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Pseudospectra and structured pseudospectra are subsets of the complex plane which give a geometric representation, via eigenvalues, of the effects of perturbations to a matrix. We survey the historical development of the subject, and the definitions and characterizations of the various sets of pseudospectra. Motivated by the fact that a nonnormal matrix in the 2-norm can become normal in a different norm, we describe a numerical investigation into the question of characterizing which perturbations have the greatest effect on the eigenvalues of the matrix.

PSEUDOSPECTRA AND STRUCTURED PSEUDOSPECTRA

by

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Approved by

____
Committee Chair

This thesis is dedicated to my family and friends, the faculty of the University of North Carolina at Greensboro mathematics department, and to the Brevard College mathematics department.

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION AND DEFINITIONS

I.I Introduction

The research in this thesis is centralized in the study of error in mathematical models of physical phenomena in the sciences and engineering. In this context, error refers to the difference between a value computed with the mathematical model and the corresponding actual value in the real-world phenomenon being modeled. There are many sources of error in mathematical models, such as inaccuracy in measurement of data, simplifying assumptions in the modeling process, round-off error in computer calculations, approximation error in numerical methods, etc. Therefore for any mathematical model it is important to understand the effect of small perturbations. For linear mathematical models, particularly linear differential equations and difference equations, an important issue is the effect of perturbations on the stability of solutions. The stability of solutions is in turn determined by the location in the complex plane of the eigenvalues of the relevant system matrix. This observation motivates the issues investigated in this thesis, namely, the effects of small perturbations on the location of eigenvalues of a matrix. The study of such issues lies within the general area of perturbation theory, and leads naturally to the notion of pseudospectra. In the remainder of this introductory chapter, notations and definitions will be given while a more explicit understanding of the motivation of this paper is shown.

I.II Notation and Motivation

The set of all complex numbers will be denoted by $\mathbb{C}$, the set of all vectors of length n with complex entries by $\mathbb{C}^n$, and the set of all matrices of size $m \times n$ with complex entries by $\mathbb{C}^{m\times n}$. For notational purposes, all matrices will be denoted by a bold uppercase English alphabet letter, all vectors will be denoted by a lowercase English alphabet letter, and all complex scalars will be denoted by a lowercase Greek alphabet letter. All of the definitions and results in the thesis generalize to linear operators on finite dimensional vector spaces. The identity matrix in $\mathbb{C}^{n\times n}$ will be denoted by $\mathbf{I}$ unless otherwise defined. For a matrix $\mathbf{A}$, the range is denoted by $\mathcal{R}(\mathbf{A})$, and the null space by $\mathcal{N}(\mathbf{A})$. The orthogonal complement of a subspace Z of $\mathbb{C}^n$ will be denoted by $Z^\perp$. The norm of a vector in $\mathbb{C}^n$ will be denoted by $\|\cdot\|$. Any norm on $\mathbb{C}^n$ defines an induced matrix norm on a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ by $\|\mathbf{A}\| = \max\limits_{x\neq 0} \frac{\|\mathbf{A}x\|}{\|x\|} = \max\limits_{\|x\|=1} \|\mathbf{A}x\|$ [5, pp.56].

Definition 1. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. A non-zero vector v is an <u>eigenvector</u> of the matrix $\mathbf{A}$ corresponding to the <u>eigenvalue</u> λ if $\mathbf{A}v = \lambda v$. The set of all eigenvalues of $\mathbf{A}$ is called the <u>spectrum</u> of $\mathbf{A}$, and will be denoted by $\Lambda(\mathbf{A})$ [5, pp.332-333].

