k \rangle = \sum_{k=1}^{K} \langle \Delta c(x_k) \rangle .$$

This result follows immediately from the mean value theorem of functions applied for all agent-specific differentiable functions separately. Contributions of the separate *K* explanatory variables to the macro change of consumption are $\langle \Delta c(x_k) \rangle = \langle \overline{m}_k \Delta x_k \rangle$ and they add up. There is no approximation error in the equation (2). Equations (1) and (2) hold for all values of their high-dimensional arguments: they can be either freely variable or interrelated e.g. by economic laws. This is important and allows applying any interdependencies, regularities and correlations between the arguments in the future analysis. Note how different functional forms of agent-specific level relations (1) disappear from sight and are expressed by the marginal coefficients $\overline{m}_k(a)$. Equation (2) is still a so called MicMac-function: its output is a macro variable, but inputs are micro vectors $\overline{m}_k$ and Δx_k. We cannot know so detailed information say for millions of households, but have to concentrate on their general and statistical characteristic such as signs, rough magnitude, variability, weighted means etc. The important point is that equation (2) is easier to aggregate than the level equation (1).

From now on we concentrate on one explanatory variable only, namely on the income effects of the consumption function. We denote the current income by *x* without sub index. Contribution of the current income on the change of consumption equals in the mean value case[2]

(2b) $\langle \Delta c(x) \rangle = \langle \overline{m} \Delta x \rangle$.

In dynamic setting, there appears similar contributions of the lagged incomes, $\langle \Delta c(x_{-l}) \rangle = \langle \overline{m}_{-l} \Delta x_{-l} \rangle$, $l = 1,...,L$. The principal question is the following. How can we express the inner product $\langle \overline{m} \Delta x \rangle$ in terms of meaningful macro characteristics? This information cannot be exact and detailed, but qualitative and of summary character, such as weighted means, variances and covariances. Uniform proportional and absolute changes (and sum of them) of the input variable are important possible ways of change and these lead to different weighting of the mc's as shown in chapter 3. The remarkable point is that aggregation results derived for the income contribution (2b) apply as such for all contributions $\langle \Delta c(x_k) \rangle = \langle \overline{m}_k \Delta x_k \rangle$ of the other variables. For intuitive reasons we discuss contributions of income in this paper and the reader can extend our results to any other similar variable.

It is rather surprising that it is difficult to decompose the inner product in terms of its two variables. The vector analytic representation $\langle \overline{m} \Delta x \rangle = \|\overline{m}\| \cdot \|\Delta x\| \cos\alpha$ is as such only rarely used in economic analysis. Instead of the length $\|\Delta x\| = (\frac{1}{n} \sum \Delta x(a)^2)^{1/2}$ the average $\langle \Delta x \rangle = \frac{1}{n} \sum \Delta x(a)$ of the input vector is the relevant observable economic variable. Sums and averages are the main observables in official statistics (and especially in the System of National Accounts, SNA) and there are no intentions to change this practice. The Basic Lemma of Aggregation given below gives a useful decomposition $\langle \overline{m} \Delta x \rangle = \langle \overline{m} \rangle \langle \Delta x \rangle +$ $s(\overline{m}) s(\Delta x) corr(\overline{m}, \Delta x) \; = \langle \overline{m} \rangle \langle \Delta x \rangle + \|\delta \overline{m}\| \cdot \|\delta \Delta x\| \cos\alpha(\delta \overline{m}, \delta \Delta x)$, where the standard deviations are the lengths of the deviation vectors, say $\delta \Delta x = \Delta x - \langle \Delta x \rangle$, $\|\delta \Delta x\| = s(\Delta x)$ etc. Another vector analytic concept which may be useful in aggregation is the orthogonal complement. For the vector of marginal coefficients $\overline{m} \in R^n$ it is defined by the vectors *y* orthogonal to it: $\overline{m}^{\perp} = \{ y \,|\, \langle \overline{m} y \rangle = 0, y \in R^n \}$. It is a subspace and has dimension $\dim(\overline{m}^{\perp}) = n - 1$, while the vectors $\lambda \overline{m}$ have the dimension $\dim(\lambda \overline{m}) = 1$. Of all the vectors Δx the great majority lies in the orthogonal complement, $\Delta x \in \overline{m}^{\perp}$, or $\langle \overline{m} \Delta x \rangle = 0$. A similar condition is the uncorrelation: $\mathrm{cov}(\overline{m}, \Delta x) = 0$.

This decomposition task reminds of the index number problem where the value relative is to be expressed as a product of the price and quantity indices, see Vartia (1976). A simple but rather realistic way to decompose $\langle \overline{m} \Delta x \rangle$ is to divide and multiply by $\langle \Delta x \rangle \neq 0$ to get $\dfrac{\langle \overline{m} \Delta x \rangle}{\langle \Delta x \rangle} \langle \Delta x \rangle = \langle \overline{m} \rangle_{\Delta x} \langle \Delta x \rangle$. Here $\langle \overline{m} \rangle_{\Delta x} = \dfrac{\langle \overline{m} \Delta x \rangle}{\langle \Delta x \rangle} = \sum_{\Omega} \dfrac{\Delta x(a)}{n \langle \Delta x \rangle} \overline{m}(a)$ is an *affine combination* of

[2] In the differential case this would be $\langle m^0 \Delta x \rangle$ and in the income change only situation $\langle m^* \Delta x \rangle$. All these expressions have the same functional form, the inner product of the marginal coefficient and the income change.

the marginal coefficients and the input changes. This theoretically important representation shows at once that the effective *MC* depends on the distribution of input *changes* and how they are connect to the micro mc´s. Different vectors Δx imply a different $\langle \overline{m} \rangle_{\Delta x}$. In the case of uniform absolute changes, where $\Delta x(a) = \Delta x^{UA}(a) = A =$ constant for all agents,

$$\langle \overline{m} \rangle_{\Delta x} = \frac{\langle \overline{m} A \rangle}{A} = \langle \overline{m} \rangle = \text{an unweighted average.}$$

In the case of uniform proportional changes, where $\Delta x(a) = \Delta x^{UP}(a) = Px^0(a)$,

$$\langle \overline{m} \rangle_{\Delta x} = \langle \overline{m} \rangle_{Px^0} = \frac{\langle \overline{m} Px^0 \rangle}{\langle Px^0 \rangle} = \langle \overline{m} \rangle_{x^0} = x^0\text{-weighted average.}$$

State income taxation provides a good example. If large positive input changes are connected to large marginal coefficients (here large marginal taxes), then the macro marginal coefficient $\langle \overline{m} \rangle_{\Delta x}$ is larger than if all the income changes had been the same. Income weighted marginal tax rate is much higher than the unweighted one. Turkkila (2011, p. 216-221) calculates these as person and income weighted averages and for 2000-2009 these marginal tax rates vary between 0.1117 - 0.1389 and 0.2531 - 0.2893. For every year the income weighted marginal tax is more than twice the person (or unweighted) marginal tax.

This strong dependence of weighting has to be admitted and the aggregation problem reduces to explaining how actual Δx affects $\langle \overline{m} \rangle_{\Delta x}$. This view differs radically from the old aggregation approach where the behaviours were restricted in such a way that the weighting does not matter. The condition of *perfect aggregation* in the old approach is that $\overline{m}(a)$ does not depend on *a*, quite a triviality and highly unrealistic. The expression of $\langle \overline{m} \rangle_{\Delta x}$ does not make good sense if $\langle \Delta x \rangle$ is almost zero, because the individual mc´s would get large negative and positive weights.

This is only one path of $\langle \Delta x \rangle$ and Δx towards zero. If all the components approach zero at the same velocity, $\Delta x(t) = \Delta x \cdot t$, then $\Delta x(0) = \Delta x \cdot 0 = 0$ and $\langle \overline{m} \rangle_{\Delta x(t)} = \langle \overline{m} \rangle_{\Delta x} =$ t-constant. But this is a very special and unrealistic way of how $\langle \Delta x \rangle \to 0$. However, this path is a realistic virtual way of changing small non-zero changes.

The basic lemma of aggregation presented below may be applied in this situation. We will get $\langle \overline{m} \Delta x \rangle = \text{cov}(\overline{m}, \Delta x) = s(\overline{m}) s(\Delta x) r(\overline{m}, \Delta x)$, because $\langle \overline{m} \rangle \langle \Delta x \rangle = \langle \overline{m} \rangle \cdot 0 = 0$. This shows that the expression behaves regularly.

A variable with both positive and negative mc's on the micro level may lead to essentially zero macro marginal coefficient $|MC_k| \approx 0$. Such a variable vanishes on the macro level and is called a *vanishing* explanatory variable. Attitudes evenly for and against cancel each other on the macro level which is neutral in this respect. *A potential* explanatory variable: $|MC_k| >> 0$. *An actual* explanatory variable has measurable macro effects: $|MC_k \cdot \langle \Delta x_k \rangle| >> 0$. For instance male indicator is only potential and not actual because the gender distribution (and still less actual individual gender values!) does not change in normal conditions, war and gender specific diseases excluded. Same applies for family size, age, education etc. Education has positive influences in wage and employment functions and also positive macro *MC´*s. But average education grows so slowly that education is an actual variable only in the long run, as an important variable in the technical development. Old age pensions have been an actual macro variable in Finland during recent years because of large retiring age groups. In macro behaviour, many actual micro variables are frozen and therefore only potential. This strongly simplifies macro behaviours for which most of the micro controlling variables are only potential. Potential macro variables are multicollinear with the constant term, because they vary so little. Their coefficients cannot be estimated by regression.

