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Stress intensity factor histories of a steadily propagating crack under 3-D combined mode traction

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Abstract. Elastodynamic stress intensity factor histories of an unbounded solid containing a semi-infinite plane crack that propagates at a constant velocity under 3-D time-independent combined mode loading are considered. The fundamental solution, which is the response of point loading, is obtained. Then, stress intensity factor histories of a general loading system are written out in terms of superposition integrals. The methods used here are the Laplace transform methods in conjunction with the Wiener–Hopf technique.

Key words: stress intensity factor history, Wiener–Hopf technique.

Introduction

Some data on dynamic fracture measurements show that a number of three-dimensional effects are of sufficient importance to warrant further investigation. Freund (1971) first investigated the reflection of Rayleigh waves traveling on the faces of a crack and obliquely incident on the crack edge. Achenbach and Gautesen (1977) solved the elastodynamic steady-state problem for a semi-infinite crack under 3-D loading. Freund (1987) dealt with the case of incident stress-wave loading, and some extensions have also been considered by Ramirez (1987) and Champion (1988). Most recently, Li Xiang-Ping and Liu Chun-Tu (1994; 1995) published a series of papers for 3-D elastodynamic crack problems. They presented the method that will be employed in this paper to deal with the coupled Wiener–Hopf equations.

In this paper, a combined mode loading problem is considered. That is, the dynamic stress intensity factor histories for a half-plane crack extending uniformly in an otherwise unbounded elastic body under a combined loading system are to be found. This problem is the three-dimensional analogue of the plane strain problems solved by Fossum and Freund (1975). Due to the fact that the phenomena under II and III loading are combined in 3-D problems, II and III loading cases cannot be easily treated separately in three-dimensional problems. One may note in the results of this paper that both the II and III loading can lead to both II and III responses.

In the fundamental problem the crack is subjected to two pairs of suddenly applied point loading, which are time independent and tend to tear and shear open the crack. The problem yields two coupled Wiener–Hopf equations, which will be treated via the aforementioned method used by Li et al. (1994). Finally, the stress intensity factors for general loading are obtained by means of superposed integrals.

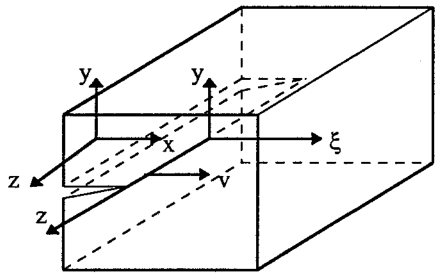

Figure 1. The geometry of the elastic body.

Fundamental Solution

1. Statement of problem

Consider the elastic body containing a half-plane crack depicted in Figure 1. The body is initially stress free and at rest. At time $t = 0$, two pairs of point forces appear on the crack tip tending to slide and tear open the crack separately. Then, while the crack tip moves in the x-direction, the forces remain acting at the origin.

Two coordinate systems are employed in this paper as shown in Figure 1. It is assumed that the crack tip passes $x = 0$ at time $t = 0$. The moving $o' - \xi yz$ system translates at speed $v(v < c_R)$ in the x or ξ direction, and the crack edge is on the z-axis of the moving coordinate system. The relation between these two coordinate systems is:

$$\xi = x - \nu t; \quad y = y; \quad z = z.$$

Because this problem is antisymmetric about the plane $y = 0$, attention is restricted to the upper half-space $y \geqslant 0$. The boundary conditions in terms of displacements and stresses in the $o - xyz$ coordinate system are:

$$\begin{aligned}
&\sigma_{yy}(x,0,z,t) = 0,\\
&\sigma_{xy}(x,0,z,t) = P_1 H(t)\delta(x)\delta(z), \quad x < vt,\\
&\sigma_{yz}(x,0,z,t) = P_2 H(t)\delta(x)\delta(z), \quad x < vt,\\
&u_x(x,0,z,t) = 0, \quad x > vt,\\
&u_z(x,0,z,t) = 0, \quad x > vt,
\end{aligned} \tag{1}$$

where $H(t)$ is the Heaviside step function and $\delta(z)$ is the Dirac delta function. The boundary conditions listed above are valid for $z \in (-\infty, \infty)$ and $t \in (0, \infty)$.

All the initial conditions are zero.

This problem can be expressed in terms of the scalar dilatational wave potential ϕ and the vector shear wave potential $\boldsymbol{\psi} = (\psi_x, \psi_y, \psi_z)$ (see Appendix A). The potentials ϕ and $\boldsymbol{\psi}$ satisfy the following wave equations

$$\begin{aligned}
&\nabla^2\phi - a^2\frac{\partial^2\phi}{\partial t^2} = 0,\\
&\nabla^2\boldsymbol{\psi} - b^2\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = 0, \quad \text{for } y > 0,\\
&\nabla\cdot\boldsymbol{\psi} = 0,
\end{aligned} \tag{2}$$

where $a = 1/c_d, b = 1/c_s, c_d$ and c_s are the velocities of dilatational wave and shear wave, respectively.

It proves convenient to work with the moving coordinate system. Let $d = 1/v$, and Equations (2) become, in the coordinate system (ξ, y, z):

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + \left(1 - \frac{a^2}{d^2}\right)\frac{\partial^2\phi}{\partial \xi^2} + 2\frac{a^2}{d}\frac{\partial^2\phi}{\partial\xi\partial t} - a^2\frac{\partial^2\phi}{\partial t^2} = 0,$$
$$\frac{\partial^2\boldsymbol{\psi}}{\partial y^2} + \frac{\partial^2\boldsymbol{\psi}}{\partial z^2} + \left(1 - \frac{b^2}{d^2}\right)\frac{\partial^2\boldsymbol{\psi}}{\partial \xi^2} + 2\frac{b^2}{d}\frac{\partial^2\boldsymbol{\psi}}{\partial\xi\partial t} - b^2\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = 0, \tag{3}$$
$$\frac{\partial\psi_y}{\partial y} + \frac{\partial\psi_z}{\partial z} + \frac{\partial\psi_x}{\partial \xi} = 0.$$

After being extended to the full range of ξ the boundary conditions take the following forms in the $o' - \xi yz$ system:

$$\lambda\left(\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\psi_x}{\partial y\partial z} - \frac{\partial^2\psi_z}{\partial y\partial\xi}\right) = 0,$$
$$\mu\left(2\frac{\partial^2\phi}{\partial y\partial\xi} + \frac{\partial^2\psi_z}{\partial y^2} - \frac{\partial^2\psi_z}{\partial\xi^2} - \frac{\partial^2\psi_y}{\partial y\partial z} + \frac{\partial^2\psi_x}{\partial\xi\partial z}\right) = P_1 H(t)\delta(\xi + vt)\delta(z) + \sigma_{xy}^+,$$
$$\mu\left(2\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\psi_x}{\partial z^2} - \frac{\partial^2\psi_x}{\partial y^2} - \frac{\partial^2\psi_z}{\partial\xi\partial z} + \frac{\partial^2\psi_y}{\partial\xi\partial y}\right) = P_2 H(t)\delta(\xi + vt)\delta(z) + \sigma_{yz}^+, \tag{4}$$
$$\frac{\partial\phi}{\partial\xi} + \frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z} = u_x^-(\xi, z, t),$$
$$\frac{\partial\phi}{\partial z} + \frac{\partial\psi_y}{\partial\xi} - \frac{\partial\psi_x}{\partial y} = u_z^-(\xi, z, t),$$

where μ, λ are the positive *Lamé* elastic constants. The superscripts $+$ and $-$ are used at this point to indicate the half ξ-plane on which the superscribed function is non-zero. σ_{xy}^+ and σ_{yz}^+ are the unknown stresses on the half-plane $y = 0, \xi > 0$. u_x^- and u_z^- are the unknown displacements on the half-plane $y = 0, \xi < 0$. In the following parts of this article these symbols are carried over into the transformed domain, where they are used to designate a particular half-plane of analyticity.

2. Transform techniques

Transform techniques are used here to solve this problem. The following equations are written out in terms of the dilatational wave potential ϕ. The shear wave potential can be worked out in the same way.

First, the one-sided Laplace transform over time is introduced. The parameter is s, and the transformed function is denoted by a superposed hat

$$\widehat{\phi}(\xi, y, z, s) = \int_0^\infty \phi(\xi, y, z, t)\,\mathrm{e}^{-st}\,\mathrm{d}t. \tag{5}$$

For the present, s may be considered as a real positive parameter.