Thus $\Lambda(\mathbf{A}) \subset \mathbb{C}$, and it is known that there are at most n distinct eigenvalues in $\Lambda(\mathbf{A})$ [5, pp.332]. The Euclidean inner product on $\mathbb{C}^n$ is defined by $(x, y) = \sum_{j=1}^{n} x_j \overline{y_j} = y^* x$ where y^* denotes the conjugate transpose of y [18, pp.17]. The vector 2-norm on a vector $x \in \mathbb{C}^n$ defined by the Euclidean inner product is $\|x\|_2^2 = (x, x) = \sum_{j=1}^{n} |x_j|^2$ [18, pp.9,17]. It is known that the induced matrix 2-norm $\|\mathbf{A}\|_2$ is equal to the largest singular value of $\mathbf{A}$ [5, pp.71].

One motivation for this paper is the issue of the stability of linear differential equations. An important mathematical model is a system of first order linear differential equations, which can be formulated as the matrix ordinary differential equation

$$\frac{d}{dt}x(t) = \mathbf{A}x(t),\ x(0) = x_0 \in \mathbb{C}^n. \tag{1}$$

A solution is an $n \times 1$ vector-valued function $x(t)$. The system (1) is said to be exponentially stable if there exist $M \geq 1$ and $\alpha > 0$ such that for any $x_0 \in \mathbb{C}^n$, $\|x(t)\| \leq M\mathrm{e}^{-\alpha t}\|x(0)\|$ for all $t \geq 0$. In this case the solutions satisfy $\lim_{t\to\infty} \|x(t)\| = 0$. It is known that (1) is exponentially stable if and only if the spectrum lies in the left half complex plane. In this case we also say that $\mathbf{A}$ is stable. Because of the sources of error in the mathematical model, it is often desirable to understand how small perturbations to $\mathbf{A}$ affect its stability. For a system like (1), this means understanding whether perturbations to $\mathbf{A}$ have eigenvalues in the right half of the the complex plane. Figure 1 illustrates the possibilities for the built-in MATLAB [14] gallery matrix grcar. The grcar matrices are $n \times n$ matrices, $n \geq 4$, with 1's down the main diagonal, -1's down the first subdiagonal, 1's on the first three superdiagonals and zero's as all other elements. Figure 1 shows the eigenvalues of a matrix $\mathbf{A}$ denoted by $\boxdot$ which are all in the left half complex plane, and the eigenvalues of the matrix $\mathbf{A} + \mathbf{E}$ denoted by $\odot$. With $\|\mathbf{E}\|_2 = 0.01$, $\mathbf{A} + \mathbf{E}$ has eigenvalues in the right half complex plane.

