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ORIGINAL ARTICLE

Comments on 'On Hadamard powers of polynomials'

Stanisław Białas[1] · **Leokadia Białas-Cież**[2]

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Abstract Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real positive coefficients and $p \in \mathbb{R}$. The pth Hadamard power of f is the polynomial $f^{[p]}(x) := a_n^p x^n + a_{n-1}^p x^{n-1} + \cdots + a_1^p x + a_0^p$. We give sufficient conditions for $f^{[p]}$ to be a Hurwitz polynomial (i.e., to be a stable polynomial) for all $p > p_0$ or $p < p_1$ with some positive p_0 and negative p_1 (without any assumption about stability of f). Theorem 5 by Gregor and Tišer (Math Control Signals Syst 11:372–378, 1998) asserts that if f is a stable polynomial with positive coefficients then $f^{[p]}$ is stable for every $p \geq 1$. We construct a counterexample to this statement.

Keywords Hadamard powers of polynomials · Hurwitz matrix · Stability of polynomials

Mathematics Subject Classification Primary 11C08 · Secondary 26C10

1 Introduction

For a positive integer number n we consider

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \text{with} \quad a_0, \ldots, a_n > 0. \tag{1}$$

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✉ Leokadia Białas-Cież
leokadia.bialas-ciez@uj.edu.pl

1 The School of Banking and Management, Armii Krajowej 4, 30-150 Kraków, Poland

2 Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

Let $\mathbb{R}^+[n]$ be the family of all polynomials of the form (1). The polynomial

$$f^{[p]}(x) := a_n^p x^n + a_{n-1}^p x^{n-1} + \cdots + a_1^p x + a_0^p \tag{2}$$

where $p \in \mathbb{R}$ is called the pth *Hadamard power* of $f \in \mathbb{R}^+[n]$. We say that the polynomial f with real coefficients is *stable* (f is a *Hurwitz polynomial*) if every zero of f has strictly negative real part. A necessary condition for a polynomial f with real coefficients to be stable is that f has all coefficients of the same sign. Let H_n be the family of all stable polynomials of degree n with positive coefficients.

In 1996 J.Garloff and D.G.Wagner proved in [1] that $f \in H_n$ implies $f^{[p]} \in H_n$ for all $p \in \{1, 2, 3, \ldots\}$. The natural question arises of a set of real numbers p for which $f^{[p]}$ is stable where $f \in \mathbb{R}^+[n]$. We give some conditions on p and on f for $f^{[p]}$ to belong to H_n. Moreover, we show that $f^{[p]}$ does not need to be stable for a stable polynomial f and an exponent $p > 1$, contrary to Theorem 5 in [2].

Observe that if $n = 1$ or $n = 2$ then $f^{[p]}$ is stable for every $p \in \mathbb{R}$ and for all polynomials $f \in \mathbb{R}^+[n]$. The case of $n \geq 3$ is much more complicated, e.g., for $f(x) = x^3 + x^2 + x + 1$ we have $f^{[p]} \notin H_n$ for any $p \in \mathbb{R}$. Therefore, we will consider only the case $n \geq 3$.

1.1 Basic information

For relevant background material concerning Hurwitz polynomials and related topics see [5, Sec.11]. We list below selected theorems that will be useful in the paper.

The *Hurwitz matrix* $H(f)$ associated to the polynomial $f \in \mathbb{R}^+[n]$ is given as follows

$$H(f) := \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ 0 & a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & 0 & a_{n-1} & a_{n-3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Denote by $D_i(p)$ for $i = 1, \ldots, n$ the ith leading principal minor of the Hurwitz matrix $H(f^{[p]})$, i.e.,

$$D_1(p) = a_{n-1}^p, \quad D_2(p) = \det \begin{bmatrix} a_{n-1}^p & a_{n-3}^p \\ a_n^p & a_{n-2}^p \end{bmatrix}, \ \ldots\ , \ D_n(p) = \det H(f^{[p]}).$$

To simplify the writing, we put $D_i := D_i(1)$.

Theorem 1 Routh–Hurwitz criterion
If $f \in \mathbb{R}^+[n]$ then $f \in H_n$ if and only if $D_i > 0$ for all $i = 1, \ldots, n$.