Variable	**Micro level speciality**	**Actual macro variable?**
Age	Same changes, upwards only	No
Old age pension (indicator)	Changes only $0 \rightarrow 1$	Possibly
Cumulative education	Changes only upwards	Yes
Gender	Constant in time	In special situations
Work experience	Changes only upwards	Yes
Family size		Yes, in developing countries
Profession		No, possibly yes
Dwelling place		Not usually
Time	Same for all	Yes
Interest rate	Same for all	Yes
Price	Same for all	Yes
Income		Yes
Consumption		Yes
Wages	Sticky wages	Yes
Employment		Yes
Ethnic background	Immigration?	Possibly
Nationality		Not normally
Mother tongue		Not normally
Political view		Possibly
New technology (IT)		Yes
Taxation	Known function	Yes

We return to our principal question: How can we express the inner product $\langle \overline{m}\Delta x \rangle$ in terms of meaningful macro characteristics? Changes in input, namely income here, can and should be expressed as a sum of different terms, $\Delta Income(a) = \sum_{l=1}^{L} \Delta Income^{(l)}(a) + \varepsilon(a)$. Especially change in income is the sum of uniform absolute and uniform proportional terms plus an error. This is then inserted in the inner product and its additivity gives

$$\text{(3a)} \qquad \langle \overline{m}\Delta Income \rangle = \left\langle \overline{m}\left(\sum_{l=1}^{L} \Delta Income^{(l)} + \varepsilon\right) \right\rangle = \left\langle \sum_{l=1}^{L} \overline{m}\Delta Income^{(l)} + \overline{m}\varepsilon \right\rangle$$
$$= \sum_{l=1}^{L} \langle \overline{m}\Delta Income^{(l)} \rangle + \langle \overline{m}\varepsilon \rangle = \sum_{l=1}^{L} MC^{(l)} \langle \Delta Income^{(l)} \rangle + \tilde{\varepsilon}$$

The desired representation of the inner product $\langle \overline{m}\Delta x \rangle$ is of the form

$$\text{(3)} \qquad \langle \overline{m}\Delta Income \rangle = \sum_{l=1}^{L} MC^{(l)} \langle \Delta Income^{(l)} \rangle + \tilde{\varepsilon}$$
$$= \text{systematic + non-systematic parts.}$$

Non-systematic parts are deterministic at the micro level, but appear as random-like terms in the macro behaviour. It is our conjecture that random effects on the macro level arise in this way. In the most common way of aggregation we set L = 1, which reduces in its simplest form by the basic lemma of aggregation $\langle \overline{m}\Delta Income\rangle = \langle \overline{m}\rangle\langle \Delta Income\rangle + \tilde{\varepsilon}$. More generally, one type of income changes $\Delta Income^{(1)}$ gives rise to their specific effective macro coefficient $MC^{(1)}$ and $\langle \overline{m}\Delta Income\rangle = MC^{(1)}\langle \Delta Income^{(1)}\rangle + \tilde{\varepsilon}$. It will be shown that for the uniform proportional changes $\Delta Income^{(1)} = P\cdot Income^{0}$ the effective marginal coefficient is $MC^{(1)} = \langle \overline{m}\rangle_{Income^{0}}$. We do not require that the micro and macro equations have the same functional form while we allow the non-systematic part $\tilde{\varepsilon}$. It arises because we cannot forecast (or estimate) the input changes without error. It forecasts were accurate, we could calculate the following decomposition $\langle \overline{m}\Delta x\rangle = \langle \overline{m}\rangle_{\Delta x}\langle \Delta x\rangle = MC\langle \Delta x\rangle$. More generally, we allow for several (L > 1) additive income components in the macro level.

An essential methodological point is that this macro equation expressed in changes does not correspond to any equation expressed in levels, which will be seen later. The level equations in standard macro theory and text-books do not exist. Behaviors must be expressed not in levels but as changes (like in index numbers) which give the needed flexibility.

For arbitrary real weights (not necessary non-negative) define for $\langle w\rangle \neq 0$

$$\langle x\rangle_w = \frac{\langle wx\rangle}{\langle w\rangle} = \frac{1}{n\langle w\rangle}\sum_\Omega w(a)x(a) = \sum_\Omega \frac{w(a)}{n\langle w\rangle}x(a),$$

$$\langle xy\rangle_w = \frac{\langle wxy\rangle}{\langle w\rangle} = \frac{1}{n\langle w\rangle}\sum_\Omega w(a)x(a)y(a),$$

$$\mathrm{cov}_w(x,y) = \frac{\langle w(x-\langle x\rangle_w)(y-\langle y\rangle_w)\rangle}{\langle w\rangle} = \frac{1}{n\langle w\rangle}\sum_\Omega w(a)(x(a)-\langle x\rangle_w)(y(a)-\langle y\rangle_w).$$

For instance, $\langle x\rangle_w$ is the *affine combination* of x-variable with weights w. It is evident that multiplication of the weights by the same non-zero real constant does not change the affine combination: $\langle x\rangle_{\lambda w} = \langle x\rangle_w$. It is also a linear function in its x-argument:

$$\langle ax+by\rangle_w = a\langle x\rangle_w + b\langle y\rangle_w.$$

In terms of these the Basic Lemma of Aggregation (*BLA*, originally introduced as a tool in aggregation in Edgren, C., Turkkila, J. and Vartia, Y. (1978) or Vartia (2008b)) is expressed as Proposition 1. Its verbal content is: Inner product equals the outer product plus the covariance.

Proposition 1:
BLA $\quad \forall x,y,w\in R^n : \langle w\rangle \neq 0 \Rightarrow \langle xy\rangle_w = \langle x\rangle_w\langle y\rangle_w + \mathrm{cov}_w(x,y)$.

Proof: Denote $(x)_p = \sum_\Omega p(a)x(a)$ and $(y)_p = \sum_\Omega p(a)y(a)$.
By straightforward calculation

$$\sum_\Omega p(a)(x(a)-(x)_p)(y(a)-(y)_p) =$$
$$\sum_\Omega p(a)x(a)y(a) - \sum_\Omega p(a)x(a)(y)_p - \sum_\Omega p(a)y(a)(x)_p + \sum_\Omega p(a)(x)_p(y)_p =$$
$$\sum_\Omega p(a)x(a)y(a) - (x)_p(y)_p - (y)_p(x)_p + (\sum_\Omega p(a))(x)_p(y)_p =$$
$$\sum_\Omega p(a)x(a)y(a) - (2 - \sum_\Omega p(a))(x)_p(y)_p =$$
$$\sum_\Omega p(a)x(a)y(a) - (x)_p(y)_p \text{ when } \sum_\Omega p(a) = 1.$$

Inserting $p(a) = w(a)/n\langle w\rangle$, we have $\sum_\Omega p(a) = \frac{1}{n\langle w\rangle}\sum_\Omega w(a) = \frac{\langle w\rangle}{\langle w\rangle} = 1$ and the expressions above become $\mathrm{cov}_w(x,y) = \frac{1}{n\langle w\rangle}\sum_\Omega w(a)(x(a)-\langle x\rangle_w)(y(a)-\langle y\rangle_w) = \langle xy\rangle_w - \langle x\rangle_w\langle y\rangle_w$. This is equivalent to *BLA*. QED.

Note that some of the weights $w(a)$ are allowed to be negative though they normally are all non-negative. Generally, linear combinations with respect to *w´*s are thus *affine combinations* if the sum of weights equals one, see e.g. Wikipedia. For positive weights the terms of *BLA* are literally weighted averages (or convex combinations) of *x* and *y*, their inner product and covariance. The sub-index of the weights *w* is dropped when the weights are equal.
The covariance has the well-known representation for non-negative weights

$$\mathrm{cov}_w(x,y) = \sqrt{\mathrm{var}_w(x)}\sqrt{\mathrm{var}_w(y)}\,corr_w(x,y) = s_w(x)s_w(x)corr_w(x,y).$$

The covariance is small if either of the standard deviations or the correlation coefficient is small. Because $|corr_w(x,y)| \le 1$, we have always $|\mathrm{cov}_w(x,y)| \le s_w(x)s_w(x)$.