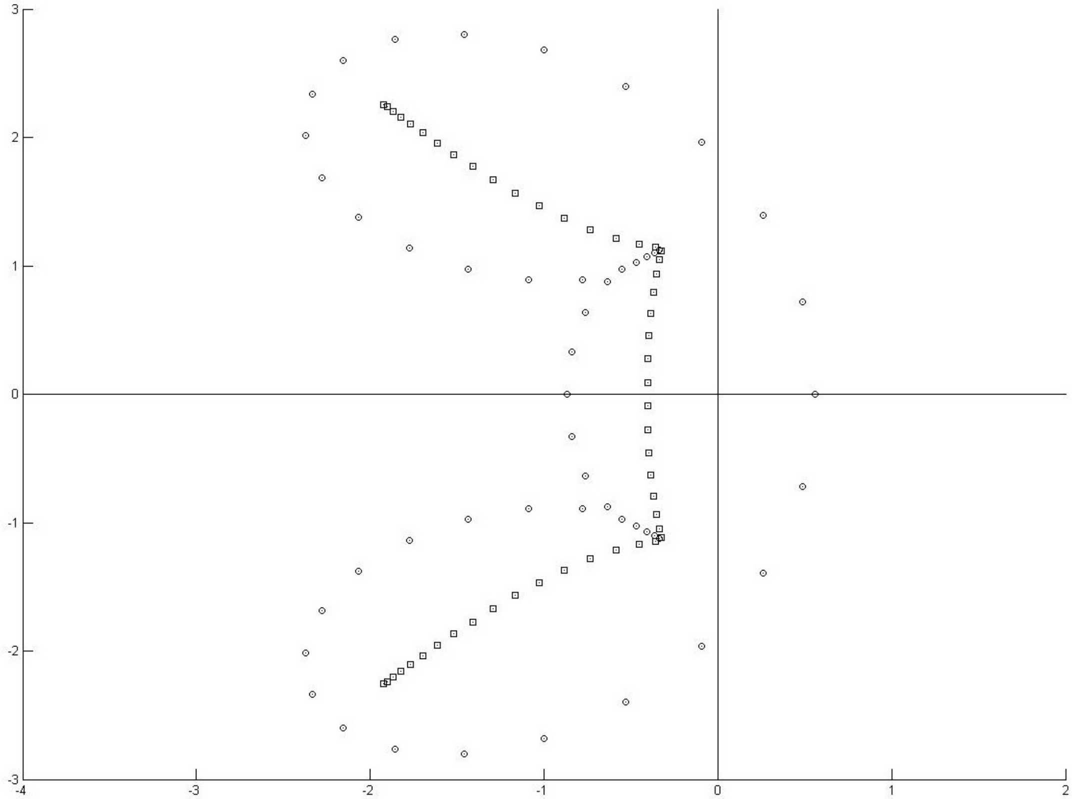

Figure 1. Matrix showing how perturbations can cause instability

Another motivating example is the stability of linear difference equations or recursion equations. Given an initial vector $x_0 \in \mathbb{C}^n$ a recursive sequence can be defined as

$$x_{j+1} = \mathbf{A}x_j. \tag{2}$$

The system is said to be stable if the solutions satisfy $\lim_{n\to\infty} \|x_n\| = 0$. It is known that (2) is stable if and only if the spectrum of $\mathbf{A}$ lies inside the unit circle. Just as with (1), for (2) we may ask if small perturbations to $\mathbf{A}$ can have eigenvalues outside the unit circle in $\mathbb{C}$. Figure 2 illustrates the possibilities plotting the eigenvalues of a matrix

$\mathbf{A}$ denoted by $\boxdot$ which are all in the left half complex plane, and the eigenvalues of the matrix $\mathbf{A}+\mathbf{E}$ denoted by $\odot$. With $\|\mathbf{E}\|_2 = 0.01$, $\mathbf{A}+\mathbf{E}$ has eigenvalues outside the unit circle.

