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# From the octagon to the SFT vertex — gluing and multiple wrapping

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Abstract: We compare various ways of decomposing and decompactifying the string field theory vertex and analyze the relations between them. We formulate axioms for the octagon and show how it can be glued to reproduce the decompactified pp-wave SFT vertex which in turn can be glued to recover the exact finite volume pp-wave Neumann coefficients. The gluing is performed by resumming multiple wrapping corrections. We observe important nontrivial contributions at the multiple wrapping level which are crucial for obtaining the exact results.

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JHEP06(2017)058

## Contents



## 1 Introduction

The AdS/CFT correspondence relates string theories on anti de Sitter backgrounds to conformal gauge theories on the boundary of these spaces [1]. The energies of string states correspond to the scaling dimensions of local gauge invariant operators which determine the space time dependence of the conformal 2- and 3-point functions completely. In order to build all higher point correlation functions of the CFT one needs to determine the 3-point couplings, which is in the focus of recent research.

String theories on many AdS backgrounds are integrable [2–5] and this miraculous infinite symmetry is the one which enables us to solve the quantum string theory dual to the strongly coupled gauge theory.[1] In the prototypical example the type IIB superstring theory on the $AdS_5 \times S^5$ background is dual to the maximally supersymmetric 4D gauge theory. Integrability shows up in the planar limit and interpolates between the weak and strong coupling sides. The spectrum of string theory, i.e. the scaling dimensions of local gauge-invariant operators are mapped to the finite volume spectrum of the integrable theory, which has been determined by adapting finite size techniques such as Thermodynamic Bethe Ansatz [7–9] (consequently developed into a NLIE [10] and the quantum spectral

[1] For a review see the collection in ref. [6].



JHEP06(2017)058

curve [11, 12]). Further important observables such as 3-point correlation functions or nonplanar corrections to the dilatation operator are related to string interactions. A generic approach to the string field theory vertex was introduced in [13] which can be understood as a sort of finite volume form factor of non-local operator insertions in the integrable worldsheet theory. There is actually one case when the 3-point function corresponds to a form factor of a *local* operator insertion. In the case of heavy-heavy-light operators the string worldsheet degenerates into a cylinder and the SFT vertex is nothing but a diagonal finite volume form factor, see [14–16]. Another approach through cutting the string worldsheet corresponding to a 3-point correlation function into two hexagons was introduced in [17], see also [18–23] for further developments.

The string field theory vertex describes a process in which a big string splits into two smaller ones. In light-cone gauge fixed string sigma models on $AdS_5 \times S^5$ and some similar backgrounds, the string worldsheet theory is integrable and the conserved J-charge serves as the volume, so that the size[2] of the incoming string exactly equals the sum of the sizes of the two outgoing strings. Initial and final states are characterized as multiparticle states of the worldsheet theory on the respective cylinders and we are interested in the asymptotic time evolution amplitudes, which can be essentially described as finite volume form factors of a non-local operator insertion representing the emission of the third string. In order to be able to obtain functional equations for these quantities we suggested in [13] to analyze the decompactification limit, in which the incoming and one outgoing volume are sent to infinity, such that their difference is kept fixed. We called this quantity the decompactified string field theory (DSFT) vertex or decompactified Neumann coefficient. We formulated axioms for such form factors, which depend explicitly on the size of the small string, and determined the relevant solutions in the free boson (plane-wave limit) theory. Taking a natural Ansatz for the two particle form factors we separated the kinematical and the dynamical part of the amplitude and determined the kinematical Neumann coefficient in the AdS/CFT case [24], too. These solutions automatically contain all wrapping corrections in the remaining finite size string, which makes it very difficult to calculate them explicitly in the interacting case, especially for more than two particles. It is then natural to send the remaining volume to infinity and calculate the so obtained *octagon* amplitudes. One can go even further and introduce another cut between the front and back sheets leading to two hexagons, which were introduced and explicitly calculated in [17]. Since we are eventually interested in the string field theory vertex, we have to understand how to glue back the cut pieces. This paper is an attempt going into this direction. Clearly, gluing two hexagons together we should recover the octagon amplitude.[3] Gluing two edges of the octagon we get the DSFT vertex, while gluing the remaining two edges we would obtain the *finite* volume SFT vertex, which would be the ultimate goal for the interacting theory.

The study of various observables in integrable quantum field theories in *finite* volume in a natural way can be decomposed into a number of stages. Firstly, the problem posed in infinite volume typically yields a set of axioms or functional equations for the observable

[2]In this paper we will use terms size and volume interchangeably to mean the circumference of the cylinder on which the worldsheet QFT of the string is defined.

[3]We will not consider this case, however, in the current paper.



JHEP06(2017)058

in question which often can be solved explicitly. The key property of the infinite volume formulation is the existence of analyticity and crossing relations which allow typically for formulating functional equations [25–28]. Secondly one considers the same problem in a large finite volume neglecting exponential corrections of order $e^{-mL}$. In this case the answers are mostly known like for the energy levels, generic form factors[4] and diagonal form factors [29, 30]. However, some of these answers are still conjectural and are not known in various interesting cases. Thirdly, one should incorporate the exponential corrections of order $e^{-mL}$, which are often termed as wrapping corrections as they have the physical interpretation of a virtual particle wrapping around a noncontractible cycle. The key example here are the Lüscher corrections for the mass of a single particle [31] and their multiparticle generalization [32]. Once one wants to incorporate multiple wrapping corrections, the situation becomes much more complicated however in some cases this can be done [33].

In the case of the spectrum of the theory on a cylinder, fortunately one does not need to go through the latter computations as there exists a Thermodynamic Bethe Ansatz formulation which at once resums automatically all multiple wrapping corrections and provides an exact finite volume answer [34]. Unfortunately for other observables like the string interaction vertex we do not have this technique at our disposal and we may hope that understanding the structure of multiple wrapping corrections will shed some light on an ultimate TBA like formulation. This is another motivation for the present work. In fact one of the new results of the present paper is an integral representation for the exact pp-wave Neumann coefficient which involves a measure factor reminiscent of various TBA formulas.

In [17], a formula for gluing two hexagons was proposed: insert a complete basis of particles on the mirror edge[5] and sum over them

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int \mu_n e^{-\sum_i E_i L} \tag{1.1}$$

This is in fact a rather formal expression as the observable in question is divergent. Also we allowed for a generic measure factor. It will indeed turn out that the measure factor is nontrivial for multiple wrapping. In this paper we analyze the multiple wrapping terms for a massive free boson theory which corresponds to the relevant quantities being evaluated for the pp-wave geometry.

The outline of this paper is as follows. In section 2 we will review the decompactified SFT vertex axioms as well as introduce the axioms for the octagon. We will also deduce the measure factor by requiring that gluing the octagon through (1.1) reproduces the exact pp-wave decompactified SFT vertex. Then in section 3, we will revisit the gluing procedure (1.1) from the point of view of cluster expansion (or equivalently compactification in the mirror channel) and isolate the key ingredients which are necessary for obtaining the finite volume answer for a generic observable. We will also illustrate this structure with the well known relativistic examples of ground state energy and LeClair-Mussardo formula for the finite volume 1-point expectation value [36]. In the following two sections we will

[4]By this we mean form factors with no coinciding rapidities in any channel.

[5]In the relativistic case this would correspond to inserting particles in the channel with space and time interchanged i.e. with rapidities $\theta + i\pi/2$.

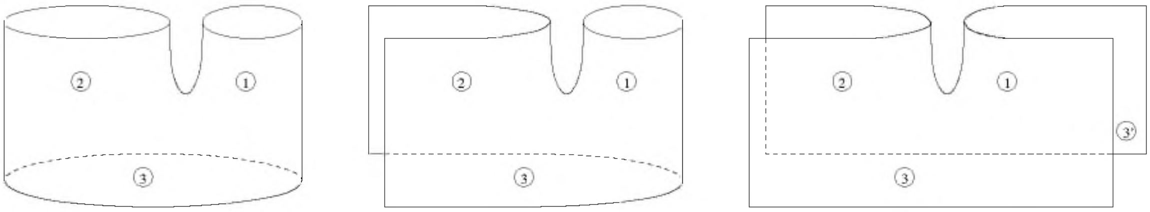


**Figure 1**. Splitting of a big string into two smaller ones and its decompactified versions.

show that one can provide natural choices for these ingredients which enable us to glue the octagon into the decompactified SFT vertex and then glue the decompactified SFT vertex into the exact finite volume pp-wave vertex.[6] We close the paper with a discussion and two appendices, one of which contains the derivation of the integral representation of the pp-wave Neumann coefficient, and the other a discussion of the relation between octagon axioms and decompactified SFT vertex axioms in the context of the gluing formula.

