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CLOSED TRANS-SCALE STATISTICAL MICRODAMAGE MECHANICS*

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ABSTRACT: Damage and failure due to distributed microcracks or microvoids are on the challenging frontiers of solid mechanics. This appeals strongly to tools not yet fully developed in continuum damage mechanics, in particular to irreversible statistical thermodynamics and a unified macroscopic equations of mechanics and kinetic equations of microstructural transformations. This review provides the state of the art in statistical microdamage mechanics.

(1) It clarifies on what level of approximation continuum damage mechanics works. Particularly, D-level approximation with dynamic function of damage appears to be a proper closed trans-scale formulation of the problem.

(2) It provides physical foundation of evolution law in damage mechanics. Essentially, the damage-dependent feature of the macroscopic evolution law is due to the movement of microdamage front, resulting from microdamage growth.

(3) It is found that intrinsic Deborah number D^*, a ratio of nucleation rate over growth rate of microdamage, is a proper indication of critical damage in damage mechanics, based on the idea of damage localization.

(4) It clearly distinguishes the non-equilibrium damage evolution from equilibrium phase transition, like percolation.

Finally, some comments on its limitations are made.

KEY WORDS: damage, microdamage, statistical microdamage mechanics, damage evolution

1 INTRODUCTION

1.1 Essential Features of Damage Evolution

If there is a crack in solid, fracture mechanics successfully characterizes the solid. However, for most heterogeneous materials, such as alloys, ceramics, composites, rocks, etc. there might be distributed microcracks or microvoids rather than a single macroscopical

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crack. Roughly speaking, microdamage is formed at mesoscopic inhomogeneity, for example, particulates in alloy. So, the nucleation of microdamage usually has the size of particulates or grains, like micrometres, and the density of such microdamages on the surface of metals is in the range of $(10^2 \sim 10^4)/\text{mm}^2$. Moreover, some microdamage may lead to eventual failure owing to growth and coalescence. Hence, the main issues in damage evolution are nucleation, growth and coalescence of microdamage. This means that a trans-scale (from meso- to macroscopic) understanding of damage evolution is badly needed[1,2].

The distinct features of the evolution are as follows[1~5]:

(1) The non-linear interactions involved in the evolution concern various scales, from micro- to macroscopic. A noticeable feature is the richness of structures and processes at mesoscopic length scales. In this case, mesoscale structures, such as grains, microvoids, microcracks, etc. play significant role in the evolution.

(2) The damage and failure of solids under external loading are usually far from quilibrium state. Moreover, there are various mesoscopic kinetics with various time-scales. So, not only size-scales but also various time-scales play significant roles in the evolution.

(3) The evolution is governed by the collective effects of numerous microdamages, rather than the singularity of an individual crack as traditional fracture mechanics deal with.

Based on these characteristics of microdamage evolution in solids, statistical approach to the evolution on meso-scales becomes a necessity. Therefore, this is the main concern of this review, in particular, closed trans-scale formulation of microdamage evolution in solids.

1.2 Continuum Damage Mechanics and Micromechanics

For a piece of solid containing preexisting inhomogeneities, defects or flaws, what kind of successive changes will appear under external loadings? Obviously, this problem is the focus in engineering and the goal of damage mechanics.

In last mid-fifties, continuum damage mechanics develops[6~11]. Continuum damage mechanics ignores both microscopic details and governing mechanisms. As a practical tool in engineering, it simply takes the average variations of mechanical properties in damaged materials as the measure of damage, for example, the degradation of elastic modulus. Therefore, continuum damage mechanics consists of two parts: evolution law and critical damage D_c.

Kachanov[6] introduced a damage variable D. $D = 0$ and 1 denote intact and failed states of solids, respectively. Furthermore, he assumed that the damage variable D is an internal variable in constitutive formulation and follows an assigned evolution law

$$\dot{D} = \frac{K}{(1-D)^\gamma} \tag{1}$$

where γ is an undetermined parameter and K can be stress-dependent $K(\sigma)$ or merely a second parameter. The stress in the damaged solids σ is calculated as

$$\sigma = \sigma_s(1-D) \tag{2}$$

where σ and σ_s are nominal stress and the stress in intact matrix, respectively, because the effective load-supporting area is reduced in damaged solids.

To deal with dynamic response of solids to impulsive loading, Davison and Stevens[12] extended the concept of continuum damage to spallation. Particularly, they pointed out

the significance of compound damage in failure. In last nineties, Meyers[13] and Grady and Kipp[14] made comprehensive and critical review on the subject. On one hand, they noticed that "the current availability of high-speed computers and shock-wave propagation codes makes possible the development of continuum models of fracture and fragmentation to include in these codes". On the other hand, they stressed that "we still need quantitative/predictive models based on continuum measure of spalling and nucleation-and-growth of microcracks"[13] and "the continuum models based on the statistical nucleation and growth of brittle and ductile fracture appear to be an attractive approach"[14].

In order to forecast failure, the other part of continuum damage mechanics is a threshold of damage from gradual to catastrophic accumulation. Because of lacking proper understanding of the transition, an empirical critical damage D_c is usually assumed as a material property, for example, Lemaitre suggested the range of the critical damage[8]

$$D_c \sim (0.2 \sim 0.5) \tag{3}$$

Clearly, this is quite arbitrary.