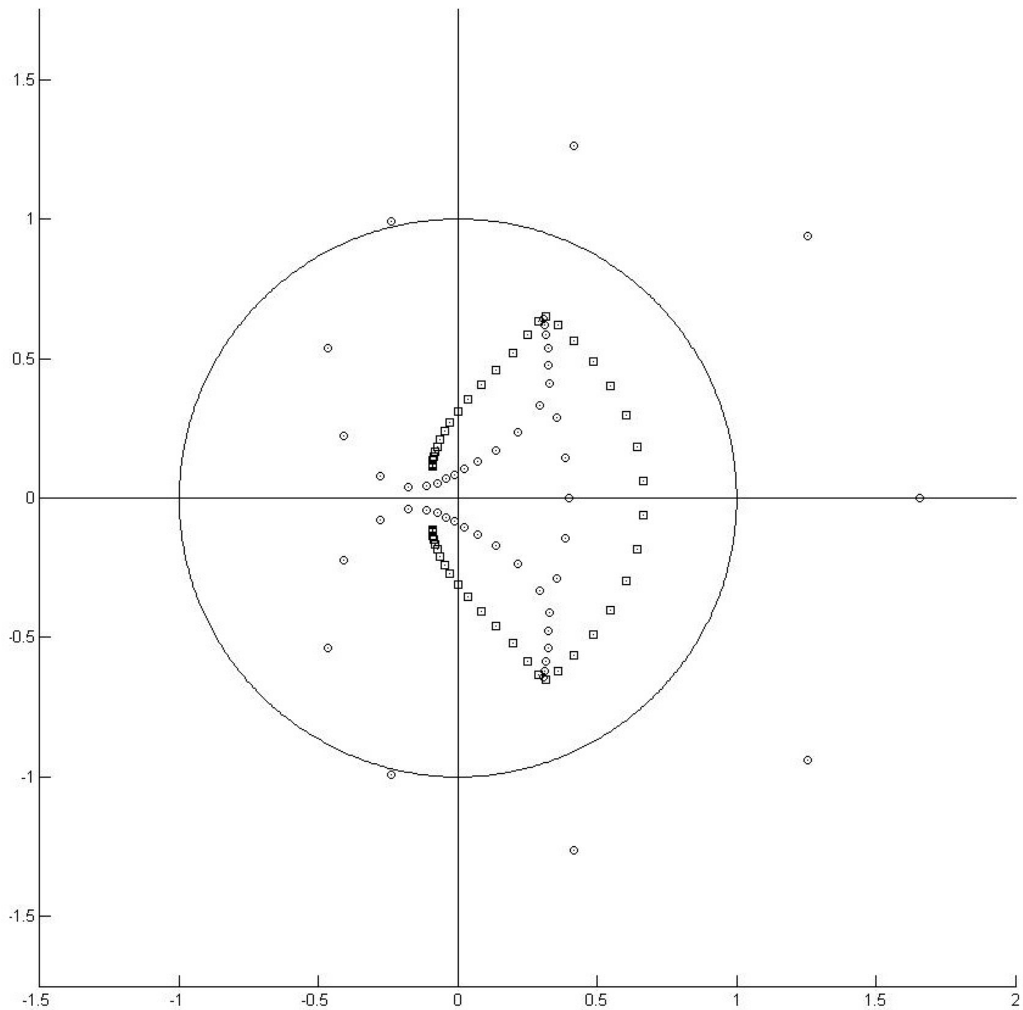

Figure 2. Matrix showing how perturbations can cause instability

I.III Pseudospectra

The notion of pseudospectra is used to describe the set of eigenvalues which arise from small perturbations to a matrix.

Definition 2. Let $\epsilon > 0$ be arbitrary. The ϵ-pseudospectrum of $\mathbf{A}$, denoted by $\mathcal{T}(\mathbf{A};\epsilon)$ [18, pp.14], is the set of $\lambda \in \mathbb{C}$ such that $\lambda \in \Lambda(\mathbf{A}+\mathbf{E})$ for some $\mathbf{E} \in \mathbb{C}^{n\times n}$ with $\|\mathbf{E}\| \leq \epsilon$:

$$\mathcal{T}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{E}) \text{ for some } \mathbf{E} \text{ with } \|\mathbf{E}\| \leq \epsilon\}. \tag{3}$$

It should be noted that the ϵ-pseudospectrum depends on the choice of norm, even though this is not explicitly indicated in the notation. In 1967, J.M. Varah introduced the idea of ϵ-pseudoeigenvalue in his thesis where he defined r-approximate eigenvalues [21]. His motivation came from his analysis of the accuracy of eigenpairs produced by computer implementations of the inverse iteration algorithm [18]. In between Varah's thesis [21] and his 1979 paper [22] where he defines the term ϵ-spectrum, H.J. Landau introduced ϵ-pseudoeigenvalues under the term ϵ-approximate eigenvalues [13]. Though for Varah in [21] the first definition of pseudospectra is said to be defined, the first published definition of pseudospectra is thought to be by Landau in [13]. Throughout the 1980s, following their work in in the study of how nonnormality affects the numerical stability of discretized differential equations [4], the Novosibirsk group including Sergei Godunov and colleagues conducted research related to pseudospectra, and published the first sketch of pseudospectra which were called 'spectral portraits of matrices' containing various 'patches of spectrum' [11]. Other research on pseudospectra in the mid-1980s was conducted by J.W. Demmel, of which includes a paper that presents the first published computer plot of pseudospectra [2]. In 1986, J.H. Wilkinson introduces the modern interpretation of ϵ-pseudospectrum where he defines it for an arbitrary matrix norm induced by a vector norm [23]. L.N. Trefethen published his first works on pseudospectra in 1990, [16] and [15], where he talks of ϵ-approximate eigenvalues and ϵ-pseudospectrum. Following their mid-1980s work on the stability radius of a matrix, in 1992 D. Hinrichsen and A.J. Pritchard introduced the term spectral value set which categorized real structured ϵ-pseudospectrum of nonnormal matrices [8][9]. In 1993, Hinrichsen and B. Kelb extended on [8] to complex perturbations and also more general structured perturbations [7]. Finally, in 2005, Trefethen and M. Embree published their book *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators* [18] which is the basis of citations and references throughout this thesis.