## 2 Cutting pants into DSFT vertex, octagon

The string field theory vertex describes the amplitude of the process in which a big string (#3) splits into two smaller ones (#1 and #2), see left of figure 1. In light-cone gauge fixed string sigma models the conserved J-charge serves as the volume, which adds up in the process $J_3 = J_2 + J_1$. Initial and final states are characterized as finite volume multiparticle states and the asymptotic time evolution amplitudes can be understood as finite volume form factors of a non-local operator insertion. In calculating these quantities we go to the decompactification limit, in which the volumes $J_3$ and $J_2$ are sent to infinity, such that their difference $J_3 - J_2 = L$, which is the size of the remaining closed string, is kept fixed leading to infinite volume form factors, see the middle of figure 1. These form factors automatically contain all wrapping corrections in the remaining finite size string, which makes very difficult to calculate them explicitly. It is then natural to send the remaining volume to infinity and calculate the so obtained *octagon* amplitudes. See the right of figure 1 for the geometry. Since eventually we are interested in the string field theory vertex we have to glue back the cut edges. Gluing two edges of the octagon we get the decompactified SFT vertex, while gluing the remaining two edges we obtain the seeked for SFT vertex.

### 2.1 The decompactified SFT vertex

In our previous paper [13] we formulated the axioms of the DSFT vertex, which we also called the generalized Neumann coefficients. Here for simplicity we quote the axioms for a relativistic theory with a single type of particle. For initial particles living on string #2 with rapidities $\theta_i$ they read as follows:

- The exchange axiom is

$$N_L(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S(\theta_i - \theta_{i+1}) N_L(\theta_1, \dots, \theta_{i+1}, \theta_i \dots, \theta_n) \tag{2.1}$$

[6] Here we are concerned with just the bosonic case so we do not consider issues related with the prefactor.

JHEP06(2017)058

- The periodicity axiom explicitly includes the volume of the small emitted string:

$$N_L(\theta_1+2i\pi,\theta_2,\ldots,\theta_n)=e^{ip_1L}N_L(\theta_2,\ldots,\theta_n,\theta_1) \tag{2.2}$$

- The kinematical singularity axiom, which relates form factors with different particle numbers, takes the form:

$$-i\mathrm{Res}_{\theta'=\theta}N_L(\theta'+i\pi,\theta,\theta_1,\ldots,\theta_n)=\left(1-e^{-ipL}\prod_j S(\theta-\theta_j)\right)N_L(\theta_1,\ldots,\theta_n) \tag{2.3}$$

We have determined in [13] the 2-particle solution for the free boson theory $S(\theta)=1$, which reads as[7]

$$N_L(\theta_1,\theta_2)=-\frac{1}{2\cosh\frac{\theta_1-\theta_2}{2}}d_L(\theta_1)d_L(\theta_2) \tag{2.4}$$

where the functions $d_L(\theta)$ involve all order wrapping terms. They are given explicitly in terms of deformed Gamma functions which have a rather nontransparent definition. The above formula exactly coincides with the decompactification limit of the pp-wave Neumann coefficient [35]. Remarkably enough, there exists a very compact and transparent integral formula for $d_L(\theta)$ which we derive in appendix A. It takes the form

$$d_L(\theta)=e^{-\int_{-\infty}^{\infty}\frac{du}{2\pi}k(u-\theta)\log(1-e^{-mL\cosh u})}\,; \qquad k(\theta)=-\frac{1}{\cosh(\theta)} \tag{2.5}$$

The multiparticle solutions can be fixed from the kinematical residue equation and have the form

$$N_L(\theta_1,\ldots,\theta_n)=\sum_{\text{pairings}}\prod_{(i,j)\text{ pairs}}N_L(\theta_i,\theta_j) \tag{2.6}$$

Thus we sum for all possible pairings of the rapidities $\{\theta_i\}$ and take the product for the pairs of the 2-particle expressions. Clearly this form is compatible with Wick theorem in the free boson theory.

From the decompactified string vertex one can go in two opposite directions. Either one can glue together the remaining two mirror edges (the dashed lines between #2 and #3 and between #2 and #3' in figure 2) thus obtaining the finite size SFT vertex, which is really the ultimate goal of this program, or one can go in the opposite direction and send the remaining volume $L$ to infinity thus obtaining the octagon.

In the case of the free massive boson (the pp-wave) the exact finite size SFT vertex Neumann coefficient (up to an overall normalization) can be expressed very compactly as

$$N_{L_1}^{L_2}(\theta_1,\theta_2)=N_{L_1}(\theta_1,\theta_2)\cdot\frac{d_{L_2}(\theta_1)}{d_{L_3}(\theta_1)}\cdot\frac{d_{L_2}(\theta_2)}{d_{L_3}(\theta_2)} \tag{2.7}$$

In section 5 we will describe how this form can be obtained by gluing together the decompactified SFT vertex.

[7] Here we choose the $N_L(\emptyset)=1$ normalization.



JHEP06(2017)058

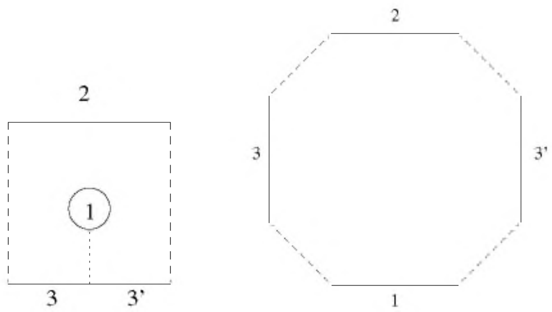


**Figure 2**. The kinematical domains of the DSFT vertex is on the left, while that of the octagon amplitude is on the right. The glued mirror edges between #31 and #3'1 are indicated by a dotted line. The finite size string represented by the circle serves as a non-local operator insertion in the square topology.

Now, however, we will concentrate on the octagon which appears when we send the remaining volume, $L$, to infinity. Effectively, this limit not only sends the volume of string #1 to infinity but also cuts the space of string #3 into two disconnected pieces, which we denote by #3 and #3'. They are connected by crossing through string #1 on one side and through string #2 on the other. This suggests the octagon description as shown of figure 2. Let us formulate the functional relations for this quantity.

## 2.2 The octagon

The octagon amplitude, when particles with rapidities $\theta_i$ are in string #2, satisfy the following axioms:

- The exchange axiom relates particles on the same kinematical edge to each other thus is not changed compared to the DSFT vertex axioms:

$$O(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S(\theta_i - \theta_{i+1}) O(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) \tag{2.8}$$

- In the periodicity properties we have to cross a particle from domain #2 to #3 first, then from #3 to #1, then from #1 back to #3' and finally to #2 leading to a $4i\pi$ periodicity:

$$O(\theta_1 + 4i\pi, \theta_2, \dots, \theta_n) = O(\theta_2, \dots, \theta_n, \theta_1) \tag{2.9}$$

- In the kinematical singularity axiom particles in domain #2 can feel particles in domain #3 only by crossing with $i\pi$, (and not by crossing with $-i\pi$ ), thus we have

$$-i\mathrm{Res}_{\theta'=\theta} O(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) = O(\theta_1, \dots, \theta_n) \tag{2.10}$$

  i.e no S-matrix factors appear, which make their determination easier.

Particles on different edges of the octagon can be obtained by analytical continuation, what we describe in detail in the appendix B.

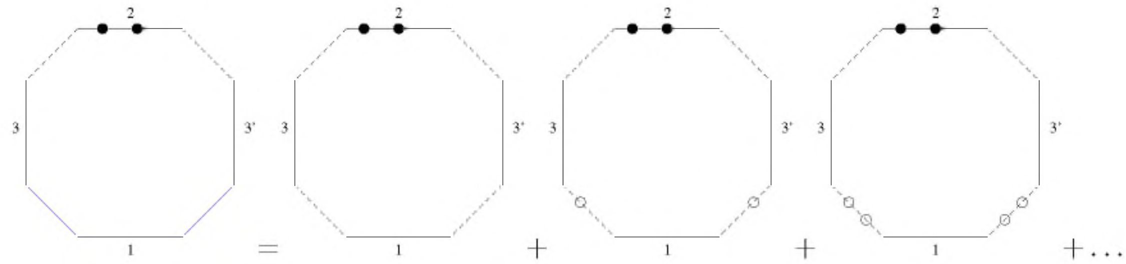


**Figure 3**. Summing up octagons to get the DSFT vertex. The blue edges are glued together by summing up for all multi-particle mirror states, represented by empty circles. Physical particles are represented by solid circles.

The two particle octagon solution for the free boson theory is

$$O(\theta_1,\theta_2) = -\frac{1}{2\cosh\frac{\theta_1-\theta_2}{2}} \tag{2.11}$$

The multiparticle solutions can be fixed from the kinematical singularity axiom and take the form

$$O(\theta_1,\dots,\theta_n) = \sum_{\text{pairings}} \prod_{(i,j)\,\text{pairs}} O(\theta_i,\theta_j) \tag{2.12}$$

Our main problem now is to understand how to obtain the DSFT vertex with string #1 having a finite size $L$, by gluing together the two mirror edges between #1 and #3 and between #1 and #3' (see figure 2).