We end the introductory chapter to a representation of affine combination. Set for the normalized weights $c(a) = w(a)/n\langle w\rangle$ so that $\sum_\Omega c(a) = \sum_{i=1}^n c_i = 1$. Because $c_1 = 1 - \sum_{i=2}^n c_i$, we can write for the affine combination

$$X = \sum_{i=1}^n c_i x_i = c_1 x_1 + \sum_{i=2}^n c_i x_i = x_1 + \sum_{i=2}^n c_i(x_i - x_1),$$

where the weights $c_2,\dots,c_n$ are arbitrary real numbers. For $n = 2$, we have the representation $X = (1-c)x_1 + cx_2 = x_1 + c(x_2 - x_1)$. Because *c* is arbitrary, this attains all real numbers, if $(x_2 - x_1) \ne 0$. This will be applied in chapter 3.3.

3.1 Uniform absolute changes

Applying *BLA* to (2b) gives a version of (3):

Proposition 2:
(4) $\langle \Delta c(x)\rangle = \langle \overline{m}\Delta x\rangle = \langle \overline{m}\rangle\langle \Delta x\rangle + \mathrm{cov}(\overline{m},\Delta x)$.

This is an identity and thus it holds for all its variables. The inner product $\langle \overline{m}\Delta x\rangle$ equals the outer product of its terms plus the covariance between them, all calculated here as unweighted statistics. Especially, $\langle \overline{m}\Delta x\rangle = \langle \overline{m}\rangle\langle \Delta x\rangle$ cannot hold unless $\operatorname{cov}(\overline{m},\Delta x)=0$. This holds for arbitrary slopes only when the absolute changes are constant. Proposition 2 is a starting point and prototype for similar but more complicated derivations.

Though we have used consumption function as a model for integration of micro and macro behaviors, this equation holds as well for any other interpretations of input and output variables. Consumption function is just a case where the components of (4) have well-known interpretations. The formal simplicity of the equation may suggest that this gives a general solution for aggregation. This assertion is, however, unwarranted. However, if absolute changes are approximately uniform, $\Delta x(a)\approx A$, this is the equation showing how the macro aggregates $\langle \Delta c\rangle, \langle \Delta x\rangle$ are connected.

The covariance vanishes identically if either the MPC´*s* or income changes do not vary or are constants. These are rare special cases. In the first case the individual consumption functions are parallel lines, clearly an unrealistic case except in textbook treatments. The covariance is small (and essentially a random term) if the marginal coefficients and changes of inputs are roughly uncorrelated. This is not necessarily the case as the covariance may include systematic information. In the case of the consumption function ordinarily *large* changes of income are connected to *small* marginal propensities to consume which makes the covariance $\operatorname{cov}(\overline{m},\Delta x)$ systematically negative. Before concentrating to this case, we state a proposition.

Proposition 3:
The covariance in (4) vanishes identically for all *Uniform Absolute* changes for which $\Delta x^{UA}(a)=A=$constant for all agents. For Uniform Absolute changes the effective *Marginal Coefficient MC* equals

(5) $MC^{UA}=\langle \overline{m}\rangle=$ equally weighted arithmetic average of $\overline{m}(a)$´s. Conversely, this equation holds *only for UA*-changes, because the covariance in (4) vanishes identically only when absolute changes are constant (for arbitrary $\overline{m}(a)$´*s*).

Uniform absolute changes satisfy

(6) $\langle \overline{m}\Delta x^{UA}\rangle = \langle \overline{m}\rangle\langle \Delta x^{UA}\rangle = MC^{UA}\langle \Delta x^{UA}\rangle$.

This holds for all n-vectors $\overline{m}$ and all *UA*-vectors Δx^{UA}.

This can be derived also as follows: $\langle \overline{m}\Delta x^{UA}\rangle = \langle \overline{m}A\rangle = \langle \overline{m}\rangle A = \langle \overline{m}\rangle\langle \Delta x^{UA}\rangle = MC^{UA}\langle \Delta x^{UA}\rangle$.

Equation (4) can be written for relative changes as follows: $\dfrac{\langle \Delta c\rangle}{\langle c^0\rangle}=\dfrac{\langle x^0\rangle}{\langle c^0\rangle}\langle \overline{m}\rangle\dfrac{\langle \Delta x\rangle}{\langle x^0\rangle}+\dfrac{\operatorname{cov}(\overline{m},\Delta x)}{\langle c^0\rangle}$,

where $e^{UA}=\dfrac{\langle x^0\rangle}{\langle c^0\rangle}\langle \overline{m}\rangle$ is the effective macro elasticity. These equations are as such not very relevant in practical applications, because *UA*-changes in income are rare in practise. For other variables such as experiment or education (in labour economics) it is much more natural. In income policy uniform absolute changes are referred as “euro line” and lump sum changes. If not important as such uniform absolute changes are very important when they are combined with uniform proportional changes in chapter 3.3.

3.2 Uniform proportional changes

More relevant type of changes are *Uniform*[3] *Proportional* changes $\Delta x^{UP}(a) = Px^0(a)$, where the relative changes $\Delta x^{UP}(a)/x^0(a)$ are equal to some constant *P*, say 0.03 = 3%. In income policy such uniform proportional changes of income are referred as "percentage line", all wages are raised 3%. For simplicity, we consider here the ordinary indicator of relative change, where the denominator is the old or base value of the variable. Alternative indicators of relative change and especially log-change are considered in Vartia (1976) and Törnqvist et al (1985). If all inputs (say incomes) change proportionally, $\Delta x^{UP}(a) = Px^0(a)$, the effective macro marginal coefficient is the x^0-weighted average of the micro marginal coefficients. This can be derived easily by inserting: $\langle \overline{m}\Delta x^{UP}\rangle = \langle \overline{m}Px^0\rangle = \langle \overline{m}\, x^0\rangle P = \langle \overline{m}\rangle_{x^0}\langle P\, x^0\rangle = \langle \overline{m}\rangle_{x^0}\langle \Delta x^{UP}\rangle = MC^{UP}\langle \Delta x^{UP}\rangle$.

Next we derive a sufficient condition for the macro *MC* being a x^0-weighted average of the micro marginal coefficients. Divide and multiply by the positive base income $x^0(a)$ and write (2b) in terms of income weighted average and relative changes as follows

$$(7)\qquad \langle \Delta c(x)\rangle = \langle \overline{m}\Delta x\rangle = \left\langle x^0\, \overline{m}\,\frac{\Delta x}{x^0}\right\rangle = \langle x^0\rangle \frac{1}{n\langle x^0\rangle}\sum\nolimits_{\Omega} x^0(a)\overline{m}(a)\frac{\Delta x(a)}{x^0(a)} = \langle x^0\rangle \left\langle \overline{m}\,\frac{\Delta x}{x^0}\right\rangle_{x^0} .$$

The inner product $\langle \overline{m}\Delta x\rangle$ is written here in terms of relative changes, because they are normally much more uniform in actual situations than the absolute changes. Absolute changes normally increase as the income increases, but the proportional changes are usually more or less constant. Next apply *BLA* to the x^0-weighted inner product term

$$(8)\qquad \left\langle \overline{m}\,\frac{\Delta x}{x^0}\right\rangle_{x^0} = \langle \overline{m}\rangle_{x^0}\left\langle \frac{\Delta x}{x^0}\right\rangle_{x^0} + \mathrm{cov}_{x^0}(\overline{m},\frac{\Delta x}{x^0}) = \langle \overline{m}\rangle_{x^0}\frac{\langle \Delta x\rangle}{\langle x^0\rangle} + \mathrm{cov}_{x^0}(\overline{m},\frac{\Delta x}{x^0})\ .$$

Inserting this back to (7) gives the important identity (9).

Proposition 4:

$$(9)\qquad \langle \Delta c(x)\rangle = \langle \overline{m}\Delta x\rangle = \langle \overline{m}\rangle_{x^0}\langle \Delta x\rangle + \langle x^0\rangle \mathrm{cov}_{x^0}(\overline{m},\frac{\Delta x}{x^0}).$$

This rather strange identity seems to be a novelty. The output $\langle \overline{m}\Delta x\rangle$ is equally weighted, but the right hand side contains both x^0-weighted and equally weighted components, especially in the term $\langle \overline{m}\rangle_{x^0}\langle \Delta x\rangle$. The covariance is x^0-weighted and it compares mpc´s and *relative* changes.

The covariance term in (9) tends to vary around zero, because usually relative changes of income are roughly constant. Covariance vanishes if either of its variables becomes a constant

[3] Alternative and shorter expressions are equiproportional (EP) and equiabsolute (EA) changes. These shorter terms are used in our illustrations.

or are uncorrelated with each other. On the other hand, according to the conventional wisdom, the covariance in (4) is systematically negative in case of consumption, because large absolute changes in income are connected with small marginal propensities to consume. The covariance in (9) vanishes identically for all uniform proportional changes $\Delta x^{UP}(a) = Px^0(a)$ or $\Delta x^{UP}(a)/x^0(a) \; = P =$ constant (say 0.03 = 3%). We have proved the following

Proposition 5:
The effective marginal coefficient *MC* for uniform proportional changes equals

(10) $MC^{UP} = \langle \overline{m} \rangle_{x^0} =$ income weighted arithmetic average of $\overline{m}(a)$´s. Conversely, this equation holds *only for UP*-changes (for arbitrary $\overline{m}(a)$*´s*), because the covariance in (9) vanishes only when proportional changes are constant .