Second, the dependence on z is suppressed by introducing a two-sided Laplace transform with parameter $s\zeta$. The transformed function is denoted by a superposed bar, i.e.

$$\bar{\hat{\phi}}(\xi, y, \varsigma, s) = \int_{-\infty}^{\infty} \hat{\phi}(\xi, y, z, s)\, \mathrm{e}^{-s\varsigma z}\, \mathrm{d}z. \tag{6}$$

Finally, a two-sided transform over ξ is taken with a parameter $s\eta$

$$\Phi(\eta, y, \varsigma, s) = \int_{-\infty}^{\infty} \bar{\hat{\phi}}(\xi, y, \varsigma, s)\, \mathrm{e}^{-s\eta\xi}\, \mathrm{d}\xi. \tag{7}$$

The domain of this transform is (see Appendix B)

$$-a_2 < \mathrm{Re}(\eta) < a_1 \tag{8}$$

in which

$$a_{2,1} = \frac{\pm a^2 d + d[a^4 + (a^2 - \varsigma^2)(d^2 - a^2)]^{1/2}}{d^2 - a^2} \tag{9}$$

and

$$-a_0 < \varsigma < a_0; \qquad a_0 = ad/(d^2 - a^2)^{1/2}. \tag{10}$$

In obtaining the above transform domain, s and ζ are considered to be real for convenience. The transform can be analytically continued in the complex s and ζ planes.

The governing equations for the potentials are transformed into the following ordinary differential equations

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} - s^2\alpha^2\Phi = 0, \qquad \frac{\mathrm{d}^2\boldsymbol{\Psi}}{\mathrm{d}y^2} - s^2\beta^2\boldsymbol{\Psi} = 0, \tag{11}$$

where

$$\alpha^2 = a^2\left(1 - \frac{\eta}{d}\right)^2 - \eta^2 - \varsigma^2, \qquad \beta^2 = b^2\left(1 - \frac{\eta}{d}\right)^2 - \eta^2 - \varsigma^2. \tag{12}$$

Noting that the wave functions should be bounded as $y \to +\infty$, the solutions to Equations (11) are

$$\Phi(\eta, y, \varsigma, s) = \frac{A(\eta, \varsigma, s)}{s^3}\, \mathrm{e}^{-\alpha s y}, \qquad \boldsymbol{\Psi}(\eta, y, \varsigma, s) = \frac{\boldsymbol{B}(\eta, \varsigma, s)}{s^3}\, \mathrm{e}^{-\beta s y}, \tag{13}$$

where $\boldsymbol{B}(\eta, \varsigma, s) = (B_x, B_y, B_z)$.

The boundary conditions and the equation $\nabla \cdot \boldsymbol{\Psi} = 0$ are altered into

$$\begin{aligned}
&[b^2(1 - \eta/d)^2 - 2\eta^2 - 2\varsigma^2]A - 2\beta\varsigma B_x + 2\beta\eta B_z = 0,\\
&-2\eta\alpha A + \eta\varsigma B_x + \beta\varsigma B_y + (\beta^2 - \eta^2)B_z = \frac{1}{\mu}(\Sigma_{xy}^- + \Sigma_{xy}^+),\\
&2\alpha\varsigma A + (\beta^2 - \varsigma^2)B_x + \eta\beta B_y + \eta\varsigma B_z = -\frac{1}{\mu}(\Sigma_{yz}^- + \Sigma_{yz}^+),\\
&\eta A - \beta B_z - \varsigma B_y = U_x^-(\eta, \varsigma, s),\\
&\varsigma A + \eta B_y + \beta B_x = U_z^-(\eta, \varsigma, s),\\
&\eta B_x - \beta B_y + \varsigma B_z = 0,
\end{aligned} \tag{14}$$

where

$$\textstyle\sum_{xy}^{+} = s\int_0^\infty \overline{\widehat{\sigma}}_{xy}^{+}(\xi,\varsigma,s)\,\mathrm{e}^{-s\eta\xi}\,\mathrm{d}\xi, \qquad \sum_{yz}^{+} = s\int_0^\infty \overline{\widehat{\sigma}}_{yz}^{+}(\xi,\varsigma,s)\,\mathrm{e}^{-s\eta\xi}\,\mathrm{d}\xi,$$

$$U_x^{-} = s^2\int_{-\infty}^{0} \overline{\widehat{u}}_x^{-}(\xi,\varsigma,s)\,\mathrm{e}^{-s\eta\xi}\,\mathrm{d}\xi, \qquad U_z^{-} = s^2\int_{-\infty}^{0} \overline{\widehat{u}}_z^{-}(\xi,\varsigma,s)\,\mathrm{e}^{-s\eta\xi}\,\mathrm{d}\xi, \tag{15}$$

$$\textstyle\sum_{xy}^{-} = -\frac{P_1 d}{\eta-d}, \qquad \sum_{yz}^{-} = -\frac{P_2 d}{\eta-d}.$$

The transformation over t and z are implicated here in the functions $\overline{\widehat{\sigma}}_{xy}^{+}, \overline{\widehat{\sigma}}_{yz}^{+}, \overline{\widehat{u}}_x^{-}$ and $\overline{\widehat{u}}_y^{-}$. Set $\varsigma = i\gamma$ with γ is real, we obtain

$$\mathbf{E}^{+} + \mathbf{E}^{-} = -\frac{\mu}{\eta^2-\gamma^2}\mathbf{X}(\eta)\mathbf{D}(\eta)\mathbf{X}(\eta)\mathbf{U}^{-}, \tag{16}$$

where

$$\mathbf{E}^{+} = \begin{bmatrix}\Sigma_{xy}^{+}\\ \Sigma_{yz}^{+}\end{bmatrix}, \qquad \mathbf{E}^{-} = \begin{bmatrix}\Sigma_{xy}^{-}\\ \Sigma_{yz}^{-}\end{bmatrix}, \qquad \mathbf{D}(\eta) = \begin{bmatrix}\dfrac{R(\eta,i\gamma)}{\beta(\beta^2+\eta^2-\gamma^2)} & 0\\ 0 & \beta\end{bmatrix},$$

$$\mathbf{U}^{-} = \begin{bmatrix}U_x^{-}\\ U_z^{-}\end{bmatrix}, \qquad \mathbf{X}(\eta) = \begin{bmatrix}\eta & i\gamma\\ i\gamma & -\eta\end{bmatrix}, \tag{17}$$

$$R(\eta,i\gamma) = 4\alpha(\eta,i\gamma)\beta(\eta,i\gamma)(\eta^2-\gamma^2) + [b^2(1-\eta/d)^2 - 2\eta^2 + 2\gamma^2]^2.$$

(16) is the vector form of the two coupled Wiener–Hopf equations.

3. Wiener–Hopf technique

In this section the Wiener–Hopf technique is employed to solve the Wiener–Hopf equations obtained in the previous section.

Function **R** is factored as (see Appendix C)

$$R(\eta,i\gamma) = -\kappa(\eta-c_1)(\eta+c_2)(\eta-d)^2 S_{+}(\eta,i\gamma)S_{-}(\eta,i\gamma), \tag{18}$$

where

$$S_{\pm}(\eta,i\gamma) = \exp\left\{-\frac{1}{\pi}\int_{a_{2,1}}^{b_{2,1}} \operatorname{arctg}\left(\frac{4\,|\alpha(\mp\eta,i\gamma)|\;|\beta(\mp\eta,i\gamma)|\,(\upsilon^2-\gamma^2)}{b^2(1\pm\upsilon/d)^2 - 2\upsilon^2 + 2\gamma^2}\right)\frac{d\upsilon}{\upsilon\pm\eta}\right\},$$

$$c_{2,1} = \frac{\pm c^2 d + d[c^4 + (c^2+\gamma^2)(d^2-c^2)]^{1/2}}{d^2-c^2},$$

$$b_{2,1} = \frac{\pm b^2 d + d[b^4 + (b^2+\gamma^2)(d^2-b^2)]^{1/2}}{d^2-b^2}, \tag{19}$$

$$\kappa = -\lim_{\eta\to\infty}\frac{R(\eta,i\gamma)}{\eta^4} = 4(1-a^2/d^2)^{1/2}(1-b^2/d^2)^{1/2} - (2-b^2/d^2)^2.$$