### 2.3 Naive resummation of the octagon

A very formal definition of gluing two mirror edges was proposed in [17]. We demonstrate this idea on the example how the DSFT could be obtained from the resummation of octagons. The idea of the gluing is to interpret the cutting as a resolution of the identity

$$\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int_{-\infty}^{\infty}\frac{du_i}{2\pi}\mu(\{u_i\})e^{-\sum_i E(u_i)L}|u_1,\dots,u_n\rangle\langle u_n,\dots,u_1| \tag{2.13}$$

where $|u_1,\dots,u_n\rangle$ denotes an infinite volume mirror state living between the spaces #3 and #1, while $\langle u_n,\dots,u_1|$ is its dual mirror space living between the space #1 and #3'. In formulas it means for a two particle DSFT vertex that

$$\begin{aligned} N_L(\theta_1,\theta_2) = \frac{1}{\text{norm}}\Big\{ & O(\theta_1,\theta_2) + \int_{-\infty}^{\infty}\frac{du}{2\pi}\mu_1(u)O(\theta_1,\theta_2,u^+,u^-)e^{-LE(u)} + \\ & +\frac{1}{2}\int_{-\infty}^{\infty}\frac{du_1}{2\pi}\int_{-\infty}^{\infty}\frac{du_2}{2\pi}\,\mu_2(u_1,u_2)O(\theta_1,\theta_2,u_1^+,u_2^+,u_2^-,u_1^-)e^{-L(E(u_1)+E(u_2))} + \dots \Big\} \end{aligned} \tag{2.14}$$

where $u^{\pm} = u \pm \frac{i3\pi}{2}$. Graphically it can be represented as on figure 3. Since the mirror particle-anti-particle pairs come on the opposite edges of the octagon the amplitude is singular due to the kinematical singularity axioms. However, it is very natural to normalize



JHEP06(2017)058

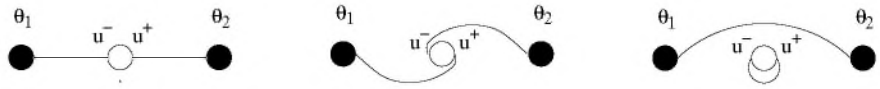


**Figure 4**. $O(\theta_1, \theta_2, u^+, u^-)$ in graphical notation. The first two diagrams are connected, while the last is disconnected.

the amplitude by the "infinite" empty glued octagon:

$$\begin{aligned}\text{norm} &= 1 + \int_{-\infty}^{\infty} \frac{du}{2\pi}\mu_1(u) O(u^+, u^-) e^{-LE(u)} + \\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{du_1}{2\pi}\int_{-\infty}^{\infty}\frac{du_2}{2\pi}\mu_2(u_1,u_2)O(u_1^+,u_2^+,u_2^-,u_1^-)e^{-L(E(u_1)+E(u_2))} + \dots\end{aligned} \tag{2.15}$$

which exactly suffers from the same divergences. Indeed, the special "free" form of the octagon amplitudes guarantees that the normalization in the denominator removes all the disconnected singular terms and only finite regular expressions remain:

$$\begin{aligned}N_L(\theta_1,\theta_2) &= O(\theta_1,\theta_2) + \int_{-\infty}^{\infty}\frac{du}{2\pi}\mu_1(u)O^c(\theta_1,\theta_2,u^+,u^-)e^{-LE(u)} + \\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{du_1}{2\pi}\int_{-\infty}^{\infty}\frac{du_2}{2\pi}\mu_2(u_1,u_2)O^c(\theta_1,\theta_2,u_1^+,u_2^+,u_2^-,u_1^-)e^{-L(E(u_1)+E(u_2))} + \dots\Big\}\end{aligned} \tag{2.16}$$

where $O(\theta_1, \theta_2, u_1^+, u_1^-, \dots, u_n^+, u_n^-)_c$ denotes the connected part, i.e. the one which is connected with the following graphical rules: put the first vertex for $\theta_1$ and the last for $\theta_2$, while in between $n$ vertices for each $u_i$. Left side of the $u_i$ represents $u_i^-$, while the right $u_i^+$. For each propagator $O(\theta_1, u_j^\pm)$ draw an edge from $\theta_1$ to the right/left of $u_j$. For a propagator $O(u_j^{\epsilon_j}, u_k^{\epsilon_k})$ leave the $\epsilon_j$ side of vertex $u_j$ and arrive at the $\epsilon_j$ side of $u_j$. See figure 4 for the diagrams representing $O(\theta_1, \theta_2, u^+, u^-)$. The connected part of the multiparticle octagon consists of exactly those graphs which are connected. Actually the sum of these terms are not singular and takes a very special finite form. For one pair of mirror particles we have

$$\begin{aligned}O^c(\theta_1,\theta_2,u^+,u^-) &= O(\theta_1,u^+)O(\theta_2,u^-) + O(\theta_1,u^-)O(\theta_2,u^+) \\ &= O(\theta_1,\theta_2)(k(u-\theta_1)+k(u-\theta_2))\,; \qquad k(u) = -\frac{1}{\cosh u}\end{aligned} \tag{2.17}$$

which generalize to $n$ particles as

$$O^c(\theta_1,\theta_2,u_1^+,u_1^-,\dots,u_n^+,u_n^-) = O(\theta_1,\theta_2)\prod_{i=1}^{n}(k(u_i-\theta_1)+k(u_i-\theta_2)) \tag{2.18}$$

This can be checked by noticing that the connected form factors satisfy the kinematical singularity axiom at $\theta_i + i\pi/2 = u_j$. Resolving the related recursion leads to the formula of the connected form factors.

Taking naively the measures to be trivial $\mu(\{u_i\}) = 1$ leads to the naive and "wrong" result

$$N_L(\theta_1,\theta_2) = O(\theta_1,\theta_2)d_n(\theta_1)d_n(\theta_2)\,; \qquad d_n(\theta) = e^{\int_{-\infty}^{\infty}\frac{du}{2\pi}k(u-\theta)e^{-mL\cosh u}} \tag{2.19}$$

At the leading wrapping order the naive result is correct, indicating that we are missing some relevant contributions from multiple wrappings. Actually the missing terms come from the domains of integrations, when $u_i = u_j$ for some $i$ and $j$. As a guiding principle one could demand that $N_L(\theta_1, \theta_2)$ should satisfy the DSFT vertex axioms. In appendix B we show that it is equivalent to the teleportation requirement, which can be rephrased as that after an analytical continuation $\theta \to \theta - i\pi$ we have

$$d_L(\theta + i\pi) = (1 - e^{-ipL})\frac{1}{d_L(\theta)} \tag{2.20}$$

Expanding both sides and taking into account that $k(\theta - i\pi) = -k(\theta)$ we can see that the residue terms must sum up to $-e^{-ipL}/d_L(\theta)$. Evaluating at leading order gives

$$\mu_1(u) = 1 + O(e^{-mL\cosh u}) \tag{2.21}$$

At next order, assuming $\mu_2(u_1, u_2) = 1 + O(e^{-mL\cosh u})$, the direct continuation of $d_L(\theta)$ from the double pole term produces a term $\frac{1}{2}e^{-2ipL}$, which can be canceled choosing

$$\mu_1(u) = 1 + \frac{1}{2}e^{-mL\cosh u} + O(e^{-2mL\cosh u}) \tag{2.22}$$

Calculating systematically the higher order terms we can find that

$$\mu_1(u) = \sum_{n=0}^{\infty} \frac{1}{n+1} e^{-nmL\cosh u}; \qquad \mu_n(\{u_i\}) = \prod_i \mu_1(u_i) \tag{2.23}$$

which gives the expected results:

$$\mu_1(u)e^{-mL\cosh u} = \log\left(1 - e^{-mL\cosh u}\right) \tag{2.24}$$

Clearly the relevant nontrivial terms are kinds of "diagonal" contribution associated with multiple wrapping. In order to understand better their role and origin we will now look at the gluing process from the point of view of so-called cluster expansion in relativistic integrable field theories. Then we will revisit again the gluing of the octagon as well as describe how one can glue the decompactified SFT vertex into the finite volume one.

## 3 The structure of multiple wrapping corrections

In this section we first exhibit explicitly the exactly known observables for a free massive boson which will give insight to the multiple wrappings, starting from the completely standard examples of free energy and going on to the quite intricate formulas for the exact string vertex. We then analyze the general structure of the wrapping corrections.

**The ground state energy.** The ground state energy can be obtained from the large $R$ limit of the torus partition function

$$Z \sim e^{-RE_0(L)} \tag{3.1}$$

with
$$E_0(L) = \pm m \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \log\left(1 \mp e^{-mL\cosh\theta}\right) \tag{3.2}$$
with the upper/lower signs corresponding to a free boson/fermion. Expanding the above formula in a power series in $e^{-mL\cosh\theta}$ gives multiple wrapping contributions to the ground state energy.

Incidentally the exact equation which holds in the interacting case has a very similar form
$$E_0(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \log\left(1 + e^{-\varepsilon(\theta)}\right) \tag{3.3}$$
where $\varepsilon(\theta)$ is a solution of the relevant TBA equation.