Theorem 2 (see [3, Th.2 and (1.10)])
If $f \in H_n$ with $n \geq 3$ then

$$\det \begin{bmatrix} a_{n-i} & a_{n-i-2} \\ a_{n-i+1} & a_{n-i-1} \end{bmatrix} > 0 \quad \text{for all} \quad i = 1, \ldots, n-2.$$

Theorem 3 (see [4])
Let $f \in \mathbb{R}^+[n]$ with $n \geq 5$ and γ be the unique real root of the equation

$$\gamma(\gamma + 1)^2 = 1.$$

If $\gamma\, a_{n-i}\, a_{n-i-1} > a_{n-i+1}\, a_{n-i-2}$ for every $i = 1, \ldots, n-2$ then $f \in H_n$.

1.2 Counterexample

Theorem 5 in [2] asserts that if $f \in H_n$ then $f^{[p]} \in H_n$ for all $p \geq 1$. We construct below a counterexample to this statement.

For a fixed polynomial $f \in \mathbb{R}^+[n]$ with $n \geq 3$ consider the following decomposition

$$f(x) = g(x^2) + x\, h(x^2), \text{ where } g \text{ and } h \text{ are polynomials of positive coefficients} \quad (3)$$

It may be worth reminding the reader that g and h are called *interlacing* if

- all zeros of g and h are real, negative and distinct,
- between every two zeros of g there exists a zero of h and vice versa.

Among variants of Hermite–Biehler theorem we will apply the following one to construct a counterexample.

Theorem 4 (see [5, Chapter 6.3]) *Every polynomial $f \in \mathbb{R}^+[n]$ decomposed as in (3) is stable if and only if g and h are interlacing.*

Counterexample 1 Let

$$g(t) = t^4 + 46\, t^3 + 791\, t^2 + 6026\, t + 17160 = (t+10)(t+11)(t+12)(t+13).$$

Y. Wang and B. Zhang considered g in [6] and observed that for $p = 1.139$ the polynomial $g^{[p]}$ has two nonreal zeros: $-16.0617 \pm 0.178468\, i$ (approximated value). Take now

$$h(t) = t^3 + 34.5\, t^2 + 395.75\, t + 1509.375 = (t+10.5)(t+11.5)(t+12.5)$$

and put

$$\begin{aligned}f(x) &= g(x^2) + xh(x^2)\\ &= x^8 + x^7 + 46\,x^6 + 34.5\,x^5 + 791\,x^4 + 395.75\,x^3\\ &\quad +6026\,x^2 + 1509.375\,x + 17160.\end{aligned}$$

It is easy to verify that f is stable (e.g., by the Routh–Hurwitz criterion). We have $f^{[p]}(x) = g^{[p]}(x^2) + x\,h^{[p]}(x^2)$ and thus, by Theorem 4 the polynomial $f^{[p]}$ is not stable for $p = 1.139$. By means of Wolfram Mathematica 10.4 we found two zeros of $f^{[1.139]}$ that have positive real part: $0.00179025 \pm 4.01279\,i$ (approximated value).

2 Main results

Now we will state and prove some sufficient conditions for $f^{[p]}$ to be a Hurwitz polynomial for all $p > p_0$ or $p < p_1$ with some positive p_0 and negative p_1 depending only on coefficients of f. The polynomial f is assumed to be of the form (1) but need not to be stable. We will discuss separately three cases: $n = 3$, $n = 4$ and $n \geq 5$. We start with a lemma and some necessary conditions for the Hurwitz stability.

2.1 Notations and preliminary results

For a fixed polynomial $f \in \mathbb{R}^+[n]$ with $n \geq 3$ and $p \in \mathbb{R}$ we put

$$w_i(p) := a^p_{n-i-1}\,a^p_{n-i} - a^p_{n-i-2}\,a^p_{n-i+1}, \qquad i = 1, \ldots, n-2. \tag{4}$$

Moreover, for ease of notation, throughout the paper we write w_i for $w_i(1)$. Let

$$\begin{aligned}\overline{d} &:= \max_{1\leq i\leq n-2} \frac{a_{n-i-2}a_{n-i+1}}{a_{n-i-1}a_{n-i}},\\ \underline{d} &:= \min_{1\leq i\leq n-2} \frac{a_{n-i-2}a_{n-i+1}}{a_{n-i-1}a_{n-i}}.\end{aligned}$$

It is worth noticing that

- if $w_i > 0$ for all i then $\overline{d} < 1$,
- if $w_i < 0$ for all i then $\underline{d} > 1$.