We can generate a crude but reasonable numerical approximation of $\mathcal{T}(\mathbf{A};\epsilon)$ by simply plotting the eigenvalues of $\mathbf{A}+\mathbf{E}$ for a large number of randomly generated matrices $\mathbf{E}$ satisfying $\|\mathbf{E}\| \leq \epsilon$ as seen in Figure 3. The plot on the left is the pseudospectra generated by `PM(A,[1 1E-1 1E-2],500)`, Appendix C, and the plot on the right is the pseudospectra generated by `PM(A,[1 1E-1 1E-2],5000)`. In both, $\boxdot$ represents the eigenvalues of the matrix `A=grcar(50);`. This 'brute force' approach is sometimes referred to as the 'poor man's' method, and yields a reasonably accurate picture of $\mathcal{T}(\mathbf{A};\epsilon)$ [18, pp.378-379].

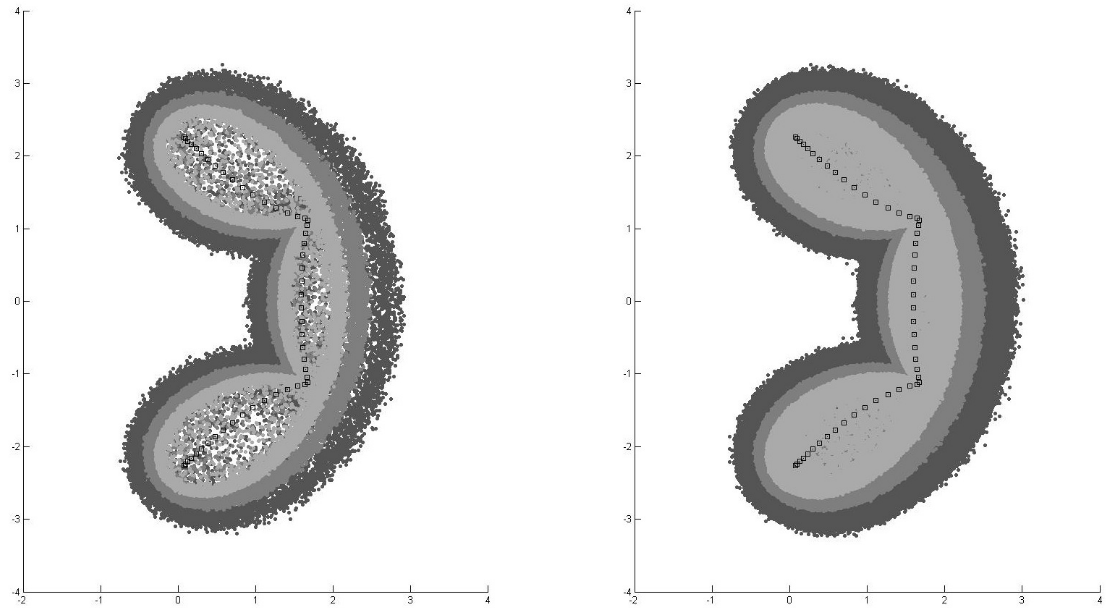

Figure 3. 'Poor Man's' method of estimating pseudospectra

One can observe whether the numerically generated approximation of $\mathcal{T}(\mathbf{A};\epsilon)$ has any parts outside the known stability region. We note that pseudospectra have the following property: for $\epsilon_1 > \epsilon_2 > 0$, the respective pseudospectra are nested sets: $\mathcal{T}(\mathbf{A};\epsilon_2) \subseteq \mathcal{T}(\mathbf{A};\epsilon_1)$. Thus $\bigcap_{\epsilon>0} \mathcal{T}(\mathbf{A};\epsilon) = \Lambda(\mathbf{A})$ [18, pp.15]. Therefore restricting the ϵ to a smaller value can in turn restrict the pseudospectrum to a certain stability region. In other words, it is possible to use the 'poor man's' method to estimate the largest value of ϵ for which $\mathcal{T}(\mathbf{A};\epsilon)$ stays inside the stability region.

I.IV Structured Pseudospectra

In some applications it might be known that only certain types of perturbations can arise. In this case it is reasonable to modify the definition of ϵ-pseudospectrum to consider only the eigenvalues which arise from these certain specified types of perturbations. In other cases, it might be necessary and sometimes convenient to be able to control how the perturbation effects the given matrix. This leads to the notion of structured pseudospectra. In 1992-1993, Hinrichsen and Pritchard developed the idea of spectral value sets which leads to one possible method for defining structured pseudospectra [7–9].

Definition 3. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times s}$, $\mathbf{C} \in \mathbb{C}^{t\times n}$. Let $\epsilon > 0$. The structured ϵ-pseudospectrum of $\mathbf{A}$, denoted by $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$ [7, pp.2], is the set of $\lambda \in \mathbb{C}$ such that $\lambda \in \Lambda(\mathbf{A}+\mathbf{BEC})$ for some $\mathbf{E} \in \mathbb{C}^{s\times t}$ with $\|\mathbf{E}\| \leq \epsilon$:

$$\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{BEC}) \text{ for some } \mathbf{E} \text{ with } \|\mathbf{E}\| \leq \epsilon\}. \quad (4)$$

It should be noted that as for the ϵ-pseudospectrum, the structured pseudospectrum depends on the choice of norm, even though this is not explicitly indicated in the notation. The idea behind this definition of structured pseudospectra is to manipulate the perturbing matrices such that the pseudospectra produces the results that are specific to certain requirements. By constructing the matrices $\mathbf{B}$ and $\mathbf{C}$ in certain ways, the perturbations of the pseudospectra can be restricted which can be useful in certain situations. The following example illustrates this idea.

Example 1. Given a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, to perturb $\mathbf{A}$ only at the i, jth entry, define $\mathbf{B} \in \mathbb{C}^{n\times n}$ and $\mathbf{C} \in \mathbb{C}^{n\times n}$ by

$$\mathbf{B}_{k,l} = \begin{cases} 1 & \text{if } k = i \text{ and } l = i \\ 0 & \text{otherwise} \end{cases} \quad \mathbf{C}_{k,l} = \begin{cases} 1 & \text{if } k = j \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}.$$

Therefore the set $\mathcal{T}(\mathbf{A}; \mathbf{B}, \mathbf{C}, \epsilon)$ consists of all eigenvalues of matrices which arise from perturbations only to the i, jth entry of $\mathbf{A}$: for all $\mathbf{E} \in \mathbb{C}^{n\times n}$ such that $\|\mathbf{E}\| \leq \epsilon$

$$(\mathbf{A} + \mathbf{BEC})_{k,l} = \begin{cases} \mathbf{A}_{k,l} & \text{if } k \neq i \text{ or } l \neq j \\ \mathbf{A}_{i,j} + \mathbf{E}_{i,j} & \text{if } k = i \text{ and } l = j \end{cases}.$$

This comes from the fact that $|\mathbf{E}_{i,j}| \leq \|\mathbf{E}\| \leq \epsilon$. Other choices of $\mathbf{B}$ and $\mathbf{C}$ can give the same pseudospectra. For example if we define $\mathbf{B} \in \mathbb{C}^{n\times 1}$ and $\mathbf{C} \in \mathbb{C}^{1\times n}$ by

$$\mathbf{B}_{k,1} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases} \quad \mathbf{C}_{1,l} = \begin{cases} 1 & \text{if } l = j \\ 0 & \text{otherwise} \end{cases},$$

we get the same structured pseudospectra as above. Another expansion on the example is to allow $\mathbf{B}$ or $\mathbf{C}$ to be the identity in the example.

