

On Linear and Nonlinear Beltrami Systems

Jarmo Jääskeläinen

Academic dissertation

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Department of Mathematics and Statistics
Faculty of Science
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A person can find anything if he takes the time, that is, if he can afford to look. And while he's looking, he's free, and he finds things he never expected.

Tove Jansson: A Summer Book

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Helsinki, October 2012

Jarmo Jääskeläinen

The thesis consists of the introductory part and the following three articles, referred to in the text by Roman numerals [I]–[III]. The papers are reproduced with the permission of their respective copyright holders.

[I] K. Astala and J. Jääskeläinen. Homeomorphic solutions to reduced Beltrami equations. *Annales Academiæ Scientiarum Fennicæ. Mathematica* 34(2):607-613, 2009.



[II] J. Jääskeläinen. On reduced Beltrami equations and linear families of quasiregular mappings. *Journal für die Reine und Angewandte Mathematik*, advance online publication doi:10.1515/crelle-2012-0041, 2012.



[III] K. Astala, A. Clop, D. Faraco, J. Jääskeläinen, and L. Székelyhidi, Jr. Uniqueness of normalized homeomorphic solutions to nonlinear Beltrami equations. *International Mathematics Research Notices* 2012(18):4101–4119, doi:10.1093/imrn/rnr178, 2012.



All authors had an equal role in the analysis of joint articles [I] and [III].

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Introduction

A route to elliptic PDEs in the theory of planar quasiregular mappings goes through the *classical Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}, \quad |\mu(z)| \leq k = \frac{K-1}{K+1} < 1, \quad (1.1)$$

for almost every $z \in \Omega \subset \mathbb{C}$. Here we use partial derivatives in the complex notation, that is,

$$\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \partial_z = \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),$$

and suppose that Ω is a domain, i.e., an open and connected set in the plane \mathbb{C} . For (1.1) to make sense, we have to assume suitable regularity properties on $f : \Omega \rightarrow \mathbb{C}$; namely, a Sobolev regularity $f \in W_{\text{loc}}^{1,2}(\Omega)$ that inaugurates the *natural domain of definition*. $W_{\text{loc}}^{1,2}$ -solutions to (1.1) are called *K -quasiregular mappings*—and *K -quasiconformal*, if they are homeomorphisms. Note that if the *complex dilatation* μ equals zero, we acquire the Cauchy-Riemann equations.

Quasiregular maps have very strong geometric properties that resemble those of analytic (holomorphic) mappings in the plane. As a matter of fact there are three main definitions for quasiconformal mappings in Euclidean setting: metric (they map infinitesimal balls onto infinitesimal ellipsoids with controlled eccentricity), geometric (mainly based on the concept of the modulus of a path family), and analytic (which is built on distortion

bounds, as in (1.1)). The interplay of the features is one of the key points of the theory. The three definitions date from different times, e.g., [Lav35], [Geh60], [Pfl51], [Ahl54], [Mor38], and the equivalence of them was achieved in the 1950s and early 1960s by Frederick Gehring, Olli Lehto, Lipman Bers, Albert Pfluger, and many others, e.g., in [GL59], [Geh60], [Ber57], [Pfl59].

The quasiconformal theory in the complex plane has a long distinct history starting from Herbert Grötzsch 1928, [Grö28]. He asked, given a square Q and a rectangle (not a square) R , what is the *most nearly conformal mapping* of Q on R which maps vertices on vertices. One needs to measure the approximate conformality and, by giving such a measure, Grötzsch took the first step toward quasiconformality.

The importance of quasiconformal mappings in the complex analysis was realized by Lars Ahlfors and Oswald Teichmüller in the 1930s. For Ahlfors, the quasiconformality provided an important tool for his approach to Nevanlinna's value distribution theory, [Ahl35]. Teichmüller found an integral connection between quasiconformal mappings and quadratic differentials in his studies concerning Riemann surfaces, starting a theory that is nowadays named after him, [Tei39].

There have been numerous cooks involved in making the quasicake, far too many to name even few of them. Since the influential studies of Charles B. Morrey Jr [Mor38], [Mor52], [Mor66], the publication of Ahlfors's book [Ahl66], and the classical text of Olli Lehto and Kalle Virtanen [LV73], profound developments have been made with wide-ranging applications to conformal and holomorphic dynamics, holomorphic motions, surface topology, fluid mechanics, elliptic PDEs, and nonlinear analysis, to mention a few. A modern approach to planar quasiconformal theory is presented in a recent monograph by Kari Astala, Tadeusz Iwaniec, and Gaven Martin [AIM09]. It is used as a background reference in this introductory part together with [IM01] and [AC05]. Survey [AC05] is a quite broad historical review of the theory of quasiconformal mappings.

There are two homotopy classes of the first-order elliptic systems, those represented by the Cauchy-Riemann operator $\partial_{\bar{z}}$ and those by its formal adjoint ∂_z . The most general first-order linear uniformly elliptic systems

(that can be deformed to the Cauchy-Riemann equation) take the form of an \mathbb{R} -linear Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \quad |\mu(z)| + |\nu(z)| \leq k = \frac{K-1}{K+1} < 1, \quad (1.2)$$

for almost every $z \in \Omega$. Note that the classical Beltrami equation (1.1) is simply linear over the complex numbers \mathbb{C} . The \mathbb{R} -linear equations have arisen as a correct framework for concrete applications, for example, a solution u to a uniformly elliptic equation of divergence type and its conjugate v together define a solution $f = u + iv$ to an \mathbb{R} -linear general Beltrami equation. Other examples include the hodograph transformations of certain energy minimizers and the hodograph transformations of a complex gradient of a p -harmonic function.

The classical Stoilow factorization in the complex plane states that every quasiregular solution $g : \Omega \rightarrow \mathbb{C}$ to the classical Beltrami equation (1.1) can be factorized in the following way: let f be a K -quasiconformal homeomorphic solution to (1.1), then there is an analytic function φ such that $g = \varphi \circ f$. It is somewhat surprising that a similar complete factorization of solutions is possible for \mathbb{R} -linear Beltrami equations, that is, every solution g to (1.2) takes the form $g = F \circ f$, where f is a K -quasiconformal solution to (1.2) and F solves a so-called *reduced Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \lambda(z) \operatorname{Im} \left(\frac{\partial f}{\partial z} \right), \quad |\lambda(z)| \leq \kappa < 1, \quad (1.3)$$

for almost every $z \in \Omega$, where λ naturally depends on μ and ν , [AIM09, Theorem 6.1.1] or see Theorem 2.5.

The peculiarity of reduced Beltrami equation (1.3) is that its solutions create an \mathbb{R} -linear space of quasiregular mappings. We establish the following fundamental fact, which is the main theorem in [II].

Theorem (Theorem 1.1 in [II], Theorem 2.1). *Suppose $f : \Omega \rightarrow \mathbb{C}$, $f \in W_{\text{loc}}^{1,2}(\Omega)$, is a solution to the reduced Beltrami equation (1.3). If the solution is not flat, i.e., $f(z) \neq az + b$, where $a \in \mathbb{R}$ and $b \in \mathbb{C}$, then*

$$\operatorname{Im} \left(\frac{\partial f}{\partial \bar{z}} \right) \neq 0 \quad \text{almost everywhere in } \Omega.$$

The statement plays the same role in Geometric Function Theory as the nonvanishing of the Jacobian determinant, $J(z, f) = |\partial_z f|^2 - |\partial_{\bar{z}} f|^2 \neq 0$ a.e., of a general quasiregular mapping f . The null Lagrangian $\text{Im}(\partial_z f)$, see Proposition 2.4, appears naturally in the denominator of an algebraic fraction and is vital in the G -compactness of Beltrami operators and questions about linear families, e.g, a Wronsky-type theorem for (1.2), see Chapter 3.

An early application of the reduced equation (1.3) can be found in [Boj57]. In that paper the reduced equation is used for uniqueness properties of \mathbb{R} -linear Beltrami equation (1.2) for self-mappings of the unit disk. Lately, the reduced equation has generated a considerable new-found interest, see [AN09], [AIM09], [GIK⁺04], [IKO09], [IKO11], [KO09], and [KO11].