$\beta(\eta, i\gamma)$ can be factored as

$$\beta(\eta, i\gamma) = \sqrt{\frac{d^2 - b^2}{d^2}}\sqrt{b_1(i\gamma) - \eta}\sqrt{b_2(i\gamma) + \eta}. \tag{20}$$

Noting the factorization of (18) and (20), one completes the Wiener–Hopf factorization (see Appendix D)

$$\begin{aligned} &\frac{1}{\eta - |\gamma|}\mathbf{B}^+\mathbf{E}^- - \boldsymbol{\beta}^+ + \frac{1}{\eta - |\gamma|}\{\mathbf{B}^+\mathbf{E}^+ - (\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|}\} \\ &\quad = -\mu\mathbf{A}^-\mathbf{U}^- - \frac{1}{\eta - |\gamma|}(\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|} - \boldsymbol{\beta}^+, \end{aligned} \tag{21}$$

where

$$\begin{aligned} &\mathbf{E}^- = -\frac{d}{\eta - d}\mathbf{P}, \qquad \mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \\ &\mathbf{B}^+ = [\mathbf{D}_+(\eta)]^{-1}\mathbf{X}(\eta), \qquad \mathbf{A}^- = \frac{1}{\eta - |\gamma|}\mathbf{D}_-(\eta)\mathbf{X}(\eta), \\ &\mathbf{D}_+ = \begin{bmatrix} \dfrac{W_1}{\Omega(\eta)} & 0 \\ 0 & -W_2\sqrt{b_2 + \eta} \end{bmatrix}, \\ &\mathbf{D}_- = \begin{bmatrix} \dfrac{W_1(c_1 - \eta)S_-(\eta, i\gamma)}{\sqrt{b_1 - \eta}} & 0 \\ 0 & -W_2\sqrt{b_1 + \eta} \end{bmatrix}, \\ &W_1 = \frac{d\sqrt{k}}{b\sqrt[4]{1 - \frac{b^2}{d^2}}}, \qquad W_2 = -\sqrt[4]{1 - \frac{b^2}{d^2}}, \qquad \Omega(\eta) = \frac{\sqrt{b_2 + \eta}}{(c_2 + \eta)S_+(\eta, \varsigma)}, \\ &\boldsymbol{\beta}^+ = -\frac{d}{\eta - d}\left[\frac{\mathbf{B}^+}{\eta - |\gamma|}\right]\Bigg|_{\eta=d}\mathbf{P} - \frac{d}{\eta - |\gamma|}\left[\frac{\mathbf{B}^+}{\eta - d}\;\right]\Bigg|_{\eta=|\gamma|}\mathbf{P}. \end{aligned} \tag{22}$$

Now the left and right sides of this equation are analytic in the planes $\mathrm{Re}(\eta) > -a_2$ and $\mathrm{Re}(\eta) < a_1$, respectively. As already noted, each side is the unique analytic continuation of the other into a complementary half-plane, and together the two sides represent a single entire function. After some asymptotic analysis on each side of Equation (21), one can find that both sides approach zero as $\eta \to \infty$ in the respective half-plane. Hence, by Liouville's theorem, the entire function represented by each side of Equation (21) is zero. Thus, one obtains

$$\begin{cases} -\mu\mathbf{A}^-\mathbf{U}^- = \dfrac{1}{\eta - |\gamma|}(\mathbf{B}^+\mathbf{E}^+)|_{n=|\gamma|} + \boldsymbol{\beta}^+, \\[2ex] \dfrac{1}{\eta - |\gamma|}\mathbf{B}^+\mathbf{E}^+ = \dfrac{1}{\eta - |\gamma|}(\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|} + \boldsymbol{\beta}^+ - \dfrac{1}{\eta - |\gamma|}\mathbf{B}^+\mathbf{E}^-. \end{cases} \tag{23}$$

As already known, $\mathbf{U}^-$ is analytic in the half-plane $\mathrm{Re}(\eta) < a_1$, so the singular point at $\eta = -|\gamma|$ must be removed.

After some analysis similar to that employed by Li et al. (1994; 1995), one finds that

$$\mathbf{U}^- = -\frac{1}{\mu}\frac{\mathbf{B}^-}{\eta+|\gamma|}\left[\frac{1}{\eta-|\gamma|}(\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|} + \boldsymbol{\beta}^+\right], \tag{24a}$$

$$\mathbf{E}^+ = \mathbf{A}^+\left[\frac{1}{\eta-|\gamma|}\frac{2|\gamma|\mathbf{e}_2\cdot(\boldsymbol{\beta}^+)|_{\eta=-|\gamma|}}{\mathbf{e}_2\cdot\mathbf{e}_1}\mathbf{e}_1 + \boldsymbol{\beta}^+ - \frac{1}{\eta-|\gamma|}\mathbf{B}^+\mathbf{E}^-\right], \tag{24b}$$

in which

$$\begin{aligned}
&\mathbf{e}_1 = \begin{bmatrix} M_2 \\ i\ \mathrm{sgn}(\gamma)M_1\end{bmatrix}, \qquad M_1 = \frac{W_1}{\Omega(|\gamma|)}, \qquad M_2 = -W_2\sqrt{b_2+|\gamma|},\\
&\mathbf{e}_2 = [-N_2, i\ \mathrm{sgn}(\gamma)N_1], \qquad \mathbf{B}^- = \mathbf{X}(\eta)[\mathbf{D}_-(\eta)]^{-1},\\
&N_1 = W_1\frac{(c_1+|\gamma|)S_-(-|\gamma|, i\gamma)}{\sqrt{b_1+|\gamma|}}, \qquad N_2 = -W_2\sqrt{b_1+|\gamma|}.
\end{aligned} \tag{25}$$

4. Stress intensity factor histories

Because the problem is antisymmetric to the plane $z = 0$ if $P_1 = 0$ and symmetric to the plane $z = 0$ when $P_2 = 0$, attention is restricted to the half-space $z > 0$ in this section.

The stress intensity factor histories along the crack edge can be written out as

$$\begin{aligned}
K_{\mathrm{II}}(z,t) &= \operatorname*{Lim}_{\xi\to 0^+}\sqrt{2\pi\xi}\,\sigma_{xy}(\xi,0,z,t),\\
K_{\mathrm{III}}(z,t) &= \operatorname*{Lim}_{\xi\to 0^+}\sqrt{2\pi\xi}\,\sigma_{yz}(\xi,0,z,t),
\end{aligned} \tag{26}$$

The transformed stress intensity factors are

$$\begin{aligned}
\overline{\widehat{K}}_{\mathrm{II}}(\gamma,s) &= \operatorname*{Lim}_{\xi\to 0^+}[\overline{\widehat{\sigma}}_{xy}(\xi,0,\gamma,s)\sqrt{2\pi\xi}],\\
\overline{\widehat{K}}_{\mathrm{III}}(\gamma,s) &= \operatorname*{Lim}_{\xi\to 0^+}[\overline{\widehat{\sigma}}_{yz}(\xi,0,\gamma,s)\sqrt{2\pi\xi}].
\end{aligned} \tag{27}$$

Using Abel's theorem on the asymptotic properties of transformed terms one gets

$$\begin{aligned}
\overline{\widehat{K}}_{\mathrm{II}}(\gamma,s) &= \frac{1}{s}\operatorname*{Lim}_{\eta\to\infty}[\sqrt{2s\eta}\ \Sigma^+_{xy}(\eta,0,\gamma,s)],\\
\overline{\widehat{K}}_{\mathrm{III}}(\gamma,s) &= \frac{1}{s}\operatorname*{Lim}_{\eta\to\infty}[\sqrt{2s\eta}\ \Sigma^+_{yz}(\eta,0,\gamma,s)].
\end{aligned} \tag{28}$$

Thus, utilizing a similar technique used by Li et al. (1994; 1995), one obtains

$$\begin{pmatrix}\overline{\widehat{K}}_{\mathrm{II}}(\gamma,s)\\ \overline{\widehat{K}}_{\mathrm{III}}(\gamma,s)\end{pmatrix} = \sqrt{\frac{2}{s}}\begin{bmatrix} W_1 & 0\\ 0 & W_2\end{bmatrix}$$

$$\times \left\langle \frac{2|\gamma|\mathbf{e}_1}{\mathbf{e}_2\cdot\mathbf{e}_1}\mathbf{e}_2 \cdot \left(\frac{\mathbf{B}^{+}_{\eta=d}}{|\eta|+d} - \frac{\mathbf{B}^{-}_{\eta=|\gamma|}}{2|\eta|} \right) \mathbf{P} - (\mathbf{B}^{+}_{\eta=d} - \mathbf{B}^{+}_{\eta=|\gamma|})\mathbf{P} \right\rangle$$

$$\times \frac{d}{d-|\gamma|}. \tag{29}$$

After relaxing the constraint that γ is real, the inversion of the above transformed stress intensity factors over **z** gives

$$\begin{bmatrix} \widehat{K}_{\mathrm{II}}(z,s) \\ \widehat{K}_{\mathrm{III}}(z,s) \end{bmatrix} = \frac{s}{2\pi} \int_{-\infty+\gamma_0 i}^{\infty+\gamma_0 i} \begin{bmatrix} \overline{\overline{K}}_{\mathrm{II}}(\gamma,s) \\ \overline{\overline{K}}_{\mathrm{III}}(\gamma,s) \end{bmatrix} \exp(is\gamma z)\,\mathrm{d}\gamma, \tag{30}$$

where $\gamma_0 \in (-b_0, b_0)$.

