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MULTIPLE ISOPARAMETRIC FINITE ELEMENT METHOD FOR NONHOMOGENEOUS MEDIA

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Introduction

Many engineering materials, such as composites, bonded materials and geophysical materials, are generally modeled as nonhomogeneous continua and usually assumed to be piecewise homogeneous for simplicity. However, in some of these materials the mechanical properties do vary continuously and the continuity has to be considered. For examples, hydraulic fracturing of geophysical materials, interface crack problems in bonded materials [1,2], functionally graded materials [3] and so on, the continuity of the material properties should be considered undoubtedly in analysis.

There have been a number of papers that studied the nonhomogeneous problem [4-6]. However, the analytical approach used by these studies can only deal with unbounded media and simple distribution of material properties, such as linear form, power form or exponential form. For a great deal of finite dimensional structural components and/or various distributions of material properties, it is difficult, if not impossible, to find the analytical solution. Therefore, numerical methods have to be developed.

In numerical methods, the most versatile method is the Finite Element Method (FEM) and it has been used to solve many practical problems. However, almost all of these finite element approaches mainly concentrated on homogeneous materials or piecewise homogeneous materials. The finite element formulation relating to nonhomogeneous materials with continuously varying properties was very few.

In this paper, we propose a more simple and more versatile finite element formulation. The concept of isoparametric transformation is adopted for simulating the variations of the material properties in individual finite elements. The continuity requirement of the material properties is satisfied. The feasibility and the versatility of this method are verified by examples.

Multiple isoparametric elements

The finite element stiffness equations can be written as

$$\mathbf{K}^e\mathbf{u}^e = \mathbf{F}^e \tag{1}$$

where

$$\mathbf{K}^e = \int_{\Omega_e} \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega_e \tag{2}$$

and

$$\mathbf{F}^e = \int_{\Omega_e} \mathbf{N}^{eT}\mathbf{b}^e d\Omega_e + \int_{\Gamma_e} \mathbf{N}^{eT}\bar{\mathbf{t}}^e d\Gamma_e + \int_{\Omega_e} \mathbf{B}^{eT}\mathbf{D}^e\ \boldsymbol{\varepsilon}_T d\Omega_e \tag{3}$$

where $\mathbf{B}^e = \mathbf{L}\mathbf{N}^e$ is the strain shape function with $\mathbf{L}$ is a suitable linear operator, which is different for one-, two- or three-dimensional problems; $\mathbf{D}^e$ is the constitutive matrix containing the appropriate material properties; $\mathbf{b}^e$, $\bar{\mathbf{t}}^e$ and $\boldsymbol{\varepsilon}_T$ are the body force vector, the traction vector and the thermal strain vector respectively; Ω_e is the domain of the element and Γ_e is the boundary on which traction is prescribed.

In above finite element formulation, the constitutive matrix $\mathbf{D}^e$, the body force vector $\mathbf{b}^e$ and the thermal strain ε_T have relations with the material properties E (elastic modulus), ν (Poisson's ratio), ρ (density) and α (the coefficient of linear thermal expansion) respectively. For homogeneous materials, E, ν , ρ and α are constants. However, for nonhomogeneous materials, these material properties are functions of spatial coordinates.

In this paper, we adopt the concept of the well-known isoparametric transformation to describe the variations of material properties in nonhomogeneous media. It is stated below for two-dimensional problem. The procedures for axisymmetric or three-dimensional problem are similar.

Consider a m-node plane element as shown in Fig. 1. The global coordinates of a point on the element at (ξ,η) are given by

$$x = \sum_{i=1}^{m} N_i(\xi,\eta)x_i, \quad y = \sum_{i=1}^{m} N_i(\xi,\eta)y_i. \tag{4}$$