First steps in proving the above theorem were made by F. Giannetti, T. Iwaniec, L. Kovalev, G. Moscarillo, and C. Sbordone in [GIK⁺04]. They proved the statement for global homeomorphisms, that is, homeomorphisms of the plane \mathbb{C} , when $\kappa < \frac{1}{2}$ in (1.3). Next, combining results and methods of G. Alessandrini and V. Nesi [AN09] and B. Bojarski, L. D’Onofrio, T. Iwaniec, and C. Sbordone [BDIS05] one can prove the assertion for global homeomorphisms. Direct, and substantially simplified, proof of this result in the global case can be found in [I].

Previously mentioned sets of equations are particular cases of the genuinely nonlinear first-order system, the *nonlinear Beltrami equation*,

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}\left(z, \frac{\partial f}{\partial z}\right), \quad \text{for almost every } z \in \Omega, \quad (1.4)$$

where $\mathcal{H} : \Omega \times \mathbb{C} \rightarrow \mathbb{C}$ is assumed to be Lipschitz in the second variable,

$$|\mathcal{H}(z, w_1) - \mathcal{H}(z, w_2)| \leq k|w_1 - w_2|, \quad 0 \leq k < 1.$$

The principal aspect of (1.4) is that the difference of two solutions need not solve the same equation but it still is K -quasiregular mapping. The study of nonlinear Beltrami equations was introduced in the seminal works by Tadeusz Iwaniec and Bogdan Bojarski, [Iwa76], [BI74], and [Boj76].

One of the major achievements, [Mor38], in Geometric Function Theory was the existence and the uniqueness of a so-called *normalized solution*

to the classical Beltrami equation (1.1), that is, a homeomorphic solution $f : \mathbb{C} \rightarrow \mathbb{C}$ normalized by $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$. One can measurably set beforehand the eccentricity and the angle of the infinitesimal ellipses. In [III], we establish the uniqueness problem for nonlinear Beltrami equations.

Theorem (Theorem 1.1 in [III], Theorem 4.1). *Assume $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies the following:*

(H1) *For every $w \in \mathbb{C}$, the mapping $z \mapsto \mathcal{H}(z, w)$ is measurable on \mathbb{C} .*

(H2) *For $w_1, w_2 \in \mathbb{C}$,*

$$|\mathcal{H}(z, w_1) - \mathcal{H}(z, w_2)| \leq k(z)|w_1 - w_2|, \quad 0 \leq k(z) \leq k < 1,$$

for almost every $z \in \mathbb{C}$.

(H3) $\mathcal{H}(z, 0) \equiv 0$.

If

$$\limsup_{|z| \rightarrow \infty} k(z) < 3 - 2\sqrt{2} = 0.17157\dots,$$

then the nonlinear Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}\left(z, \frac{\partial f}{\partial z}\right), \quad \text{for almost every } z \in \Omega, \quad (1.5)$$

admits a unique homeomorphic solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$.

Furthermore, the bound on k is sharp: for each $k > 3 - 2\sqrt{2}$, there are functions $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ for which (H1)–(H3) hold, such that (1.5) admits two normalized homeomorphic solutions.

There are several possibilities to ensure uniqueness under weaker hypotheses, see [III]. We discuss a couple of them in Chapter 4.

The most general genuinely nonlinear system takes the form

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H} \left(z, f, \frac{\partial f}{\partial z} \right), \quad \text{for almost every } z \in \Omega, \quad (1.6)$$

where we assume, as in (1.4), that \mathcal{H} is a contraction in $\partial_z f$ -variable. The existence of normalized homeomorphic solutions can be established in great generality, see [AIM09, Theorem 8.2.1], but no matter how small is the distortion k , the uniqueness of normalized solutions need not hold for (1.6), see [III, pages 5–6].

Reduced Beltrami Equations

In this chapter we discuss the *reduced Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \lambda(z) \operatorname{Im} \left(\frac{\partial f}{\partial z} \right), \quad |\lambda(z)| \leq k < 1, \quad (2.1)$$

for almost every $z \in \Omega$, and its solutions, *reduced quasiregular mappings*. The main matter is to outline papers [I], [II]. Their application to linear families of quasiregular mappings and G -compactness problems is done in Chapter 3.

Studies of the reduced Beltrami equation (2.1) indicate that for its solutions the null Lagrangian

$$\mathcal{J}(z, f) = \operatorname{Im} \left(\frac{\partial f}{\partial z} \right),$$

see Proposition 2.4, has many properties similar to the Jacobian determinant of a Sobolev function. The key result in that direction is the nonvanishing property almost everywhere. More precisely

Theorem 2.1 (Theorem 1.1 in [II]). *Suppose $f : \Omega \rightarrow \mathbb{C}$, $f \in W_{\text{loc}}^{1,2}(\Omega)$, is a solution to the reduced Beltrami equation (2.1). If the solution is not flat, i.e., $f(z) \neq az + b$, where $a \in \mathbb{R}$ and $b \in \mathbb{C}$, then*

$$\operatorname{Im} \left(\frac{\partial f}{\partial z} \right) \neq 0 \quad \text{almost everywhere in } \Omega.$$

We discuss the proof of the above statement in Section 2.3.

For global solutions, that is, solutions to (2.1) in the whole complex plane, we can actually say something more. We combine the results from [BDIS05], [IKO09], [AN09], and [I]—and sketch a proof in Section 2.2.

Theorem 2.2. *Assume that $f : \mathbb{C} \rightarrow \mathbb{C}$, $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$, is a global homeomorphic solution to the reduced Beltrami equation (2.1). Then either $\partial_z f$ is a constant or else $\text{Im}(\partial_z f)$ has a strict constant sign, i.e.,*

$$\text{Im}\left(\frac{\partial f}{\partial z}\right) > 0 \quad \text{or} \quad \text{Im}\left(\frac{\partial f}{\partial z}\right) < 0 \quad \text{almost everywhere.}$$

Thus, if $\text{Im}(\partial_z f)$ vanishes on a set of positive measure, then $f(z) = az + b$, where $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

Conversely, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a solution to the reduced equation (2.1) and $\text{Im}(\partial_z f)$ does not change sign (namely, $\text{Im}(\partial_z f) \leq 0$ or $\text{Im}(\partial_z f) \geq 0$), then it is a homeomorphism. Further, $\text{Im}(\partial_z f) \neq 0$ almost everywhere; thus $\text{Im}(\partial_z f)$ has a strict constant sign.

Note that, if we know the sign of $\text{Im}(\partial_z f)$, we can still say something about injectivity even in proper subdomains.

Theorem 2.3 (Theorem 1.1 in [IKO09]). *If $f : \Omega \rightarrow \mathbb{C}$ is a nonconstant quasiregular mapping and $\text{Im}(\partial_z f) \geq 0$ almost everywhere in Ω , then f is a local homeomorphism.*

The proof of the above result uses the theorem of Poincaré-Bendixson and its extension by Brouwer on local structure of integral curves of a continuous planar vector field near its critical point.

2.1 Basic Properties and Examples

Proposition 2.4. *Expression $\text{Im}(\partial_z f)$ is a null Lagrangian.*

Proof. Let $f, g \in W^{1,2}(\Omega)$ and $f - g \in W_0^{1,2}(\Omega)$. Approximating with C^∞ - and C_0^∞ -functions f and $f - g$, respectively, we may assume that f and g are smooth up to the boundary and $f = g$ on $\partial\Omega$. Then

$$\begin{aligned} \int_{\Omega} \operatorname{Im} \left(\frac{\partial f}{\partial z} \right) dm(z) &= \frac{1}{2i} \int_{\Omega} \left[\frac{\partial f}{\partial z} - \overline{\frac{\partial f}{\partial z}} \right] dm(z) \\ &= \frac{1}{2i} \int_{\Omega} \left[\frac{\partial g}{\partial z} - \overline{\frac{\partial g}{\partial z}} + \frac{\partial}{\partial z} (f - g) - \frac{\partial}{\partial \bar{z}} \overline{(f - g)} \right] dm(z) \\ &= \int_{\Omega} \operatorname{Im} \left(\frac{\partial g}{\partial z} \right) dm(z) + \frac{1}{4} \int_{\partial\Omega} \left[(f - g) d\bar{z} + \overline{(f - g)} dz \right] \\ &= \int_{\Omega} \operatorname{Im} \left(\frac{\partial g}{\partial z} \right) dm(z). \end{aligned}$$

The second to the last equality follows from Green's formula. \square

Generalized Stoilow Factorization, [AIM09, Theorem 6.1.1] is the connection between reduced Beltrami equations and linear families of quasiregular mappings.