After knowing the two parts, like Eqs.(1) and (3), damage evolution can be described in the framework of continuum mechanics and constitutive theories. Obviously, continuum damage mechanics is essentially an empirical description. So, great efforts are made in terms of micromechanics, among which the works on void growth by Rice and Tracy, McClintock and Gurson are the well-known[15~17]. Roughly speaking, the basic idea of these approaches is to perform stress analysis of individual hole or void in a deformed solid and then to extrapolate to array of holes or voids as average constitutive laws. Since these formulations are quite parallel to the paradigm of plasticity, they are widely applied and need not to be reviewed here. However, one point is worthy emphasizing. This is that the micromechanical analysis of a hole or a void is just a miniature of traditional mechanics applied to a hole or cavity and is not actual kinetics of microstructural evolution. What really needs understanding is, as Barenblatt declared at 18th ICTAM[4], "the governing influence of the variations of the material structure on the macroscopic behaviour of bodies". "In the mathematical models of such phenomena, the macroscopic equations of mechanics and the kinetic equations of the microstructural transformations form a unified set that should be solved simultaneously." This inevitably appeals statistical considerations of microdamage ensemble.

As a matter of fact, continuum damage should be a collective representation of microdamage. Therefore, the evolution law of damage should be based on some mesoscopically physical essence and critical damage is also by no means arbitrarily assumed values. As overviewed by McDowell at ASTM STP[2] "rigorous treatment of non-uniformly distributed defects requires tools not yet fully developed in continuum damage mechanics". "Weighing the influence of distributed damage at the microscale on the collective macroscale stiffness and evolution of damage is a challenge as well." "In fact, the overall framework of continuum damage mechanics, based on the use of internal state variables to represent evolving structure of the material, appeals strongly to irreversible statistical thermodynamics."

1.3 Statistical Approaches

Actually, microstructural engineering of materials has become a necessity in modern technology and engineers intend to consider the application, preparation and processing of materials comprehensively, as shown in the recent special issue of "Science". It is said in

its editorial[18] that "A piece of blackboard chalk and a clam shell are chemically almost identical in composition, but the chalk will snap far more easily. The difference in the performance of these two materials originates in their microstructure." Hence, to develop a statistical theory of microdamage evolution, including the necessary knowledge of the underlying mesoscopical mechanisms, is not only an academic interest, but also an urgent practical task.

In last fifties, Weibull[19,20] proposed a well-known Weibull distribution

$$W(\sigma) = 1 - \exp(-aL^d\sigma^m) \tag{4}$$

where W is the cumulative probability of strength less than σ, L and d are the size and dimension of the sample and a and m are two parameters, respectively. In practice, it was found that Weibull distribution is suitable for solids with quite random microscopic structures.

In order to correlate Weibull's statistical idea to the reality of damage in various solids, there are several approaches to various objectives: fibers and composites, brittle heterogeneous materials (ceramics and rocks), crazing in polymers, cumulative fatigue damage, and so on.

For fibres, Deniels, even earlier than Weibull, derived that the strength of fibre bundle follows normal distribution[21]. Later, Coleman justified Weibull distribution in fibre[22]. The essential idea in the approaches is simple load-sharing rule. Following this idea, Harlow and Phoenix developed statistical theories of composites[23]. For more details on this approach, it can be referred to a comprehensive overview recently made by Du and Wang[24]. For brittle materials, combining Weibull's idea, Griffith's concept of fracture and distribution function of cracks, McClintock[25] and Batdorf[26] formed a statistical fracture theory. For rocks, some critical reviews are included in the papers in Atkinson's book[27] and the article by Wu et al.[28].

Since seventies of 20^{th} century, based on various experimental data of microcracks and microvoids, Curran, Seaman and Shockey[3] proposed their microstatistical fracture mechanics, NAG (nucleation and growth) model. According to their statistics of microvoids and microcracks, they assumed that the cumulative distribution of microdamage remains exponential in the course of damage evolution. Additionally, they adopted time-independent but stress-dependent mesoscopic kinetics of nucleation rate and viscous growth rate of microdamage. Then, they put all these into a continuum constitutive framework and a numerical code of wave propagation. However, the NAG model involves too many empirical parameters.

Instead of assumed distribution functions, the other trend is to transplant some statistical formulations into the study. Bogdanoff[29] initiated a B-model of cumulative damage based on Markovian process. Xing[30,31] applied Fokker-Planck equation to the study of microcrack distribution and then calculated fracture probability with Griffith's concept.

In order to characterize the non-equilibrium nature of damage evolution, Sih emphasized three fundamental parameters: the time rate of nonequilibrium temperature, dissipation energy density and volume-change to surface-change ratio[32]. Yuan and his group, based on their experimental results of local temperature field and pyromagnetic effect in the evolution of defects, examined the rupture process from a unified micro-meso-macroscopic standpoint[33].

Another distinctive aspect of the concerned subject is scaling and size-effect of failure. Recently, Bazant and Chen gave an extensive review on the subject[34]. They discussed three types of size effect: statistical size effect due to randomness of strength, the energy release size effect and the size effect due to fractality of fracture or microcracks.

Now, it appears to be more and more clear that to develop a sound framework of statistical formulation of microdamage evolution becomes more significant. The models should be a unified set of macroscopic equations of mechanics and mesoscopically kinetic equations.