**The LeClair-Mussardo formula.** The finite volume expectation value of a local operator is given by the following formula [36]
$$\langle \mathcal{O} \rangle_L = \sum_{n=0}^{\infty} \int \prod_{k=1}^{n} \frac{d\theta_k}{2\pi} \frac{1}{1 \mp e^{mL\cosh\theta_k}} F_n^c(\theta_1, \ldots, \theta_n) \tag{3.4}$$
Here $F_n^c(\theta_1, \ldots, \theta_n)$ is the infinite volume (connected) diagonal form factor of the operator $\mathcal{O}$. Remarkably enough the above formula again generalizes to the interacting case through the simple substitution $mL\cosh\theta \to \varepsilon(\theta)$ [36]:
$$\langle \mathcal{O} \rangle_L = \sum_{n=0}^{\infty} \int \prod_{k=1}^{n} \frac{d\theta_k}{2\pi} \frac{1}{1 + e^{\varepsilon(\theta_k)}} F_n^c(\theta_1, \ldots, \theta_n) \tag{3.5}$$

For completeness let us quote here the finite volume expansions for the decompactified SFT vertex as well as the finite size string vertex. It is illuminating to recognize the structural similarity of the multiple wrapping terms appearing in these expressions with the relativistic formulas given above.

**The decompactified string vertex.** The formula for the decompactified string vertex with two particles on string #2 takes the form:
$$N_L(\theta_1, \theta_2) = O(\theta_1, \theta_2) d_L(\theta_1) d_L(\theta_2) \tag{3.6}$$
where the logarithm of the function $d_L(\theta)$ is
$$\log d_L(\theta) = -\int_{-\infty}^{\infty} \frac{du}{2\pi} k(u-\theta) \log(1 - e^{-mL\cosh u}) \tag{3.7}$$
We note a surprising similarity with the ground state energy formula.

**The finite size string vertex.** The formula for the string vertex with all the three strings being of finite size $L_i$ has been derived by a direct calculation in [35]. The formulas there can be recast into a simpler form when expressed in terms of rapidities and take the form (again up to an overall normalization):
$$N_{L_1}^{L_2}(\theta_1, \theta_2) = N_{L_1}(\theta_1, \theta_2) \cdot \frac{d_{L_2}(\theta_1)}{d_{L_3}(\theta_1)} \cdot \frac{d_{L_2}(\theta_2)}{d_{L_3}(\theta_2)} \tag{3.8}$$



JHEP06(2017)058

### 3.1 Mirror channel compactification — cluster expansion

Let us now review the approach of mirror channel compactification *aka* cluster expansion. Let us first consider the case of the partition function evaluated on a torus of size $L \times R$ where $R$ is very large in order to extract the ground state. As in the derivation of the TBA, it is convenient to perform this calculation in the mirror channel (which is compactified to the large size $R$). Then the partition function is by definition the summation over all states in the mirror theory weighted with $e^{-EL}$ where $E$ is the mirror channel energy:

$$1 + \sum_n e^{-E_n L} + \sum_{n_1 \geq n_2} e^{-(E_{n_1}+E_{n_2})L} + \dots \tag{3.9}$$

Here for the free boson, mode numbers can coincide hence we have $n_1 \geq n_2$. In contrast for an interacting theory or for a free fermion we would have a sharp inequality. In the next step one takes the continuum limit

$$\sum_n \to R \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta \tag{3.10}$$

however, and this is the key point, one has to take care of the diagonal terms and first separate them out

$$\sum_{n_1 \geq n_2} = \frac{1}{2} \sum_{n_1, n_2} + \frac{1}{2} \sum_{n_1 = n_2} \tag{3.11}$$

Writing all the contributions appearing in (3.9) gives

$$\begin{aligned} &1 + R \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta \, e^{-mL\cosh\theta} + \frac{1}{2} R \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta \, e^{-2mL\cosh\theta} + \\ &+ \frac{1}{2} \left( R \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta \, e^{-mL\cosh\theta} \right)^2 + \dots \end{aligned} \tag{3.12}$$

We see that this coincides with the first terms of the expansion of

$$e^{-mR \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \log\left(1 - e^{-mL\cosh\theta}\right)} \tag{3.13}$$

The diagonal terms like the one with $n_1 = n_2$ are exactly responsible for the nontrivial measure factor $\log\left(1 - e^{-mL\cosh\theta}\right)$. Let us emphasize that their interpretation is not as straightforward as it may seem. Indeed, for an interacting theory such states with $n_1 = n_2$ are *not* even part of the spectrum. Provisionally a useful interpretation of these terms is that they represent *multiply wrapped* single particles. This interpretation seems to provide the correct intuition for the treatment of such terms in all cases considered in this paper.

The LeClair-Mussardo formula arises when we insert a local operator into the above expansion. We thus have to evaluate the diagonal expectation values of the type

$$\langle n_1 n_2 | \mathcal{O} | n_2 n_1 \rangle_R \tag{3.14}$$

for asymptotically large $R$, i.e. neglecting any wrapping terms in $R$. In this limit the expectation value can be written[8] as a linear combination of appropriate measure factors

[8] This formula is still conjectural for more than two particles but there is overwhelming evidence that it is correct [30, 37].

(with the only explicit $R$ dependence) and *infinite volume* diagonal form factors of the local operator $\mathcal{O}$. For the case of a free massive boson we have the following explicit formulas for up to two particles

$$\langle n_1 | \mathcal{O} | n_1 \rangle_R = \frac{F_1^c(\theta_1)}{RE_1} + F_0^c \tag{3.15}$$

$$\langle n_1 n_2 | \mathcal{O} | n_2 n_1 \rangle_R = \frac{F_2^c(\theta_1, \theta_2)}{R^2 E_1 E_2} + \frac{F_1^c(\theta_1)}{RE_1} + \frac{F_1^c(\theta_2)}{RE_2} + F_0^c \tag{3.16}$$

$$\langle n_1 n_1 | \mathcal{O} | n_1 n_1 \rangle_R = 0 + \mathbf{2} \cdot \frac{F_1^c(\theta_1)}{RE_1} + F_0^c \tag{3.17}$$

Note the factor of 2 in the diagonal double wrapping term. It is exactly this factor (and other factors of this type at higher orders) that cancels the 1/2 appearing in front of the diagonal term in (3.11) and effectively transforms the measure factor

$$\log\left(1 - e^{-mL\cosh\theta}\right) \tag{3.18}$$

into

$$\frac{-1}{1 - e^{mL\cosh\theta}} \tag{3.19}$$

appearing in the LeClair-Mussardo formula.

### 3.2 The structure of the multiple wrappings

Looking at the above two examples, we see that the computation of the finite volume observable $\langle \mathcal{X} \rangle_L$ can be summarized as regularizing the mirror channel (e.g. by compactifying it on a finite but large volume $R$), decomposing the summation over a complete basis of states into independent sums of single and multiple wrapped particles with appropriate combinatorial factors, and finally providing an expression for the *diagonal* finite volume *asymptotic*[9] expectation values, namely

$$\begin{aligned} Z \langle \mathcal{X} \rangle_L = {} & \langle \emptyset | \mathcal{X} | \emptyset \rangle_R + \sum_{n_1} \langle n_1 | \mathcal{X} | n_1 \rangle_R \, e^{-E_{n_1} L} + \frac{1}{2} \sum_{n_1, n_2} \langle n_1 n_2 | \mathcal{X} | n_2 n_1 \rangle_R \, e^{-(E_{n_1} + E_{n_2})L} \\ & + \frac{1}{2} \sum_{n_1} \left\langle {n_1}^{(\times 2)} \middle| \mathcal{X} \middle| {n_1}^{(\times 2)} \right\rangle_R e^{-2E_{n_1} L} + \frac{1}{6} \sum_{n_1, n_2, n_3} \langle n_1 n_2 n_3 | \mathcal{X} | n_3 n_2 n_1 \rangle_R \, e^{-(E_{n_1} + E_{n_2} + E_{n_3})L} \\ & + \frac{1}{2} \sum_{n_1, n_2} \left\langle n_1 n_2^{(\times 2)} \middle| \mathcal{X} \middle| n_2^{(\times 2)} n_1 \right\rangle_R e^{-(E_{n_1} + 2E_{n_2})L} + \frac{1}{3} \sum_{n_1} \left\langle {n_1}^{(\times 3)} \middle| \mathcal{X} \middle| {n_1}^{(\times 3)} \right\rangle_R e^{-3E_{n_1} L} + \ldots \end{aligned} \tag{3.20}$$

In an interacting or fermionic theory one has to flip some signs as there all quantized mode numbers must be distinct. The key remaining information are the above $R$-regularized diagonal expectation values. We may expect that they have the following general form

$$\left\langle \{ n_i^{(\times k_i)} \} \middle| \mathcal{X} \middle| \{ n_i^{(\times k_i)} \} \right\rangle_R = \sum_{\alpha \cup \bar{\alpha}} \mu(\alpha, \bar{\alpha}, R) \cdot \mathcal{F}_{\mathcal{X}}(\{ n_i^{(\times k_i)} \}_{i \in \alpha}) \tag{3.21}$$

[9] I.e. neglecting all $e^{-mR}$ terms.



JHEP06(2017)058

The measure factor is the only place with explicit $R$ dependence. For the free boson we expect it to take a simple form

$$\mu(\alpha,\bar{\alpha},R)=\frac{1}{\prod_{i\in\alpha}RE_i} \tag{3.22}$$

The second factor in (3.21) should be a quantity defined in infinite volume associated to the observable $\mathcal{X}$ which should follow from some appropriate functional equations.