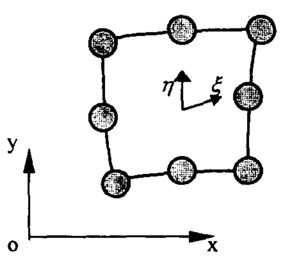

Fig. 1 A multiple isoparametric element

where N_i are the shape functions corresponding to the node i, whose coordinates are (x_i, y_i) in the global system and (ξ_i, η_i) in the local system. As an isoparametric element, the displacements within the element are interpolated as follows

$$u = \sum_{i=1}^{m} N_i(\xi,\eta)u_i, \quad \upsilon = \sum_{i=1}^{m} N_i(\xi,\eta)\upsilon_i. \tag{5}$$

where (u, υ) are the nodal displacements in the x and y directions, respectively. Now, we let the material properties E, ν, ρ and α at the point (ξ,η) be expressed as

$$
\begin{aligned}
E &= \sum_{i=1}^{m} N_i(\xi,\eta)E_i, \quad \nu = \sum_{i=1}^{m} N_i(\xi,\eta)\nu_i, \\
\rho &= \sum_{i=1}^{m} N_i(\xi,\eta)\rho_i, \quad \alpha = \sum_{i=1}^{m} N_i(\xi,\eta)\alpha_i,
\end{aligned}
\tag{6}
$$

where $(E_i, \nu_i, \rho_i, \alpha_i)$ stand for the material properties at the node i of the element. By using equation (6), the actual variations of the material properties in a specified element can be approximated by polynomial forms. The degree of the polynomial depends on the number of nodes in the element. Thus, in the element, the simulating accuracy for coordinates, displacements and material properties is identical, because the same shape functions are used. This kind of elements may be called as multiple isoparametric elements.

Substituting (6) into the element constitutive matrix $\mathbf{D}^e(E,\nu)$, the body force vector $\mathbf{b}^e(\rho)$ and the thermal strain $\boldsymbol{\varepsilon}_T(\alpha)$ in equations (2) and (3) respectively, we obtain the element stiffness matrix and the load vector in the domain of the reference element, that are

$$
\mathbf{K}^e = \int_{A_e} \mathbf{B}^{e\,T}(\xi,\eta)\, \mathbf{D}^e(\xi,\eta)\mathbf{B}^e(\xi,\eta) \det \mathbf{J}(\xi,\eta)\; dA_e \tag{7}
$$

$$
\begin{aligned}
\mathbf{F}^e = &\int_{A_e} \mathbf{N}^{eT}(\xi,\eta)\, \mathbf{b}^e(\xi,\eta) dA_e + \int_{S_e} \mathbf{N}^{eT}(\xi,\eta)\, \bar{\mathbf{t}}^e(\xi,\eta) ds_e \\
&+ \int_{A_e} \mathbf{B}^{eT}(\xi,\eta)\, \mathbf{D}^e(\xi,\eta)\, \boldsymbol{\varepsilon}_T(\xi,\eta)\, dA_e
\end{aligned}
\tag{8}
$$

where A_e is the area of the reference element and s_e is the part of the boundary of the reference element over which $\bar{\mathbf{t}}^e$ is specified.

Because the variations of material properties have been simulated properly in the elements, it is expected that the higher accuracy may be obtained with relative coarse mesh for the stress analysis of nonhomogeneous media, whose material properties vary continuously. Clearly, the finite element formulation presented above is also suitable for homogeneous medium and piecewise homogeneous medium merely by making the value of each material property be the same at all of nodes on an element. Of course, if necessary, we could also have multiple superparametric elements and multiple subparametric elements corresponding to the superparametric elements and the subparametric elements in conventional FEM [7] respectively.

Numerical integration

Some basic requirements for above so-called multiple isoparametric elements, such as the geometrical conformability, the displacement continuity and the convergence aspects, are undoubtfully satisfied. The continuity of material properties, if required (for nonhomogeneous materials with varying continuously properties), can also be satisfied. The proof is similar to that for the continuity of displacement functions.

The problem to which should be pay attention is the numerical integration of the element stiffness matrices and the element load vectors. For a multiple isoparametric element, the order of the integrand is generally higher than the corresponding isoparametric element, because the matrix $\mathbf{D}^e$ and the vectors $\mathbf{b}^e$, $\boldsymbol{\varepsilon}_T$ are functions of the coordinates ξ and η, as shown in (7) and (8). The terms of the matrix $\mathbf{D}^e$ are rational fractions because they are calculated by E and ν, for example,

$$\mathbf{D}^e = \frac{E}{(1+\nu)[1-(1+\beta)\nu]}\begin{bmatrix} 1-\beta\nu & \nu & 0 \\ \nu & 1-\beta\nu & 0 \\ 0 & 0 & \frac{1-(1+\beta)\nu}{2} \end{bmatrix} \tag{9}$$