Lemma 1 *Let* $\lambda \in (0, 1)$ *and* $f \in \mathbb{R}^+[n]$ *with* $n \geq 3$. *Put*

$$p_0 := \frac{\log \lambda}{\log \overline{d}}, \qquad p_1 := \frac{\log \lambda}{\log \underline{d}}.$$

1. If $w_i > 0$ *for all* $i = 1, \ldots, n-2$

$$\textit{then} \quad \lambda a^p_{n-i-1}\,a^p_{n-i} - a^p_{n-i-2}\,a^p_{n-i+1} > 0 \quad \textit{for all}$$
$$i = 1, \ldots, n-2 \quad \textit{and} \quad p > p_0 > 0.$$

2. *If* $w_i < 0$ *for all* $i = 1, \dots, n-2$

$$\begin{aligned} &then \quad \lambda a^p_{n-i-1}\, a^p_{n-i} - a^p_{n-i-2}\, a^p_{n-i+1} > 0 \quad for\ all \\ &i = 1, \dots, n-2 \quad and \quad p < p_1 < 0. \end{aligned}$$

Proof Firstly we show statement 1. Since $w_i > 0$ for all i, it follows that $\overline{d} < 1$ and hence for a fixed $p > p_0$ we have $\overline{d}^p < \lambda$. From the definition of $\overline{d}$ we can easily conclude that $\overline{d} a_{n-i-1} a_{n-i} \geq a_{n-i-2} a_{n-i+1}$ for all i and so

$$0 \leq \overline{d}^p a^p_{n-i-1} a^p_{n-i} - a^p_{n-i-2} a^p_{n-i+1} < \lambda a^p_{n-i-1} a^p_{n-i} - a^p_{n-i-2} a^p_{n-i+1}.$$

In an analogous manner we can prove statement 2. Indeed, in this case we have $\underline{d} > 1$ and $\underline{d}^p < \lambda$ for $p < p_1$. From the definition of $\underline{d}$ we get $\underline{d}\, a_{n-i-1}\, a_{n-i} \leq a_{n-i-2}\, a_{n-i+1}$ for all i. Hence

$$0 \leq \underline{d}^p\, a^p_{n-i-1}\, a^p_{n-i} - a^p_{n-i-2}\, a^p_{n-i+1} < \lambda\, a^p_{n-i-1}\, a^p_{n-i} - a^p_{n-i-2}\, a^p_{n-i+1}$$

and the proof is completed. □

We give below some sufficient conditions for $f^{[p]}$ not to be stable. This is a direct consequence of Theorem 2.

Theorem 5 *Let* $f \in \mathbb{R}^+[n]$ *with* $n \geq 3$.

1. *If* $w_i \geq 0$ *for some* $i \in \{1, \dots, n-2\}$ *then* $f^{[p]} \notin H_n$ *for all* $p \leq 0$.
2. *If* $w_i \leq 0$ *for some* $i \in \{1, \dots, n-2\}$ *then* $f^{[p]} \notin H_n$ *for all* $p \geq 0$.

2.2 Case $n = 3$

In this subsection we consider $f(x) = a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0$ with positive coefficients $a_3,\ a_2,\ a_1,\ a_0$. For $n = 3$ the family of w_i's [see (4)] is reduced to the unique element $w_1 = a_1\, a_2 - a_0\, a_3$.

Theorem 6 *For any polynomial* $f \in \mathbb{R}^+[n]$ *with* $n = 3$ *we have*

1. *If* $w_1 > 0$ *then* $f^{[p]} \in H_3$ *for all* $p > 0$.
2. *If* $w_1 < 0$ *then* $f^{[p]} \in H_3$ *for all* $p < 0$.