Then, one obtains

$$\widehat{K}_{\mathrm{II}} = \frac{1}{\pi}\sqrt{\frac{s}{2}} \int_{b_0}^{\infty} \times \left\{ 2\,\mathrm{Re}\left[\frac{2ud\left(\frac{b_0^2}{d} + \frac{ib_0}{b}\sqrt{u^2-b_0^2} + iu\right)}{b(d-iu)g(iu)} f(u) \right] + 2\,\mathrm{Im}\left[\frac{d}{d-iu} \frac{\sqrt{\frac{b_0^2}{d} + \frac{ib_0}{b}\sqrt{u^2-b_0^2} + d}}{\left(\frac{c_0^2}{d} + \frac{c_0}{c}\sqrt{c_0^2-u^2} + d\right) S_+(d,-u)} (dP_1 - uP_2) \right] \right\} \mathrm{e}^{-suz}\,\mathrm{d}u, \tag{31a}$$

$$\widehat{K}_{\mathrm{III}} = \frac{1}{\pi}\sqrt{\frac{s}{2}} \int_{b_0}^{\infty} \times \left\{ 2\,\mathrm{Im}\left[\frac{2ud\left(\frac{c_0^2}{d} + \frac{c_0}{c}\sqrt{c_0^2-u^2} + iu\right)}{b(d-iu)g(iu)} \times S_+(iu,-u)\sqrt{-\frac{b_0^2}{d} + \frac{ib_0}{b}\sqrt{u^2-b_0^2} + iu}\; f(u) \right] + 2\,\mathrm{Im}\left[\frac{d}{d-iu} \frac{(uP_1 + dP_2)}{\sqrt{\frac{b_0^2}{d} + \frac{ib_0}{b}\sqrt{u^2-b_0^2} + d}} \right] \right\} \mathrm{e}^{-suz}\,\mathrm{d}u, \tag{31b}$$

where

$$f(u) = -\frac{b(1-b^2/d^2)\left(-\frac{b_0^2}{d}+\frac{ib_0}{b}\sqrt{u^2-b_0^2}+iu\right)}{\left(\frac{c_0^2}{d}+\frac{c_0}{c}\sqrt{c_0^2-u^2}+d\right)S_+(d,-u)(d+iu)}$$

$$\times\sqrt{\frac{b_0^2}{d}+\frac{ib_0}{b}\sqrt{u^2-b_0^2}+d}\,(dP_1-uP_2)$$

$$-\frac{\mathrm{i}\,\mathrm{d}^2\kappa\left(-\frac{c_0^2}{d}+\frac{c_0}{c}\sqrt{c_0^2-u^2}+iu\right)S_-(-iu,-u)(uP_1+dP_2)}{b(d+iu)\sqrt{\frac{b_0^2}{d}+\frac{ib_0}{b}\sqrt{u^2-b_0^2}+d}}, \quad (32a)$$

$$g(iu) = (1-b^2/d^2)(b_1+iu)(b_2+iu)$$

$$+\frac{d^2\kappa}{b^2}(c_1+iu)(c_2+iu)S_+(iu,-u)S_-(-iu,-u) \quad (32b)$$

and $b_0 = db/\sqrt{d^2-b^2}$ and $c_0 = cd/\sqrt{d^2-c^2}$.

In order to be able to apply the convolution theorem for Laplace transforms, $\widehat{K}_{\mathrm{II}}$ and $\widehat{K}_{\mathrm{III}}$ must be written out as the products of two such transforms. One must introduce two other functions to rectify the order of s

$$\begin{cases} K_{\mathrm{II}}(z,t) = \dfrac{\partial H_2(z,t)}{\partial t}, \quad H_2(z,t) = 0, \\ K_{\mathrm{III}}(z,t) = \dfrac{\partial H_3(z,t)}{\partial t}, \quad H_3(z,t) = 0, \end{cases} \quad (33)$$

Thus,

$$\widehat{K}_{\mathrm{II}}(z,s) = s\hat{H}_2(z,s), \qquad \widehat{K}_{\mathrm{III}}(z,s) = s\hat{H}_3(z,s). \quad (34)$$

If let

$$g_1(\tau) = -b'^2\left(\frac{1}{d'^2}+\left(\frac{1}{b'}\sqrt{\tau^2-1}+\tau\right)^2\right)$$

$$+\frac{d'^2\kappa}{b'^2}\left(-\frac{c_0^2}{db_0}+\frac{c_0}{c}\sqrt{c'^2-\tau^2}+i\tau\right)$$

$$\times\left(\frac{c_0^2}{db_0}+\frac{c_0}{c}\sqrt{c'^2-\tau^2}+i\tau\right)S_+(i\tau,\tau)S_-(i\tau,\tau) \quad (35a)$$

$$f_1(\tau) = -\frac{b'^2\left(-\frac{1}{d'}+\frac{i}{b'}\sqrt{\tau^2-1}+i\tau\right)\sqrt{\frac{1}{d'}+\frac{i}{b'}\sqrt{\tau^2-1}+d'}}{\left(\frac{c_0^2}{db_0}+\frac{c_0}{c}\sqrt{c'^2-\tau^2}+d'\right)S_+(d',-\tau)}(d'P_1-\tau P_2)$$

$$-i\frac{d'^2\kappa}{b'^2}\frac{\left(-\frac{c_0^2}{db_0}+\frac{c_0}{c}\sqrt{c'^2-\tau^2}+i\tau\right)S_-(-i\tau,-\tau)(\tau P_1+d'P_2)}{\sqrt{\frac{1}{d'}+\frac{i}{b'}\sqrt{\tau^2-1}+d'}}, \tag{35b}$$

where $d'=d/b_0, b'=b/b_0$ and $c'=c_0/b_0$.

K_{II} and K_{III} can be written out as

$$\begin{aligned} K_{\mathrm{II}} = {} & \frac{\sqrt{2}}{(\pi|z|)^{3/2}}\frac{\partial}{\partial T}\int_1^T \\ & \times\left\{\mathrm{Re}\left[\frac{2d'\tau\left(\frac{1}{d'}+\frac{i}{b'}\sqrt{\tau^2-1}+i\tau\right)}{(d'^2+\tau^2)g_1(\tau)}f_1(\tau)\right]\right. \\ & \left. +\mathrm{Im}\left[\frac{d'}{d'-i\tau}\frac{\sqrt{\frac{1}{d'}+\frac{i}{b'}\sqrt{\tau^2-1}+d'}(d'P_1-\tau P_2)}{\left(\frac{c_0^2}{db_0}+\frac{c_0}{c}\sqrt{c'^2-\tau^2}+d'\right)S_+(d',-\tau)}\right]\right\} \\ & \times\frac{\mathrm{d}\tau}{\sqrt{T-\tau}}H(T-1), \end{aligned} \tag{36a}$$

$$\begin{aligned} K_{\mathrm{III}} = {} & \frac{\sqrt{2}}{(\pi|z|)^{3/2}}\frac{\partial}{\partial T}\int_1^T \\ & \times\left\{\mathrm{Im}\left[\frac{2d'\tau\left(\frac{c_0^2}{db_0}+\frac{c_0}{c}\sqrt{c'^2-\tau^2}+i\tau\right)S_+(i\tau,-\tau)}{(d'^2+\tau^2)g_1(\tau)}f_1(\tau)\right]\right. \\ & \left. +\mathrm{Im}\left[\frac{d'}{d'-i\tau}\frac{(\tau P_1+d'P_2)}{\sqrt{\frac{1}{d'}+\frac{i}{b'}\sqrt{\tau^2-1}+d'}}\right]\right\} \\ & \times\frac{\mathrm{d}\tau}{\sqrt{T-\tau}}H(T-1), \end{aligned} \tag{36b}$$

where

$$T=\frac{t}{b_0|z|},\qquad \tau=\frac{u}{b_0|z|}.$$

(36) is valid for $z>0$. In order to get the stress intensity factors for those points that satisfy $z<0$, one should utilize the symmetric and antisymmetric condition mentioned at the beginning of this section.

