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ONCE AGAIN – ABOUT VELOCITY OF LIGHT

Basing on the quantum mechanics and the modern model of physical vacuum we offer the consistent method of understanding one of the postulates of Special Theory of Relativity – invariability of speed expansion of electromagnetic wave in physical vacuum in all inertial systems.

Keywords: *physical vacuum, photon, special theory of relativity.*

УДК 538.97+517.9

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THE ELECTROMAGNETIC LORENTZ PROBLEM AND THE HAMILTONIAN STRUCTURE ANALYSIS OF THE MAXWELL-YANG-MILLS TYPE DYNAMICAL SYSTEMS WITHIN THE REDUCTION METHOD

Based on analysis of reduced geometric structures on fibered manifolds, invariant under action of an abelian functional symmetry group, we construct the symplectic structures associated with connection forms on the related principal fiber bundles with abelian functional structure groups. The Marsden-Weinstein reduction procedure is applied to the Maxwell and Yang-Mills type dynamical systems. The geometric properties of Lorentz type constraints, describing the electromagnetic field properties in vacuum and related with the well known Dirac-Fock-Podolsky problem, are discussed.

Keywords: *symplectic structures, principal fiber bundles, Lorentz constraints.*

1. Introduction

It is well known [4, 2] that Hamiltonian theory of electromagnetic Maxwell equations aces with very important classical problem of introducing into the unique formalism the well known Lorentz conditions, ensuring both the wave structure of propagating quanta and the positivity of energy. To the regret, in spite of classical studies on this problem given by Dirac, Fock and Podolsky [5], the problem still stays open, and the Lorentz condition is imposed within the modern electrodynamics "by hands" as the external constraint not entering a priori the initial Hamiltonian (or Lagrangian) theory. Moreover, when trying to quantize the electromagnetic theory, as it was shown by Pauli, Dirac, Bogolubov and Shirkov and others, within the existing approach the quantum Lorentz condition could not be satisfied, but only in average sense, since it becomes not compatible. This and related problems stimulated our studying of this problem from so called symplectic reduction theory [1, 3, 9, 12], which, in addition, allows systematically to introduce in the Hamiltonian formalism the external charge and current boundary conditions as well as to suggest a solution to the Lorentz condition problem mentioned above. Some applications of the method are given to Yang-Mills type equations, interacting with a point charged particle. We study the related Poissonian structures on the corresponding reduced symplectic manifolds, which are used in various problems of dynamics in modern mathematical physics, and apply them to studying the nonstandard Hamiltonian properties of the Maxwell and Yang-Mills type dynamical systems. We also analyze from symplectic point of view the important Lorentz type constraints, describing the electrodynamic vacuum properties.

2. The symplectic analysis of the maxwell and yang-mills type electromagnetic dynamical systems

2.1 The Hamiltonian analysis of the Maxwell electromagnetic dynamical systems

Under the Maxwell electromagnetic equations we will understand the following relationships

$$\partial E/\partial t = \nabla \times B - J, \qquad \partial B/\partial t = -\nabla \times E, \quad (2.1)$$

$$<\nabla, E> = \rho, \qquad <\nabla, B> = 0,$$

on the cotangent phase space $T^*(N)$ to $N \subset T(D;\mathbb{E}^3)$ being the smooth manifold of smooth vector fields on an open domain $D \subset \mathbb{R}^3$, all expressed in the light speed units. Here $(E,B) \in T^*(N)$ is a vector of electric and magnetic fields, $\rho : D \to \mathbb{R}$ and $J : D \to \mathbb{E}^3$ are, simultaneously, fixed density and current functions for a smeared in the domain D electric charge, satisfying the continuity relationship

$$\partial p/\partial t + <\nabla, J> = 0, \qquad (2.2)$$

holding for all $t \in \mathbb{R}$, where we denoted by sign "∇" the gradient operation with respect to a variable $x \in D$, by sign "×" the usual vector product in $\mathbb{E}^3 := (\mathbb{R}^3, <\cdot,\cdot>)$, being the standard three-dimensional Euclidean vector space $\mathbb{R}^3$, endowed with the usual scalar product $<\cdot,\cdot>$.