Alternatively we could calculate $\langle\mathcal{X}\rangle_L$ instead of $Z\,\langle\mathcal{X}\rangle_L$. By this we remove many disconnected terms (as $Z^{-1}$ has the same structure as $\langle\mathcal{X}\rangle_L$.) The modified quantity appearing in the expansion is denoted by $\mathcal{F}^c_{\mathcal{X}}(\{n_i^{(\times k_i)}\}_{i\in\alpha})$ where the superscript $^c$ indicates that one would have to take just the connected part. Note that care should be taken to define these generalized form factors also for the multiply wrapped particles. How to do it in general is by no means obvious. The main result of this paper is to provide the relevant expressions both for the octagon and for the decompactified string vertex with two external particles such that summing (3.20) for the octagon yields the decompactified string vertex and summing (3.20) for the decompactified string vertex gives the exact finite volume string vertex.

Before doing so, we summarize the relevant quantities both for the ground state energy and for the LeClair-Mussardo formula.

**Ground state energy.** The ground state energy is related to the torus partition function as $Z\sim e^{-RE_0(L)}$, thus we basically analyze the $\mathcal{X}=\mathbb{I}$ situation. In this case the $\alpha\cup\bar{\alpha}$ decomposition degenerates only to one "trivial" term

$$\left\langle\{n_i^{(\times k_i)}\}\middle|\,\mathcal{X}\,\middle|\{n_i^{(\times k_i)}\}\right\rangle_R=1=\frac{1}{\prod_i RE_i}\mathcal{F}_{\mathcal{X}}(\{n_i^{(\times k_i)}\}) \tag{3.23}$$

In particular, in our normalization it implies that

$$\mathcal{F}_{\mathcal{X}}(\{n_i^{(\times k_i)}\})=\prod_i RE_i \tag{3.24}$$

**LeClair-Mussardo formula.** In the case of the LeClair-Mussardo formula we analyze the expansion of $\langle\mathcal{O}\rangle_L$ instead of $Z\,\langle\mathcal{O}\rangle_L$, since by this trick we can remove all disconnected terms and the $\alpha\cup\bar{\alpha}$ decomposition degenerates only to one term

$$\mathcal{F}^c_{\mathcal{X}}(\{n_i^{(\times k_i)}\})=F^c_n(\{n_i\})\prod_i k_i \tag{3.25}$$

where $F^c_n(\{n_i\})$ is the connected diagonal form factor. Thus the wrapping order appears only as a combinatorial factor.

## 4 Resumming the octagon

Here we normalize the decompactified string vertex as $N_L(\emptyset)=1$, i.e. we calculate only the connected contributions (see section 2.3 for their definition). We propose the following form of the finite volume expectation values which, when inserted into (3.20) will exactly reproduce the decompactified string vertex with string #1 being of length $L$:

$$\begin{aligned}
\langle\emptyset|\,\mathcal{O}^{\theta_1,\theta_2}\,|\emptyset\rangle_R &= O(\theta_1,\theta_2)\\
\langle n_1|\,\mathcal{O}^{\theta_1,\theta_2}\,|n_1\rangle_R &= \frac{1}{RE_1}O^c(\theta_1,\theta_2,u_1^-,u_1^+) \equiv \frac{1}{RE_1}O(\theta_1,\theta_2)\,(k(u_1-\theta_1)+k(u_1-\theta_2))\\
\langle n_1n_2|\,\mathcal{O}^{\theta_1,\theta_2}\,|n_2n_1\rangle_R &= \frac{1}{R^2E_1E_2}O^c(\theta_1,\theta_2,u_1^-,u_2^-,u_2^+,u_1^+)\\
&= \frac{1}{R^2E_1E_2}O(\theta_1,\theta_2)\prod_{i=1}^{2}(k(u_i-\theta_1)+k(u_i-\theta_2))
\end{aligned} \tag{4.1}$$

The key assumption now involves the expectation values when some of the mirror particles wrap multiple times. The exact answer implies that the relevant expectation value does not depend on the wrapping order. E.g. we have

$$\left\langle {n_1}^{(\times 2)}\right|\mathcal{O}^{\theta_1,\theta_2}\left|{n_1}^{(\times 2)}\right\rangle_R = \langle n_1|\,\mathcal{O}^{\theta_1,\theta_2}\,|n_1\rangle_R = \frac{1}{RE_1}O(\theta_1,\theta_2)\,(k(u_1-\theta_1)+k(u_1-\theta_2)) \tag{4.2}$$

With the above formulas in place, it is simple to generalize as

$$\begin{aligned}
\left\langle \{{n_i}^{(\times k_i)}\}\right|\mathcal{O}^{\theta_1,\theta_2}\left|\{{n_i}^{(\times k_i)}\}\right\rangle_R &= \langle\{n_i\}|\,\mathcal{O}^{\theta_1,\theta_2}\,|\{n_i\}\rangle_R\\
&= O(\theta_1,\theta_2)\prod_i \frac{1}{RE_i}\,(k(u_i-\theta_1)+k(u_i-\theta_2))
\end{aligned} \tag{4.3}$$

As then one can easily convince oneself (looking e.g. at all terms up to $3^{rd}$ order or comparing to the free boson ground state energy) that (3.20) can be summed up to give exactly the decompactified string vertex

$$\begin{aligned}
N_L(\theta_1,\theta_2) &= O(\theta_1,\theta_2)\cdot\underbrace{e^{-\int_{-\infty}^{\infty}\frac{du}{2\pi}k(u-\theta_1)\log(1-e^{-mL\cosh u})}}_{d_L(\theta_1)}\cdot(\theta_1\to\theta_2)\\
&= O(\theta_1,\theta_2)d_L(\theta_1)d_L(\theta_2)
\end{aligned} \tag{4.4}$$

## 5 Resumming the string vertex

Passing from the decompactified string vertex with string #1 being of size $L_1$ and strings #2 and #3 being infinite to the finite volume string vertex with all strings having finite size $L_i$ can be done following closely the strategy employed for the octagon. We will again consider a configuration with two particles on string #2. In order to do the infinite volume part of the calculation it is convenient to transport the mirror particles up to string #2. Now the rapidities will have to be shifted by $\pm i\pi/2$ so in this section we will denote

$$u^{\pm} = u \pm i\frac{\pi}{2} \tag{5.1}$$

Since we are dealing with a free theory the decompactified vertex with multiple particles will be obtained by Wick contractions but now with pairing performed with

$$N_L^{\infty}(\theta_1,\theta_2) = N_{L_1}(\theta_1,\theta_2) = O(\theta_1,\theta_2)d_{L_1}(\theta_1)d_{L_1}(\theta_2) \tag{5.2}$$



JHEP06(2017)058

So we see that the nontrivial part of the computation is almost exactly the same as for the octagon (up to the redefinition of $u^\pm$ here) and the $L_1$-dependent factors will appear only as an overall product for all particles entering the amplitude. It is clear that we thus get the following expressions:

$$\langle\emptyset|\,\mathcal{N}_{L_1}^{\theta_1,\theta_2}\,|\emptyset\rangle_R = N_{L_1}^\infty(\theta_1,\theta_2) \tag{5.3}$$

$$\langle n_1|\,\mathcal{N}_{L_1}^{\theta_1,\theta_2}\,|n_1\rangle_R = \frac{1}{RE_1} N_{L_1}^\infty(\theta_1,\theta_2)\left(k(u_1-\theta_1)+k(u_1-\theta_2)\right) d_{L_1}(u_1^+) d_{L_1}(u_1^-) \tag{5.4}$$

$$\begin{aligned}\langle n_1 n_2|\,\mathcal{N}_{L_1}^{\theta_1,\theta_2}\,|n_2 n_1\rangle_R &= \frac{1}{R^2E_1E_2} N_{L_1}^\infty(\theta_1,\theta_2)\\ &\quad\times\prod_{i=1}^{2}\left(k(u_i-\theta_1)+k(u_i-\theta_2)\right) d_{L_1}(u_i^+) d_{L_1}(u_i^-)\end{aligned} \tag{5.5}$$

The product of the $d_{L_1}(.)$ functions can be simplified using the functional equations

$$d_{L_1}(u^+) d_{L_1}(u^-) = \left(1-e^{-mL_1\cosh u}\right) \tag{5.6}$$

In order to see the crucial role of the above remaining $u$-dependent factor let us consider the expression (3.20) up to the single particle term. We have

$$N_{L_1}^\infty(\theta_1,\theta_2)\left[1+\int_{-\infty}^{\infty}\frac{du}{2\pi}\left(k(u-\theta_1)+k(u-\theta_2)\right)e^{-mL_2\cosh u}\left(1-e^{-mL_1\cosh u}\right)+\ldots\right] \tag{5.7}$$

The 1 particle term thus splits into a difference of two terms: one with a wrapping factor $e^{-mL_2\cosh u}$ and the other with the wrapping factor $e^{-m(L_1+L_2)\cosh u}\equiv e^{-mL_3\cosh u}$, where we used the conservation of lengths in the light cone gauge $L_1+L_2=L_3$. But these are indeed exactly the first terms in the expansion of

$$\frac{d_{L_2}(\theta_1)}{d_{L_3}(\theta_1)}\cdot\frac{d_{L_2}(\theta_2)}{d_{L_3}(\theta_2)} \tag{5.8}$$