Equation (10) can be expressed as

(11) $\left\langle \overline{m}\Delta x^{UP} \right\rangle = \langle \overline{m} \rangle_{x^0} \left\langle \Delta x^{UP} \right\rangle = MC^{UP} \left\langle \Delta x^{UP} \right\rangle$.

These equations hold for all n-vectors $\overline{m}$ and all *UP*-vectors $\Delta x^{UP} = Px^0$.

All components in (6) and (11) are natural macro statistics. Emergency of effective *MC´s* (5) and (10) are shown in figures 3 in our illustrations.

3.3 Arbitrary income changes

Now we turn our attention to arbitrary changes in the input. It is written as the sum of *UA*- and *UP*-components and a non-systematic or random term *u*. This representation corresponds to the combination of the *percentage and euro line terms* in income policy. Wages may be raised by three percent annually $\Delta x^{UP}(a) = 0.03x^0(a)$ plus a constant euro-level increase in hourly wages. The effective marginal coefficient in equation (15) becomes expressed in terms of these uniform components. For actual yearly income changes the percentage line (*UP*-component) normally dominates this representation: the percentage increase in income is more constant than the absolute changes. In principle, the two uniform components may have different signs and arbitrary magnitudes. Also non-systematic income changes may have large effects.

Proposition 6:
Arbitrary changes of income decompose uniquely into its uniform absolute and uniform proportional parts plus a non-systematic component as follows

(12) $\Delta x(a) = \Delta x^{UA}(a) + \Delta x^{UP}(a) + u(a) = \; A + Px^0(a) + u(a)$.

This decomposition and constants (*A*, *P*) follow from regression analysis, where $\Delta x(a)$ is explained by a single variable $x^0(a)$. These are $P = \mathrm{cov}(\Delta x, x^0)/\mathrm{cov}(\Delta x, \Delta x)$ and $A = \langle \Delta x \rangle - P\left\langle x^0 \right\rangle$.The non-systematic component is orthogonal to both uniform components: $u \perp \Delta x^{UA}$ and $u \perp \Delta x^{UP}$. These can be expressed also as familiar conditions $\langle u \rangle = 0$ and $\mathrm{cov}(u, x^0) = \mathrm{cov}(u, \Delta x^{UP}) = 0$ presented in all intermediate text-books of regression analysis.

This decomposition is shown graphically in figures 2 in our illustrations. Because $\langle u \rangle = 0$,

(13) $\langle \Delta x \rangle = \left\langle \Delta x^{UA} \right\rangle + \left\langle \Delta x^{UP} \right\rangle$.

Dividing both sides by $\langle \Delta x \rangle$ gives the weights in $w^{UA} + w^{UP} = 1$, where $w^{UA} = \left\langle \Delta x^{UA} \right\rangle / \langle \Delta x \rangle$ and $w^{UP} = \left\langle \Delta x^{UP} \right\rangle / \langle \Delta x \rangle$. They sum to unity, but need not have the same sign. One of them may be negative and the other positive. They cannot be both negative, because their sum is unity. The weights are closely connected to relative income inequality *RII* and its changes. If $w^{UP} = 1$, then incomes change proportionally and *RII* stays constant. If the weights are both positive, $0 < w^{UP} < 1$, relative changes of income of "poorer" increase more than incomes of "richer". As a result, *RII* decreases. If $w^{UP} > 1$ while w^{UA} is negative, *RII* increases. These results require that the average change in incomes is positive. If it is negative, similar conclusions apply. These results are easier to grasp in concrete examples presented later.

It took a long time to realize that the following equations essentially summarize the situation when $|\langle \Delta x \rangle| >> 0$. The equation (16) where two explanatory income variables are used can be used generally, i.e., even when $|\langle \Delta x \rangle| \approx 0$.

Proposition 7:

(14) $\langle \Delta c(x) \rangle = MC \cdot \langle \Delta x \rangle + \mathrm{cov}(\overline{m}, u)$, where

(15) $MC = MC^{UP} \cdot w^{UP} + MC^{UA} \cdot (1 - w^{UP})$.

Effective marginal propensity to consume MC is not necessarily a weighted average of its components MC^{UP} and MC^{UA} but an **affine combination** of them. Only for positive weights it is a weighted average of these marginal coefficients and lies between them.

It is conjectured that normally most from the income change (13) come from the proportional part and the weight of the *UP*-component $w^{UP} \approx 1$. This makes the yearly values of $MC \approx MC^{UP}$ roughly constant. Income distribution approximately shifts in the log-scale and *RII* such as logarithmic variance of income stays roughly constant. For nearly constant and slowly changing marginal coefficients $MC \approx MC^{UP}$ time series estimations make sense. Sample period estimate of the regression coefficient would be an average of these yearly changing parameters. Although regression analysis is not accurate, it gives sensible results. This is our best explanation and rationalization of time series regression analysis of consumption, which cannot produce accurate estimates.

If the weights have different signs, MC is an extrapolation of its uniform components. The covariance $\mathrm{cov}(\overline{m}, u)$ between the mpc´s and the residual *u* in (12) is the unsystematic component of (3) and it varies more or less randomly around zero. For all sums $\Delta x(a) = \Delta x^{UA}(a) + \Delta x^{UP}(a) \ = A + Px^0$ the residual term $u(a) \equiv 0$ in (12) and thus the covariance $\mathrm{cov}(\overline{m}, u)$ in (14) vanishes identically. If the residual term is small also its covariance with the *mpc´s* remains small.

Emergency of (15) is shown in figures 3 and 5 in our illustrations.

Proof: $\langle \Delta c(x)\rangle = \langle \overline{m}\Delta x\rangle = \langle \overline{m}(\Delta x^{UA} + \Delta x^{UP} + u)\rangle$ Insert (12) into (2b)

$= \langle \overline{m}\Delta x^{UA}\rangle + \langle \overline{m}\Delta x^{UP}\rangle + \langle \overline{m}u\rangle$ linearity of the inner product

$= MC^{UA}\langle \Delta x^{UA}\rangle + MC^{UP}\langle \Delta x^{UP}\rangle + \mathrm{cov}(\overline{m},u)$ by (6) and (11), $\langle u\rangle = 0$

$= \dfrac{MC^{UA}\langle \Delta x^{UA}\rangle + MC^{UP}\langle \Delta x^{UP}\rangle}{\langle \Delta x^{UA}\rangle + \langle \Delta x^{UP}\rangle}\langle \Delta x\rangle + \mathrm{cov}(\overline{m},u)$ by (13) when $|\langle \Delta x\rangle| > 0$

Here $\mathrm{cov}(\overline{m},u) = \langle \overline{m}u\rangle - \langle \overline{m}\rangle\langle u\rangle = \langle \overline{m}u\rangle$. Using the definition of the weights $w^{UA} + w^{UP} = 1$ gives us equations (14-15). QED.

Proposition 8:
An equivalent alternative to (14)-(15) is

(16) $$\langle \Delta c\rangle = MC^{UA}\langle \Delta x^{UA}\rangle + MC^{UP}\langle \Delta x^{UP}\rangle + \mathrm{cov}(\overline{m},u) .$$

It has two macro inputs (13) instead of single input and a more variable *MC* in (14). Both these decompositions are illustrated in figures 4 below.

3.4 Effect of income transfers on macro consumption

Instead of economics, we may illustrate this problem by an example from medicine. What happens if the dose of medicine is decreased for patients whose effect of medicine are exceptionally low and the changes in doses are transformed to patients with higher effects? The average effect, of course, increases even though the total amount of medicine remains the same! The average effect of the medicine increases.

In economics and consumption, this corresponds to income transfers from high income earners (with low marginal propensity to consume) to low income earners. For every such individual income transfer, the consumption increases while the total and average income remains constant. In symbols: $\langle \overline{m}\Delta x\rangle$>0 while $\langle \Delta x\rangle = 0$. This effect does not occur only if all the marginal coefficients are the same, which implies that the agent-specific behavioural functions are *parallel lines* in respect to the relevant variable (income). This is extremely restrictive.

As a summary, macro output (consumption) changes even though macro income is constant. This cannot be explained by a text-book model, where income is represented by total or mean income. Other information on the income distribution (say its coefficient of variation or logarithmic variance) is not utilized – partly because they are not part of SNA or have not been produced otherwise. SNA directs the building of econometric models more generally than only in text-books, because its time-series are used in specification and estimating the models.

This cannot be explained by text-book models, which are based on the information of SNA only. SNA contains only value sums and time series based (usually) on Laspeyres price or quantity indices. (Tell me if it is explained somewhere else.)

In progressive income taxation: Income transfer from high income earners to low ones has mean zero but negative tax effects. Every high income earner has a larger marginal tax than the income receiver.

In consumption function: An income transfer from high to low income would have mean zero (because positive and negative parts cancel each other) but positive effect on mean consumption. These effects cannot be hidden anywhere. Zero-mean input variable is connected to (or "causes") non-negative outputs. In this situation effective macro MC is either plus or minus infinity. More generally, zero-mean input variable can be planned to have non-zero output effects (and normally has such without any planning if micro mc's differ from each other) and MC gets infinitely small or large values. For non-zero mean input such effects are included ("below the surface") and do not cause infinities. As a summary: MC varies from one situation to another, especially if input variable has zero mean.