Aiming to represent equations (2.1) as those on reduced symplectic space, define an appropriate configuration (base) space $M \subset \mathcal{T}(D;\mathbb{E}^3)$ with a vector potential field coordinate $A \in M$. The cotangent space $T^*(M)$ may be identified with pairs $(A;Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D;\mathbb{E}^3)$ is a suitable vector field density in D. On the space $T^*(M)$ there exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M))$, allowing, owing to definition of the Liouville from

$$\alpha^{(1)}(A;\tilde{Y}) = \int_D d^3x(<Y, dA> := (Y, dA), \qquad (2.3)$$

the canonical expression

$$\omega^{(2)} := dpr^*\alpha^{(1)} = (dY, \wedge dA). \qquad (2.4)$$

Here we denoted by "$\wedge$" the standard external differentiation, by d^3x, $x \in D$, the Lebesgue measure in the domain D and by $pr : T^*(M) \to M$ the standard projection upon the base space M. Define now a Hamiltonian function $\tilde{H} \in \mathcal{D}(T^*(M))$ as

$$H(A,Y) = \\ = 1/2\left[(Y,Y) + (\nabla \times A, \nabla \times A) + (<\nabla, A>, <\nabla, A>)\right], \qquad (2.5)$$

describing the well known Maxwell equations in vacuum, if the densities $\rho = 0$ and $J = 0$. Really, owing to (2.4) one easily obtains from (2.5) that

$$\partial A/\partial t := \delta H/\partial Y = Y, \qquad (2.6)$$

$$\partial Y/\partial t := -\delta H/\delta A = -\nabla \times B + \nabla <\nabla, A>,$$

being true wave equations in vacuum, where we put, by definition,

$$B := \nabla \times A, \qquad (2.7)$$

being the corresponding magnetic field. If now to define

$$E := -Y - \nabla W, \qquad (2.8)$$

for some function $W : M \to \mathbb{R}$ as the corresponding electric field, the system of equations (2.6) transforms, owing to definition (2.7), into $\partial B/\partial t = -\nabla \times E$, $\partial E/\partial t = \nabla \times B$, exactly coinciding with the Maxwell equations in vacuum, if the Lorentz condition

$$\partial W/\partial t + <\nabla, A> = 0 \qquad (2.9)$$

is involved.

Since definition (2.8) was not foreseen *a priori* from the very beginning within the Hamiltonian approach and our equations fit only for vacuum, we will try to proceed with analysis our electrodynamic model making use of the reduction approach devised in [1, 3, 9, 12]. Namely, we start with the Hamiltonian (2.5) and observe that it is invariant with respect to the following abelian functional symmetry group $G := \exp\mathcal{G}$ where $\mathcal{G} \simeq C^{(1)}(D;\mathbb{R})$, acting on the base manifold M naturally lifted to $T^*(M)$: for any $\square \in \mathcal{G}$ and $(A,Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \qquad \varphi_\psi(Y) = Y. \qquad (2.10)$$

Under transformation (2.10) 1-form (2.3) is, evidently, invariant too since $\varphi_\psi^*\alpha^{(1)}(A,Y) = (Y, dA + \nabla d\psi) = (Y, dA) - (<\nabla, Y>, d\psi) = \alpha^{(1)}(A,Y)$, where we made use of the condition $d\psi \simeq 0$ in $\Lambda^1(T^*(M))$ for $\psi \in \mathcal{G}$. Thus, the corresponding momentum mapping [1, 3, 9, 12] is given as $l(A,Y) = -<\nabla, Y>$ for all $(A,Y) \in T^*(M)$. If $\rho \in \mathcal{G}^*$ is fixed, one can define the reduced phase space $\bar{\mathcal{M}}_\rho := l^{-1}(\rho)/G$ since evidently, the isotropy group $G_\rho = G$ due to its commutivity and condition (2.10). Consider now a principal fiber bundle $p{:}M \to N$ with the abelian structure group G and a base manifold N taken as $N := \{B \in \mathcal{T}(D;\mathbb{E}^3) : <\nabla, B> = 0, <\nabla, E(S)> = \rho\}$, where, by definition, $p(A) = B = \nabla \times A$.

Over this bundle one can build [10, 3] a connection 1-form $\mathcal{A} \in \Lambda^1(M) \otimes \mathcal{G}$, where for all $A \in M$ $\mathcal{A}(A) \cdot \hat{A}_*(l) = 1$, $d <\mathcal{A}(A), \rho>_{\mathcal{G}} = \Omega_\rho^{(2)}(A) \in H^2(M;\mathbb{Z})$, where $\mathcal{A}(A) \in \Lambda^1(M)$ is some differential 1-form, which we choose in the following form: $\mathcal{A}(A) := -(W, d<\nabla, A>)$, where $W \in C^{(1)}(D;\mathbb{R})$

is some scalar function, still not defined. As a result, the Liouville form (2.3) transforms into

$$\tilde{\alpha}_\rho^{(1)} := (Y, dA) - (W, d<\nabla, A>) = = (Y + \nabla W, dA) := (\tilde{Y}, dA), \quad \tilde{Y} := Y + \nabla W, \tag{2.11}$$

giving rise to the corresponding canonical symplectic structure on $T^*(M)$ as

$$\tilde{\omega}_\rho^{(2)} := d\tilde{\alpha} = (d\tilde{Y}, \wedge dA). \tag{2.12}$$