It is clear that the two particle term coming from (5.5) will contribute to the exponentiation of the above structure. However the contribution of the doubly wrapped particle is quite subtle and requires some care. In order to motivate our proposal, let us recall that the decompactified string vertex axioms introduced in [13] involve an overall monodromy factor $e^{ipL_1}$. Now since we are considering a particle which wraps twice across the vertex, we expect that it would effectively feel a factor $e^{2ipL_1}$. Thus it is very natural to expect that the generalization of formula (5.4) to a doubly wrapped particle takes the form

$$\left\langle n_1{}^{(\times 2)}\middle|\mathcal{N}_{L_1}^{\theta_1,\theta_2}\middle|n_1{}^{(\times 2)}\right\rangle_R = \frac{1}{RE_1}N_{L_1}^\infty(\theta_1,\theta_2)\left(k(u_1-\theta_1)+k(u_1-\theta_2)\right)d_{2L_1}(u_1^+)d_{2L_1}(u_1^-) \tag{5.9}$$

Now we can examine the doubly wrapped particle contribution in (3.20):

$$N_{L_1}^\infty(\theta_1,\theta_2)\cdot\frac{1}{2}\int_{-\infty}^{\infty}\frac{du}{2\pi}\left(k(u-\theta_1)+k(u-\theta_2)\right)e^{-2mL_2\cosh u}\left(1-e^{-2mL_1\cosh u}\right) \tag{5.10}$$

We see that this yields the first nontrivial double wrapping terms in the expansion of the logarithms in (5.8). In order for this to work it was absolutely crucial that the double wrapped particle feels effectively the double factor $e^{2ipL_1}$. It is clear that analogous property should hold for multiple wrapped particles.

$$\left\langle \{n_i{}^{(\times k_i)}\} \middle| \mathcal{N}_{L_1}^{\theta_1,\theta_2} \middle| \{n_i{}^{(\times k_i)}\} \right\rangle_R = N_{L_1}^{\infty}(\theta_1,\theta_2) \prod_i \frac{d_{k_i L_1}(u_i^+) d_{k_i L_1}(u_i^-)}{RE_i} \left(k(u_i-\theta_1)+k(u_i-\theta_2)\right) \tag{5.11}$$

Repeating the above for higher number of particles we see that we obtain the exact finite volume Neumann coefficient

$$N_{L_1}^{L_2}(\theta_1,\theta_2) = N_{L_1}(\theta_1,\theta_2) \cdot \frac{d_{L_2}(\theta_1)}{d_{L_3}(\theta_1)} \cdot \frac{d_{L_2}(\theta_2)}{d_{L_3}(\theta_2)} \equiv O(\theta_1,\theta_2) \cdot \frac{d_{L_1}(\theta_1) d_{L_2}(\theta_1)}{d_{L_3}(\theta_1)} \cdot \frac{d_{L_1}(\theta_2) d_{L_2}(\theta_2)}{d_{L_3}(\theta_2)} \tag{5.12}$$

## 6 Conclusions

The nontrivial topology of the SFT vertex allows for various lines of approach towards determining it exactly. By cutting the vertical edges various number of times and decompactifying one obtains the decompactified string vertex of [13], the octagon and two hexagons of [17]. Although the final goal is the determination of the exact finite volume vertex, i.e. with all the three strings being of finite size, the necessity of passing through this intermediate decompactified stage is that only then we can formulate functional equations for the relevant quantities which incorporate analyticity and various variations of crossing symmetry. One of the contributions of the present paper was to formulate appropriate axioms for the octagon in the interacting case.

Hence a key question is to understand the procedure of gluing back the decompactified answers into the final finite volume result. In [17] a formal expression for gluing back was suggested by a summation over a complete set of mirror particles living on the edge which is being glued. This expression is, however, rather formal as it stands and suffers from divergences. The subtleties arise at the multiple wrapping level which is in general difficult to study.

The case of the pp-wave vertex (essentially a free massive boson on the string pants diagram) is a very interesting theoretical laboratory for studying these issues as we have at our disposal exact finite volume answers for the finite size SFT vertex as well as its various decompactified variations — the decompactified SFT vertex and the octagon. As these expressions are exact and incorporate an infinite set of multiple wrapping corrections we may quantitatively explore the subtleties of the gluing procedure.

We argue that the quantitative structure of the gluing procedure may be efficiently understood within the so-called cluster expansion (equivalently compactification in the mirror channel). There the main ingredient is the asymptotic large mirror volume expectation value for the observable in question which should decompose into a linear combination of measure factors and appropriate infinite volume quantities. This is a standard way to understand ground state energy and the LeClair-Mussardo formula for one point expectation values in relativistic integrable theories. In the present paper we adopt this framework to the case of the octagon and the decompactified SFT vertex. Note, however, that even in



JHEP06(2017)058

the classical case of LeClair and Mussardo there is no proof of the general large mirror volume expectation value formula for more than two particles. In the case of the vertex we also do not provide a proof, however our proposed formulas are very natural from the physical point of view. Also *a-posteriori* it is very nontrivial that any such formulas exist which reproduce the apparently very complicated finite volume Neumann coefficients.

We demonstrated that one can resum the multiple wrapping corrections for the octagon into the exact decompactified SFT vertex. This necessitates a nontrivial, but quite natural modification of the multiple wrapping measure. We then proceed to interpret this modification through the cluster expansion where it turns out to arise from certain diagonal terms. We then show that similarly one can resum the decompactified SFT vertex and recover the exact finite volume pp-wave Neumann coefficients.

There are numerous further questions to investigate. A key question, and one of the long term motivations of this work, would be to guess some underlying exact TBA-like formulation for the SFT vertex. The integral expression for the pp-wave Neumann coefficient obtained in the present paper is very intriguing in that respect. Also in this paper we did not discuss the hexagons at all. It would be interesting to understand this better, as well as the differences w.r.t. [17].[10]

In this paper we focused on the 3-point functions and on the way how they could be described by gluing octagons and the DSFT vertex. The 4-point functions, however, are even more interesting and recently there have been activities using integrable methods in their descriptions [38–41]. It would be very challenging to figure out how two octagons (or their modifications) could be glued together to describe the four point functions. Actually the geometry of the 4-point function allows for two different cuttings into two octagons. Demanding their compatibility might lead to non-trivial constraint on the gluing procedure or on the octagon themselves.

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## A Large volume expansion of the plane-wave DSFT vertex

In this appendix we rewrite the plane-wave DSFT vertex into the form, in which multiple wrapping terms can be easily identified. Recall from [13, 35] that the DSFT vertex takes the form[11]

$$N(\theta_1,\theta_2) = O(\theta_1,\theta_2)d_L(\theta_1)d_L(\theta_2) \tag{A.1}$$

[10] A main difference between the approach of [17] and the considerations of the present paper is that there the light cone gauge choice is different for each of the three strings, while here we concentrate on the conventional light cone SFT vertex picture where we have a single gauge choice, and hence e.g. the total size of the strings is conserved.

[11] Here we normalized the DSFT vertex as $N_L(\emptyset) = 1$.



JHEP06(2017)058

where

$$d_L(\theta) = \pi\sqrt{\frac{2}{ML}}\frac{e^{\frac{\theta}{2\pi}pL}}{\sinh\frac{\theta}{2}}\frac{1}{\tilde{\Gamma}_{\frac{mL}{2\pi}}(\theta)} \tag{A.2}$$

We are interested in the large $L$ expansion of $\tilde{\Gamma}_{\frac{mL}{2\pi}}$. The large $L$ expansion of $\Gamma_\mu(z)$ was calculated in [35] and rephrased in [13] as

$$\tilde{\Gamma}_\mu(\theta) = \sqrt{\frac{\pi}{\mu}}\frac{e^{\frac{\theta}{2\pi}pL}}{\sinh\frac{\theta}{2}}\tilde{\Gamma}_{\exp}(\theta)\,; \qquad \mu = \frac{ML}{2\pi} \tag{A.3}$$

where $\tilde{\Gamma}_{\exp}(\theta)$ vanishes exponentially for large $L$ in the following way:

$$\partial_\mu \log\tilde{\Gamma}_{\exp}(z) = -2\sum_{n=1}^{\infty}\frac{\mu}{\omega(z)}K_0(2\mu n\pi) = -\frac{\mu}{\omega(z)}\int_{-\infty}^{\infty}du\frac{1}{e^{mL\cosh u}-1} \tag{A.4}$$

We can integrate this equation as

$$\log\tilde{\Gamma}_{\exp}(z) = -\int_{-\infty}^{\infty}\frac{du}{2\pi}\frac{\log(1-e^{-2\pi\mu\cosh u})}{\sqrt{1+\frac{z^2}{\mu^2}\cosh u - \frac{z}{\mu}\sinh u}} \tag{A.5}$$

and the constant of integration is fixed from the vanishing large volume limit. By introducing the rapidity variable via $z = \mu\cosh\theta$ we get

$$\tilde{\Gamma}_{\exp}(\theta) = \exp\left\{\int_{-\infty}^{\infty}\frac{du}{2\pi}k(u-\theta)\log(1-e^{-mL\cosh u})\right\}\,; \qquad k(\theta) = -\frac{1}{\cosh(u-\theta)} \tag{A.6}$$

This implies that

$$\log d_L(\theta) = -\log\tilde{\Gamma}_{\exp}(\theta) = \sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}\frac{du}{2\pi}k(u-\theta)e^{-nmL\cosh u} \tag{A.7}$$

In order to check this expression we first perform an analytical continuation in $\theta$ as $\theta \to \theta + i\pi$. In doing so a pole singularity of the kernel $k(u-\theta)$ crosses the integration contour, which contribute to the functional relation giving

$$d_L(\theta + i\pi) = (1-e^{-ipL})\frac{1}{d_L(\theta)} \tag{A.8}$$

which is required by the kinematical singularity axiom. Continuing further to $\theta \to \theta + 2i\pi$ another singularity crosses the integration contour, which contributes with an opposite residue leading to

$$d_L(\theta + 2i\pi) = \frac{1-e^{ipL}}{1-e^{-ipL}}d_L(\theta) = -e^{ipL}d_L(\theta) \tag{A.9}$$

which is the required monodromy property of the function.