5. Results and discussion

One can note in the Equations (36a,b) that the responses of P_1 and P_2 are coupled together. That is, II and III responses cannot be separated in 3-D problems.

One useful check on the results obtained is that they should reduce to the stress intensity factor histories for the two-dimensional line load problem solved by Fossum and Freund (1975) and Freund (1990) when integrated over the range $-\infty < z < \infty$. (Here we should mention again that II and III responses can be separated only in these 2-D problems.) If this integration is performed on (30) the results are

$$\begin{pmatrix} \widehat{K}_{\mathrm{II}} \\ \widehat{K}_{\mathrm{III}} \end{pmatrix} = -\sqrt{\frac{2}{s}} \begin{bmatrix} \dfrac{d-c}{\sqrt{d-b}} \dfrac{P_1}{S_+(d,0)} \\ P_2\sqrt{d-b} \end{bmatrix}. \tag{37}$$

Then invert (37) over time

$$\begin{pmatrix} \widehat{K}_{\mathrm{II}}(z,t) \\ \widehat{K}_{\mathrm{III}}(z,t) \end{pmatrix} = -\sqrt{\frac{2}{\pi t}} \begin{bmatrix} \dfrac{d-c}{\sqrt{d-b}} \dfrac{P_1}{S_+(d,0)} \\ P_2\sqrt{d-b} \end{bmatrix}. \tag{38}$$

This agrees with the results obtained by Fossum and Freund (1975) and Freund (1990).

A numerical integration of the SIF is now carried out. Before the calculation, the functions $S_+(\eta, i\gamma)$ and $S_-(\eta, i\gamma)$ are written out in a more convenient way. Taking the substitutions

$$w^2 = (v^2 - \gamma^2)/(1 \pm v/d)^2, \tag{39}$$

leads to

$$S_\pm(\eta, i\gamma) = \exp\left\{ -\frac{1}{\pi} \int_{a'}^{b'} \tan^{-1}\left[\frac{4w^2(w^2-a^2)^{1/2}(b^2-w^2)^{1/2}}{(b^2-2w^2)^2} \right] f(w, i\gamma)\, dw \right\}, \tag{40}$$

where

$$f(w, i\gamma) = \frac{dw(d^2 \pm e(w, i\gamma))^2}{e(w, i\gamma)(d^2 - w^2)(d(e(w, i\gamma) \pm w^2) \pm \eta(d^2 - w^2))}, \tag{41}$$

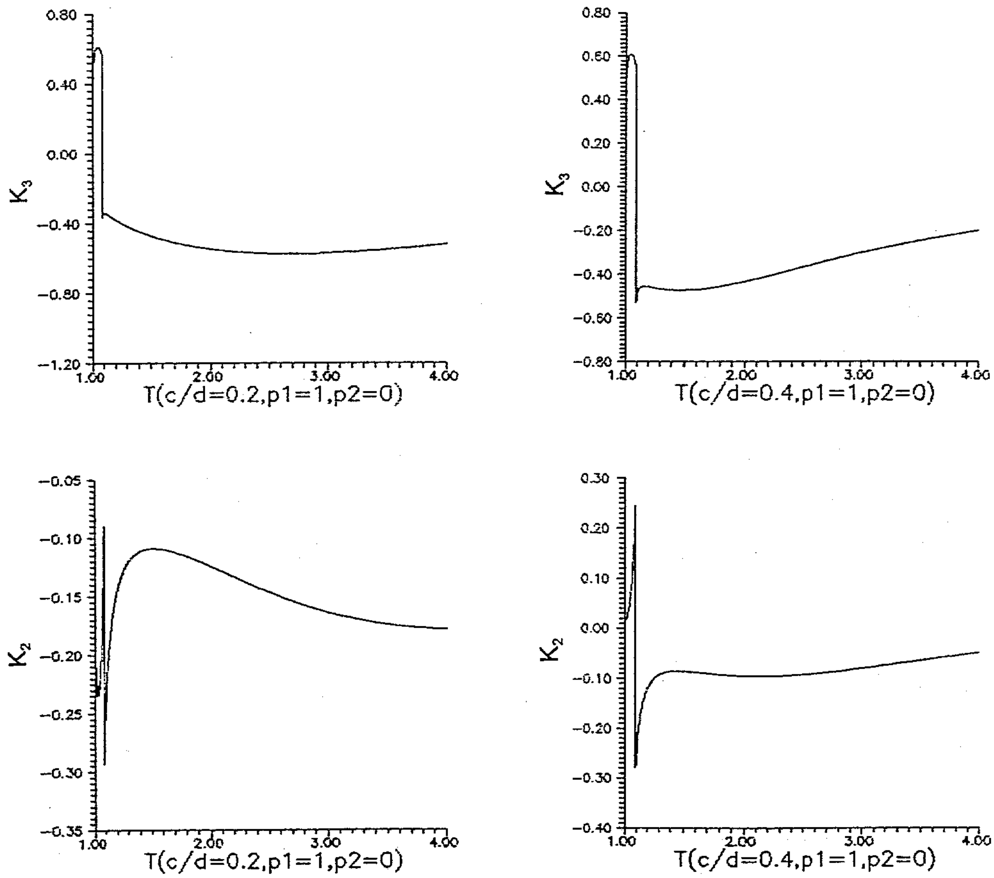

Figure 2a. The scaled stress intensity factor histories $K_2 = (\pi|z|)^{3/2}K_{\mathrm{II}}/\sqrt{2}$ and $K_3 = (\pi|z|)^{3/2}K_{\mathrm{III}}/\sqrt{2}$ vs. $T = t/(b_0|z|)$ for the fundamental solution when $P_1 = 1, P_2 = 0$ for a relatively short period of time.

in which

$$e(w, i\gamma) = \sqrt{w^2(d^2 - \gamma^2) + d^2\gamma^2}. \tag{42}$$

In this paper the scaled stress intensity factor histories $(\pi|z|)^{3/2}K_{\mathrm{II}}/\sqrt{2}$ and $(\pi|z|)^{3/2}K_{\mathrm{III}}/\sqrt{2}$ are calculated and plotted against scaled time $T = t/b_0|z|$ for various values of c/d with Poisson's ratio $\nu = 0.3$ ($b = 1.87a, c = 2.02a$).

From Figure 2 one can find that, for any value $z > 0$, the stress intensity factors are zero up until the arrival of the first shear wave. The first dilatational wave has no contribution to the stress concentration factors because the contribution from the first dilatational waves traveling in the upper and lower half-spaces are canceled out in the case along the crack front. The reason for this is that the driving forces for these waves are in opposite direction.

The energy that first Rayleigh wave carries is so large that it causes a singular change in the stress level upon its arrival at the crack front. One may notice that the integrands in Equations (36a, b) have an order of $1/\tau$, therefore the singularity that the stress intensity factors have is logarithmic singularity.

Upon the arrival of the Rayleigh wave the curves turn down sharply. The so-defined scaled stress intensity factors K_2 and K_3 change about $0.2 - 1.1$ in just about 0.02 normalized time.