Respectively, the Hamiltonian function (2.5), as a function on $T^*(M)$, transforms into

$$\tilde{H}_\rho(A, \tilde{Y}) = = 1/2\left[(\tilde{Y}, \tilde{Y}) + (\nabla \times A, \nabla \times A) + (<\nabla, A>, <\nabla, A>)\right], \tag{2.13}$$

coinciding with the well known Dirac-Fock-Podolsky [4, 5] Hamiltonian expression. The corresponding Hamiltonian equations on the cotangent space $T^*(M)$ $\partial A/\partial t := \delta\tilde{H}/\delta\tilde{Y}$, $\tilde{Y} := -E - \nabla W$, $\partial\tilde{Y}/\partial t := -\delta\tilde{H}/\delta A = -\nabla \times (\nabla \times A) + \nabla <\nabla, A>$, describe true wave processes related with Maxwell equations in vacuum, but not take into account boundary charge and current densities conditions. Really, from (2.13) we obtain that $\partial^2 A/\partial t^2 - \nabla^2 A = 0 \Rightarrow \partial E/\partial t + \nabla(\partial W/\partial t + <\nabla, A>) = -\nabla \times B$, giving rise to the true vector potential wave equation, but the electromagnetic Farady induction law equation satisfies if one to impose additionally the Lorentz condition (2.9).

To remedy this situation, we will apply to this symplectic space the reduction technique [1, 3, 9, 12]. Namely, the constructed above cotangent manifold $T^*(N)$ is symplectomorphic to the corresponding reduced phase space $\bar{\mathcal{M}}_\rho$, that is

$$\bar{\mathcal{M}}_\rho \simeq \{(B; S) \in T^*(N) : <\nabla, E(S)> = \rho, <\nabla, B> = 0\} \tag{2.14}$$

with the reduced canonical symplectic 2-form

$$\omega_\rho^{(2)}(B, S) = (dB, \wedge dS) = d\alpha_\rho^{(1)}(B, S), \quad \alpha_\rho^{(1)}(B, S) := -(S, dB), \tag{2.15}$$

where we put, by definition,

$$\nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, \quad <\nabla, F> := \rho, \tag{2.16}$$

for some fixed vector mapping $F \in C^{(1)}(D; \mathbb{E}^3)$, depending on the imposed boundary conditions. The result (2.15) follows right away, if to substitute the expression for the ellectric fileld $E = \nabla \times S + F$ into the symplectic structure (2.12), having taken into account that the external differenatial $dF = 0$ in $\Lambda^1(M)$. The Hamiltonian function (2.13) reduces, respectively, to the following symbolic form:

$$H_\rho(B, S) = 1/2\left[(B, B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + \left(<\nabla, (\nabla\times)^{-1} B>, <\nabla, (\nabla\times)^{-1} B>\right)\right], \tag{2.17}$$

where "$(\nabla\times)^{-1}$" means, by definition, the corresponding inverse curl-operation, mapping the divergence-free subspace $C^{(1)}_{div}(D; \mathbb{E}^3) \subset C^{(1)}(D; \mathbb{E}^3)$ into itself. As a result from (2.17), the Maxwell equations (2.1) become a canonical Hamiltonian system upon the reduced phase space $T^*(N)$, endowed with the canonical symplectic structure (2.15) and the modified Hamiltonian function (2.17). Really, one obtains easily that

$$\partial S/\partial t := \delta H/\delta B = B - (\nabla\times)^{-1} \nabla <\nabla, (\nabla\times)^{-1} B>, \quad \partial B/\partial t := -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) := -\nabla \times E, \tag{2.18}$$

where we made use of the definition $E = \nabla \times S + F$ and the elementary identity $\nabla \times \nabla = 0$. Thus, the second equation of (2.18) coincides with the second Maxwell equation of (2.1) in the classical form $\partial B/\partial t = -\nabla \times E$. Moreover, from (2.16), owing to (2.18) one obtains via the differentiation with respect to $t \in \mathbb{R}$ that

$$\partial E/\partial t = \partial F/\partial t + \nabla \times \partial S/\partial t = \partial F/\partial t + \nabla \times B, \tag{2.19}$$

as well as, owing to (2.2),

$$<\nabla, \partial F/\partial t> = \partial\rho/\partial t = -<\nabla, J>. \tag{2.20}$$