In the MicMac-function (2) we have possibly millions of terms $\overline{m}(a)\Delta x(a)/n =$ effect coefficient*change in input. The effect coefficients and input changes both vary normally from agent to another. We cannot aggregate these millions of equations into *one term* of the form $MC\langle \Delta x \rangle + \varepsilon =$ macro effect coefficient*macro change in input + error term, because for income transfers its main term becomes $MC * \langle \Delta x \rangle = (\pm\infty) * 0$. This is undefined mathematically. But we will demonstrate in chapter 3.3 that the sum of these millions of equations can be sensibly aggregated into a sum of *two terms*, namely as $\langle m \rangle \langle \Delta x^{UA} \rangle + \langle m \rangle_{x^0} \langle \Delta x^{UP} \rangle + \varepsilon$. Here the macro input change $\langle \Delta x \rangle$ is decomposed as the sum of Uniform Absolute and Uniform Proportional parts *UA* and *UP*. All the terms in this decomposition are intuitively and mathematically well defined. In this way in the macro equation an input variable gives rise to two macro variables, its of Uniform Absolute and Uniform Proportional parts. We anticipate, that this idea will not be understood at once but will be accepted after some time.

The affine combination is not a weighted average, unless all the input changes have the same sign. The most important methodological implication is that the macro MPC $\langle \overline{m} \rangle_{\Delta x}$ is not a constant but varies from one distribution of input changes to another. Compared to the index number problem, difficulties arise because neither the marginal coefficients nor the input changes are measured in the ratio scale. They both may get both positive and negative values. The peculiar property of $\langle \overline{m} \rangle_{\Delta x}$ is that its value may lie outside the range (minimum and maximum) of the marginal coefficients. This implies that the macro reaction is not necessarily the average of the micro reactions. As an example, suppose that the marginal coefficients are all positive and the input changes are both positive and negative in such a way that these two vectors are orthogonal, i.e. $\langle \overline{m}\Delta x \rangle = 0$. Then $\langle \Delta c \rangle = \langle \overline{m} \rangle_{\Delta x} \langle \Delta x \rangle = 0 \cdot \langle \Delta x \rangle = 0$ for all mean values $\langle \Delta x \rangle$ of the inputs.

It is even possible for positive marginal coefficients, that $\langle \overline{m} \rangle_{\Delta x}$ is negative and therefore positive $\langle \Delta x \rangle$ implies $\langle \Delta c \rangle = \langle \overline{m} \rangle_{\Delta x} \langle \Delta x \rangle < 0$. This will first feel strange.
An even better way to reveal the behaviour in this situation is to decompose the input change into uniform proportional and additive parts (plus a deviation). This leads to expression (16):

$$\langle \Delta c(x) \rangle = MC^{UA} \langle \Delta x^{UA} \rangle + MC^{UP} \langle \Delta x^{UP} \rangle + \mathrm{cov}(\overline{m}, u).$$

This is a non-linear function of the original inputs due to the decomposition input changes, see (12-14). If $\langle \Delta x \rangle = \langle \Delta x^{UA} \rangle + \langle \Delta x^{UP} \rangle = 0$, the affine combination (15) of the *MC´s* is either plus or minus infinity. This feels first as a serious instability (say, in the Keynesian multiplier effects). Rewrite this equation by adding $0 = MC^{UA}\left\langle \Delta x^{UP} \right\rangle - MC^{UA}\left\langle \Delta x^{UP} \right\rangle$. Regrouping terms gives

$$\left\langle \Delta c(x) \right\rangle = MC^{UA}\left\langle \Delta x \right\rangle + (MC^{UP} - MC^{UA})\left\langle \Delta x^{UP} \right\rangle + \operatorname{cov}(\overline{m}, u).$$

In this situation, we have $\left\langle \Delta x^{UA} \right\rangle = -\left\langle \Delta x^{UP} \right\rangle$ (pure income transfers) and the equation transforms to

$$\left\langle \Delta c(x) \right\rangle = (MC^{UP} - MC^{UA})\left\langle \Delta x^{UP} \right\rangle + \operatorname{cov}(\overline{m}, u).$$

Macro reaction in respect to this input variable shows no instabilities. The natural variables in macro behavioural equation are $\left\langle \Delta x^{UA} \right\rangle, \left\langle \Delta x^{UP} \right\rangle$ and not $\left\langle \Delta x \right\rangle$, which would show instable behaviour for nearly zero values. In this way we will take into account the *change* in input distribution.

3.5 Summary list of key concepts

Equations (14)-(16) are remarkably simple considering the generality of the problem. But there is a fairly long list of intermediate concepts involved. As a summary, we list the definitions of our concepts for easy reference.

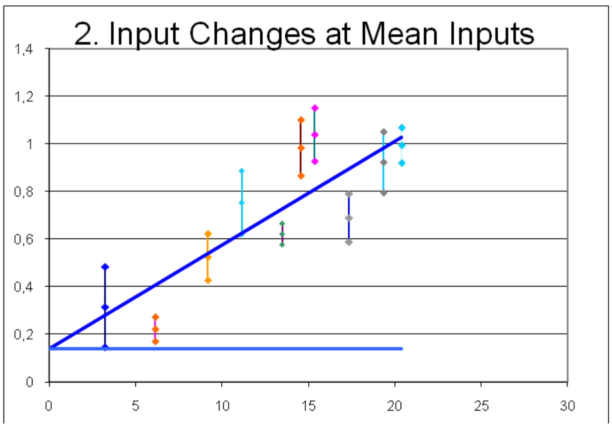
- $c(a) = f^a(x(a)) = f^a(x_1(a), \ldots, x_K(a))$ — Agent-specific differentiable behaviours
- $\left\langle c \right\rangle = \frac{1}{n}\sum_{\Omega} c(a) = \frac{1}{n}\sum_{\Omega} f^a(x(a))$ — Mean output in terms of micro inputs
- $\Delta c(a) = \sum_{k=1}^{K} \overline{m}_k(a)\Delta x_k(a)$ — Mean value theorem
- $\left\langle \Delta c \right\rangle = \sum_{k=1}^{K} \left\langle \overline{m}_k \Delta x_k \right\rangle$ — Change in macro consumption
- $\left\langle \Delta c(a) \right\rangle = \left\langle \overline{m}_k \Delta x_k \right\rangle$ — Contribution of the variable x_k
- $\left\langle \Delta c(x) \right\rangle = \frac{1}{n}\sum_{\Omega} \overline{m}(a)\Delta x(a) = \left\langle \overline{m}\Delta x \right\rangle$ — The inner product of $\overline{m}$ - and Δx -vectors
- $\left\langle xy \right\rangle_w = \left\langle x \right\rangle_w \left\langle y \right\rangle_w + \operatorname{cov}_w(x, y)$ — The basic lemma of aggregation *BLA*
- $MC^{UA} = \left\langle \overline{m} \right\rangle = \frac{1}{n}\sum_{\Omega} \overline{m}(a)$ — Uniform absolute marginal coefficient
- $MC^{UP} = \left\langle \overline{m} \right\rangle_{x^0} = \frac{1}{n\left\langle x^0 \right\rangle}\sum_{\Omega} x^0(a)\overline{m}(a)$ — Uniform proportional marginal coefficient
- $\Delta x(a) = \Delta x^{UA}(a) + \Delta x^{UP}(a) + u(a)$ — Decomposition into uniform components
- $\left\langle \Delta x \right\rangle = \left\langle \Delta x^{UA} \right\rangle + \left\langle \Delta x^{UP} \right\rangle$ — Decomposition in the macro level
- $w^{UP} = \left\langle \Delta x^{UP} \right\rangle / \left\langle \Delta x \right\rangle$ — Weight of the *UP*-component
- $MC = MC^{UP} \cdot w^{UP} + MC^{UA} \cdot (1 - w^{UP})$ — The effective macro marginal coefficient
- $\left\langle \Delta c \right\rangle = MC \cdot \left\langle \Delta x \right\rangle + \operatorname{cov}(\overline{m}, u)$ — The final macro equation

4. Illustrative calculations

Illustrations in chapter 4 deal with three cases: Relative income inequality *RII* stays almost constant, increases and decreases. All the cases are based on five figures which illustrate the algebraic theory presented above. First figure shows small segments of 10 heterogeneous agent behaviors (behavioral functions). Input changes of the three illustrations differ as the figures 2 show. The horizontal and upwards or downwards sloping lines show how the input changes are decomposed into its uniform components by equation (12). Figures 3 show how the micro and macro marginal coefficients are related: uniform macro coefficients are shown by triangles and horizontal lines and the effective *MC* determined in equation (15) is marked by a circle. All the macro components of our decompositions (14)-(16) are illustrated in one figure 4. The important conclusion of equation (15) appears in figures 5, where in the illustration 1 the circle is between the triangles, but below or above them in illustrations 2 and 3, respectively.