This will give the following respectfully:

$$(\mathbf{A}+\mathbf{BEC})_{k,l} = \begin{cases} \mathbf{A}_{k,l} & \text{if } k \neq i \\ \mathbf{A}_{i,l}+\mathbf{E}_{i,l} & \text{if } k = i \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{A}_{k,l} & \text{if } l \neq j \\ \mathbf{A}_{k,j}+\mathbf{E}_{k,j} & \text{if } l = j \end{cases}.$$

Here we are only perturbing the ith row or jth column respectfully.

In the next chapter, we will discuss the characterization of the pseudospectrum and structured pseudospectrum in terms of the resolvent operator of a given matrix $\mathbf{A}$, and how the behavior of these sets is related to the departure from normality of $\mathbf{A}$. We will then investigate how changing the norm can affect normality, and hence the behavior of the pseudospectrum and structured pseudospectrum of $\mathbf{A}$.

CHAPTER II

CHARACTERIZATIONS AND NORMALITY

II.I Equivalent Characterizations of Pseudospectral Sets

A different characterization of the pseudospectrum of a matrix can be obtained from the resolvent of the given matrix.

Definition 4. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$. For $\lambda \in \mathbb{C} \setminus \Lambda(\mathbf{A})$, the resolvent of $\mathbf{A}$ is defined as the matrix $(\lambda \mathbf{I} - \mathbf{A})^{-1}$.

The following equivalent characterization of $\mathcal{T}$, which we will denote by the set $\mathcal{S}$, is constructed through the use of the norm of the resolvent matrix, and gives a more tangible understanding of the movement of the eigenvalues.

Definition 5. Let $\epsilon > 0$ be arbitrary. The set $\mathcal{S}(\mathbf{A};\epsilon)$ [18, pp.13] is the set of $\lambda \in \mathbb{C}$ such that $\|(\lambda \mathbf{I} - \mathbf{A})^{-1}\| \geq \epsilon^{-1}$:

$$\mathcal{S}(\mathbf{A};\epsilon) = \Lambda(\mathbf{A}) \cup \{\lambda \in \mathbb{C} : \|(\lambda \mathbf{I} - \mathbf{A})^{-1}\| \geq \epsilon^{-1}\}. \tag{5}$$

It should be noted that as for the ϵ-pseudospectrum, the set $\mathcal{S}$ depends on the choice of norm, even though this is not explicitly indicated in the notation. It is known that $\mathcal{S}(\mathbf{A};\epsilon) = \mathcal{T}(\mathbf{A};\epsilon)$ [18, pp.16][17, `http://www.cs.ox.ac.uk/pseudospectra/thms/thm1.pdf`]. In the Appendix B we provide a proof of this fact for a couple of reasons. First, the ideas in the proof provide a motivation for our proof of theorems to come later in this thesis. Second, in our later numerical investigations we shall make use of certain rank one matrices similar to those constructed in the proof. In this set as λ approaches an eigenvalue of the given matrix, the norm of the resolvent will grow large without bound. Therefore as the norm of the resolvent approaches $\frac{1}{\epsilon}$, λ approaches the boundary of the pseudospectrum for ϵ. These ideas lead to methods of generating approximations of the set $\mathcal{S}$. One method is to generate a grid of points in the complex plane that spans over the range of the expected pseudospectrum, and then calculate the norm of the resolvent of the points in this grid. Therefore plotting the points that satisfy the restriction on the norm of their respective resolvent i.e. the points such that the norm of the resolvent of is greater than or equal to $\frac{1}{\epsilon}$, will give an approximation of the span of the pseudospectrum.