Panin and his group in Russia proposed a systematic model named as physical mesomechanics. In their theory, there are two mesoscopic levels. On level I, vortex plastic flow is characterized by the scheme "shear + rotation", whereas on level II new defects appear irrespective of crystallographic orientations[35].

In recent years, Bai, Xia and their co-workers made endeavour to work out a formulation of statistical microdamage mechanics, based on their experimental observations of microdamage evolution. This theory combines macroscopic equations of mechanics and mesoscopic kinetics and then forms a closed trans-scale formulation of damage evolution[5,36]. In next sections, we will review the formulation in more details. Further applications of the statistical formulation have been made to damage and failure of glassy polymers by Li et al.[37] , to evolution of microvoids in metals by Huang et al.[38], to short fatigue cracks in metals by Hong et al.[39,40], to brittle solids by Fang and Zhou[41] and so on.

2 MICRODAMAGE——ITS FUNDAMENTAL EQUATION AND SOLUTION

2.1 Essential Features of Microdamage

In this review, we skip the experimental observations and relevant measurements in details[3,5,42,43]. But, as mentioned at the beginning of the review, the number of microcracks in tested specimens ranges over $(10^2 \sim 10^4)/\text{mm}^2$ on sectional surface, roughly corresponding to $(10^3 \sim 10^6)/\text{mm}^3$ or $(10^{12} \sim 10^{15})/\text{m}^3$ in volume. Hence, when a millimeter element of metal is examined, a statistical formulation of these microcracks is proper. In this case, the damaged element can be considered macroscopically infinitesimal representative one. Actually, the length scale of a concerned element of damaged material is closely dependent on the intrinsic size of the examined medium. For example, the sizes of crystal, second phase particles and microcracks in metals are in the order of $(10^0 \sim 10^2)\,\mu\text{m}$. In this sense, a damaged element of mm^2 can be representative one statistically. Clearly, for rock, the size of microdamage and the assumed continuum element can both be much greater.

The statistical microdamage mechanics is a description of the state of microdamage and corresponding mesoscopic kinetics. The key to the issue is to find out a proper description of microdamage, not oversimplified and not too detailed as well. That is to say, we should develop such a description that the physical model of microdamage is concise enough to deal with mathematically. On the other hand, the model should be relatively inclusive, containing all necessary information for further statistical average.

The main concern in the study is how the number density n of microdamage evolves with time under a certain loading, where n is the number of microdamage within unit phase

space. Since the total of microdamage changes in the evolution owing to nucleation and annihilation, it is more convenient to use the number density of microdamage n in this study, rather than the probability density usually used in statistical physics.

2.2 Fundamental Equation of Microdamage[5,44,45]

We merely pay attention to sensitive variables of microdamage $p_i(i = 1, \cdots, I)$, such as crack length, with $P_i = \dot{p}_i \neq 0$, where subscript i covers all sensitive mesoscopic variables of microdamage. The letter with "." above it denotes its corresponding rate. Let us examine an element $\mathrm{d}\Omega$ in the phase space of microdamage, where $\mathrm{d}\Omega = (p_i, p_i + \mathrm{d}p_i)$. The flux of microdamage flowing into and out of the element in the phase space $\mathrm{d}\Omega$ in unit time should be balanced by the change of microdamage number within this element owing to nucleation and annihilation of microdamage there. This balance rule of microdamage in the phase space leads to a general evolution equation of microdamage

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{I} \frac{\partial (n \cdot P_i)}{\partial p_i} = n_{\mathrm{N}} - n_{\mathrm{A}} \tag{5}$$

where n is the number density of microdamage, t is generalized time, like actual time, nominal deformation, cycles, etc. n_{N} and n_{A} are the nucleation and annihilation rates of number density of microdamage, respectively.

In Eq.(5), there are three kinds of mesoscopic kinetics. They are the nucleation rate n_{N}, the rates of all sensitive variables P_i and the annihilation rate n_{A} of microdamage, respectively. Generally speaking, they are all functions of independent variables, such as p_i, and loading and environmental parameters.

2.3 Basic Solution of Microdamage Number Density[5,45~47]

Clearly, Eq.(5) still contains too much information about the evolution of microdamage for further deduction of macroscopic mechanical behaviour of damaged materials. So, further simplification of the general evolution equation of microdamage is a prerequisite to the evaluation of macroscopic behaviours. Since the size of microdamage predominates the evolution and the macroscopic behaviour, phase space of microdamage could be limited to $\{c\}$, where c is current size of microdamage in a representative volume where microdamage locates. For instance, $n(c) = 10/\mu\mathrm{m}\,\mathrm{mm}^3$ means that a volume of $1\,\mathrm{mm}^3$ has 10 microdamages with size between c and $(c + 1)\,\mu\mathrm{m}$.

From mesoscopic point of view, the growth rate of microdamage may not only depend on their current size c but also on their initial size at nucleation c_0[3]. Additionally, a number of micromechanical analysis show this effect[3,48]. Though the initial size of microdamage remains fixed during the course of its extension, the distribution of initial size is usually diverse. Hence, instead of number density of microdamage n, we should examine the evolution of number density of microdamage $n_0(t, c, c_0)$, in the phase space $\{c, c_0\}$.