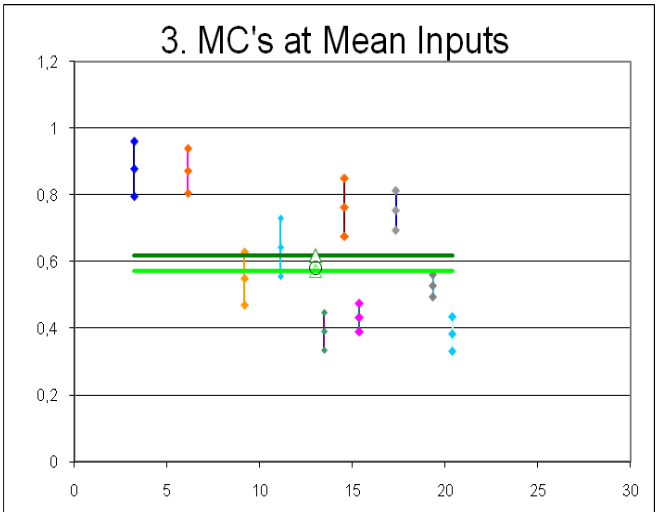
Illustration 1: Relative income inequality *RII* stays almost constant

Agent specific behavioural functions $(x(a), c(a)) = (x(a), f^a(x(a)))$ are shown for input intervals $x(a) \in [\bar{x} - \frac{1}{2}\Delta x(a), \bar{x} + \frac{1}{2}\Delta x(a))]$ in figure 1. The heavier part of the segments shows the direction of the change and all the "arrows" are directed upwards here. The horizontal axis is the income $x(a)$ and the vertical axis is the consumption $c(a) = f^a(x(a))$. All the changes $\Delta x(a)$ in this figure are positive and they increase as the level of income $\bar{x}(a)$ increases as shown in figures 1 and 2. (Vertical lines in figures 2 and 3 are drawn in order to aid vision and have no concrete meaning.) All figures correspond roughly to stylized facts of the consumption function. Consumption is a positive proportion of income and the marginal propensity to consume is also positive and smaller than one.



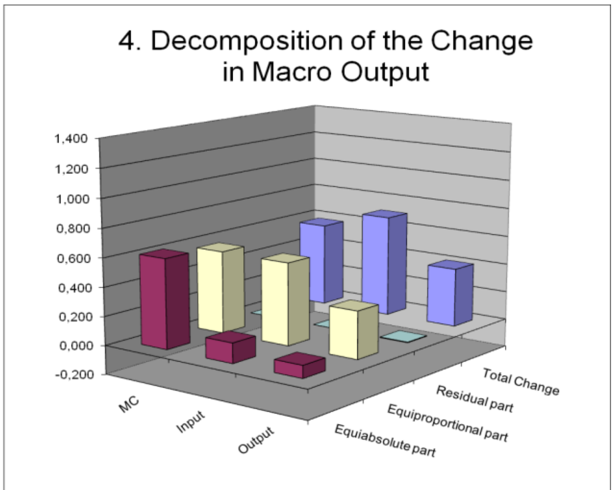
Observations $(\bar{x}, \Delta x(a))$ and the decomposition of the income changes in (12) are shown in figure 2, where the horizontal axis is the income $x^0(a)$. Vertical axis measures the horizontal input changes in figure 1. The change in income is typically bigger the bigger the level of income. Income changes of high income earners are almost five times those of low income earners. In the first illustration the uniform absolute and proportional parts are both positive: $\langle \Delta x^{UA} \rangle = 0{,}138$ and $\langle \Delta x^{UP} \rangle = 0{,}569$. These are determined from the regression analysis shown by the lines in figure 2. The uniform proportional change *P* in (12) equals 4.4% and it is the slope of the upwards sloping regression line. The horizontal line gives the intercept of the regression line as shown algebraically in equation (12).

This situation corresponds to a centralized income policy agreement *CIPA* where the general percentage increase in wages is 4.4% ("percentage line") accompanied with "the euro line" or lump sum increase of 0,138. Using equation (12) and figure 2 we infer that relative income inequality *RII* measured e.g. by Gini-coefficient or log-variance slightly decreases. Small income earners have greater *relative* changes of income than the high income earners, because of positive uniform *absolute* changes of income. (Vertical segments of agent-data in figures 2-3 are used for visualization purposes only.)

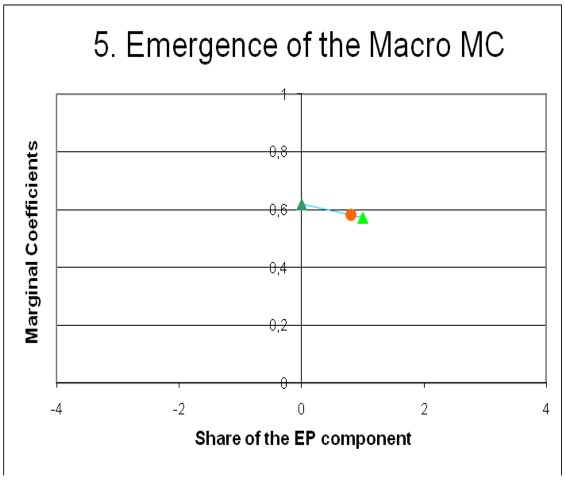


Marginal propensity to consume $\overline{m}(a)$ decreases as a function of income $\overline{x}(a)$ as the third figure shows. From the horisontal lines and the triangles on them in figure 3, figure 5 and table in figure 4 we note that $MC^{UA} = 0{,}619$ and $MC^{UP} = 0{,}572$. The latter is smaller as the smaller high income mpc´s $\overline{m}(a)$ are weighted more in the income weighted average (10) than in the equally weighted one (5). The circle shows the effective macro MC = 0,581 and in this illustration 1 it is between its uniform components but nearer $MC^{UP} = 0{,}572$.

Figure 4 shows all the macro components of our decompositions (14)-(16). It deserves a careful consideration.



	MC	Input	Output
Equiabsolute part	0,619	0,138	0,085
Equiproportional part	0,572	0,569	0,325
Residual part	0,000	0,000	-0,004
Total Change	0,581	0,707	0,407



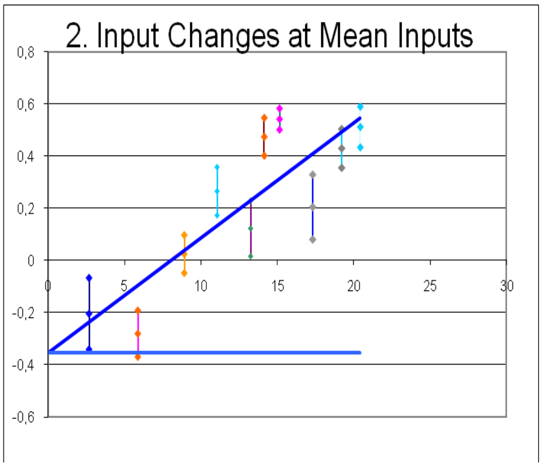
In figure 5 the equation (15) is illustrated. By equation (15) and figures 4 and 5 the effective *MC* (circle in figures 3 and 5) moves slightly from its uniform proportional part towards its uniform absolute component:

$$MC = MC^{UA} \cdot w^{UA} + MC^{UP} \cdot w^{UP} = 0{,}619 \cdot w^{UA} + 0{,}572 \cdot w^{UP} = 0{,}581\,.$$

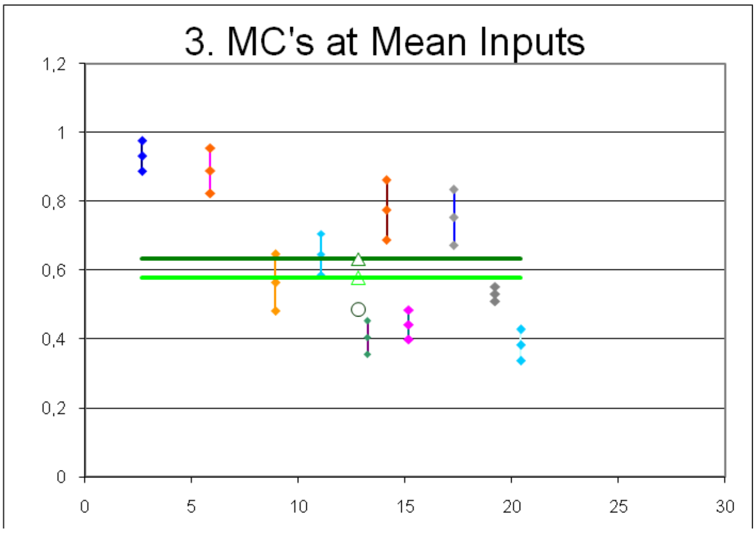
Here both weights are positive: $w^{UP} = 0{,}76$ and $w^{UA} = 0{,}24$, see figures 3, 4 and especially 5.