Theorem 2.5 (Generalized Stoilow Factorization). *Let $f \in W_{\text{loc}}^{1,2}(\Omega)$ be a homeomorphic solution to the equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \quad |\mu(z)| + |\nu(z)| \leq k < 1, \quad (2.2)$$

for almost every $z \in \Omega$, and so, in particular, f is K -quasiconformal with $K = \frac{1+k}{1-k}$.

Then any other solution $g \in W_{\text{loc}}^{1,2}(\Omega)$ to this equation takes the form

$$g(z) = F(f(z))$$

where F is a K^2 -quasiregular mapping satisfying almost everywhere

$$\frac{\partial F}{\partial \bar{w}} = \lambda(w) \operatorname{Im} \left(\frac{\partial F}{\partial w} \right), \quad w \in f(\Omega), \quad (2.3)$$

where

$$\lambda(w) = \frac{-2i\nu(z)}{1 + |\nu(z)|^2 - |\mu(z)|^2}$$

with

$$|\lambda(w)| \leq \frac{2k}{1+k^2} < 1, \quad z = f^{-1}(w).$$

Conversely, if $F \in W_{\text{loc}}^{1,2}(f(\Omega))$ satisfies the equation (2.3), then $g = F \circ f$ solves the equation (2.2).

Examples 2.6. Reduced Beltrami equation admits always so-called *flat solutions*, that is, $f(z) = az + b$, $a \in \mathbb{R}$, $b \in \mathbb{C}$ is always solution to (2.1).

Remark 2.7. If a homeomorphic solution to (2.1), $f : \mathbb{C} \rightarrow \mathbb{C}$, fixes two points, then $f(z) \equiv z$. This is proven in [AIM09, Theorem 6.2.2]. The claim also follows from a more general statement on normalized solutions to Beltrami systems, see Corollary 4.5.

The reduced Beltrami equation with a constant dilatation

$$\frac{\partial f}{\partial \bar{z}} = \kappa \operatorname{Im} \left(\frac{\partial f}{\partial z} \right), \quad |\kappa| < 1, \quad \text{almost everywhere,} \quad (2.4)$$

holds for f if and only if $f = f_0^{-1} \circ h \circ f_0$ where h is analytic and $f_0(z) = z + k\bar{z}$, $|k| < 1$. The relation between κ and k is $\kappa = \frac{2ki}{1+|k|^2}$.

It is a straightforward calculation to check that $f_0^{-1} \circ h \circ f_0$ solves (2.4). For the other direction, it is enough to note that f_0 is a K -quasiconformal solution to the classical Beltrami equation with constant dilation, i.e., $\partial_{\bar{z}}g(z) = k\partial_zg(z)$, and $f_0 \circ f$ is another solution to the same equation. Then the claim follows by classical Stoïlow factorization.

Radial stretchings usually give extremal type behaviour. Set $f : \mathbb{C} \rightarrow \mathbb{C}$

$$f(z) = \frac{z}{|z|} \rho(|z|), \quad f(0) = 0,$$

where we assume $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to be continuous and strictly increasing with $\rho(0) = 0$.

Let ρ be differentiable, then the complex partial derivatives are fairly easy to compute using identity $\partial_{\bar{z}}|z| = z/(2|z|)$; namely,

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \frac{z}{\bar{z}} \left[\rho'(|z|) - \frac{\rho(|z|)}{|z|} \right], \\ \frac{\partial f}{\partial z} &= \frac{1}{2} \left[\rho'(|z|) + \frac{\rho(|z|)}{|z|} \right] \in \mathbb{R}.\end{aligned}$$

Basic examples are choices $\rho(|z|) = |z|^{1/K}$ and $\rho(|z|) = |z|^K$ which give K -quasiconformal homeomorphisms $f_1(z) = z|z|^{1/K-1}$ and $f_2(z) = z|z|^{K-1}$. They solve a classical Beltrami equation,

$$\frac{\partial f_j}{\partial \bar{z}} = \mu_j(z) \frac{\partial f_j}{\partial z}, \quad \mu_j(z) = (-1)^j \frac{K-1}{K+1} \frac{z}{\bar{z}}, \quad \text{a.e.}$$

Thus, *rotated* radial stretching $if_j(z)$ solves a reduced Beltrami equation with $\lambda_j = i\mu_j$, since z -derivative of f_j is real-valued as we remarked above.

It is worth noting that reduced quasiregular mappings admit the same Hölder-regularity as quasiregular maps, precisely $C^{1/K}$. In general, this cannot be improved for reduced quasiregular mappings as seen by the rotated radial stretching if_1 .

2.2 Global Homeomorphic Solutions, Theorem 2.2

Let us study more closely the pieces that form Theorem 2.2. Firstly, [BDIS05, Lemma 7.1] or [AIM09, Theorem 6.3.2]

Theorem 2.8. *If $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ is a homeomorphic solution to the reduced equation*

$$\frac{\partial f}{\partial \bar{z}} = \lambda(z) \text{Im} \left(\frac{\partial f}{\partial z} \right), \quad |\lambda(z)| \leq k < 1, \quad (2.5)$$

for almost every $z \in \Omega$, then $\text{Im}(\partial_z f)$ does not change sign:

- if $f(1) - f(0) \in \mathbb{R}$, then $\operatorname{Im}(\partial_z f) \equiv 0$,
- if $\operatorname{Im}(f(1) - f(0)) > 0$, then $\operatorname{Im}(\partial_z f) \geq 0$ almost everywhere,
- if $\operatorname{Im}(f(1) - f(0)) < 0$, then $\operatorname{Im}(\partial_z f) \leq 0$ almost everywhere.

Idea of the Proof. Note that if the quotient $\frac{f(z_0) - f(w_0)}{z_0 - w_0} := t$ is real for some $z_0, w_0 \in \mathbb{C}$, then the mapping

$$z \mapsto \frac{f(z)}{t} - \frac{c_0}{t}, \quad \text{where } c_0 := f(z_0) - tz_0 = f(w_0) - tw_0,$$

solves the reduced Beltrami equation (2.5) and fixes two points $z = z_0$ and $z = w_0$. Hence, $f(z) \equiv tz + f(0)$, by the uniqueness of the normalized solution, see Remark 2.7. This is the first case of the theorem.

Otherwise the incremental quotients of f do not take real values and therefore

$$(z, w) \mapsto \operatorname{Im} \left(\frac{f(z) - f(w)}{z - w} \right) : (\mathbb{C} \times \mathbb{C}) \setminus \{(z, w) : z = w\} \rightarrow \mathbb{R}$$

is continuous, nonvanishing, real-valued function on the connected domain, so it cannot change sign. To see that it is enough to check the sign of $\operatorname{Im}(f(1) - f(0))$ one uses Taylor's first-order expansion at the points of differentiability.

Secondly, we sketch the nonvanishing of $\operatorname{Im}(\partial_z f)$ for global homeomorphic solutions that are not flat, i.e, the inequalities are strict in Theorem 2.8. We follow the idea of [AN09], [I, Theorem 1.2].