Proof In order to prove statement 1, we observe that $w_1 > 0$ implies $w_1(p) = a_1^p\, a_2^p - a_0^p\, a_3^p > 0$ for every $p > 0$. By the Routh–Hurwitz criterion we get the stability of $f^{[p]}$ for $p > 0$, because

$$D_1(p) = a_2^p > 0, \quad D_2(p) = w_1(p) \quad \text{and} \quad D_3(p) = a_0^p\, w_1(p).$$

In an analogous manner we can prove statement 2. □

2.3 Case $n = 4$

We start this subsection with a simple characterization of stable polynomials of degree 4 with positive coefficients.

Proposition 7 *Let $f \in \mathbb{R}^+[n]$ with $n = 4$. The polynomial f is stable if and only if*

$$\frac{a_1\, a_4}{a_2\, a_3} + \frac{a_0\, a_3}{a_1\, a_2} < 1. \tag{5}$$

Proof It is easily computed that

$$D_1 = a_3, \quad D_2 = a_2\, a_3 - a_1\, a_4, \quad D_3 = a_1\, a_2\, a_3 - a_0\, a_3^2 - a_1^2\, a_4, \quad D_4 = a_0\, D_3.$$

By the Routh–Hurwitz criterion, $f \in H_4$ implies $D_3 > 0$, i.e.,

$$a_1 a_2 a_3 > a_0 a_3^2 + a_1^2 a_4.$$

Dividing by $a_1\, a_2\, a_3$ we obtain inequality (5).

For the reverse implication, we can conclude from (5) that

$$\frac{a_1\, a_4}{a_2\, a_3} < 1$$

and hence $D_2 > 0$. Moreover, an immediate consequence of (5) is $D_3 > 0$, and so $D_4 > 0$. Once again we use the Routh–Hurwitz criterion and get the stability of f. □

Note that for $n = 4$ and any function $f(x) = a_4\, x^4 + a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0$ we have only two w_i's defined by (4):

$$w_1 = a_2 a_3 - a_1 a_4, \qquad w_2 = a_1 a_2 - a_0 a_3$$

and

$$\overline{d} := \max\left\{\frac{a_1 a_4}{a_2 a_3}, \frac{a_0 a_3}{a_1 a_2}\right\} \qquad \underline{d} := \min\left\{\frac{a_1 a_4}{a_2 a_3}, \frac{a_0 a_3}{a_1 a_2}\right\}.$$

It is worth recalling from the beginning of Sect. 2.1 that for f with positive coefficients we have $\overline{d} < 1$ if all w_i's are positive and $\underline{d} > 1$ whenever all w_i's are negative.

Theorem 8 *Let $f \in \mathbb{R}^+[n]$ with $n = 4$ and*

$$p_0 := \frac{\log 0.5}{\log \overline{d}} \qquad p_1 := \frac{\log 0.5}{\log \underline{d}}.$$

1. *If $w_1, w_2 > 0$ then $f^{[p]} \in H_4$ for all $p > p_0 > 0$.*
2. *If $w_1, w_2 < 0$ then $f^{[p]} \in H_4$ for all $p < p_1 < 0$.*

Moreover, the constants p_0 and p_1 are the best possible, i.e., for p_0 it means that there exists a polynomial f of degree 4 with positive coefficients and $w_1, w_2 > 0$ such that $f^{[p]}$ is not stable for every $p \le p_0$.

Proof For the proof of statement 1, we use Lemma 1. For $\lambda = 1/2$ and $p > p_0$ we have

$$\frac{1}{2}\,a_2^p\,a_3^p - a_1^p\,a_4^p \;>\; 0, \qquad \frac{1}{2}\,a_1^p\,a_2^p - a_0^p\,a_3^p \;>\; 0.$$

Consequently,

$$\frac{a_1^p\,a_4^p}{a_2^p\,a_3^p} < \frac{1}{2} \quad \text{and} \quad \frac{a_0^p\,a_3^p}{a_1^p\,a_2^p} < \frac{1}{2}$$

and therefore,

$$\frac{a_1^p\,a_4^p}{a_2^p\,a_3^p} + \frac{a_0^p\,a_3^p}{a_1^p\,a_2^p} \;<\; 1. \tag{6}$$

By Proposition 7 we get the stability of $f^{[p]}$ for $p > p_0$. Statement 2 can be proved in an analogous fashion.