For a macroscopically representative element, the solution of the number density of microdamage, $n_0(t, c, c_0)$ is derived on the following essential assumptions:

(1) The coalescence and healing of microdamage are ignored and microdamages are independent of each other statistically.

(2) The nucleation and growth of microdamage are governed by its state variables: current and initial sizes c and c_0 as well as macroscopically local average stress σ[3,48]

$$n_{\mathrm{N}} = n_{\mathrm{N}}(c_0; \sigma) \tag{6}$$

$$V = \dot{c} = V(c, c_0; \sigma) \tag{7}$$

(3) The macroscopically local field can be altered by the average interaction between microdamages. Therefore, stress σ is not coupled with individual microdamage but serve as a macroscopically local mean field parameter.

The fundamental equation of number density of microdamage, $n_0(t, c, c_0)$ is

$$\frac{\partial n_0}{\partial t} + \frac{\partial (n_0 V)}{\partial c} = n_{\mathrm{N}}(c) \cdot \delta(c - c_0) \tag{5a}$$

where $\delta(c - c_0)$ is Dirac δ-function and it has the same dimension as the reciprocal of its argument has, i.e. the reciprocal of microdamage size c. The first order quasi-linear partial differential Eq.(5a) has the following characteristic ordinary equations

$$\frac{\mathrm{d}(n_0 V)}{\mathrm{d}c} = n_{\mathrm{N}} \delta(c - c_0) \tag{8}$$

$$\frac{\mathrm{d}t}{\mathrm{d}c} = \frac{1}{V} \tag{9}$$

Then, by assuming stress σ be a parameter, the solution to Eq.(5a) is

$$n_0(t, c, c_0) = \frac{n_{\mathrm{N}}(c_0; \sigma)}{V(c, c_0; \sigma)} \qquad \text{when} \quad c_0 < c \le c_{\mathrm{f}} \tag{10}$$

$$n_0(t, c, c_0) = 0 \qquad \text{otherwise}$$

$$t = \int_{c_0}^{c_{\mathrm{f}}} \frac{\mathrm{d}c}{V(c, c_0; \sigma)} \tag{11}$$

where c_{f} is the moving front of microdamage[45,46]. The implication of the front $c_{\mathrm{f}} = c_{\mathrm{f}}(t, c_0)$ is that the microdamage nucleated at time $t = 0$ with the initial size c_0 will extend to current size c_{f} at time t (Fig.1). On the other hand, for fixed time t and current size c, there is a corresponding minimum initial size $c_{0\mathrm{f}} = c_{0\mathrm{f}}(t, c)$ and the initial microdamage less than ($c_0 < c_{0\mathrm{f}}$) can not reach the current size c within time t (Fig.1). c_{f} and $c_{0\mathrm{f}}$ are inverse functions each other

$$t = \int_{c_{0\mathrm{f}}}^{c} \frac{\mathrm{d}c}{V(c, c_0; \sigma)} \tag{11a}$$

By definition, the relation between number densities of microdamage n and n_0 is the integration of n_0 with respect to initial size c_0

$$n(t, c) = \int_0^c n_0(t, c, c_0) \mathrm{d}c_0 \tag{12}$$

Substitution of solution $n_0(t, c, c_0; \sigma)$ (10) into this definition (12) gives the expression of number density of microdamage n by means of the kinetics of nucleation and growth of microdamage, n_{N} and V

$$n(t, c) = \begin{cases} \displaystyle\int_0^c \frac{n_{\mathrm{N}}(c_0; \sigma)}{V(c, c_0; \sigma)} \mathrm{d}c_0 & c < c_{\mathrm{f},0} \\ \displaystyle\int_{c_{0\mathrm{f}}}^c \frac{n_{\mathrm{N}}(c_0; \sigma)}{V(c, c_0; \sigma)} \mathrm{d}c_0 & c > c_{\mathrm{f},0} \end{cases} \tag{13}$$

where $c_{\mathrm{f},0} = c_{\mathrm{f}}(t, c_0 = 0)$ is the front of microdamage at infinitesimal initial size (Fig.1).

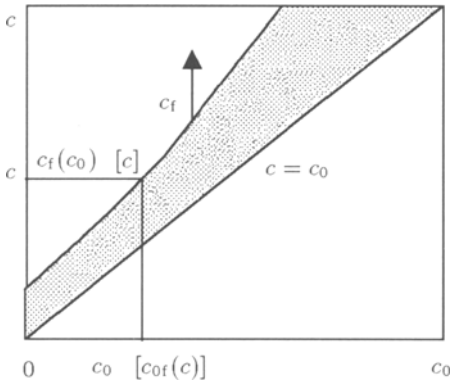

Fig.1 The solution region of microdamage number density $n(t, c)$. The shaded indicates where non-zero solution locates. c_f is the microdamage front moving upward. Also, $c_f(c_0)$ and $c_{0f}(c)$ (shown in blanket [ ]) are inverse each other for the microdamage front.

The solution of number density of microdamage, consisting of integrals (13) and (11a), is unsteady, namely time-dependent, but the time-dependent feature appears only on the microdamage front $c_f(t, c_0)$. Within the area (the shaded one in Fig.1) enclosed by the front c_f and two natural boundaries $c_0 = 0$ and $c = c_0$, the solution remains steady, i.e. fixed. So, essentially, the evolution of damage is due to the movement of microdamage front (Fig.2)[45,47].