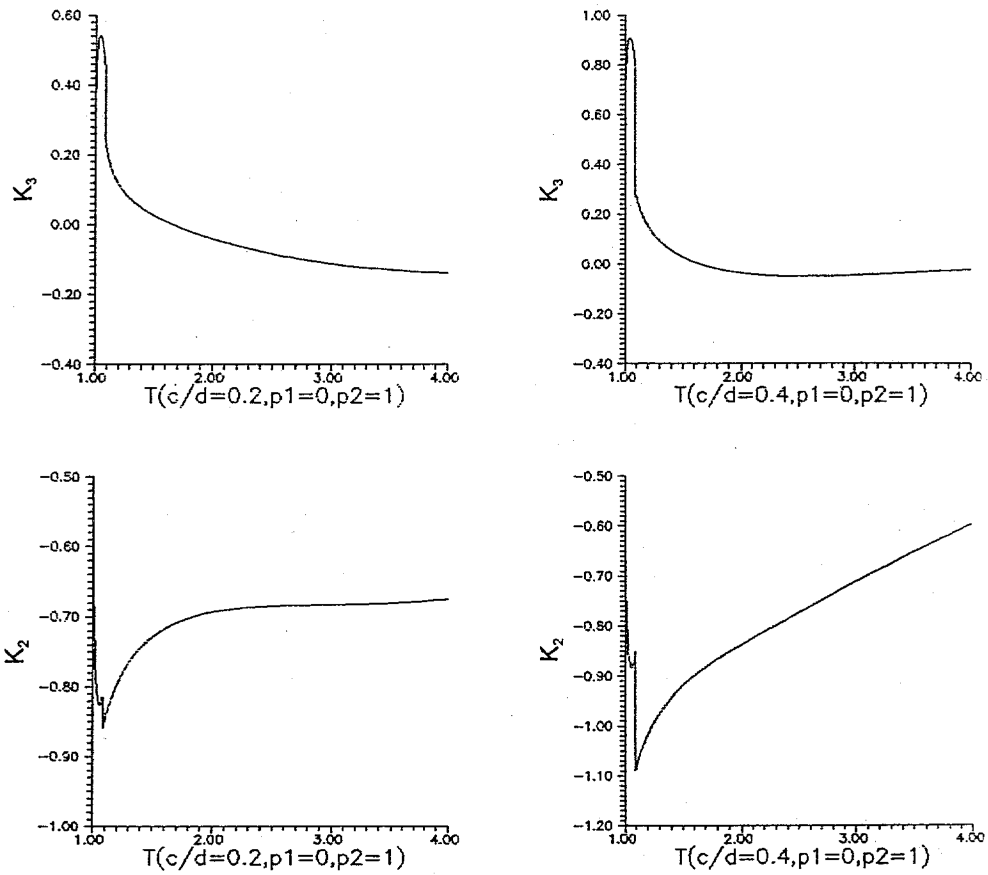

Figure 2b. The scaled stress intensity factor histories $K_2 = (\pi|z|)^{3/2}K_{\rm II}/\sqrt{2}$ and $K_3 = (\pi|z|)^{3/2}K_{\rm III}/\sqrt{2}$ vs. $T = t/(b_0|z|)$ for the fundamental solution when $P_1 = 0, P_2 = 1$ for a relatively short period of time.

After that, the curves gradually rise. The factors will go to zero as time approaches infinity, but this property varies largely with the ratio c/d. In some cases the curves go to zero much slower, while in other cases they go somewhat quicker. That is, the velocity at which the crack travels and the property of the material in conjunction have an important influence on the SIF histories in this problem.

The curves of this problem have some different properties from those of the I loading case for 3-D problem solved by Champion (1988) and the II and III loading cases in the 2-D problems obtained by Fossum and Freund (1975) and Freund (1990). In the I loading case of the 3-D problem, the curve has no downward movement at the moment of the arrival of the Rayleigh wave. It continues to rise behind it and in time returns to zero. In the 2-D problems the singular points at the arrival of the Rayleigh wave have disappeared anyway and the curves go on smoothly after the first shear wave.

General Loading

Similar to those obtained by Freund (1972) for the mode I plane strain problem, the solutions for general combined mode loading are written out as a superposition integral of those

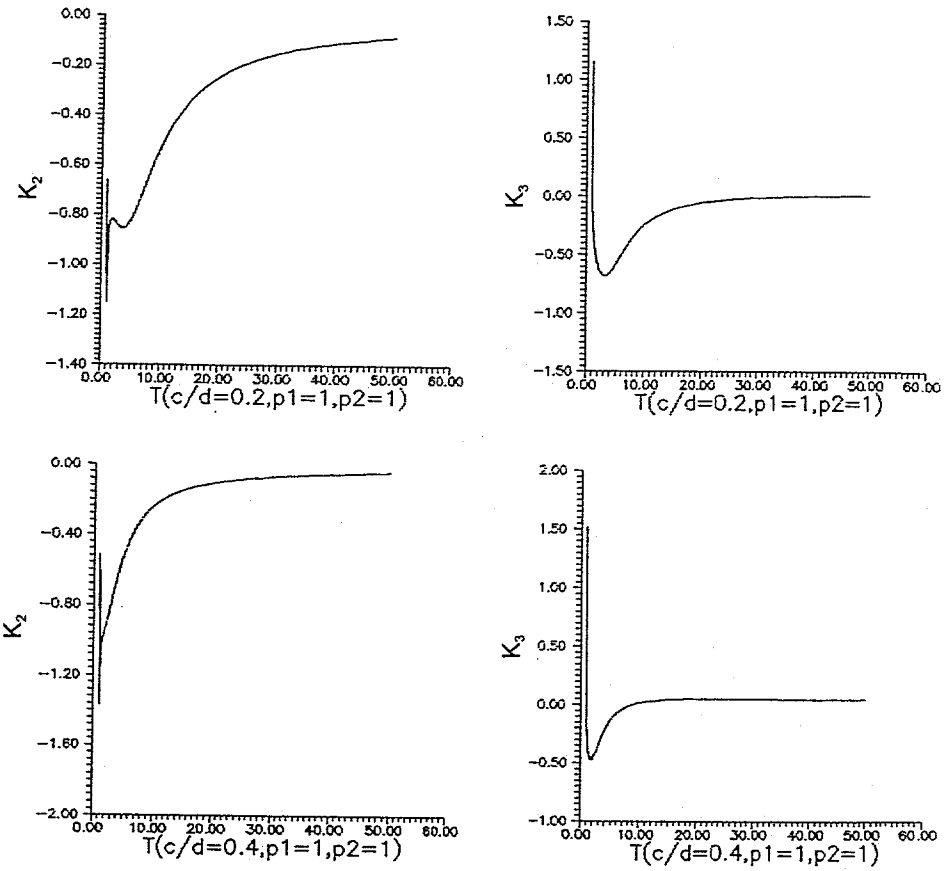

Figure 2c. The scaled stress intensity factor histories $K_2 = (\pi|z|)^{3/2}K_{\rm II}/\sqrt{2}$ and $K_3 = (\pi|z|)^{3/2}K_{\rm III}/\sqrt{2}$ vs. $T = t/(b_0|z|)$ for the fundamental solution when $P_1 = 1, P_2 = 1$ for a relatively long period of time.

obtained in the previous section, with the fact that the effects under II and III loading are coupled together.

Suppose that the crack is stationary for $t < 0$ under equilibrium loading, with a stress distribution given by

$$\begin{cases} \sigma_{xy}(x,0,z) = -P_1(x,z) \\ \sigma_{yz}(x,0,z) = -P_2(x,z) \end{cases} \tag{43}$$

on the half-plane $y = 0$ ahead of the crack. Beginning at time $t = 0$, the crack moves in the positive x-direction at speed v, and creates new stress-free surfaces. Then, the results may be considered by superposing a dynamic field created by imposing traction $P_i(x,z)$ $(i = 1,2, 0 < x < \nu t, -\infty < z < \infty)$ on the newly created crack faces and a static field corresponding to the equilibrium load. The solution to the dynamic field can be obtained as follows in the integral form.

If the distribution $P_i(x,z)(i=1,2)$ is known, the full dynamic stress intensity factors can be written out as

$$\begin{aligned} KP_{\mathrm{II}}(z,t) &= \int_{-\infty}^{\infty}\int_0^{vt} K_{22}(z-z',t-x'/v)P_1(x',z')\,\mathrm{d}x'\,\mathrm{d}z' \\ &\quad + \int_{-\infty}^{\infty}\int_0^{vt} K_{23}(z-z',t-x'/v)P_2(x',z')\,\mathrm{d}x'\,\mathrm{d}z', \\ KP_{\mathrm{III}}(z,t) &= \int_{-\infty}^{\infty}\int_0^{vt} K_{32}(z-z',t-x'/v)P_1(x',z')\,\mathrm{d}x'\,\mathrm{d}z' \\ &\quad + \int_{-\infty}^{\infty}\int_0^{vt} K_{33}(z-z',t-x'/v)P_2(x',z')\,\mathrm{d}x'\,\mathrm{d}z'. \end{aligned} \tag{44}$$

Here $KP_{\mathrm{II}}(z,t)$ and $KP_{\mathrm{III}}(z,t)$ are the stress intensity factor histories of this general loading distribution. And $K_i(z,t)$ $(i=22,23,32,33)$ provide weights for the general impact loads defined above.