## B DSFT vertex axioms from octagon axioms

In this appendix we show how the DSFT vertex axioms could be obtained from the octagon axioms. This will shed also light, how we need to use the octagon amplitude to describe particles in the split #3 and #3' domains. For simplicity we present the ideas for the free theory and for 2 particles only. The generalization for the interacting theory can be easily done at the formal level similarly to eq. (2.14). At a less formal level one has to understand how to regularize the kinematical singularities for the mirror particle-anti-particle pairs.

Recall that the DSFT vertex can be written in terms of the connected octagons as

$$N_L(\theta_1,\theta_2) = O(\theta_1,\theta_2) + \int_{-\infty}^{\infty}\frac{du}{2\pi}\mu_1(u)O^c(\theta_1,\theta_2,u^+,u^-)e^{-LE(u)} + \\ +\frac{1}{2}\int_{-\infty}^{\infty}\frac{du_1}{2\pi}\int_{-\infty}^{\infty}\frac{du_2}{2\pi}\mu_2(u_1,u_2)O^c(\theta_1,\theta_2,u_1^+,u_2^+,u_2^-,u_1^-)e^{-L(E(u_1)+E(u_2))}+\dots \tag{B.1}$$

where $u^{\pm}=u\pm\frac{3i\pi}{2}$. Let us see now how the DSFT axioms are satisfied.

- The exchange axiom is trivially reproduced as each term in the expansion has this property. In the free boson theory the connected and the disconnected terms are mapped to each other under the exchange $\theta_1\leftrightarrow\theta_2$, thus the connected terms are symmetric themselves.

- In order to show the kinematical singularity axiom we continue analytically $\theta_1\to\theta_1+i\pi$. As a first step we continue it into the mirror domain between space #2 and #3: $\theta_1\to\theta_1+\frac{i\pi}{2}$. When it is exactly in the mirror domain it will hit a kinematical singularity of the octagon coming from integrals for mirror particles of type $u_i^+$. We can avoid this singularity by slightly deforming the contours. However, when we continue the particle's rapidity further to domain #3 we cross the integration contour by a pole singularity. Thus we will have two types of contributions: the direct continuations and the pole contributions. See figure 5 for a graphical representation. The direct term, denoted by $N_{\rm sum}(\theta_1+i\pi,\theta_2)$ is simply

$$O(\theta_1+i\pi,\theta_2) + \int_{-\infty}^{\infty}\frac{du}{2\pi}\mu_1(u)O^c(\theta_1+i\pi,\theta_2,u^+,u^-)e^{-LE(u)} + \\ +\frac{1}{2}\int_{-\infty}^{\infty}\frac{du_1}{2\pi}\int_{-\infty}^{\infty}\frac{du_2}{2\pi}\mu_2(u_1,u_2)O^c(\theta_1+i\pi,\theta_2,u_1^+,u_2^+,u_2^-,u_1^-)e^{-L(E(u_1)+E(u_2))}+\dots \tag{B.2}$$

Let us explain the notation a bit. In the following we understand by $N_{\rm sum}(\theta_1+i\pi,\theta_2)$ the above sum. So whatever is the argument of $N_{\rm sum}(\theta_1,\theta_2)$ it means we evaluate the octagon sum at that rapidities and we *do not continue it analytically.* With this notation the residue term is $e^{-ip_1L}N_{\rm sum}(\theta_2,\theta_1+3i\pi)$ and as we explained $N_{\rm sum}(\theta_2,\theta_1+3i\pi)$ now denotes the sum

$$O^c(\theta_2,\theta_1+3i\pi) + \int_{-\infty}^{\infty}\frac{du}{2\pi}\mu_1(u)O(\theta_2,\theta_1+3i\pi,u^+,u^-)e^{-LE(u)} + \\ +\frac{1}{2}\int_{-\infty}^{\infty}\frac{du_1}{2\pi}\int_{-\infty}^{\infty}\frac{du_2}{2\pi}\mu_2(u_1,u_2)O^c(\theta_2,,\theta_1+3i\pi,u_1^+,u_2^+,u_2^-,u_1^-)e^{-L(E(u_1)+E(u_2))}+\dots \tag{B.3}$$

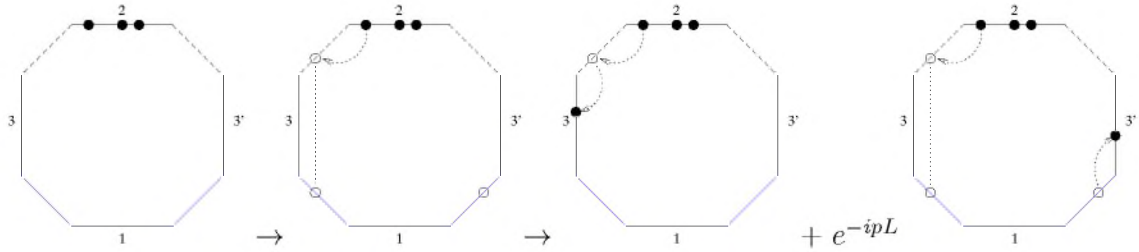


**Figure 5**. Analytical continuation from domain #2 to domain #3.

For the DSFT axioms to be fulfilled in the general case it is crucial that the factor $e^{-ip_1L}$ can be factored out from each term. This can be guaranteed by demanding

$$-i\mathrm{res}_{\theta_1+i\pi=u_1}\mu_2(u_1,u_2)O^c(\theta_1+i\pi,\theta_2,u_1^+,u_1^-,u_2^+,u_2^-)=\mu_1(u_2)O^c(\theta_2,\theta_1+3i\pi,u_2^+,u_2^-) \tag{B.4}$$

and assuming that higher order poles do not contribute. The mechanism producing the extra residue term was called teleportation in [17]. This also indicates, how we should split the particles between regions #3 and #3': we should sum for all possible distributions with an additional $e^{-ipL}$ factor, whenever a particle is moved to region #3'. Observe that it is crucial that we do not have any contributions from double or higher order integrations as they would spoil the above structure. Actually in the free boson case we know that there is a double pole contribution, which can be compensated by an appropriately chosen measure factor. Thus the existence of higher order poles leads to non-trivial measure factors to guarantee

$$N_L(\theta_1+i\pi,\theta_2)=N_{\mathrm{sum}}(\theta_1+i\pi,\theta_2)+e^{-ip_1L}N_{\mathrm{sum}}(\theta_2,\theta_1+3i\pi) \tag{B.5}$$

This the equation we should satisfy in the general interacting case. Demanding it for the continued rapidities will give restrictions on the definition of the connected octagon form factors and the measure.

Now, assuming that $O^c(\theta_1,\theta_2,u_1^+,\dots,u_1^-)$ is non-singular at $\theta_1+i\pi=\theta_2$ (actually it is true in the free boson theory as $k(\theta+i\pi)=-k(\theta)$ and follows from our normalization $N_L(\emptyset)=1$ in general) we obtain the kinematical singularity equation

$$-i\mathrm{res}_{\theta'=\theta}N_L(\theta'+i\pi,\theta)=(1-e^{-ipL}) \tag{B.6}$$

- In order to show the periodicity axiom we need to continue further $\theta_1+i\pi\to\theta_1+2i\pi$. In doing the continuation in each term of the sum $N_{\mathrm{sum}}(\theta_1+i\pi,\theta_2)$ we do not expect any teleportation as the $\theta_1^++i\pi=u_1^-$ singularity is regularized in the connected part. In continuing in terms of the sum $N_{\mathrm{sum}}(\theta_2,\theta_1+3i\pi)$ we expect both the direct and the teleported terms, such that the teleported residue term is proportional to the direct term, see figure 6:

$$\begin{aligned}N_L(\theta_1+2i\pi,\theta_2)&=N_{\mathrm{sum}}(\theta_1+2i\pi,\theta_2)+e^{ip_1L}N_{\mathrm{sum}}(\theta_2,\theta_1+4i\pi)\\&\quad+e^{ip_1L}(-e^{-ip_1L})N_{\mathrm{sum}}(\theta_1+2i\pi,\theta_2)\end{aligned} \tag{B.7}$$