Theorem 2.9. *Assume that $f : \mathbb{C} \rightarrow \mathbb{C}$, $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$, is a global homeomorphic solution to the reduced Beltrami equation (2.1). Then either $\partial_z f$ is a constant or else $\operatorname{Im}(\partial_z f)$ has a strict constant sign, i.e,*

$$\operatorname{Im} \left(\frac{\partial f}{\partial z} \right) > 0 \quad \text{or} \quad \operatorname{Im} \left(\frac{\partial f}{\partial z} \right) < 0 \quad \text{almost everywhere.}$$

Thus, if $\operatorname{Im}(\partial_z f)$ vanishes on a set of positive measure, then $f(z) = az + b$, where $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

Idea of the Proof. The proof is based on the fact that $\text{Im}(\partial_z f) \simeq \partial_y(\text{Re } f)$, where the uniform bounds depend only on the uniform ellipticity of the reduced equation (2.1), that is, on k . By Theorem 2.8, we can assume that $\partial_y(\text{Re } f)$ is nonnegative.

The expression $\partial_y(\text{Re } f)$ is actually a weak solution to $L^*(\partial_y(\text{Re } f)) = 0$, where L is a second-order uniformly elliptic operator of non-divergence type. Work of E. B. Fabes and D. W. Stroock, [FS84] or [AIM09, Theorem 6.4.2], shows that nonnegative solutions to such equations satisfy a reverse Hölder inequality. Thus, they cannot vanish in a set of positive measure. More precisely, consider an operator

$$L = \sum_{i,j=1}^2 \sigma_{ij}(z) \frac{\partial^2}{\partial x_i \partial x_j},$$

where $\sigma_{ij} = \sigma_{ji}$ are measurable and the matrix

$$\sigma(z) = \begin{bmatrix} \sigma_{11}(z) & \sigma_{12}(z) \\ \sigma_{12}(z) & \sigma_{22}(z) \end{bmatrix}$$

is uniformly elliptic,

$$\frac{1}{K} |\xi|^2 \leq \langle \sigma(z)\xi, \xi \rangle = \sigma_{11}(z)\xi_1^2 + 2\sigma_{12}(z)\xi_1\xi_2 + \sigma_{22}(z)\xi_2^2 \leq K |\xi|^2$$

for all $\xi \in \mathbb{C}$ and $z \in \Omega$. Above K is the *ellipticity constant*. The mapping $\omega \in L^2_{\text{loc}}(\Omega)$ is a weak solution to the adjoint equation $L^*(\omega) = 0$ if

$$\int_{\Omega} \omega L(\varphi) \, dm = 0, \quad \text{for every } \varphi \in C_0^\infty(\Omega). \quad (2.6)$$

To identify $\partial_y(\text{Re } f)$ as a weak solution to an adjoint equation of the type (2.6), one uses the fact that $\partial_y(\text{Re } f)$ satisfies a divergence type second-order equation.

We mention the method for the converse direction.

Theorem 2.10 (Corollary 1.5 in [IKO09]). *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a solution to the reduced Beltrami equation (2.1) and $\text{Im}(\partial_z f)$ does not change sign (namely, $\text{Im}(\partial_z f) \leq 0$ or $\text{Im}(\partial_z f) \geq 0$), then it is a homeomorphism.*

Idea of the Proof. It is enough to show that mappings

$$f^\lambda(z) = f(z) + i\lambda z$$

are quasiconformal when $\lambda > 0$ is small enough, because then f is a local uniform limit of K -quasiconformal mappings f^λ , and thus K -quasiconformal or constant, by a Hurwitz-type theorem [LV73, II 5.3] or [AIM09, Theorem 3.9.4].

Now, f^λ is quasiregular as a solution to the reduced Beltrami inequality

$$\left| \frac{\partial f}{\partial \bar{z}} \right| \leq k \operatorname{Im} \left(\frac{\partial f}{\partial z} \right)$$

with

$$\operatorname{Im}(\partial_z f^\lambda) = \operatorname{Im}(\partial_z f) + \lambda > 0.$$

Hence, f^λ is locally quasiconformal by Theorem 2.3. Moreover, $J(z, f^\lambda) = J(z, f) + \lambda^2 + 2\lambda \operatorname{Im}(\partial_z f) \geq \lambda^2$ and one can use the following result, [IKO09, Proposition 4.1]: if $g : \mathbb{C} \rightarrow \mathbb{C}$ is locally quasiconformal and $1/J(z, g) \in L^\infty(\mathbb{C})$, then g is quasiconformal.

2.3 Noninjective Solutions, Theorem 2.1

We will sketch the proof for Theorem 2.1. Suppose $f : \Omega \rightarrow \mathbb{C}$, $f \in W_{\text{loc}}^{1,2}(\Omega)$, is a solution to the reduced Beltrami equation (2.1). If the solution is not flat, i.e., $f(z) \neq az + b$, where $a \in \mathbb{R}$ and $b \in \mathbb{C}$, then $\operatorname{Im}(\partial_z f) \neq 0$ almost everywhere in Ω .

Idea of the Proof. Similarly, as in the global homeomorphic case, $\operatorname{Im}(\partial_z f) \simeq \partial_y(\operatorname{Re} f)$, where the uniform bounds depend only on k , and $\partial_y(\operatorname{Re} f)$ is a weak solution to the adjoint equation determined by a non-divergence type operator. The difference is that $\partial_y(\operatorname{Re} f)$ may change sign, although $\operatorname{Im}(\partial_z f)$ and u_y have still the same zeros. We do not have as strong a statement as for nonnegative weak solutions (that is, the reverse Hölder inequality by the Fabes-Stroock theorem [FS84], [AIM09, Theorem 6.4.2]), but $\partial_y(\operatorname{Re} f)$ admits a weak reverse Hölder inequality, [II, Theorem 3.1]. In fact, we have a slightly better result.

Theorem 2.11. *Let $\omega \in L^2_{\text{loc}}(\Omega)$ be a real-valued weak solution to the adjoint equation $L^*(\omega) = 0$ determined by a non-divergence type operator (2.6). Then*

$$\left(\frac{1}{r^2} \int_B \omega^p dm \right)^{1/p} \leq \frac{c}{r^2} \int_{2B} |\omega| dm,$$

for every disk $B := \mathbb{D}(a, r)$ such that $2B := \mathbb{D}(a, 2r) \subset \Omega$ and $p \in \left[2, \frac{2K}{K-1}\right)$. The constant c depends only on p and the ellipticity constant K .

We actually need only the weak reverse Hölder inequality, i.e., the case $p = 2$. But same tricks as in [II, Section 3] give the more general statement of Theorem 2.11. To derive the result, we use interpolation, appropriately chosen Dirichlet problem, and an inner regularity from [AIM06]. The difference for the general p is that one uses the dual expression of the L^p -norm

$$\left(\int_B \omega^p dm \right)^{1/p} = \sup \left\{ \int_B \omega h dm : h \in C^1_0(B), \|h\|_{L^q} \leq 1 \right\}$$

where q is the Hölder conjugate pair of p , $pq = p + q$, and [AIM06] gives the upper bound for p . Similar argumentation is used in [MS05].

A weak reverse Hölder inequality implies that almost every zero z_0 of $\partial_y(\text{Re } f)$ (or, equivalently, of $\text{Im}(\partial_z f)$) is of infinite order, that is, for every positive integer N , there is $r_0(z_0, N) > 0$ such that

$$\int_{\mathbb{D}(z_0, r)} |\partial_y(\text{Re } f)| dm \leq \frac{r^N}{r_0^N} \int_{\mathbb{D}(z_0, 2r_0)} |\partial_y(\text{Re } f)| dm = \mathcal{O}(r^N),$$

for $0 < r \leq r_0(z_0, N)$.

The claim of Theorem 2.1 follows now by studying the behaviour of reduced quasiregular mapping f at zeros of $\text{Im}(\partial_z f)$. This is done in a somewhat same spirit as smoothness at a point representations in [Dyn97].

Linear Families and G -Compactness

Given a domain $\Omega \subset \mathbb{C}$ and a constant $1 \leq K < \infty$ we study a linear family $\mathcal{F} \subset W_{\text{loc}}^{1,2}(\Omega)$ of K -quasiregular mappings. We assume it to be generated by a countable set of functions, that is,

$$\mathcal{F} = \left\{ \sum_{i \in \mathcal{I}} a_i f_i : a_i \in \mathbb{R} \right\}$$

for some \mathbb{R} -linearly independent quasiregular mappings $f_i : \Omega \rightarrow \mathbb{C}$.