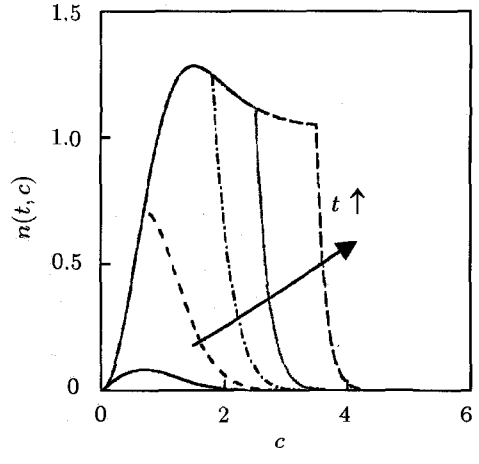

Fig.2 An example of the solution of microdamage number density $n(t, c)$ under constant stress, for dimensionless nucleation rate $mc^{m-1}e^{-c^m}$ and growth rate $[1-(c_0/c)^2]^{1/2}$. One can see the trajectory of microdamage front c_f on plane (t, c) and the tendency to saturation of microdamage.

3 DAMAGE FIELD —— ITS CLOSED TRANS-SCALE APPROXIMATIONS

3.1 n-level Formulation, Based on Number Density of Microdamage n

We mentioned before that one main point in damage mechanics is how to close the formulation of damage evolution. From practical point of view, engineers are more interested

in macroscopic evolution and damage distribution in components or structures. In these cases, the independent variable of spatial coordinates $\boldsymbol{x}$ is of significance. Then, we define number density of microdamage in a macroscopic spatial element and adopt its corresponding coordinates $\boldsymbol{x}$ as a variable describing the state of microdamage. The corresponding phase space of the inhomogeneous microdamage is $\{c, \boldsymbol{x}\}$. Certainly, the coordinates $\boldsymbol{x}$ and the corresponding particle velocity $\boldsymbol{v}$ are the representative average values of the element. That is to say, all microdamage in the element would have the same coordinates $\boldsymbol{x}$ and velocity $\boldsymbol{v}$. Therefore, for spatial inhomogeneous microdamage, the fundamental equation of microdamage (5) can be written as

$$\frac{\partial n}{\partial t} + \frac{\partial (nA)}{\partial c} + \frac{\partial (n\boldsymbol{v})}{\partial \boldsymbol{x}} = n_{\mathrm{N}} \tag{5b}$$

where A is the average growth rate of microdamage with current size c

$$A(t, c; \sigma) = \frac{\int_0^c V n_0 \mathrm{d}c_0}{n} \tag{14}$$

Clearly, this equation should be associated with other continuum equations, such as mass, momentum and energy equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \rho}{\partial \boldsymbol{x}} + \rho \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = 0 \tag{15}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \rho^{-1} \cdot \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}} \tag{16}$$

$$\frac{\partial (e - q)}{\partial t} + \boldsymbol{v} \cdot \frac{\partial (e - q)}{\partial \boldsymbol{x}} = \rho^{-1} \cdot \boldsymbol{\sigma} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} - \rho^{-1} \cdot \frac{\partial^2 \boldsymbol{h}}{\partial \boldsymbol{x} \cdot \partial \boldsymbol{x}} \tag{17}$$

where ρ is density, e and q are specific internal energy and heat source, respectively, $\boldsymbol{\sigma}$ is Cauchy stress tensor and $\boldsymbol{h}$ is heat flux. For simplicity, we ignore energy equation in this review.

Later, to illustrate the framework easily, we confine our discussion to its one-dimensional strain state version in Lagrangian coordinates

$$\frac{\partial n}{\partial T} + \frac{\partial (nA)}{\partial c} + n \frac{\rho}{\rho_0} \frac{\partial v}{\partial Y} = n_{\mathrm{N}} \tag{5c}$$

$$\frac{\partial \rho}{\partial T} + \frac{\rho^2}{\rho_0} \frac{\partial v}{\partial Y} = 0 \tag{18}$$

$$\frac{\partial v}{\partial T} - \rho_0^{-1} \frac{\partial \sigma}{\partial Y} = 0 \tag{19}$$

The transformation from Euclidean (t, y) to Lagrangian coordinates (T, Y) in one dimensional strain state is $t = T$ and $y = Y + u$ (u is displacement) or

$$\frac{\partial}{\partial t} + v \frac{\partial}{\partial y} = \frac{\partial}{\partial T} \qquad \frac{\partial}{\partial y} = \frac{\rho}{\rho_0} \frac{\partial}{\partial Y} \tag{20}$$

where $\dfrac{\rho}{\rho_0} = \dfrac{1}{1 + \varepsilon}$, ε is strain. To complete the formulation, apart from usual relations in continuum mechanics, this system of equations should be supplemented by the following relations:

- The above-mentioned relation between nominal stress and the effective stress in intact matrix, formula (2).
- Constitutive equation of intact matrix and the principle of strain equivalence[6~9]

$$\sigma_s = \sigma_s(\varepsilon) \tag{21}$$

- The relation between continuum damage D and number density of microdamage n

$$D(t, \boldsymbol{x}) = \int_0^\infty n(t, \boldsymbol{x}, c)\tau \mathrm{d}c \tag{22}$$

where τ is the failure volume of an individual microdamage with size c, for example, for a spherical microdamage $\tau \sim \pi c^3/6$.