So, we can write down from (2.20) that, up to non-essential curl-terms $\nabla \times (\cdot)$, the following relationship

$$\partial F/\partial t = -J \tag{2.21}$$

holds. Having now substituted (2.21) into (2.19), we obtain exactly the first Maxwell equation of (2.1):

$$\partial E/\partial t = \nabla \times B - J, \tag{2.22}$$

being supplemented, naturally, with the external boundary constraint conditions

$$<\nabla, B> = 0, \quad <\nabla, E> = \rho, \quad \partial\rho/\partial t + <\nabla, J> = 0, \tag{2.23}$$

owing to the continuity relationship (2.2) and definition (2.14).

Concerning the wave equations, related with Hamiltonian system (2.18), we obtain the following: the electric field E is recovered from the second eqution as $E := -\partial A/\partial t - \nabla W$, where $W \in C^{(1)}(D; \mathbb{R})$ is some smooth function, depending on the vector field $A \in M$.

To retrieve this dependence, we substitute (2.21) into equation (2.22), having taken into account that $B = \nabla \times A$:

$$\partial^2 A/\partial t^2 - \nabla(\partial W/\partial t + <\nabla, A>) = \nabla^2 A + J. \tag{2.24}$$

Thereby, if to choose now that the Lorentz condition (2.9) is involved, one obtains from (2.24) the corresponding true wave equations in the space-time, taking into account the imposed external boundary conditions (2.23).

Nonetheless, the problem of fulfilling *a priori* the Lorentz type constraint (2.9) within the canonical Hamiltonian formalism remains still not solved, that forces us to proceed to analysing the structure of the Liouville 1-form (2.11) for Maxwell equations in vacuum on a functional manifold slightly extending *M*. As the first step, we rewrite 1-from (2.11) as $\tilde{\alpha}^{(1)}_{\rho} := (\tilde{Y}, dA) = (Y + \nabla W, dA) = = (Y, dA) + (W, -d < \nabla, A >) := (Y, dA) + (W, d\eta)$, where we put, by definition, $\eta := - < \nabla, A >$. Considering now the elements $(T, A; \eta, W) \in T^*(M \times L)$ as new canonical variables on the extended cotangent phase space $T^*(M \times L)$, where $L := C^{(1)}(D; \mathbb{R})$, we can rewrite the symplectic structure (2.12) in the following canonical form

$$\omega^{(2)}_{\rho} = (dY, \wedge dA) + (dW, \wedge d\eta). \quad (2.25)$$

Subject to the Hamiltonian function (2.13) we obtain the expression

$$H(A, Y; \eta, W) = 1/2[(Y - \nabla W, Y - \nabla W) +, + (\nabla \times A, \nabla \times A) + (\eta, \eta)] \quad (2.26)$$

with respect to which the corresponding Hamiltonian equations look as follows:

$$\partial A/\partial t := \delta H/\delta Y = Y - \nabla W, \quad Y := -E,$$
$$\partial Y/\partial t := -\delta H/\delta A = -\nabla \times (\nabla \times A),$$
$$\partial \eta/\partial t := \delta H/\delta W = < \nabla, Y - \nabla W >,$$
$$\partial W/\partial t := -\delta H/\delta \eta = -\eta. \quad (2.27)$$

From (2.27) we obtain, owing to external boundary conditions (2.23), sucessively that $\partial B/\partial t + \nabla \times E = 0$, $\partial^2 W/\partial t^2 - \nabla^2 W = \rho$, $\partial E/\partial t - \nabla \times B = 0$, $\partial^2 A/\partial t^2 - \nabla^2 A = -\nabla(\partial W/\partial t + < \nabla, A >)$.