Generally, linear combinations do not preserve quasiregularity; consider, for example, $f(z) = k\bar{z} + z$, $g(z) = k\bar{z} - z$. However, if the mappings are solutions to the same \mathbb{R} -linear Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \quad |\mu(z)| + |\nu(z)| \leq k < 1, \quad (3.1)$$

almost everywhere, then their linear combinations are quasiregular. Conversely, [BDIS05] associates to a two-dimensional linear class \mathcal{F} of quasiregular mappings an \mathbb{R} -linear Beltrami equation of the type (3.1) that is satisfied by every $f \in \mathcal{F}$. For the idea behind the proof, see the next section.

The associated equation is actually unique, [II, Theorem 1.3]. We discuss this in Section 3.2 with the following key ingredient, the *Wronsky-type theorem*.

Theorem 3.1 (Theorem 1.2 in [II]). *Suppose $\Phi, \Psi \in W_{\text{loc}}^{1,2}(\Omega)$ are solutions to*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \quad |\mu(z)| + |\nu(z)| \leq k < 1,$$

for almost every $z \in \Omega$. Solutions Ψ and Φ are \mathbb{R} -linearly independent if and only if complex gradients $\partial_z \Phi$ and $\partial_z \Psi$ are pointwise independent almost everywhere, i.e.,

$$\mathcal{J}(\Phi, \Psi) := \text{Im} \left(\frac{\partial \Phi}{\partial z} \overline{\frac{\partial \Psi}{\partial z}} \right) \neq 0 \quad \text{almost everywhere in } \Omega.$$

Above $\mathcal{J}(\Phi, \Psi)$ plays the role of Wronskian. Note that, if Ψ and Φ are \mathbb{R} -linearly dependent, then $\mathcal{J}(\Phi, \Psi) \equiv 0$.

Proof. We can assume Φ is nonconstant. As a nonconstant quasiregular mapping, Φ is discrete, open, and the branch set consists of isolated points. Thus, it is enough to study points outside the branch set. Let z_0 be such a point. There exists a ball $B := \mathbb{D}(z_0, r)$ such that $\Phi|_B : B \rightarrow \Phi(B)$ is a homeomorphism, hence quasiconformal. From the Stoilow factorization of general Beltrami equations, Theorem 2.5, we know that

$$\Psi = F \circ \Phi \quad \text{in } B,$$

where F solves the reduced Beltrami equation $\partial_{\bar{w}} F = \lambda(w) \text{Im}(\partial_w F)$ in $\Phi(B)$ with

$$\lambda(w) = \frac{-2i\nu(z)}{1 + |\nu(z)|^2 - |\mu(z)|^2}, \quad w = \Phi(z), \quad z \in B.$$

Let $z \in B$. Using the chain rule and identities $J(z, f)h_{\bar{w}}(w) = -f_{\bar{z}}(z)$, $J(z, f)h_w(w) = \overline{f_z(z)}$, where $h = f^{-1}$ and $w = f(z)$, we arrive at

$$\begin{aligned} J(z, \Phi)F_w(w) &= \Psi_z(z)\overline{\Phi_z(z)} - \Psi_{\bar{z}}(z)\overline{\Phi_{\bar{z}}(z)} \\ &= (1 - |\mu|^2)\Psi_z\overline{\Phi_z} - |\nu|^2\overline{\Psi_z}\Phi_z - 2\text{Re}(\mu\bar{\nu}\Psi_z\Phi_z), \quad w = \Phi(z). \end{aligned}$$

Thus,

$$J(z, \Phi) \text{Im}(F_w \circ \Phi) = (-1 + |\mu|^2 - |\nu|^2) \text{Im}(\Phi_z \overline{\Psi_z}).$$

Since $\Phi|_B$ preserves sets of zero measure, the statement follows by Theorem 2.1. \square

3.1 Linear Class of Injections

If we have a linear class of injections (i.e., \mathcal{F} is a linear family of K -quasiconformal mappings and constants), then $\dim \mathcal{F} \leq 2$. This is quickly seen, [BDIS05, Lemma 5.1], by considering a set of mappings $g_1, g_2, \dots, g_n \in \mathcal{F}$ with $n > 2$. Then for $a \neq b$ the vectors $[g_j(a) - g_j(b)] \in \mathbb{C}$, $j = 1, \dots, n$, are \mathbb{R} -linearly dependent. Hence,

$$\sum_{j=1}^n \alpha_j [g_j(a) - g_j(b)] = 0, \quad \alpha_j \in \mathbb{R},$$

where not all α_j are zero. Thus, $\sum_{j=1}^n \alpha_j g_j \in \mathcal{F}$ achieves the same value at two distinct points, $z = a$ and $z = b$, and the only noninjective map in the family \mathcal{F} is the trivial one, $\sum_{j=1}^n \alpha_j g_j \equiv 0$, as desired.

Viewing \mathbb{C} as one-dimensional space over complex numbers, every non-trivial linear class of injections is of type

$$\{a\Phi : a \in \mathbb{C}\}, \tag{3.2}$$

that is, generated by only one injection $\Phi : \Omega \rightarrow \mathbb{C}$. Over the field of real numbers, Φ and $i\Phi$ are naturally the generators. Of course not every \mathbb{R} -linear family of injections can be obtained this way; simply because $\Phi \in \mathcal{F}$ does not necessarily give $i\Phi \in \mathcal{F}$. Every mapping in the family (3.2) satisfies the classical Beltrami equation (1.1) with $\mu = \partial_{\bar{z}}\Phi/\partial_z\Phi$.

The study of Beltrami systems for two-dimensional \mathbb{R} -linear families is more subtle. Note that global homeomorphic solutions to

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \quad |\mu(z)| + |\nu(z)| \leq k < 1, \tag{3.3}$$

almost everywhere in Ω , are determined by their values at two distinct points ([AIM09, Corollary 6.2.4], which combines the uniqueness of the normalized solution to the reduced Beltrami equation, Corollary 4.5, and the generalized Stöilow factorization, Theorem 2.5). It follows that in case $\Omega = \mathbb{C}$, linear

combinations of solutions to (3.3) are either constants or homeomorphisms. Thus, in this case, we have two-dimensional linear family of quasiconformal mappings.

There are further situations where the injectivity of a linear family of solutions to (3.3) can be guaranteed. For example, if f satisfies (3.3) in a bounded convex domain Ω with

$$\operatorname{Re}(f(z)) = \operatorname{Re}(\mathcal{A}(z)) \quad \text{on } \partial\Omega, \quad \mathcal{A} : \mathbb{C} \rightarrow \mathbb{C} \text{ a linear isomorphism,}$$

then f is injective: the idea of the proof with the help of the classical Radó-Kneser-Choquet theorem is outlined in [I, page 609]. There are alternative proofs using properties of the Beltrami equation, see [BDIS05], [LN97], or [AN09].

For two-dimensional \mathbb{R} -linear family of quasiconformal mappings, there still exists a corresponding general Beltrami equation (3.3) satisfied by every element $f \in \mathcal{F}$. We sketch the idea from beginning of Section 5.3 in [BDIS05], alternatively, see the proof of Theorem 16.6.6 in [AIM09]. It is worth noting that, for the existence, only local properties play a part, i.e., the existence of the equation for two-dimensional linear quasiregular family follows with the same proof.