Compared to traditional continuum mechanics, there are two more variables: microdamage number density n and continuum damage D, and one more statistical evolution Eq.(5c) and an integral relation (22).

Though the above system constitutes a unified and closed formulation, the differentio-integral equations and their complicated meso- and macroscopically coupled nature make the problem too tricky to deal with.

3.2 D-level Approximation, Based on the Statistical Average of Microdamage Density Solution

The main difficulty of the above formulation comes from the integral including microdamage number density n. However, the number density contains more information than we usually need in engineering practice. Hence, the first approximation is to eliminate the microdamage number density, but to remain its function bridging mesoscopic kinetics and macroscopic formulation.

The statistical evolution equation of number density of microdamage (5c) can be converted to the following continuum damage field equation by integration under proper boundary conditions[3,49]

$$\frac{\partial D}{\partial T} + D\frac{\rho}{\rho_0}\frac{\partial v}{\partial Y} = f \tag{23}$$

where

$$f = \int_0^\infty n_{\mathrm{N}}(c;\sigma)\tau \mathrm{d}c + \int_0^\infty n(t, \boldsymbol{x}, c)A(c, \sigma)\tau' \mathrm{d}c \tag{24}$$

f is the dynamic function of damage (DFD), which represents the statistical average effects of nucleation and growth of microdamage and $\tau' = \mathrm{d}\tau/\mathrm{d}c$. Obviously, the function is an agent bridging microdamage n and continuum damage D.

Substitution of the obtained solution of number density n in Eq.(13) and growth rate A in Eq.(14) into integral Eq.(24) leads to a closed DFD without microdamage number density but still with mesoscopic kinetics. To facilitate integration, we should exchange the integrating order of the double integral in Eq.(24). The region of the double integral Eq.(24) with non-zero number density n_0 in Eq.(10) is illustrated by the shaded area in Fig.1. Then a closed DFD is expressed directly by two mesoscopic kinetic laws of nucleation and growth rates of microdamage[36,46]

$$f = \int_0^\infty n_{\mathrm{N}}(c;\sigma)\tau(c)\mathrm{d}c \left\{ 1 + \frac{\int_0^\infty \left[n_{\mathrm{N}}(c_0;\sigma) \int_{c_0}^{c_{\mathrm{f}}} \tau'(c)\mathrm{d}c \right] \mathrm{d}c_0}{\int_0^\infty n_{\mathrm{N}}(c;\sigma)\tau(c)\mathrm{d}c} \right\} =$$

$$\int_0^\infty n_{\mathrm{N}}(c_0;\sigma)\tau(c_{\mathrm{f}})\mathrm{d}c_0 \tag{25}$$

along with the time-dependent microdamage front c_{f}, (11). The approximation is a closed and trans-scale one, combining traditional continuum mechanics and mesoscopic kinetics closely and explicitly. So, the D-approximation is preferable to others. However, we have to confess there are two assumptions involved in the derivation. Firstly, although continuum damage $D(t,\boldsymbol{x})$ is assumed to be a variable depending on macroscopic spatial coordinates $\boldsymbol{x}$ in Eq.(23), the closed form of DFD in Eq.(25) is based on the local solution of number density of microdamage $n(t,c)$, formula (13), namely independent of macroscopic spatial coordinates $\boldsymbol{x}$. However, this is in accord with the concept of locality in constitutive theory. Secondly, the stress σ in the formulation is nominal stress, whereas the stress in mesoscopical kinetics Eqs.(6) and (7) should be understood as the stress in intact matrix σ_{s}. The relation between the two stresses is shown in Eq.(2). So, the decoupling in the formulation (25) is based on the assumption of $D \ll 1$. Additionally, stress is usually time-dependent, whilst the solution (13) is obtained by assuming stress to be a parameter. These are the points of the approximate closure.

There is another D-level approximation, but based on empirical DFD. One can notice that the only time-dependent part in DFD is the second term in the blanket of expression (25), because of the time-dependent front of microdamage c_{f}. Actually, it is found that in some cases this time-dependent term can be fitted into a power function of continuum damage D. Hence DFD can be expressed by a very simple formula

$$f = f(D,\sigma) = n_{\mathrm{N}}^*(c^*)^4 g_{\mathrm{N}}(\sigma)\left[1+\beta g(\sigma)D^m\right] \tag{26}$$

where g_{N} and g are two dimensionless functions of stress. Constant $n_{\mathrm{N}}^*(c^*)^4$ indicates the characteristic value of microdamage nucleation rate, where n_{N}^* is characteristic nucleation rate and c^* is the characteristic microdamage size, respectively. m and β are two dimensionless parameters. This is in accord with the concept of simple and compound damages corresponding to nucleation and growth of microdamage, respectively, proposed by Davison and Stevens[12]. Generally speaking, the DFD has $m > 1$ and appears to be concave. Whereas, β is a dimensionless ratio of compound rate over nucleation rate of damage (see Eq.(25)), so it is an intrinsic Deborah number. Additionally, it is worthy noticing that in this approximation continuum damage D is still governed by the damage field equation.

Compared to traditional continuum mechanics, there are one more variable —— continuum damage D and one more field Eq.(23), which is dependent on mesoscopic kinetics of microdamage by DFD, integral (25).