for two-dimensional problems, where $\beta=0$ for plane stress and $\beta=1$ for plane strain. Therefore, the minimum number of integration points required by the matrix $\mathbf{K}^e$ and the vector $\mathbf{F}^e$ could not be determined easily. But by considering the first term of the load vector $\mathbf{F}^e$, we can say that the errors of the integration formulae may be at least of the order as follows: linear elements, $O(h^3)$; quadratic elements, $O(h^5)$; cubic elements, $O(h^7)$. Thus, the minimum number of Gaussian integration points will be, for example, 3×3 for eight-node two-dimensional elements. The effect of different integration points on the accuracy of computed results will be demonstrated in next section. It will be seen that 3×3 Gaussian points are adequate for quadratic quadrilaterals. So, the efforts for numerical integration of the multiple isoparametric elements are little more than that for the conventional finite elements.

Numerical example and discussion

For verifying the accuracy and the efficiency of above multiple isoparametric elements, a following example is computed. Consider a plate with exponential elastic modulus and constant Poisson's ratio subjected to uniform tension (Fig. 2). Assume that the elastic modulus varies along x-direction as

$$E(x) = E_0 \exp\left(\frac{x}{w}\ln\frac{E_w}{E_0}\right), \tag{10}$$

where E_0 is the elastic modulus at $x = 0$, E_w is that at $x = w$ and that $E_w / E_0 = 5.0$. The element division is shown in Fig. 2. All the elements are eight-node elements. The computed results areillustrated in Fig. 3 and Fig. 4. It can be seen from Fig. 3 that the multiple isoparametric finite element method proposed in this paper is very effective for the stress analysis of nonhomogeneous materials with continuously varying properties. The higher accuracy has been achieved by very coarse mesh. As contrast, the result calculated by the conventional FEM can not be accepted. From Fig. 4, it is observed that 3×3 Gaussian points are adequate. The increase of the number of integration points has little effect on the accuracy. This is fortunate. It is revealed that little more efforts in integration for multiple isoparametric elements will be made than that for ordinary isoparametric elements. Thus, the advantage of multiple isoparametric elements will be brought into play.

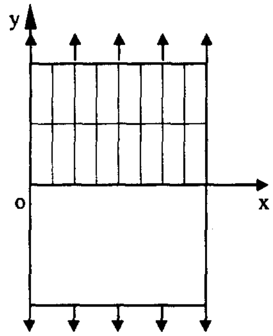

Fig. 2 An FGM plate under uniform tension

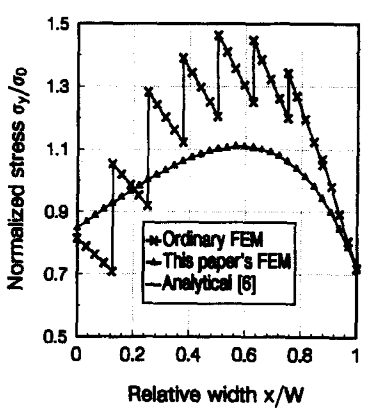

Fig. 3 The comparison of computed results

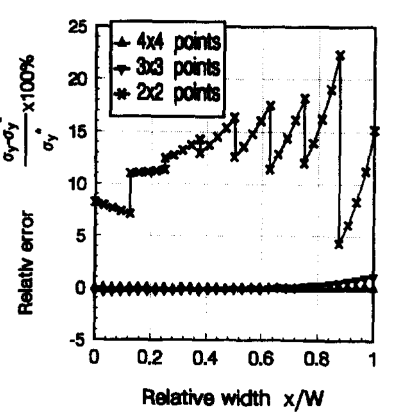

Fig. 4 The effect of Gaussian points on the accuracy

Conclusion

The technique of isoparametric transformation is adopted for simulating the variation of material properties. The multiple isoparametric finite element formulation developed in this paper has been verified to be very effective for analyzing nonhomogeneous materials with continuously varying properties. It makes the description of geometry, displacements and material properties be achieved the same accuracy. It provides enormous flexibility in meshing. Further, This method can be extended straightforwardly to three dimensional problems. The revision of the existed FEM software is very easy. The revised FEM software can be efficiently used for the analysis of various materials, which include homogeneous solids, layered composites, composites with graded interfacial zone, functionally graded materials and geophysical materials. It will enhance the FEM and make it play a more important role in practical engineering.

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