This formulation may be useful for the numerical computation for a given stress distribution. However, if a calculation for a specific distribution is desired, it is advisable to get alternative integral forms for (44).

Now consider a specific traction distribution

$$\begin{bmatrix} P_1(x,z) \\ P_2(x,z) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad \text{for } 0<x<vt, -z_0<z<z_0, \tag{45}$$

where p_1 and p_2 are constants.

The stress concentration factors of this particular traction distribution can be written out as

$$\begin{aligned} KP_{\mathrm{II}}(z,t) &= p_1k_{22}(z,t)+p_2k_{23}(z,t), \\ KP_{\mathrm{III}}(z,t) &= p_1k_{32}(z,t)+p_2k_{33}(z,t). \end{aligned} \tag{46}$$

The solution to this problem can be obtained by superposing the concentrated line loads, i.e.

$$k_i(z,t) = \int_0^{vt} K_i^*(z,t-x'/\nu)\,\mathrm{d}x', \quad i=22,23,32,33, \tag{47}$$

where $K_i^*(z,t)$ is the stress intensity factor for the unit line load of finite length passing through the crack tip at time $t=0$. And $K_i^*(z,t)$ may be written out as

$$K_i^*(z,t) = \int_{-z_0}^{z_0} K_i(z-z',t)\,\mathrm{d}z'. \tag{48}$$

Using (30) one can get

$$\widehat{K_i^*}(z,s) = \frac{1}{\pi i}\left(\frac{s}{2}\right)^{1/2} I(z,s), \tag{49}$$

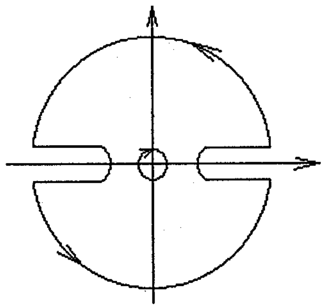

Figure 3.

where

$$I(z,s) = \frac{1}{s}[J(z+z_0,s) - J(z-z_0,s)],$$
$$J(x,s) = \int_{\varsigma_0-\infty i}^{\varsigma_0+\infty i} \frac{Q(\varsigma)}{\varsigma}\,\mathrm{e}^{s\varsigma x}\,\mathrm{d}\varsigma, \qquad Q(\varsigma) = \sqrt{\frac{s}{2}}\,\overline{\widehat{K}}_i(\varsigma,s). \tag{50}$$

Then, after some calculation, one obtains

$$J(x,s) = \begin{cases} 2\pi i Q(0) - 2iJ_1(x,s), & x>0, \\ 2iJ_1(-x,s), & x<0, \end{cases} \tag{51}$$

where

$$J_1(x,s) = \int_{b_0}^{\infty} \mathrm{Im}\,\left[\frac{Q(u+i0)}{u}\right]\,\mathrm{e}^{-sxu}\,\mathrm{d}u. \tag{52}$$

As in the previous section one only needs to take care of the part in $z>0$

$$\widehat{K}_i^*(z,s) = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{2}{s}\right)^{1/2}\{\pi Q(0) - J_1(z+z_0,s) - J_1(z_0-z,s)\}, & z<z_0, \\ \dfrac{1}{\pi}\left(\dfrac{2}{s}\right)^{1/2}\{J_1(z-z_0,s) - J_1(z_0+z,s)\}, & z>z_0. \end{cases} \tag{53}$$

It is very clear that there is a jump in $\widehat{K}_i^*$ at $z=z_0$

$$\Delta\widehat{K}_i^*(z_0,s) = \frac{1}{\pi}\left(\frac{2}{s}\right)^{1/2}\left\{2\int_{b_0}^{\infty}\mathrm{Im}\,\left[\frac{Q(u+i0)}{u}\right]\,\mathrm{d}u - \pi Q(0)\right\} \tag{54}$$

but by considering the integration of $\oint(Q(\varsigma)/\varsigma)\,\mathrm{d}\varsigma$ on the contour of Figure 3, one knows that $\Delta\widehat{K}_i^* = 0$.

Inverting Equation (53) one obtains

$$k_i(z,t) = \begin{cases} \dfrac{v}{\pi}\sqrt{\dfrac{2}{\pi}}(\pi Q(0)\sqrt{t} - J_2(z_0-z,t) - J_2(z_0+z,t)), & z<z_0, \quad (55a) \\ \dfrac{v}{\pi}\sqrt{\dfrac{2}{\pi}}(J_2(z-z_0,t) - J_2(z_0+z,t)), & z>z_0, \quad (55b) \end{cases}$$

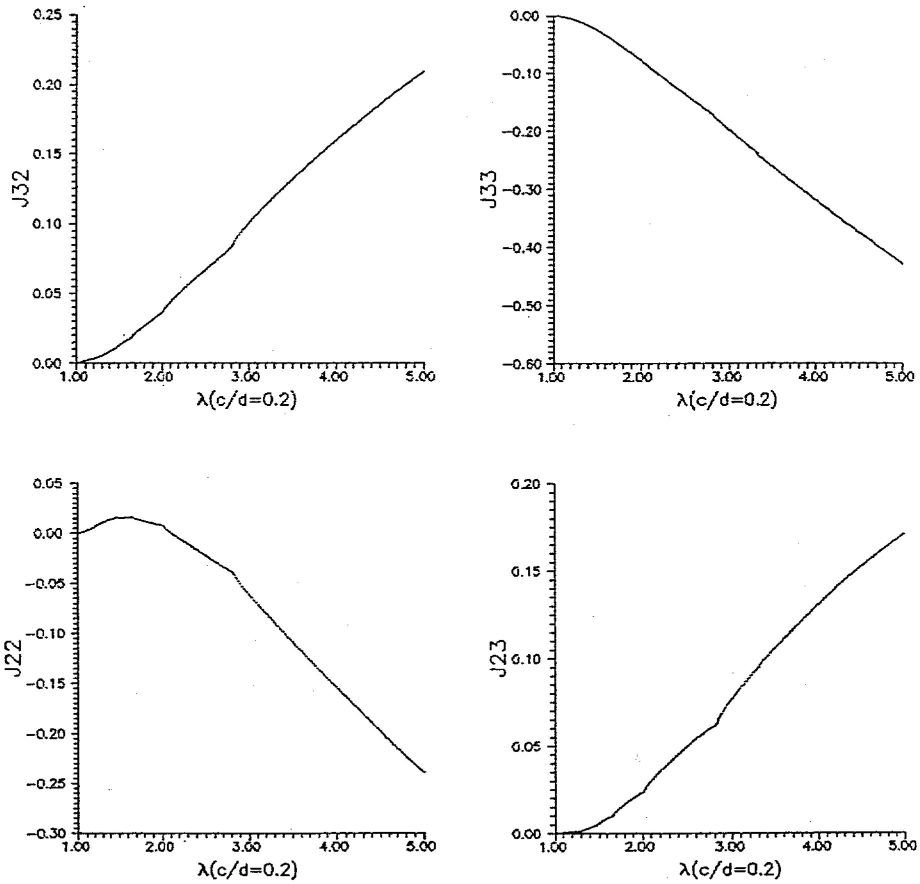

Figure 4a. The dimensional integrals $J_i(\lambda)$ vs. λ when $c/d = 0.2$.

where

$$J_2(x,t) = 2\sqrt{x}\int_{b_0}^{t/x} \mathrm{Im}\left[\frac{Q(u+i0)}{u}\right]\sqrt{t/x-u}\,\mathrm{d}u. \tag{56}$$

Here, we define a non-dimensional integral $J_i(\lambda)$ in the following form

$$J_i(\lambda) = \frac{J_2(x,t)}{2\sqrt{xb_0}}, \qquad \lambda = \frac{t}{xb_0}, \quad i = 22, 23, 32, 33. \tag{57}$$

Let us analyze Equation (55) in some more detail. The first term in (55a) represents the stress intensity factor history for a distribution of the same strength p_1 and p_2 with infinite extent in the z-direction. The second and third terms are corrections which take into account the finite distribution of the traction in the z-direction – they may be considered to correspond to waves emanating from the boundaries $z = \pm z_0$ of the traction distribution on the crack edge. For $z > z_0, k_i$ may be thought of as the superposition of two waves centered at the points $z = \pm z_0$ on the crack line.