By Example 2 given below we show that the constants p_0 and p_1 cannot be improved. □

Example 2 Consider the polynomial

$$f(x) = 2x^4 + x^3 + 5x^2 + x + 2.$$

In this case we have

$$w_1 = 5 \cdot 1 - 1 \cdot 2 = 3 > 0, \qquad w_2 = 1 \cdot 5 - 2 \cdot 1 = 3 > 0$$

and

$$\overline{d} = \max\left\{\frac{2}{5}, \frac{2}{5}\right\} = 0.4, \qquad p_0 = \frac{\log 0.5}{\log 0.4}.$$

Fix $p \le p_0$. By Proposition 7, $f^{[p]} \in H_4$ if and only if inequality (6) holds. We calculate

$$\begin{aligned}\frac{a_1^p\,a_4^p}{a_2^p\,a_3^p} + \frac{a_0^p\,a_3^p}{a_1^p\,a_2^p} &= \left(\frac{a_1\,a_4}{a_2\,a_3}\right)^p + \left(\frac{a_0\,a_3}{a_1\,a_2}\right)^p = (0.4)^p + (0.4)^p \\ &= 2 \cdot (0.4)^p \ge 2 \cdot (0.4)^{p_0} = 2 \cdot 0.5 = 1.\end{aligned}$$

We see that inequality (6) does not hold and consequently $f^{[p]}$ is not stable. Additionally, we can easily verify by Proposition 7 that polynomial f is stable.

Corollary 9 *If $f \in H_4$ then $f^{[p]} \in H_4$ for all $p \geq 1$.*

Proof Since $(t^p + s^p)^{1/p} \leq t + s$ for all $s, t \geq 0$ and $p \geq 1$, we have

$$\frac{a_1^p\, a_4^p}{a_2^p\, a_3^p} + \frac{a_0^p\, a_3^p}{a_1^p\, a_2^p} \leq \left(\frac{a_1\, a_4}{a_2\, a_3} + \frac{a_0\, a_3}{a_1\, a_2}\right)^p < 1$$

the last estimate being a consequence of the stability of f and Proposition 7. Once again we use Proposition 7 and we get the stability of $f^{[p]}$. □

2.4 Case $n \geq 5$

The main result of this subsection will be based on Theorem 3 that deals with $n \geq 5$. We remind the reader that γ denotes the unique real root of the equation $\gamma(\gamma+1)^2 = 1$. One can verify that $\gamma \in (0.4655, 0.466)$. Quantities $w_1, \ldots, w_{n-2}$ and $\overline{d}, \underline{d}$ have been defined in the beginning of Sect. 2.1.

Theorem 10 *Let $f \in \mathbb{R}^+[n]$ with $n \geq 5$ and*

$$p_0 := \frac{\log \gamma}{\log \overline{d}} \qquad p_1 := \frac{\log \gamma}{\log \underline{d}}.$$

1. *If $w_1, \ldots, w_{n-2} > 0$ then $f^{[p]} \in H_n$ for all $p > p_0 > 0$.*
2. *If $w_1, \ldots, w_{n-2} < 0$ then $f^{[p]} \in H_n$ for all $p < p_1 < 0$.*

Proof Take $p > p_0$ in the case of $w_1, \ldots, w_{n-2} > 0$ or $p < p_1$ in the case $w_1, \ldots, w_{n-2} < 0$. In both cases, by Lemma 1 used for $\lambda = \gamma$, we have $\gamma a_{n-i-1}^p\, a_{n-i}^p - a_{n-i-2}^p\, a_{n-i+1}^p > 0$ for all $i = 1, \ldots, n-2$. Thanks to Theorem 3 we obtain the stability of $f^{[p]}$ and the proof is completed. □

Let us observe that p_0 and p_1 in Theorem 10 are not far from being optimal as evidenced in the next example.