Assume $\Phi, \Psi \in W_{\text{loc}}^{1,2}(\Omega)$ are the generators of a linear family \mathcal{F} . The goal is to find coefficients μ and ν such that

$$\partial_{\bar{z}}\Phi = \mu\partial_z\Phi + \nu\overline{\partial_z\Phi} \quad \text{and} \quad \partial_{\bar{z}}\Psi = \mu\partial_z\Psi + \nu\overline{\partial_z\Psi}, \quad (3.4)$$

almost everywhere in Ω . In the *regular set* $\mathcal{R}_{\mathcal{F}}$ of \mathcal{F} , i.e., the set of points $z \in \Omega$ where the matrix

$$M(z) = \begin{bmatrix} \partial_z\Phi(z) & \overline{\partial_z\Phi(z)} \\ \partial_z\Psi(z) & \overline{\partial_z\Psi(z)} \end{bmatrix}$$

is invertible, the values $\mu(z)$ and $\nu(z)$ are uniquely determined by (3.4),

that is,

$$\mu(z) = i \frac{\Psi_{\bar{z}}(z)\overline{\Phi_z(z)} - \overline{\Psi_z(z)}\Phi_{\bar{z}}(z)}{2 \operatorname{Im}(\Phi_z(z)\overline{\Psi_z(z)})}, \quad (3.5)$$

$$\nu(z) = i \frac{\Phi_{\bar{z}}(z)\Psi_z(z) - \overline{\Phi_z(z)}\Psi_{\bar{z}}(z)}{2 \operatorname{Im}(\Phi_z(z)\overline{\Psi_z(z)})}. \quad (3.6)$$

Note that changing the generators corresponds to multiplying $M(z)$ by an invertible constant matrix. Hence, the regular set and its complement, the *singular set*

$$\mathcal{S}_{\mathcal{F}} = \{z \in \Omega : 2i \operatorname{Im}(\Phi_z(z)\overline{\Psi_z(z)}) = \det M(z) = 0\},$$

depend only on the family \mathcal{F} and not the choice of generators.

On the singular set it can be proven that for almost every $z \in \mathcal{S}_{\mathcal{F}}$ the vector $(\Phi_{\bar{z}}(z), \Psi_{\bar{z}}(z))$ lies in the range of the linear operator $M(z) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. It follows that on the singular set one may define $\nu(z) = 0$. Here one needs quasiproperties, see [GIK⁺04, Lemma 12.1] or [AIM09, page 465].

Finally, ellipticity bounds in (3.3) follow for the singular set $\mathcal{S}_{\mathcal{F}}$ by definition of μ and ν , since Φ and Ψ are K -quasiregular. For the regular set one tests the quasiregularity inequality by real-valued measurable functions, see [GIK⁺04, Lemma 12.1].

We have the uniqueness of the equation (3.3) once we have shown that the singular set, $\{z \in \Omega : \operatorname{Im}(\Phi_z(z)\overline{\Psi_z(z)}) = 0\}$, has measure zero. By generalized Stoilow factorization, Theorem 2.5, this follows from our studies of reduced Beltrami equations; namely, for injectivity classes, by Theorem 2.2 with the idea of the proof of Theorem 3.1.

3.2 Linear Families of Quasiregular Mappings

Not all \mathbb{R} -linear families of quasiregular mappings are two-dimensional, for example, 1-quasiregular family spanned by $f_i(z) = z^i$, $i = 1, 2, 3$. However, as we remarked in the previous section, for any linear two-dimensional family

\mathcal{F} of quasiregular mappings $\Omega \rightarrow \mathbb{C}$ there exists a corresponding general Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \quad |\mu(z)| + |\nu(z)| \leq k < 1,$$

almost everywhere in Ω , satisfied by every element $f \in \mathcal{F}$. With the similar argumentation as in the global injective case, i.e, using the reduced equation, Theorems 2.1 and 3.1, the singular set $\mathcal{S}_{\mathcal{F}}$ has measure zero. Hence, the associated equation is actually unique.

If the linear family has more than two generators, there is still a unique associated equation. Here we need our assumption that there are only countably many generators. The key point is to show that there exists a set $E \subset \Omega$ of full measure such that the inequality

$$\left| \sum_{i \in \mathcal{I}} a_i \partial_{\bar{z}} f_i(z) \right| \leq k \left| \sum_{i \in \mathcal{I}} a_i \partial_z f_i(z) \right|, \quad k = \frac{K-1}{K+1}, \quad (3.7)$$

holds for almost every $z \in \Omega$. The inequality (3.7) follows from the fact that the family consists of K -quasiregular mappings. For details, see [II, Section 6].

3.3 G -Compactness

As an application of the reduced Beltrami equation and linear families of quasiconformal mappings, we study the compactness properties of *Beltrami differential operators*

$$\mathcal{B} = \frac{\partial}{\partial \bar{z}} - \mu(z) \frac{\partial}{\partial z} - \nu(z) \overline{\frac{\partial}{\partial z}}.$$

As before we continue to assume the uniform ellipticity

$$|\mu(z)| + |\nu(z)| \leq k < 1 \quad \text{almost everywhere in } \mathbb{C}.$$

Now, a sequence of Beltrami differential operators

$$\mathcal{B}_j = \frac{\partial}{\partial \bar{z}} - \mu_j(z) \frac{\partial}{\partial z} - \nu_j(z) \overline{\frac{\partial}{\partial z}}$$

G -converges to a Beltrami differential operator \mathcal{B} , if $\lim_{j \rightarrow \infty} \mathcal{B}_j f_j = \mathcal{B} f$, for every sequence $(f_j)_{j=1}^\infty$ that converges weakly to f in $W_{\text{loc}}^{1,2}(\Omega)$ such that $\mathcal{B}_j f_j$ converges strongly in $L_{\text{loc}}^2(\Omega)$. The family of Beltrami differential operators is G -compact if every sequence $(\mathcal{B}_j)_{j=1}^\infty$ contains a G -converging subsequence.

If μ_j and ν_j converge almost everywhere to μ and ν , respectively, then Beltrami operators G -converge. Weak convergence of coefficients μ_j and ν_j , however, has little to do with the G -convergence, see examples in [GIK⁺04] or [AIM09, Section 16.6].

G -convergence has a long history in Italian school of PDEs, starting from S. Spagnolo, A. Marino, and E. De Giorgi, [Spa68], [MS69], [DGS73]. The very idea was concerned with second-order elliptic equations; nowadays, it has evolved further to include PDEs, see [MT97], [DM93], [GIK⁺04], [BDIS05], [AIM09, Section 16.6]. Our interest lies in Beltrami differential operators. The G -compactness in this light was first introduced and discussed in the seminal work by F. Giannetti, T. Iwaniec, L. Kovalev, G. Moscarillo, and C. Sbordone, [GIK⁺04].

Theorem 3.2 (Theorem 16.6.8 in [AIM09]). *The family of Beltrami differential operators, $1 \leq K < \infty$,*

$$\mathcal{F}_K(\Omega) = \left\{ \frac{\partial}{\partial \bar{z}} - \mu(z) \frac{\partial}{\partial z} - \nu(z) \overline{\frac{\partial}{\partial z}} : |\mu(z)| + |\nu(z)| \leq k = \frac{K-1}{K+1} < 1 \right\}$$

is G -compact.

Idea of the Proof. The proof is based on the ideas in [GIK⁺04]. Note that in [AIM09] the statement is for $\Omega = \mathbb{C}$, but the same proof applies for general domain, one just extends μ and ν outside Ω by letting them be equal to zero. In the heart of the proof is to identify a subsequence and its G -limit

by solving the following adjacent systems

$$\begin{aligned}\frac{\partial \Phi^j}{\partial \bar{z}} &= \mu_j(z) \frac{\partial \Phi^j}{\partial z} + \overline{\nu_j(z)} \frac{\partial \overline{\Phi^j}}{\partial z}, & \Phi^j(0) &= 0, & \Phi^j(1) &= 1, \\ \frac{\partial \Psi^j}{\partial \bar{z}} &= \mu_j(z) \frac{\partial \Psi^j}{\partial z} + \overline{\nu_j(z)} \frac{\partial \overline{\Psi^j}}{\partial z}, & \Psi^j(0) &= 0, & \Psi^j(1) &= i.\end{aligned}$$

Using Theorem 3.1 with the generalized Stoilow factorization (Theorem 2.5), and Theorem 2.8, we have that $\text{Im}(\Phi_z^j \overline{\Psi_z^j}) < 0$ almost everywhere. Now, we can express μ_j and ν_j in terms of complex derivatives of Φ^j and Ψ^j in the spirit of (3.5), (3.6). Next, one takes the limits of Φ^j and Ψ^j (Φ and Ψ , respectively), passing to the subsequence if necessary. Then define μ and ν in terms of Φ and Ψ by mimicking the formulas of μ_j and ν_j . The coefficients μ and ν have the right ellipticity bound, see [GIK⁺04, Lemma 12.1]. The rest is a matter of lengthy differentiation in the sense of distributions and algebraic manipulation to show that

$$\mathcal{B} = \frac{\partial}{\partial \bar{z}} - \mu(z) \frac{\partial}{\partial z} - \nu(z) \frac{\overline{\partial}}{\partial z}$$

is the G -limit of \mathcal{B}_j .