3.3 I-level Approximation, Based on Internal Variable D

When examining the second term in the equation of damage field evolution (23), one can express it as

$$D\frac{\rho}{\rho_0}\frac{\partial v}{\partial Y} = \frac{D}{1+\varepsilon}\frac{\partial \varepsilon}{\partial T} \approx D\frac{\partial \varepsilon}{\partial T} \tag{27}$$

provided strain is negligibly small. So, if damage D is also negligibly small and the two rates, strain rate $\dfrac{\partial \varepsilon}{\partial T}$ and damage rate $\dfrac{\partial D}{\partial T}$, are in the same order, the damage field Eq.(23) becomes

$$\frac{\partial D}{\partial T} = \dot{D} \approx f \tag{28}$$

This is what internal variable theory in continuum damage mechanics assumed[6~10]. So, we name this I-level approximation. Obviously, the derivation provides a ground for the usual continuum damage mechanics. In this approximation, damage D is implicitly dependent on spatial coordinates via stress field, rather than an explicit field variable as before. Hence, this I-level approximation is implicitly space-dependent. Incidentally, under this approximation, one can obtain another expression of damage D straightforwardly by its definition Eq.(22) and the solution of number density of microdamage Eq.(13), provided mesoscopic kinetics are available, regardless of damage evolution law, like Eq.(25)

$$D(t;\sigma)=\int_0^{\infty}\left[n_{\mathrm{N}}(c_0;\sigma)\int_{c_0}^{c_{\mathrm{f}}}\frac{\tau(c)}{V(c,c_0;\sigma)}\mathrm{d}c\right]\mathrm{d}c_0 \tag{29}$$

Compared to traditional continuum mechanics with internal variable, this approximation not only provides a reasoning ground of continuum damage mechanics, but also gives the expression (29) of damage D in terms of mesoscopic kinetics directly.

3.4 Intrinsic Deberah Number

In spite of various formulations of damage evolution, we can indicate that there is a characteristic dimensionless number

$$D^*=\frac{n_{\mathrm{N}}^*(c^*)^5}{V^*} \tag{30}$$

where V^* is the characteristic growth rate. D^* characterizes the damage rate ratio of two intrinsic processes: nucleation over growth. This expression can be seen clearly from expression (29) of damage D, because of $\tau\propto c^3$. Also, in the light of π theorem in dimensional analysis, since there are three independent characteristic parameters: nucleation rate n_{N}^*, growth rate V^* and microdamage size c^* but only two independent dimensions: length and time, only is there one independent dimensionless parameter that is D^*. The lower the number is, the stronger the effect of growth is. Obviously, D^* implies a certain characteristic damage. Although the significance of Deborah number, a ratio of intrinsic relaxation time over imposed one, was proposed in micromechanics[4], noticeably, D^* is an intrinsic Deborah number in damage evolution. This intrinsic Deborah number originates in multi-time-scale intrinsic processes, like nucleation and growth of microdamage in the concerned case. In this sense, the introduction of intrinsic Deborah numbers is very much natural and universal in non-equilibrium processes in heterogeneous media. We will see later that this intrinsic Deborah number is truly the decisive factor in failure prediction.

4 CRITICAL DAMAGE——DAMAGE LOCALIZATION AS MECHANISM OF FAILURE

4.1 Damage Localization Criterion[49]

Till now, we have not answered the other key question in damage mechanics: how to determine the critical threshold of damage to failure. As discussed before, damage is viewed as a field variable, i.e., explicitly dependent on spatial coordinates $\boldsymbol{x}, D=D(\boldsymbol{x})$. It is usually accepted that failure is a process of dimension reduction, like a roughly two-dimensional fracture surface formed in a three-dimensional solid. Hence, the damage field in the process of localization could be examined with one dimensional approximation, namely

only one effective spatial variable Y vertical to the premature fracture plane. In the light of this damage field concept, damage localization can be depicted by the following condition

$$\frac{\partial^2 D}{\partial Y \partial T} \Big/ \frac{\partial D}{\partial Y} \geq \frac{\partial D}{\partial T} / D \tag{31}$$

This implies that damage localization appears when the relative rate of damage gradient becomes greater than the relative rate of the damage itself. Then, what mechanical factor governs the localized failure? For the damage field, we take the form of DFD $f = f(D, \sigma)$, like Eq.(26), and differentiate damage field Eq.(22) with respect to the spatial ordinate Y. Under the assumption of quasi-static small deformation, the inequality for damage localization Eq.(31) can be expressed by DFD and its derivative[49]

$$f_D \geq \frac{f}{D} \tag{32}$$

where $f_D = \dfrac{\partial f}{\partial D}$.

This is no longer a geometric description of damage localization Eq.(31), but a critical dynamic condition for damage localization, governed by intrinsic nature of media, DFD. Most importantly, this demonstrates that critical damage should be a material constant, independent of sample size, boundary, etc. To illustrate the application of the criterion, we substitute DFD Eq.(26) into Eq.(32), then the criterion to localization becomes

$$D_{\mathrm{c}} = [(m-1) \cdot \beta]^{-1/m} \tag{33}$$

From data fitting, it is known that m is in the order of 10^0, like 2~4, but β may change from 10^1 in creep to $10^4 \sim 10^6$ in impact. From the mesoscopic measurements, D_{c} ranges from tenths to hundredths, even thousandths.