Illustration 2: Relative income inequality *RII* increases

The second set of figures 1-5 provides an illustration where small incomes decrease while large incomes increase as shown in figure 2. This causes *RII* to increase. This would be a strange income development with positive percent line accompanied with negative euro line change of income, see figure 2.



Individual *MC*´s depend negatively on income like in the previous illustration and the uniform macro components of them (MC^{UA}, MC^{UP}) shown by horizontal lines and small triangles are almost the same as before.



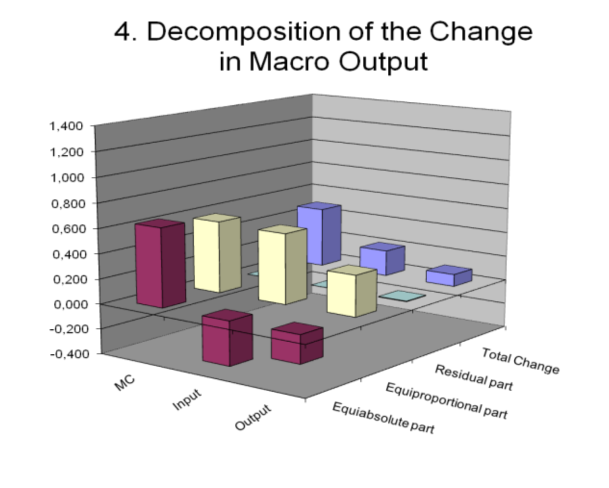
From figure 4, $w^{UP} = 0{,}563 / 0{,}209 = 2{,}694$ and the effective macro *MC* shown by the circle below the triangles becomes

$$MC = MC^{UA} \cdot w^{UA} + MC^{UP} \cdot w^{UP} = 0{,}632 \cdot (-1{,}694) + 0{,}578 \cdot 2{,}694 = 0{,}486 \, .$$

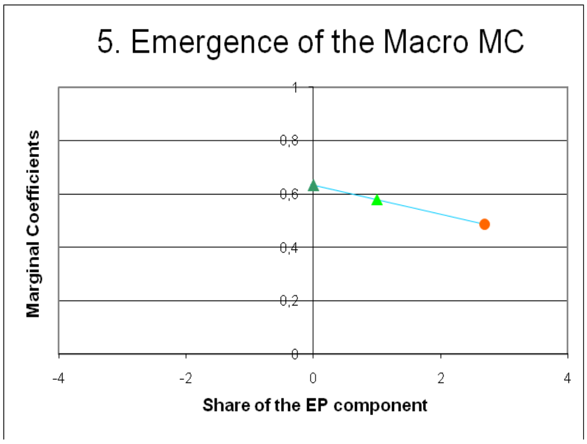
The relative income inequality *RII* increases and the affine combination of marginal coefficients 0,486 is on the extrapolation side of the uniform components as shown in figure 5. This can be also inferred qualitatively from formula

$$\langle \overline{m}\Delta x \rangle = \tfrac{1}{n}\sum\nolimits_{\Omega} \overline{m}(a)\Delta x(a) = \left(\tfrac{1}{n\langle \Delta x\rangle}\sum\nolimits_{\Omega} \Delta x(a)\overline{m}(a)\right)\langle \Delta x \rangle = \langle \overline{m} \rangle_{\Delta x} \langle \Delta x \rangle \, . \tag{20}$$

Here $\langle \overline{m} \rangle_{\Delta x}$ is an affine combination of $\overline{m}(a)$´s with rather odd weights $\Delta x(a)$. From figures 2-3 small $\overline{m}(a)$´s are weighted positively while high $\overline{m}(a)$´s are weighted negatively. This makes $\langle \overline{m} \rangle_{\Delta x}$ small.



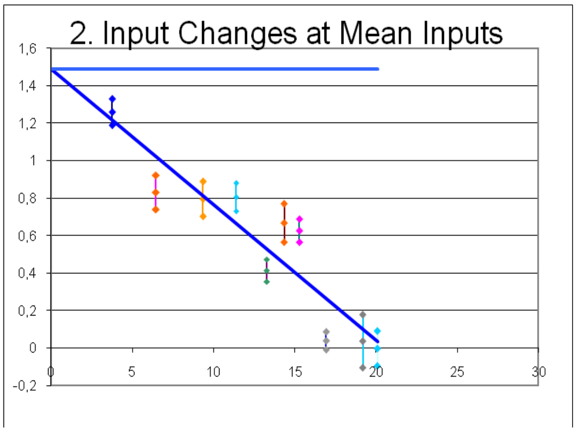
	MC	Input	Output
Equiabsolute part	0,632	-0,354	-0,224
Equiproportional part	0,578	0,563	0,326
Residual part	0,000	0,000	-0,004
Total Change	0,486	0,209	0,098



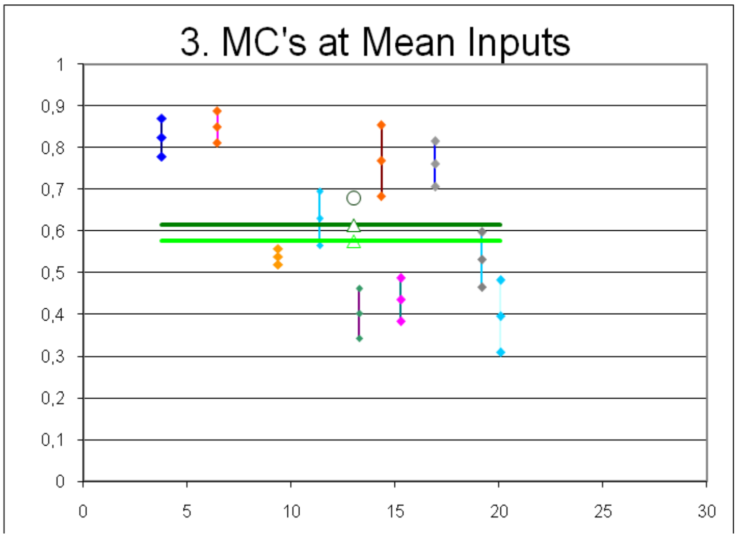
Small triangles and a circle in figures 3 and 5 show the position of the three *MC´s*. Figure 4 deserves careful consideration because it shows all the components and connections of the problem in one figure.

Illustration 3: Relative income inequality *RII* decreases

In the third illustration, *RII* strongly decreases because small incomes increase while the large incomes stay the same, see figures 1 and 2.

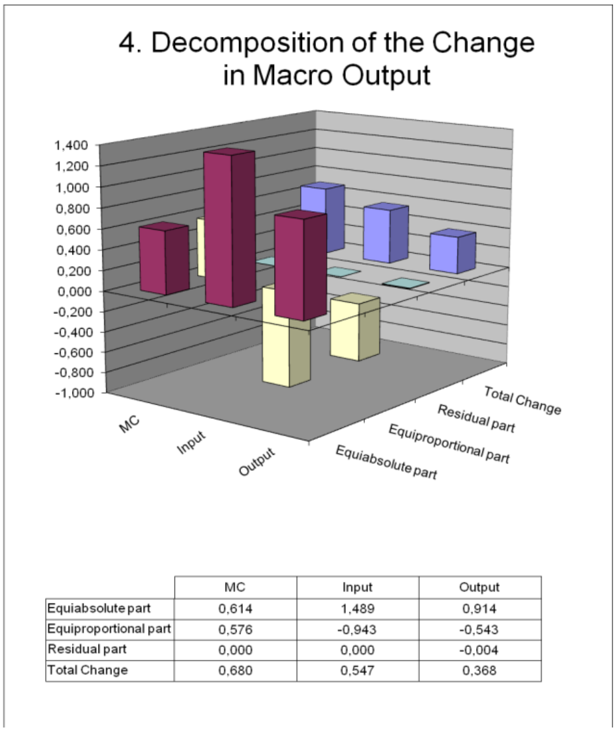


Note the two uniform *MC´s* shown by triangles and their extrapolation shown by a circle are now above them in figures 3 and 5.

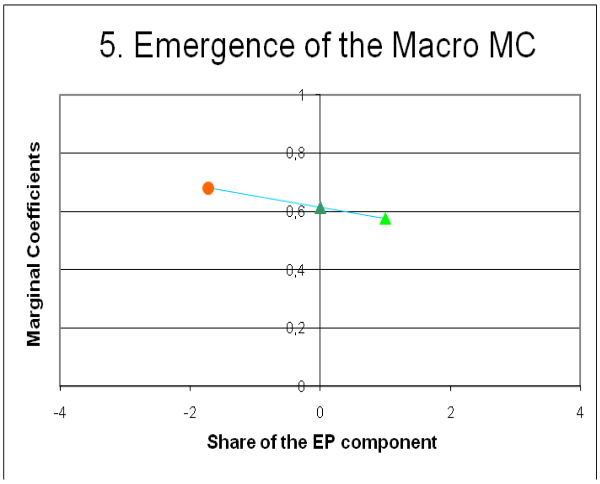


From figure 4, $w^{UP} = -0{,}943 / 0{,}547 = -1{,}724$. The affine combination of marginal coefficients is on the other extrapolation side of its uniform components:

$$MC = MC^{UA} \cdot w^{UA} + MC^{UP} \cdot w^{UP} = 0{,}614 \cdot 2{,}724 + 0{,}576 \cdot (-1{,}724) = 0{,}680 \ .$$



	MC	Input	Output
Equiabsolute part	0,614	1,489	0,914
Equiproportional part	0,576	-0,943	-0,543
Residual part	0,000	0,000	-0,004
Total Change	0,680	0,547	0,368



As a summary, the illustrations show how three different changes in income distribution produce different effective marginal propensities to consume, namely 0,581, 0,486, 0,680. These are mostly affected by the weights $(w^{UP}, w^{UA} = 1 - w^{UP})$. *MC* in (15) remains practically constant in hypothetical simulations where the macro variable $\langle \Delta x \rangle$ in (14) changes but the weight w^{UP} in (16) remains constant.