Example 3 Consider the polynomial

$$f(x) = x^5 + 5x^4 + 2x^3 + 5x^2 + x + 1.$$

We have

$$w_1 = a_3\, a_4 - a_2\, a_5 = 5 > 0, \qquad w_2 = a_2\, a_3 - a_1\, a_4 = 5 > 0,$$
$$w_3 = a_1\, a_2 - a_0\, a_3 = 3 > 0$$

and

$$\overline{d} = \max\left\{\frac{a_2\, a_5}{a_3\, a_4}, \frac{a_1\, a_4}{a_2\, a_3}, \frac{a_0\, a_3}{a_1\, a_2}\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}, \frac{2}{5}\right\} = \frac{1}{2}.$$

The Hurwitz matrix $H(f)$ associated to f is

$$H(f) = \begin{bmatrix} 5 & 5 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 5 & 5 & 1 \end{bmatrix}.$$

The leading principal minors are

$$D_1 = 5, \quad D_2 = 5, \quad D_3 = 5, \quad D_4 = -1 < 0, \quad D_5 = -1 < 0$$

and therefore, by the Routh–Hurwitz criterion, f is not stable.

Now take $p \in \mathbb{R}$ and compute the 4th leading principal minors of $H(f^{[p]})$:

$$D_4(p) = 50^p + 5^p - 25^p - 25^p - 20^p - 1 + 5^p + 10^p.$$

If we take p close to 1 then $f^{[p]}$ is not stable because of the continuity of exponential functions and since $D_4(1) < 0$.

On the other hand, by Theorem 10, $f^{[p]}$ is stable for all $p \geq 1.1032$ as

$$p_0 = \frac{\log \gamma}{\log \overline{d}} = \frac{\log \gamma}{\log 0.5} < \frac{-\log 0.4655}{\log 2} \approx 1.10315 < 1.1032.$$

We conclude that the quantity p_0 given in Theorem 10 is close to the value, where the stability of $f^{[p]}$ changes.

The above example shows also that Theorem 8 proved for $n = 4$ cannot be applied for polynomials of degree 5, because by Theorem 8 we get $f^{[p]} \in H_n$ for all $p > \frac{\log 0.5}{\log \overline{d}}$. However, for the polynomial f considered in Example 3 we have $\frac{\log 0.5}{\log \overline{d}} = 1$ and we see that $f^{[p]}$ is not stable for p close to 1.

We can show by the next example that the constant γ in Theorem 3 is close to the optimal one.

Example 4 Let

$$f(x) = x^5 + 5x^4 + \left(3 - \frac{2}{\sqrt{5}}\right) x^3 + 5x^2 + x + 1.$$

Observe that f has all positive coefficients and for

$$\lambda = 0.475 > \frac{1}{3 - \frac{2}{\sqrt{5}}} \approx 0.47493$$

that is close to $\gamma \in (0.4655, 0.466)$, we have

$$\lambda a_3 \, a_4 - a_2 \, a_5 = \lambda \left(3 - \frac{2}{\sqrt{5}}\right) \cdot 5 - 5 > 0,$$

$$\lambda a_2\, a_3 - a_1\, a_4 = \lambda \cdot 5 \left(3 - \tfrac{2}{\sqrt{5}}\right) - 5 > 0,$$

$$\lambda a_1\, a_2 - a_0\, a_3 = \lambda \cdot 5 - \left(3 - \tfrac{2}{\sqrt{5}}\right) > \frac{5}{\left(3 - \tfrac{2}{\sqrt{5}}\right)}$$

$$- \left(3 - \tfrac{2}{\sqrt{5}}\right) = \frac{12}{5\left(3 - \tfrac{2}{\sqrt{5}}\right)} (\sqrt{5} - 2) > 0.$$

By Theorem 3 analogous inequalities satisfied for γ (instead of λ) imply the Hurwitz stability of f. However, in the considered case we get

$$D_4 = \det \begin{bmatrix} 5 & 5 & 1 & 0 \\ 1 & 3 - \frac{2}{\sqrt{5}} & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 1 & 3 - \frac{2}{\sqrt{5}} & 1 \end{bmatrix} = 0$$

and therefore, by the Routh–Hurwitz criterion f is not stable.

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