Uniqueness of Normalized Solutions to Nonlinear Beltrami Equations

Assume $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies

(H1) For every $w \in \mathbb{C}$, the mapping $z \mapsto \mathcal{H}(z, w)$ is measurable on \mathbb{C} .

(H2) For $w_1, w_2 \in \mathbb{C}$,

$$|\mathcal{H}(z, w_1) - \mathcal{H}(z, w_2)| \leq k(z)|w_1 - w_2|, \quad 0 \leq k(z) \leq k < 1,$$

for almost every $z \in \mathbb{C}$.

(H3) $\mathcal{H}(z, 0) \equiv 0$.

Then $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ solves a *nonlinear Beltrami equation*, if

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}\left(z, \frac{\partial f}{\partial z}\right), \quad \text{for almost every } z \in \mathbb{C}. \quad (4.1)$$

We discuss *normalized solutions*, that is, homeomorphic solutions $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$. The goal is to consider ideas related to [III]. As mentioned in Introduction, the existence of normalized solutions can be achieved in vast generality, [AIM09, Theorem 8.2.1]. If there is no f -dependence, which is the situation in (4.1), the existence follows from the invertibility properties of nonlinear Beltrami operators.

Theorem 4.1 (Theorem 1.1 in [III]). *Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. If*

$$\limsup_{|z| \rightarrow \infty} k(z) < 3 - 2\sqrt{2} = 0.17157\dots, \quad (4.2)$$

then the nonlinear Beltrami equation (4.1) admits a unique homeomorphic solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$.

Furthermore, the bound on k is sharp: for each $k > 3 - 2\sqrt{2}$, there are functions $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ for which (H1)–(H3) hold, such that (4.1) admits two normalized homeomorphic solutions.

Note that in terms of the quasiconformal distortion the bound (4.2) reads as

$$\limsup_{|z| \rightarrow \infty} K(z) < \sqrt{2}, \quad K(z) := \frac{1 + k(z)}{1 - k(z)}.$$

We remark that (H3) asks constant maps to be solutions to the nonlinear Beltrami equation in question. If we assume, in addition, that the identity function solves (4.1) or, equivalently,

$$(H4) \quad \mathcal{H}(z, 1) \equiv 0,$$

then ellipticity bounds slightly weaker than (4.2) will suffice:

Theorem 4.2 (Theorem 1.2 in [III]). *Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies conditions (H1)–(H4) for some $k < 1$. If*

$$\limsup_{|z| \rightarrow \infty} k(z) < \frac{1}{3},$$

then the function $f(z) = z$ is the unique normalized homeomorphic solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ to the nonlinear Beltrami equation (4.1).

This is complemented with counterexamples: for any $k > 1/3$ there exists $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfying (H1)–(H4) such that (4.1) admits a normalized solution $f(z) \neq z$.

Note that there is an interesting open problem regarding what happens in the borderline case of Theorems 4.1 and 4.2. We expect that in this case (i.e., when $\limsup_{|z| \rightarrow \infty} k(z) = 3 - 2\sqrt{2}$ or $1/3$, respectively) there is a unique normalized homeomorphic solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ to the nonlinear Beltrami equation.

It turns out that the knowledge of the existence of many enough solutions gives the uniqueness of normalized solutions. This is formulated as an abstract theorem and we deduce some corollaries from it in Section 4.2.

Theorem 4.3 (Theorem 1.3 in [III]). *Assume $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. Let $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ be a normalized homeomorphic solution to the equation*

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H}\left(z, \frac{\partial f}{\partial z}\right), \quad \text{for almost every } z \in \mathbb{C}. \quad (4.3)$$

Then f is the unique normalized solution, if there exists a continuous flow of solutions $\{\psi_t : 0 \leq t \leq 1\} \subset W_{\text{loc}}^{1,2}(\mathbb{C})$ of (4.3) such that

$$(F1) \quad \psi_0 \equiv 0, \quad \psi_1 = f,$$

$$(F2) \quad f - \psi_t \text{ is quasiconformal, } 0 \leq t < 1,$$

$$(F3) \quad \text{for fixed } \varepsilon > 0, \text{ there exist } R \text{ and } \delta \text{ such that } \left| \frac{\psi_t(z) - \psi_s(z)}{\psi_t(z) - f(z)} \right| < \varepsilon, \text{ when } |z| \geq R \text{ and } |t - s| < \delta,$$

$$(F4) \quad \psi_t(0) = 0.$$

We remark that the rate of growth near the infinity point (F3) is crucial for the theorem, since it forces the solutions to have same behaviour near ∞ , see the proof in [III].

4.1 Uniqueness, Theorems 4.1 and 4.2

Idea of the Proof. Let us assume there exist two normalized homeomorphic solutions $f, g \in W_{\text{loc}}^{1,2}(\mathbb{C})$ to the nonlinear Beltrami equation (4.1). Then

conditions (H2) and (H3) imply that f, g are $K(z)$ -quasiconformal and the difference is quasiregular, but of course not necessarily injective. By the Stoilow factorization theorem, $f - g = P \circ h$, where P is an analytic mapping and h is a normalized $K(z)$ -quasiconformal homeomorphism.

The key is to find the the upper bounds for $|f(z) - g(z)|$. Since for any $K_0 > K_\infty$,

$$K_\infty := \limsup_{|z| \rightarrow \infty} K(z) < \sqrt{2}, \quad K(z) := \frac{1 + k(z)}{1 - k(z)}, \quad (4.4)$$

the mappings f and g are K_0 -quasiconformal and $f - g$ is K_0 -quasiregular outside some disk $\mathbb{D}(0, R)$, the quasismmetry forces

$$\frac{1}{C}|z|^{1/K_0} \leq |f(z)|, |g(z)|, |h(z)| \leq C|z|^{K_0}, \quad |z| \geq R.$$

Thus,

$$|P(h(z))| = |f(z) - g(z)| \leq C|z|^{K_0} = C|h^{-1}(h(z))|^{K_0} \leq C|h(z)|^{K_0^2}.$$

Hence, P is a polynomial. Since it has at least two zeroes, at $z = 0$ and $z = 1$, $\deg(P) \geq 2$. This gives us the lower bound

$$\frac{1}{C}|z|^{2/K_0} \leq |P(h(z))| = |f(z) - g(z)|.$$

Combining this with our upper bound implies $K_0 \geq \sqrt{2}$ leading to a contradiction with (4.4), when $K_0 > K_\infty$ is sufficiently close to K_∞ .

In Theorem 4.2 $g \equiv z$ and we need a topological fact to get a better upper bound for the quasiregular difference, namely, a linear growth. For details, see the proof in [III].

Counterexamples. For any $0 < t < 1$, define

$$f_t(z) = \begin{cases} (1+t)z|z|^{\sqrt{2}-1} - t(z|z|^{1/\sqrt{2}-1})^2, & \text{for } |z| > 1, \\ (1+t)z - tz^2, & \text{for } |z| \leq 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)z|z|^{\sqrt{2}-1} - tz|z|^{1/\sqrt{2}-1}, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leq 1. \end{cases}$$

Both functions are normalized at 0 and 1, and they should be considered as modifications of the radial stretching $f_2(z) = z|z|^{K-1}$ such that their difference is a polynomial vanishing at 0 and 1 combined with a normalized $\sqrt{2}$ -quasiconformal homeomorphism. Note that f_t is \mathbb{R}_+ -homogenous and g_t does not change the direction of z .