As is seen, these equations describe electromagnetic Maxwell equations in vacuum, but without the Lorentz condition (2.9). Thereby, as above, we will apply to the symplectic structure (2.25) the reduction technique devised in [1, 3, 9, 12]. We obtain that under transformations (2.16) the corresponding reduced manifold $\bar{\mathcal{M}}_{\rho}$ becomes endowed with the symplecic structure

$$\bar{\omega}^{(2)}_{\rho} := (dB, \wedge dS) + (dW, \wedge d\eta). \quad (2.28)$$

The corresponding expression for Hamiltonian (2.37) looks as

$$H(S, B; \eta, W) = = 1/2[(\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + (B, B) + (\eta, \eta)], \quad (2.29)$$

whose Hamiltonian equations

$$\partial S/\partial t := \delta H/\delta B = B, \qquad \partial W/\partial t := -\delta H/\delta \eta = -\eta,$$
$$\partial B/\partial t := -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E, \quad (2.30)$$
$$\partial \eta/\partial t := \delta H/\delta W = - < \nabla, \nabla \times S + F + +\nabla W > = - < \nabla, E > - \Delta W,$$

coincide completely with Maxwell equations (2.1) under conditions (2.16), describing true wave processes in vacuum, as well as the electromagnetic Maxwell equations, taking into account *a priori* both the imposed external boundary conditions (2.23) and the Lorentz condition (2.9), solving the problem mentioned in [4, 5]. Really, it is easy to obtain from (2.30) that

$$\partial^2 W/\partial t^2 - \nabla W = \rho, \quad \partial W/\partial t + < \nabla, A > = 0,$$
$$\nabla \times B = J + \partial E/\partial t, \quad \partial B/\partial t = -\nabla \times E, \quad (2.31)$$

Based now on (2.31) and (2.23) one can easlily calculate [7, 6] the magnetic wave equation

$$\partial^2 A/\partial t^2 - \nabla A = J, \quad (2.32)$$

supplementing the suitable wave equation on the scalar potential $W \in L$, finishing the calculations. Thus, we can formulate the following proposition.

Proposition 2.1. *The electromagnetic Maxwell equations (2.1) jointly with Lorentz condition (2.9) are equivalent to the Hamiltonian system (2.30) with respect to the canonical symplectic structure (2.28) and Hamiltonian function (2.29), which correspondingly reduce to electromagnetic equations (2.31) and (2.32) under external boundary conditions (2.23).*

The obtained above result can be, eventually, used for developing an alternative quantization procedure of Maxwell electromagnetic equations, being free of some quantum operator problems, discussed in detail in [4]. We hope to consider this aspect of quantization problem in a specially devoted study.

Remark 2.2. If one to consider a motion of a charged point particle under a Maxwell field, it is convenient to introduce a trivial fiber bundle structure *p:M→N*, such that $M = N \times G$, $N := D \subset \mathbb{R}^3$ and $G := \mathbb{R} \setminus \{0\}$, being the corresponding (abelian) structure Lie group. An analysis similar to the above gives rise to the reduced upon the space $\bar{\mathcal{M}}_{\xi} := l^{-1}(\xi)/G \simeq T^*(N)$, $\xi \in \mathcal{G}$, symplectic structure

$$\omega^{(2)}(q, p) = < dp, \wedge dq > + d < \mathcal{A}(q, g), \xi >_{\mathcal{G}}, \quad (2.33)$$

where $\mathcal{A}(q, g) := < A(q), dq > + g^{-1} dg$ is a suitable connection 1-form on phase space *M*, with $(q, p) \in T^*(N)$ and $g \in G$. The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be $\{q^i, q^i\} = 0$, $\{p_j, q^i\} = \delta^i_j$, $\{p_i, p_j\} = F_{ji}(q)$ for all $(q, p) \in T^*(N)$. If one introduces a new momentum variable $\tilde{p} := p + A(q)$ on $T^*(N) \ni (q, p)$, it is easy to verify that $\omega^{(2)}_{\xi} \to \tilde{\omega}^{(2)}_{\xi} := < d\tilde{p}, \wedge dq >$, giving rise to the following Poisson brackets [11, 13, 14]: $\{q^i, q^j\} = 0$, $\{\tilde{p}_j, q^i\} = \delta^i_j$, $\{\tilde{p}_i, \tilde{p}_j\} = 0$, where $i, j = \overline{1,3}$, iff for all $i, j, k = \overline{1,3}$ the standard Maxwell field equations are satisfied on *N*: $\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0$ with the curvature tensor $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$, $i, j = \overline{1,3}$, $q \in N$. Such a construction permits a natural generalization to the case of non-abelian structure Lie group yielding a description of Yang-Mills field equations within the reduction approach, to which we proceed below.