In Figure 4 the integrals $J_i(\lambda)$ $(i = 22, 23, 32, 33)$ are plotted against λ for different values of c/d with Poisson's ratio $\nu = 0.3$ $(b = 1.87a, c = 2.02a)$. We can see that, unlike the

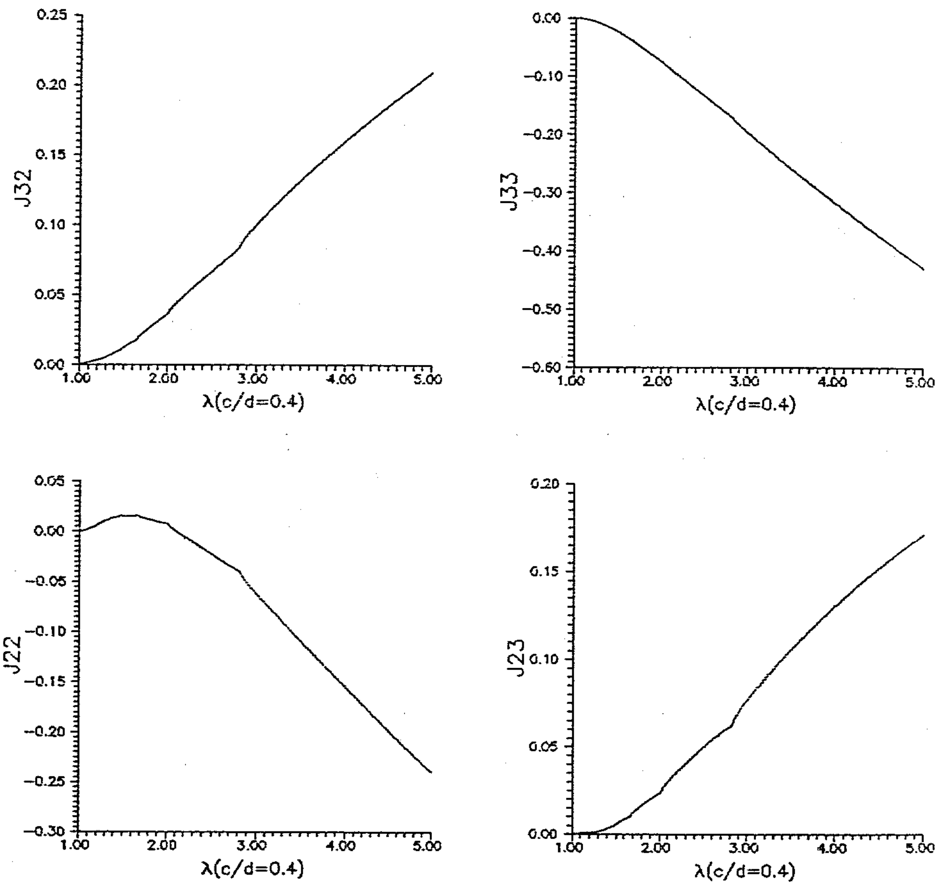

Figure 4b. The dimensional integrals $J_i(\lambda)$ vs. λ when $c/d = 0.4$.

results in the previous section, there is no singular point in the stress intensity factor when the Rayleigh wave reaches the fixed point z (here also $z > 0$).

The results for the points $z < 0$ can be obtained by the symmetric or antisymmetric conditions. This completes the analysis of the three dimensional stress intensity factor histories for the particular applied traction distribution (45). For a number of other traction distributions results could be derived in the same way.

Conclusion

Utilizing the Laplace transform methods together with the Wiener–Hopf technique, we solved the II and III combined mode constantly propagating crack problem. In dealing with the two coupled Wiener–Hopf equations, we utilized a method that was used by Li et al. (1994).

We found out that the first dilatational wave has no contribution to the stress intensity factors K_{II} and K_{III}. The reason for this is that the first dilatational waves caused by the point forces in the upper and the lower half-spaces are canceled along the crack front. The stress intensity factors are zero up until the arrival of the first shear wave. In the case of point forces, upon the arrival of the Rayleigh wave, when the curves turn down sharply, the stress intensity factors have a logarithmic singularity. This singularity is caused by the great energy that the

Rayleigh wave carries. After that the curves gradually rise. The factors will go to zero as time approaches infinity.

After the fundamental solution, a particular traction distribution for the same crack problem was analyzed. We found that, unlike the results in the point forces loading problem, there is no singular point in the stress intensity factor for the distributed loading problem.

Acknowledgments

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Appendix A

The motion of the body is governed by Navier's equation:

$$\mu \nabla \mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{A.1}$$

where λ and μ are *Lamẽ* constants.

Based on the Stokes–Helnholtz resolution theorem the displacement field can be decomposed as follows

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi} \tag{A.2}$$

and $\boldsymbol{\psi}$ should satisfy

$$\nabla \cdot \boldsymbol{\psi} = 0. \tag{A.3}$$

Substituting (A.2) into (A.1) one obtains

$$\nabla \left(c_d^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} \right) + \nabla \times \left(c_s^2 \nabla^2 \boldsymbol{\psi} - \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} \right) = \mathbf{0}. \tag{A.4}$$

If we choose ϕ and $\boldsymbol{\psi}$ as solutions to Equation (2), the above equation will be satisfied identically. The completeness of such a solution to Equation (A.1) was given by Somigliana (1892) and Duhem (1898; 1900) independently.

Appendix B

It is sufficient to take ϕ as the following function in considering the convergence of the Laplace transform (6)

$$\phi(\xi, y, z, t) = H[t^2/a^2 - (\xi + vt)^2 - y^2 - z^2]. \tag{B.1}$$

Considering the transform of this function, one finds that the convergent domain of this transform is

$$-a_0 < \text{Re}\,\zeta < a_0. \tag{B.2}$$

Then consider ζ as a real parameter in this range at present. The transform may be continued into its convergent domain in the complex ζ plane when necessary.

After taken the transform (7), utilizing the step function defined above, one may find out the convergent domain of (7) is (8).

Appendix C

Function **R** has four roots in the transformed domain. These roots are $d, d, +c_1, -c_2$.

Consider a function **S** defined by

$$S(\eta, i\gamma) = -\frac{R(\eta, i\gamma)}{\kappa(\eta - c_1)(\eta + c_2)(\eta - d)^2}. \tag{C.1}$$

Function **S** has neither poles nor zeros in the η-plane. The only singularities it has are those branch points of functions α and β. One may note that the branch cuts of all functions considered here are taken to lie on the real axis outside of the strip of analyticity, $-a_2 < \mathrm{Re}(\eta) < a_1$, of the transforms. Function **S** approaches unity as $|\eta| \to \infty$. Because **S** has all these properties, it can be factored as

$$S(\eta, i\gamma) = S_+(\eta, i\gamma)S_-(\eta, i\gamma). \tag{C.2}$$

Thus, **R** was factored as Equation (18).

Appendix D

The Wiener–Hopf factorization can be completed as follows.
Using the factorization (18) and (20) one may obtain

$$\mathbf{E}^+ + \mathbf{E}^- = -\mu\mathbf{A}^+\mathbf{A}^-\mathbf{U}^-, \tag{D.1}$$

in which

$$\mathbf{A}^+ = \frac{1}{\eta + |\gamma|}\mathbf{X}(\eta)\mathbf{D}_+(\eta). \tag{D.2}$$

Multiplying Equation (D.1) with matrix $[\mathbf{A}^+]^{-1}$ gives

$$\begin{aligned} &\frac{1}{\eta - |\gamma|}[\mathbf{B}^+\mathbf{E}^- + \{\mathbf{B}^+\mathbf{E}^+ - (\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|}\}] \\ &\quad = -\mu\mathbf{A}^-\mathbf{U}^- - \frac{1}{\eta - |\gamma|}(\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|}. \end{aligned} \tag{D.3}$$

Then one gets

$$\begin{aligned} &\frac{1}{\eta - |\gamma|}\mathbf{B}^+\mathbf{E}^- - \boldsymbol{\beta}^+ + \frac{1}{\eta - |\gamma|}\{\mathbf{B}^+\mathbf{E}^+ - (\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|}\} \\ &\quad = -\mu\mathbf{A}^-\mathbf{U}^- - \frac{1}{\eta - |\gamma|}(\mathbf{B}^+\mathbf{E}^+)|_{\eta=|\gamma|} - \boldsymbol{\beta}^+. \end{aligned} \tag{D.4}$$

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