Mappings f_t and g_t are injective by direct argumentation. It is immediate that $f_t - g_t$ is K -quasiregular with $0 < k = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2}$, $K = \frac{1+k}{1-k}$. Directly estimating complex partial derivatives of f_t and g_t , we see that letting $t \rightarrow 0$ the quasiconformal distortion K can be set as close to $\sqrt{2}$ as we wish.

Next, we construct the desired field \mathcal{H} such that f_t and g_t are both solutions to the nonlinear Beltrami equation (4.1). For each fixed $z \notin \partial\mathbb{D}$, we define $w \mapsto \mathcal{H}(z, w)$ as follows. First, set

$$\mathcal{H}(z, 0) = 0, \quad \mathcal{H}(z, \partial_z f(z)) = \partial_z f(z), \quad \mathcal{H}(z, \partial_z g(z)) = \partial_z g(z).$$

By quasiregularity, $\mathcal{H}(z, \cdot) : \{0, \partial_z f(z), \partial_z g(z)\} \rightarrow \mathbb{C}$ is $k_0(t)$ -Lipschitz, where $3 - 2\sqrt{2} < k_0(t) < 1$ and $k_0(t) \rightarrow 3 - 2\sqrt{2}$ as $t \rightarrow 0$. Using the Kirszbraun extension theorem the mapping can be extended to a $k_0(t)$ -Lipschitz map $\mathcal{H}(z, \cdot) : \mathbb{C} \rightarrow \mathbb{C}$. From an abstract use of the Kirszbraun extension theorem, however, it is not entirely clear that the map \mathcal{H} obtained is measurable in z , i.e., that (H1) is satisfied. To show this, one needs to have a constructive proof of the Kirszbraun extension theorem, see [III, pages 10–11].

For Theorem 4.2, one needs to alter slightly the counterexample or use a factorization type argument [III, Lemma 3.1].

4.2 Some Flows of Solutions

In wide-ranging systems it is somewhat problematic to construct the continuous flow of solutions as required in Theorem 4.3. One of general situations, where this is possible, assumes the existence of "L^p-connection", see [III, Theorem 1.5].

We state some corollaries of Theorem 4.3 and give the flow of solutions associated to the setting.

Corollary 4.4 (Theorem 1.4 in [III]). *Suppose $\mathcal{H} : \mathbb{C} \rightarrow \mathbb{C}$ is k -Lipschitz, $k < 1$, and $\mathcal{H}(0) = 0$. Then homeomorphic solutions $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ to the nonlinear Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H} \left(\frac{\partial f}{\partial z} \right), \quad \text{for almost every } z \in \mathbb{C},$$

are affine, that is, $f(z) = az + \mathcal{H}(a)\bar{z} + f(0)$, for some constant $a \in \mathbb{C}$.

The continuous flow is formed by linear maps

$$\psi_t(z) = taz + \mathcal{H}(ta)\bar{z}, \quad t \in [0, 1],$$

where $a \in \mathbb{C}$ is the unique fixed point of the contraction $w \mapsto 1 - \mathcal{H}(w)$.

Corollary 4.5. *Assume $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$, and $\mathcal{H}(z, tw) \equiv t\mathcal{H}(z, w)$, for $t \in \mathbb{R}$. Then there exists a unique normalized solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ to the nonlinear Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{H} \left(z, \frac{\partial f}{\partial z} \right), \quad \text{for almost every } z \in \mathbb{C}.$$

There exists a normalized solution f . Now, we take as the flow

$$\psi_t(z) = tf(z), \quad t \in [0, 1].$$

Remark that the case of \mathbb{R} -linear Beltrami equations (1.2) and, thus especially, reduced Beltrami equations (1.3) are covered with this corollary.

4.3 Remarks on Autonomous Systems

The class of all general nonlinear elliptic systems is preserved under quasiconformal change of both the z -variable and the f -variable. The inverse

map of a solution satisfies its own elliptic equation, too. The point is that the transition from nonlinear Beltrami equation that is independent of the z -variable

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{F} \left(f, \frac{\partial f}{\partial z} \right), \quad \text{for almost every } z \in \mathbb{C}, \quad (4.5)$$

where $\mathcal{F} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies conditions (H1)–(H3) for some $k < 1$, results in the autonomous equation for the inverse $h = f^{-1}$. Namely,

$$\frac{-h_{\bar{w}}}{|h_w|^2 - |h_{\bar{w}}|^2} = \mathcal{F} \left(w, \frac{\overline{h_w}}{|h_w|^2 - |h_{\bar{w}}|^2} \right), \quad w = f(z).$$

Solving it for $\partial_{\bar{w}}h$ in terms of $\partial_w h$, we obtain

$$\frac{\partial h}{\partial \bar{w}} = \mathcal{H} \left(w, \frac{\partial h}{\partial w} \right), \quad \text{for almost every } w \in \mathbb{C}.$$

Hence, uniqueness questions of (4.5) transform to uniqueness questions in the previous sections. Now, we need to know the Lipschitz ellipticity constant k of the obtained \mathcal{H} -equation.

Lemma 4.6 (Lemma 5.1 in [III]). *Suppose $\mathcal{F} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies conditions (H1)–(H3) for some $k = \frac{K-1}{K+1} < 1$. Let $f_1, f_2 \in W_{\text{loc}}^{1,2}(\mathbb{C})$ be homeomorphic solutions to the nonlinear autonomous Beltrami equation (4.5). Then there exists a function $\mathcal{H} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfying (H1)–(H3) with $k = \frac{K^3-1}{K^3+1}$, such that the inverse maps $h_i = f_i^{-1}$ solve*

$$\frac{\partial h}{\partial \bar{w}} = \mathcal{H} \left(w, \frac{\partial h}{\partial w} \right), \quad \text{for almost every } w \in \mathbb{C}.$$

In particular, for any two solutions f_1, f_2 to (4.5), the difference $h_1 - h_2 = f_1^{-1} - f_2^{-1}$ is K^3 -quasiregular.

We note that the key in the proof is quasiregularity of the difference. The wanted field \mathcal{H} can be defined by the Kirszbraun extension theorem as in the case of counterexamples in Section 4.1. Further, we remark that K^3

is the best possible in the sense that K^3 cannot be achieved, but we can be as close to it as we wish. Indeed, the following example, [Iwa11], shows that we can approach K^3 :

$$A(z) = (KK' - 1) \left(\frac{K+1}{2}z + \frac{K-1}{2}\bar{z} \right),$$

$$B(z) = (K^2 - 1) \left(\frac{K'+1}{2}z + \frac{K'-1}{2}\bar{z} \right).$$

Now, A is K -quasiconformal, B is K' -quasiconformal and their difference is K -quasiconformal, but $A^{-1} - B^{-1}$ is K^2K' -quasiconformal. To show that K^3 cannot be achieved, we can linearise the system pointwise and study just the linear K -quasiconformal mappings $A(z) = z + k\bar{z}$, $B(z) = \alpha(z + ke^{i\theta}\bar{z})$. From the proof of the above lemma, one sees that the K^3 -bound is achieved only if $A - B$ is K -quasiconformal. If the difference $A - B$ is K -quasiconformal, then $A^{-1} - B^{-1}$ is also K -quasiconformal, by a straightforward calculation.

We state the theorem concerning the autonomous systems from [III]. It is slightly sharpened from the published version taking into account the previously discussed sharp bound for the Lipschitz constant.

Theorem 4.7 (Theorem 1.6 in [III]). *Suppose $\mathcal{F} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. If*

$$k(z) \leq \frac{2^{1/4} - 1}{2^{1/4} + 1} = 0.08642\dots \quad \text{in some neighborhood of the infinity point,}$$

then the nonlinear Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mathcal{F} \left(f(z), \frac{\partial f}{\partial z} \right), \quad \text{for almost every } z \in \mathbb{C},$$

admits a unique homeomorphic solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$.

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