2.2 The Hamiltonian analysis of the Yang-Mills type dynamical systems

As above, we start with defining a phase space *M* of a particle under a Yang-Mills field in a region $D \in \mathbb{R}^3$ as $M := D \times G$, where *G* is a (not in general semisimple) Lie group, acting on *M* from the right. Over the space *M* one can define quite naturally a connection $\Gamma(\mathcal{A})$ to consider the following trivial principal fiber bundle $p: M \to N$, where $N := D$, with the structure group *G*. Namely, if $g \in G$, $q \in N$, then a connection 1-form on $M \ni (q, g)$ can be written down [3, 8, 9, 10] as

$$\mathcal{A}(q; g) := g^{-1}\left(d + \sum_{i=1}^{n} a_i A^{(i)}(q)\right) g, \quad (2.34)$$

where $\{a_i \in \mathcal{G} : i = \overline{1,n}\}$ is a basis of the Lie algebra $\mathcal{G}$ of the Lie group *G*, and $A_i : D \to \Lambda^1(D)$, $i = \overline{1,n}$, are the Yang-Mills fields on the physical space $D \in \mathbb{R}^3$.

Now one defines the natural left invariant Liouville form on *M* as $\alpha^{(1)}(q; g) = \langle p, dq \rangle + \langle y, g^{-1}dg \rangle_{\mathcal{G}}$, where $y \in T^*(G)$ and $\langle \cdot,\cdot \rangle_{\mathcal{G}}$ denotes as before the usual Ad-invariant non-degenerate bilinear form on $\mathcal{G}^* \times \mathcal{G}$, as evidently $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$. The main assumption we need to accept for further is that the connection 1-form is in accordance with the Lie group *G* action on *M*. The latter means that the condition $R_h^* \mathcal{A}(q; g) = Ad_{h^{-1}} \mathcal{A}(q; g)$ is satisfied for all $(q; g) \in M$ and $h \in G$, where $R_h : G \to G$ means the right translation by an element $h \in G$ on the Lie group *G*.

Having stated all preliminary conditions needed for the reduction to be applied to our model, suppose that the Lie group *G* canonical action on *M* is naturally lifted to that on the cotangent space $T^*(M)$ endowed due to (endowed owing to (2.3) with the following *G*-invariant canonical symplectic structure:

$$\omega^{(2)}(q, p; g, y) := d\, pr^* \alpha^{(1)}(q, p; g, y)$$
$$= \langle dp, \wedge dq \rangle + \quad (2.35)$$
$$+ \langle dy, \wedge g^{-1}dg \rangle_{\mathcal{G}} + \langle y dg^{-1}, \wedge dg \rangle_{\mathcal{G}}$$

for all $(q, p; g, y) \in T^*(M)$. Take now an element $\xi \in \mathcal{G}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is $Ad_h^* \xi = \xi$ for all $h \in G$. In the general case such an element $\xi \in \mathcal{G}^*$ can not exist but trivial $\xi = 0$, as it happens, for instance, in the case of the Lie group $G = SL_2(\mathbb{R})$. Then one can construct the reduced phase space $l^{-1}(\xi)/G$ symplectomorphic to $\left(T^*(N), \omega_\xi^{(2)}\right)$, where owing to (2.33) for any $(q, p) \in T^*(N)$

$$\omega_\xi^{(2)}(q, p) = \langle dp, \wedge dq \rangle + \langle \Omega^{(2)}(q), \xi \rangle_{\mathcal{G}} =$$
$$= \langle dp, \wedge dq \rangle + \sum_{s=1}^{n} \sum_{i,j=1}^{3} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j. \quad (2.36)$$

In the above we have expanded the element $\xi = \sum_{i=1}^{n} e_i a^i \in \mathcal{G}^*$ with respect to the bi-orthogonal basis $\{a^i \in \mathcal{G}^*, a_j \in \mathcal{G} : \langle a^i, a_j \rangle_{\mathcal{G}} = \delta^i_j,\ i, j = \overline{1,n}\}$, with $e_i \in \mathbb{R}$, $i = \overline{1,3}$, being some constants, and as well we denoted by $F_{ij}^{(s)}(q)$, $i, j = \overline{1,3}$, $s = \overline{1,n}$, the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$ components, that is $\Omega^{(2)}(q) := \sum_{s=1}^{n} \sum_{i,j=1}^{3} a_s F_{ij}^{(s)}(q) dq^i \wedge dq^j$ for any point $q \in N$. Summarizing calculations accomplished above, we can formulate the following result.