It might be interesting to compare these values to the transition thresholds of several tenths given by percolation theory. Percolation is an equilibrium theory of geometrical phase transition to form global connection[50]. Actually, the connection of broken sites resulting from random breaking resembles the failure owing to random nucleation of microdamage in process. So, the difference in transition thresholds demonstrates that damage localization to failure is by no means an equilibrium process in nature. As a matter of fact, provided growth rate of microdamage vanishes, dynamic function of damage f would become damage-independent, see Eq.(24) or (25). According to criterion Eq.(32), damage localization will not appear in this case. Furthermore, numerical simulations do show that damage localization criterion can properly predict the collapse of media under loading[51].

4.2 Intrinsic Deborah Number as Indicator of Critical Damage

Provided the critical damage for damage localization is a material constant, could it be expressed by mesoscopic kinetics, rather than empirical constants, like Eq.(33). Notice that there is differentiation with respect to damage D only in criterion (32) and stress acts as if a parameter. So, by means of the approximations of DFD (25) and damage D (29), as well as the relation among derivatives: $\dfrac{\mathrm{d}f}{\mathrm{d}D} = \dfrac{\mathrm{d}f}{\mathrm{d}c_{\mathrm{f}}} \Big/ \dfrac{\mathrm{d}D}{\mathrm{d}c_{\mathrm{f}}}$, the critical damage to localization can be derived as

$$D_{\mathrm{c}} = D^* \cdot \frac{\int_0^\infty \bar{\tau}\left(\bar{c}_{\mathrm{f}}\right) \bar{n}_{\mathrm{N}}\left(\bar{c}_0\right) \mathrm{d}\bar{c}_0 \cdot \int_0^\infty \bar{\tau}\left(\bar{c}_{\mathrm{f}}\right) \bar{n}_{\mathrm{N}}\left(\bar{c}_0\right) / V\left(\bar{c}_0, \bar{c}_{\mathrm{f}}\right) \mathrm{d}\bar{c}_0}{\int_0^\infty \bar{\tau}'\left(\bar{c}_{\mathrm{f}}\right) \bar{n}_{\mathrm{N}}\left(\bar{c}_0\right) \mathrm{d}\bar{c}_0} \tag{34}$$

where over bar denotes dimensionless and scaled variables. Actually, Eq.(34) is a transcendental equation of D_c, because of the dependence of microdamage front c_f on time t, therefore on damage D. Since variables in (34) are well-scaled, the integrals in (34) are all in order 1, i.e., O(1). Therefore

$$D_c \sim D^* \tag{35}$$

This implies that intrinsic Deborah number D^* is a proper indicator of macroscopic critical damage to localization. So, instead of the arbitrary values of critical damage (3) in continuum damage mechanics, we have had a physically meaningful evaluation of the critical damage. Based on this fact, engineers can figure out the critical threshold according to mesoscopic physics and design proper tests to measure its exact values.

Alternatively, in some cases it is found that the second term of the trans-scale formulation of DFD (25) could be fitted into a power law of damage D and then there is an approximate relation[52]

$$\beta \approx \left(\frac{V^*}{n_N^* \cdot (c^*)^5}\right)^m = (D^*)^{-m} \tag{36}$$

Then, the expression (32) of the critical damage D_c, also leads to the important relation (35).

Above all, the intrinsic dimensionless numbers D^* plays the most significant role in the prediction of damage localization to failure in heterogeneous media.

Now let us have a quick look at a practical example: spallation. For a kind of Al alloy, mesoscopic experimental measurements give $D^* = \frac{n_N^*(c^*)^5}{V^*} \sim (10^{-3} \sim 10^{-2})$. Data fitting on continuum level gives $\beta \sim (10^5 \sim 10^6)$. Suppose that damage localization criterion, Eq.(33), works, the condition leads to the critical damage to localization, $D_c \sim 4 \times 10^{-3}$ and a calculated critical time of $0.7\,\mu s$. Clearly, D_c and D^* are in the same order. Also, compared the calculated critical time $0.7\,\mu s$ to the loading duration of $(0.68 \sim 0.85)\,\mu s$ for the appearance of incipient macro-cracking in tests, the agreement is very encouraging[53].

5 CONCLUTION

The present paper reviews the progress in damage mechanics, especially, the state of the art in statistical microdamage mechanics. The main points can be summarized as follows:

(1) "Weighing the influence of distributed damage at the microscale on the collective macroscale stiffness and evolution of damage is a challenge."

(2) We reviewed its fundamental equation and basic solution of microdamage number density, which can provide an essential bridge linking mesoscopic kinetics and continuum formulation of the phenomena.

(3) Damage evolution can be described in terms of an associated system of continuum and damage field equations on various levels of approximation. The D-level approximation with dynamic function of damage (DFD) appears to be a proper closed trans-scale formulation of the problems. And, continuum damage mechanics is equivalent to I-level approximation of the trans-scale formulation.

(4) Damage localization provides a proper threshold of failure forecast. The critical damage to localization lower than the thresholds in percolation reflects the non-equilibrium essence of damage evolution to failure.

(5) Intrinsic Deborah number D^* is a significant representation of the effect of mesoscopic kinetics on macroscopic damage evolution. It is closely related to damage localization.

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