5. General conclusions

Conclusions concerning the Keynesian consumption function are as follows. The effective marginal propensity to consume MC^t in (15) for a given quarter t is not time invariant but varies systematically from one situation to another. It decomposes into uniform absolute and uniform proportional marginal coefficients and is an affine combination (16) of these. Changes of MC^t and therefore of consumption depend on the income distribution, especially on changes of the relative income inequality *RII*. The variation of MC^t is, however, in normal situations more a question of principle than of practical importance. These results generalize immediately to any output and several ratio scale input variables

(21) $\langle \Delta y^t \rangle = \sum_{k=1}^{K} MC_k^t \cdot \langle \Delta x_k^t \rangle + \varepsilon^t$

(22) $MC_k^t = MC_k^{t,UA} \cdot (1 - w_k^{t,UP}) + MC_k^{t,UP} \cdot w_k^{t,UP}$

(23) $\varepsilon^t = \sum_{k=1}^{K} \mathrm{cov}(\overline{m}_k^t, u_k^t)$,

where every MC_k^t is an affine combination of its uniform absolute and uniform proportional components. If the weight $w_k^{t,UP}$ stays stable around zero or one, also MC_k^t is a stable parameter. This is conjectured to hold normally and it would imply that variables have stable systematic marginal coefficients in the macro equation. We claim that this is the first rather realistic explanation why macro economic models expressed in terms of SNA aggregates work under normal conditions. The unsystematic part ε^t of the macro function is a sum of covariances between micro marginal coefficients $\overline{m}_k^t(a)$ and input residuals $u_k^t(a)$. Our conjecture is that ε^t will be responsible for most of the random effects observed in a macro equation.

These all are operational macro statistics, which can be estimated from a sample of the micro level information in analogy to experiments of chemistry referred on page 3. Macro behaviour reduces to its micro connections. This means that macro economics is not so strongly autonomous that it can be completely separated from micro economics. Enqvist (2007, p. 297) is in favour of Unified Theory and criticizes the concept of "autonomous science" separated from its micro connections. He describes the thinking of the proponents of autonomous chemistry as follows. "Chemistry begins when a physicist has completed his work, after which he can be given a gold watch for a long service, shake hands as farewell and be forgotten. The idea is that chemistry is so to speak closed from below and it needs no references to physics". Enqvist and other reductionists think otherwise.

According to equations (21)-(23) **no separate macro modelling is needed**. Macro behaviour reduces to micro dependences by mathematical reasoning. The macro effect of a single variable is $MC_k^t \cdot \left\langle \Delta x_k^t \right\rangle$ which becomes small if either the effective marginal coefficient or the average change of the variable is small. This explains why "frozen variables" such as gender or organization structure cannot be used in time series macro modeling although their *MC* may differ clearly from zero. They are potential explanatory variables which become effective only if large structural changes appear. For instance, gender effects cannot be estimated from macro time series, because the proportion of men (and of women) is practically constant. They are almost collinear with the intercept of the regression model.

This makes it comprehensible why important structural variables are omitted in macro economics.

Another possibility of a variable having significant effects on the micro level but practically none in the macro level occurs when micro *MC*´s distribute symmetrically around zero. Then $MC_k^t \approx 0$ and no macro effects appear even if $\left\langle \Delta x_k^t \right\rangle$ differs from zero.

Also the well-known unreliability of macro time series models is explained. Instead of quarterly changing parameters some average of them is estimated using too few (say 4*40 = 160) observations. These reasons do not disappear anywhere and thus the **knowledge does not increase**. This kind of modelling is equally hopeless than estimating the difference of social democratic and conservative parties (in Finland) in terms of 160 Gallup interviews. Supporters of these parties in a random sample of 160 could be roughly 20±6 voters or 24%±6% with 95% confidence, which are much too unreliable to give statistically significant differences. Sample size has to be at least 1000 to get any significant differences for these almost equally popular (20%) parties.

Similar unreliability holds also for our economic time series modelling, where 160 refers to the number of quarters of the last 40 years. Older past would be irrelevant ancient history, which prohibits increasing the sample size from 160 quarters. The current **macro modelling is thus a poor strategy**. Time series estimates need not be badly biased, but are inaccurate and unreliable. Mixed micro and macro strategies using panel or regional series would make better sense and be more efficient. For instant, 25 countries of EU would increase the number of observations to 4000. This amount of observations would allow us to infer more than that the parameter differs from zero at the 5 % risk level.

Unreliability of macro models arises while micro information, like panel series and integration of macro and micro behaviours, are not properly utilized. The main fault can be described as **Neglected Information Bias**. By integration of micro and macro explanations we are, however, able to remove these problems and achieve gradually a new level of accuracy in macro economics.

References:

Allen, R.G.D (1964): *Mathematical Economics*, Macmillan Co Ltd.

Apostol, Tom M. (1963): *Mathematical Analysis. A Modern Approach to Advanced Calculus.* Addison-Wesley Publishing Company. Reading, Massachusetts – Palo Alto – London.

Arrow, K.J. (1963): *Social Choice and Individual Values,* Wiley, New York.

Van Daal, J. and Merkies, A.H.Q.M. (1984): *Aggregation in Economic Research. From Individual to Macro Relations,* D. Reidel Publishing Company, Dorderch/Boston/Lancaster.

Edgren, C. - Turkkila, J. – Vartia, Y. (1978): Tuloverotuksen analysoinnin matemaattisista ongelmista. Elinkeinoelämän tutkimuslaitoksen Keskusteluaiheita No 17, Helsinki

Enqvist, Kari (2007): *Monimutkaisuus,* (Complexity in English), WSOY, Helsinki.

Gorman, W.M. (1959): Separable Utility and Aggregation, *Econometrica, 27, 53-56.*

Green, H.A.J. (1964): *Aggregation in Economic Analysis, an Introductory Survey,* Princeton University Press, Princeton.

Klein, L.R. (1946): Remarks on the Theory of Aggregation, *Econometrica, 14, 303-312.*

Klein, L.R. (1956): *The Keynesian Revolution,* Macmillan & Co LTD, London

Kolmogorov, A.N. & Fomin, S.V: (1970): *Introductory Real Analysis,* (Translated and edited by Richard A. Silverman). Dover Publications, Inc., New York.

Leontief, W.W. (1947): Introduction to a Theory of the Internal Structure of Functional Relationships, *Econometrica, 15, 361-373.*

Lintunen, J., Ropponen, O. and Vartia, Y. (2009): Micro meets Macro via Aggregation, HECER Discussion Papers No. 259, March 2009.

Monod, J. (1971): *Chance and Necessity, an Essay on the Natural Philosophy of Moderns Biology,* Albert A. Knopf, New York.

Nataf, A. (1948): Sur la Possibilite de Construction de Certains Macromodeles, *Econometrica, 16, 232-244.*

Suoperä, A. and Vartia, Y. (2011): Analysis and Synthesis of Wage Determination in Heterogeneous Cross Sections, HECER Discussion Papers No. 331, June 2011.

Theil, H. (1954): *Linear Aggregation of Economic Relations,* North-Holland Publishing Company, Amsterdam.

Turkkila, J. (2011): *Tuloverotuksen reaalinen kiristyminen Suomessa. Kuusikymmentä vuotta verojen vuoristorataa.* The Research Institute of the Finnish Economy, Series A46 Helsinki.

Törnqvist, L., Vartia, P. and Vartia, Y. (1985): How Should Relative Changes Be Measured? *The American Statistician*, February 1985, Vol. 39, No. 1.

Vartia, Yrjö (1976): *Relative Changes and Index Numbers* (Academic Dissertation in Statistics, 203 p). The Research Institute of the Finnish Economy, Series A4, Helsinki.

Vartia, Yrjö (2008a): Integration of Micro and Macro Explanations, HECER Discussion Papers No. 239, November 2008.

Vartia, Yrjö (2008b): On the Aggregation of Quadratic Micro Equations, HECER Discussion Papers No. 248, December 2008.

Vartia, Yrjö (2009): Whole and its Parts: Micro Foundations of Macro Behaviours, HECER Discussion Papers No. 257, March 2009.

Wilson, E.O. (1998): *Consilience. The Unity of Knowledge*, Albert A. Knopf, New York.