Theorem 2.3

Suppose the Yang-Mills field (2.34) on the fiber bundle p:M→N with M = D × G is invariant with respect to the Lie group G action $G \times M \to M$. *Suppose also that an element* $\xi \in \mathcal{G}^*$ *is chosen so that* $Ad_G^* \xi = \xi$. *Then for the naturally constructed momentum mapping* $l : T^*(M) \to \mathcal{G}^*$ *(being equivariant) the reduced phase space* $l^{-1}(\xi)/G \simeq T^*(N)$ *is endowed with the symplectic structure (2.36), having the following component-wise Poissoin brackets form:* $\{p_i, q^j\}_\xi = \delta_i^j$, $\{q^i, q^j\}_\xi = 0$, $\{p_i, p_j\}_\xi = \sum_{s=1}^{n} e_s F_{ji}^{(s)}(q)$ *for all* $i, j = \overline{1,3}$ *and* $(q, p) \in T^*(N)$.

The respectively extended Poisson bracket on the whole cotangent space $T^*(M)$ amounts owing to (2.10) into the following set of Poisson relationships:

$$\{y_s, y_k\}_\xi = \sum_{r=1}^{n} c_{sk}^r y_r, \quad \{p_i, q^j\}_\xi = \delta_i^j,$$
$$\{y_s, p_j\}_\xi = 0 = \{q^i, q^j\}, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^{n} y_s F_{ji}^{(s)}(q), \quad (2.37)$$

where $i, j = \overline{1,n}$, $c_{sk}^r \in \mathbb{R}$, $s, k, r = \overline{1,m}$, are the structure constants of the Lie algebra $\mathcal{G}$, and we made use of the expansion $A^{(s)}(q)=\sum_{j=1}^{n}A_j^{(s)}(q)dq^j$ as well we introduced alternative fixed values $e_i:=y_i,\ i=\overline{1,n}$. The result (2.37) can be seen easily if one to make a shift within the expression (2.35) as $\sigma^{(2)}\to\sigma_{ext}^{(2)}$, where $\sigma_{ext}^{(2)}:=\sigma^{(2)}\big|_{\mathcal{A}_0\to\mathcal{A}}$, $\mathcal{A}_0(g):=g^{-1}dg, g\in G$. Thereby one can obtain in virtue of the invariance properties of the connection $\Gamma(\mathcal{A})$ that

$$\begin{aligned}\sigma_{ext}^{(2)}(q,p;u,y)&=<dp,\wedge dq>+d<y(g),Ad_{g^{-1}}\mathcal{A}(q;e)>_{\mathcal{G}}=\\&=<dp,\wedge dq>+<dAd^*_{g^{-1}}y(g),\wedge\mathcal{A}(q;e)>_{\mathcal{G}}=\\&=<dp,\wedge dq>+\sum_{s=1}^{m}dy_s\wedge du^s+\sum_{j=1}^{n}\sum_{s=1}^{m}A_j^{(s)}(q)dy_s\wedge dq-\\&-<Ad^*_{g^{-1}}y(g),\mathcal{A}(q,e)\wedge\mathcal{A}(q,e)>_{\mathcal{G}}+\\&+\sum_{k\ge s=1}^{m}\sum_{l=1}^{m}y_lc_{sk}^{l}du^k\wedge du^s+\sum_{s=1}^{n}\sum_{i\ge j=1}^{3}y_sF_{ij}^{(s)}(q)dq^i\wedge dq^j,\end{aligned}\tag{2.38}$$

where coordinate points $(q,p;u,y)\in T^*(M)$ are defined as follows: $\mathcal{A}_0(e):=\sum_{s=1}^{m}du^ia_i$, $Ad^*_{g^{-1}}y(g)=y(e):=\sum_{s=1}^{m}y_sa^s$ for any element $g\in G$. Whence one gets right away the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables $\{u^s\in\mathbb{R}:s=\overline{1,m}\}\in\mathcal{G}^*$ and : $\{y_s\in\mathbb{R}:s=\overline{1,m}\}\in\mathcal{G}$

$$\begin{aligned}&\{y_s,u^k\}_\xi=\delta_s^k,\ \{u^k,q^j\}_\xi=0,\\&\{p_i,u^s\}_\xi=A_j^{(s)}(q),\ \{u^s,u^k\}_\xi=0,\end{aligned}\tag{2.39}$$

where $j=\overline{1,n},\ k,s=\overline{1,m}$ and $q\in N$.