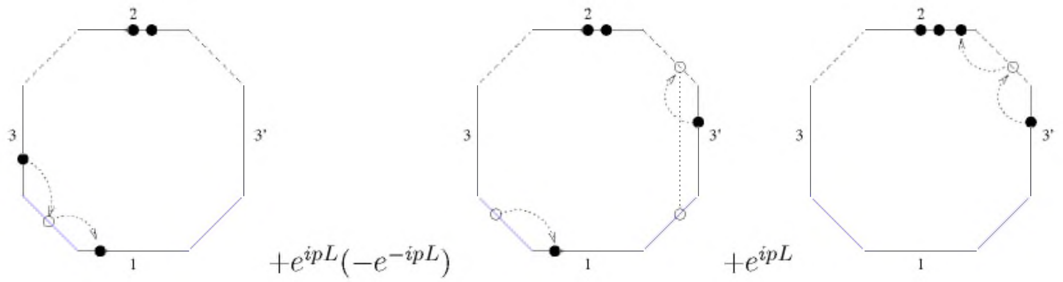


**Figure 6**. Analytical continuation from domain #3 and #3' back to domain #2.

The direct continuation from $N_{\text{sum}}(\theta_1 + i\pi, \theta_2)$ and the teleported continuation from $N_{\text{sum}}(\theta_2, \theta_1 + 3i\pi)$ cancel each other and only the direct continuation from $N_{\text{sum}}(\theta_2, \theta_1 + 3i\pi)$ remains. This result is precisely the expected relation

$$N_L(\theta_1 + 2i\pi, \theta_2) = e^{ip_1 L} N_L(\theta_2, \theta_1) \tag{B.8}$$



## References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[2] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, Phys. Rev. **D 69** (2004) 046002 [hep-th/0305116] [INSPIRE].

[3] G. Arutyunov and S. Frolov, *Superstrings on $AdS_4 \times CP^3$ as a coset σ-model*, JHEP **09** (2008) 129 [arXiv:0806.4940] [INSPIRE].

[4] B. Stefanski, jr, *Green-Schwarz action for type IIA strings on $AdS_4 \times CP^3$*, Nucl. Phys. **B 808** (2009) 80 [arXiv:0806.4948] [INSPIRE].

[5] A. Babichenko, B. Stefanski, Jr. and K. Zarembo, *Integrability and the $AdS_3/CFT_2$ correspondence*, JHEP **03** (2010) 058 [arXiv:0912.1723] [INSPIRE].

[6] N. Beisert et al., *Review of AdS/CFT integrability: an overview*, Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982] [INSPIRE].

[7] G. Arutyunov and S. Frolov, *String hypothesis for the $AdS_5 \times S^5$ mirror*, JHEP **03** (2009) 152 [arXiv:0901.1417] [INSPIRE].

[8] N. Gromov, V. Kazakov and P. Vieira, *Exact spectrum of anomalous dimensions of planar $N = 4$ supersymmetric Yang-Mills theory*, Phys. Rev. Lett. **103** (2009) 131601 [arXiv:0901.3753] [INSPIRE].

[9] D. Bombardelli, D. Fioravanti and R. Tateo, *Thermodynamic Bethe ansatz for planar AdS/CFT: a proposal*, J. Phys. **A 42** (2009) 375401 [arXiv:0902.3930] [INSPIRE].

[10] J. Balog and A. Hegedus, *Hybrid-NLIE for the AdS/CFT spectral problem*, JHEP **08** (2012) 022 [arXiv:1202.3244] [INSPIRE].



JHEP06(2017)058

[11] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for planar $N = 4$ super-Yang-Mills theory*, Phys. Rev. Lett. **112** (2014) 011602 [arXiv:1305.1939] [INSPIRE].

[12] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for arbitrary state/operator in $AdS_5/CFT_4$*, JHEP **09** (2015) 187 [arXiv:1405.4857] [INSPIRE].

[13] Z. Bajnok and R.A. Janik, *String field theory vertex from integrability*, JHEP **04** (2015) 042 [arXiv:1501.04533] [INSPIRE].

[14] Z. Bajnok, R.A. Janik and A. Wereszczyński, *HHL correlators, orbit averaging and form factors*, JHEP **09** (2014) 050 [arXiv:1404.4556] [INSPIRE].

[15] L. Hollo, Y. Jiang and A. Petrovskii, *Diagonal form factors and heavy-heavy-light three-point functions at weak coupling*, JHEP **09** (2015) 125 [arXiv:1504.07133] [INSPIRE].

[16] Z. Bajnok and R.A. Janik, *Classical limit of diagonal form factors and HHL correlators*, JHEP **01** (2017) 063 [arXiv:1607.02830] [INSPIRE].

[17] B. Basso, S. Komatsu and P. Vieira, *Structure constants and integrable bootstrap in planar $N = 4$ SYM theory*, arXiv:1505.06745 [INSPIRE].

[18] B. Eden and A. Sfondrini, *Three-point functions in $N = 4$ SYM: the hexagon proposal at three loops*, JHEP **02** (2016) 165 [arXiv:1510.01242] [INSPIRE].

[19] B. Basso, V. Goncalves, S. Komatsu and P. Vieira, *Gluing hexagons at three loops*, Nucl. Phys. **B 907** (2016) 695 [arXiv:1510.01683] [INSPIRE].

[20] Y. Jiang, S. Komatsu, I. Kostov and D. Serban, *Clustering and the three-point function*, J. Phys. **A 49** (2016) 454003 [arXiv:1604.03575] [INSPIRE].

[21] Y. Jiang and A. Petrovskii, *Diagonal form factors and hexagon form factors*, JHEP **07** (2016) 120 [arXiv:1511.06199] [INSPIRE].

[22] Y. Jiang, *Diagonal form factors and hexagon form factors II. Non-BPS light operator*, JHEP **01** (2017) 021 [arXiv:1601.06926] [INSPIRE].

[23] B. Basso, V. Goncalves and S. Komatsu, *Structure constants at wrapping order*, JHEP **05** (2017) 124 [arXiv:1702.02154] [INSPIRE].

[24] Z. Bajnok and R.A. Janik, *The kinematical $AdS_5 \times S^5$ Neumann coefficient*, JHEP **02** (2016) 138 [arXiv:1512.01471] [INSPIRE].

[25] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S matrices in two-dimensions as the exact solutions of certain relativistic quantum field models*, Annals Phys. **120** (1979) 253 [INSPIRE].

[26] G. Mussardo, *Off critical statistical models: factorized scattering theories and bootstrap program*, Phys. Rept. **218** (1992) 215 [INSPIRE].

[27] M. Karowski and P. Weisz, *Exact form-factors in $(1 + 1)$-dimensional field theoretic models with soliton behavior*, Nucl. Phys. **B 139** (1978) 455 [INSPIRE].

[28] F.A. Smirnov, *Form-factors in completely integrable models of quantum field theory*, *Adv. Ser. Math. Phys.* **14** (1992) 1 [INSPIRE].

[29] B. Pozsgay and G. Takács, *Form-factors in finite volume I: form-factor bootstrap and truncated conformal space*, Nucl. Phys. **B 788** (2008) 167 [arXiv:0706.1445] [INSPIRE].

[30] B. Pozsgay and G. Takács, *Form factors in finite volume II: disconnected terms and finite temperature correlators*, Nucl. Phys. **B 788** (2008) 209 [arXiv:0706.3605] [INSPIRE].

[31] M. Lüscher, *Volume dependence of the energy spectrum in massive quantum field theories* 1. *Stable particle states*, *Commun. Math. Phys.* **104** (1986) 177 [INSPIRE].

[32] Z. Bajnok and R.A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys.* **B 807** (2009) 625 [arXiv:0807.0399] [INSPIRE].

[33] D. Bombardelli, *A next-to-leading Lüscher formula*, *JHEP* **01** (2014) 037 [arXiv:1309.4083] [INSPIRE].

[34] A.B. Zamolodchikov, *Thermodynamic Bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models*, *Nucl. Phys.* **B 342** (1990) 695 [INSPIRE].

[35] J. Lucietti, S. Schäfer-Nameki and A. Sinha, *On the plane wave cubic vertex*, *Phys. Rev.* **D 70** (2004) 026005 [hep-th/0402185] [INSPIRE].

[36] A. Leclair and G. Mussardo, *Finite temperature correlation functions in integrable QFT*, *Nucl. Phys.* **B 552** (1999) 624 [hep-th/9902075] [INSPIRE].

[37] B. Pozsgay, *Mean values of local operators in highly excited Bethe states*, *J. Stat. Mech.* **01** (2011) P01011 [arXiv:1009.4662] [INSPIRE].

[38] B. Basso, F. Coronado, S. Komatsu, H.T. Lam, P. Vieira and D.-L. Zhong, *Asymptotic four point functions*, arXiv:1701.04462 [INSPIRE].

[39] T. Bargheer, *Four-point functions with a twist*, arXiv:1701.04424 [INSPIRE].

[40] T. Fleury and S. Komatsu, *Hexagonalization of correlation functions*, *JHEP* **01** (2017) 130 [arXiv:1611.05577] [INSPIRE].

[41] B. Eden and A. Sfondrini, *Tessellating cushions: four-point functions in $N = 4$ SYM*, arXiv:1611.05436 [INSPIRE].


JHEP06(2017)058