Note here that the suggested above transition from the symplectic structure $\sigma^{(2)}$ on $T^*(N)$ to its extension $\sigma_{ext}^{(2)}$ on $T^*(M)$ just consists formally in adding to the symplectic structure $\sigma^{(2)}$ an exact part, which transforms it into equivalent one. Looking now at the expressions (2.38), one can infer immediately that an element $\xi:=\sum_{s=1}^{m}e_sa^s\in\mathcal{G}^*$ will be invariant with respect to the Ad^*-action of the Lie group G iff $\{y_s,y_k\}_\xi\big|_{y_s=e_s}=\sum_{r=1}^{m}c_{sk}^{r}e_r=0$ identically for all $s,k=\overline{1,m},\ j=\overline{1,n}$ and $q\in N$. In this and only this case the reduction scheme elaborated above will go through.

Returning attention to the expression (2.57), one can easily write down the following exact expression:

$$\omega_{ext}^{(2)}(q,p;u,y)=\omega^{(2)}\left(q,p+\sum_{s=1}^{n}y_sA^{(s)}(q);u,y\right),\tag{2.40}$$

on the phase space $T^*(M)\ni(q,p;u,y)$, where we abbreviated for brevity $<A^{(s)}(q),dq>$ as $\sum_{j=1}^{n}A_j^{(s)}(q)dq^j$. The transformation similar to (2.40) was discussed within somewhat different context in articles [11, 14] containing also a good background for the infinite dimensional generalization of symplectic structure techniques. Having observed from (2.40) that the simple change of variable $\tilde{p}:=p+\sum_{s=1}^{m}y_sA^{(s)}(q)$ of the cotangent space $T^*(N)$ recasts our symplectic structure (2.38) into the old canonical form (2.35), one obtains that the following new set of Poisson brackets on $T^*(M)\ni(q,\tilde{p};u,y)$

$$\begin{aligned}&\{y_s,y_k\}_\xi=\sum_{r=1}^{n}c_{sk}^{r}y_r,\quad\{\tilde{p}_i,\tilde{p}_j\}_\xi=0,\quad\{\tilde{p}_i,q^j\}=\delta_i^j,\\&\{y_s,q^j\}_\xi=0=\{q^i,q^j\}_\xi,\ \{u^s,u^k\}_\xi=0,\ \{y_s,\tilde{p}_j\}_\xi=0,\\&\{u^s,q^i\}_\xi=0,\ \{y_s,u^k\}_\xi=\delta_s^k,\ \{u^s,\tilde{p}_j\}_\xi=0,\end{aligned}$$

where $k,s=\overline{1,m}$ and $i,j=\overline{1,n}$, holds iff the nonabelian Yang-Mills type field equations $\partial F_{ij}^{(s)}/\partial q^l+\partial F_{jl}^{(s)}/\partial q^i+\partial F_{li}^{(s)}/\partial q^j+\sum_{k,r=1}^{m}c_{kr}^{s}\left(F_{ij}^{(k)}A_l^{(r)}+F_{jl}^{(k)}A_i^{(r)}+F_{li}^{(k)}A_j^{(r)}\right)=0$ are fulfilled for all $s=\overline{1,m}$ and $i,j,l=\overline{1,n}$ on the base manifold N. This effect of complete reduction of gauge Yang-Mills type variables from the symplectic structure (2.38) is known in literature [11] as the principle of minimal interaction and appeared to be useful enough for studying different interacting systems as in [12, 15]. We plan further to continue the study of the geometric properties of reduced symplectic structures connected with such interesting infinite-dimensional coupled dynamical systems of Yang-Mills-Vlasov, Yang-Mills-Bogolubov and Yang-Mills-Josephson types [12, 15] as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

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ЕЛЕКТРОМАГНІТНА ПРОБЛЕМА ЛОРЕНЦА ТА АНАЛІЗ ГАМІЛЬТОНОВОЇ СТРУКТУРИ ДИНАМІЧНИХ СИСТЕМ ТИПУ МАКСВЕЛА-ЯНГА-МІЛСА НА ОСНОВІ МЕТОДУ РЕДУКЦІЇ

Ґрунтуючись на аналізі редукованих геометричних структур на розшарованих многовидах, інваріантних щодо дії звичайної групи симетрії, побудовано симплектичні структури асоційовані з формами зв`язності на відповідних головних розшаруваннях. Пропонується застосування гамільтонового аналізу та редукції Марсдена-Вейнстейна до динамічних систем Максвела і Янга-Мілса. Дано опис електродинамічних вакуумних властивостей, проаналізовано геометричну природу обмежень Лоренца.

Ключові слова: *симплектична струтура, головне розшарування, обмеження Лоренца.*