We see in Figure 4 a three dimensional representation of the norm of the resolvent. Here we are using the `surfc` function of MATLAB to calculate the norm of the resolvent of a mesh grid over the eigenvalues of the `basor(10);` matrix given as an example in the eigtool.m code. Below the three dimensional data we have used the `contourf` function of MATLAB to plot the level curves for various ϵ's. For each ϵ, we get an approximation of the boundary of the ϵ-pseudospectrum which we will denote as

$$\mathcal{C}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\| = \frac{1}{\epsilon}\}. \tag{6}$$

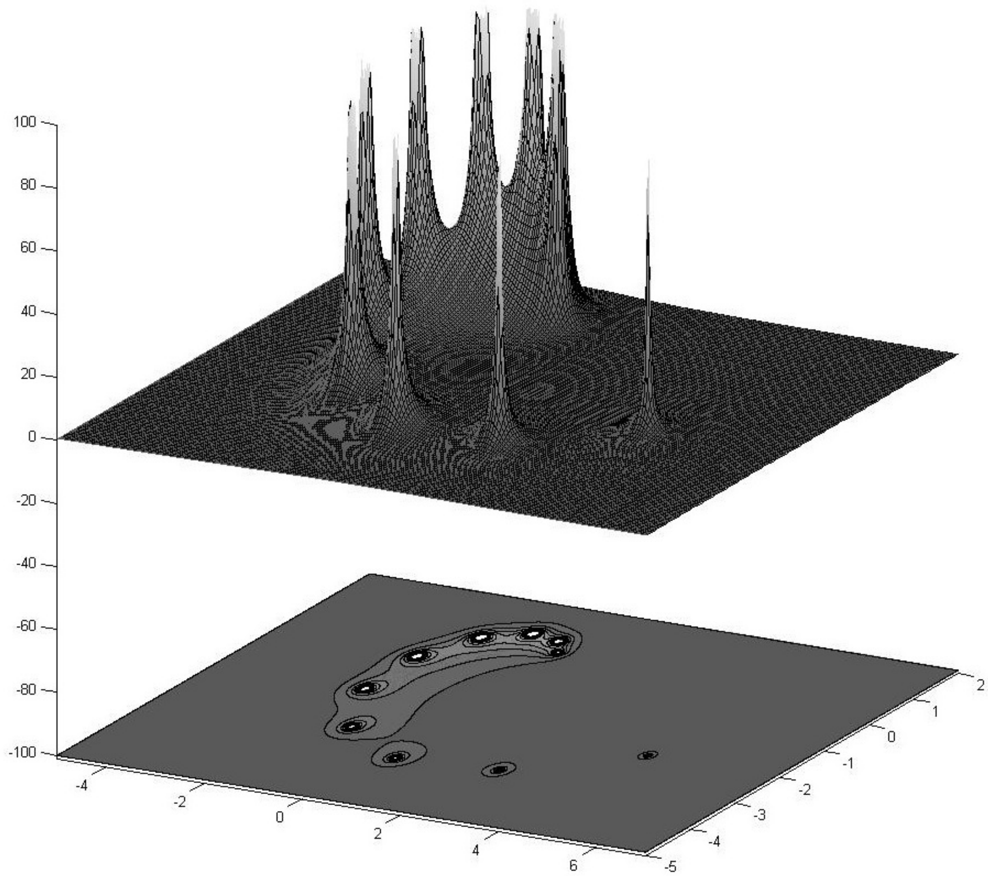

Figure 4. Plotting the norm of the resolvent over a mesh grid

This lead to another method of construction of the pseudospectrum which is to construct contour lines at specific ϵ's as seen in Figure 5. The plot on the left shows the pseudospectra generated by `PSA(A,[1 1E-1 1E-2],0.005)`, Appendix C. The plot on the right has the right plot of Figure 3 superimposed under the left plot. $\boxdot$ represents the eigenvalues of the matrix `A=grcar(50);`. This method is the basis of the eigtool.m [18, pp.375, Figure 39.3] MATLAB code algorithm in Appendix C.

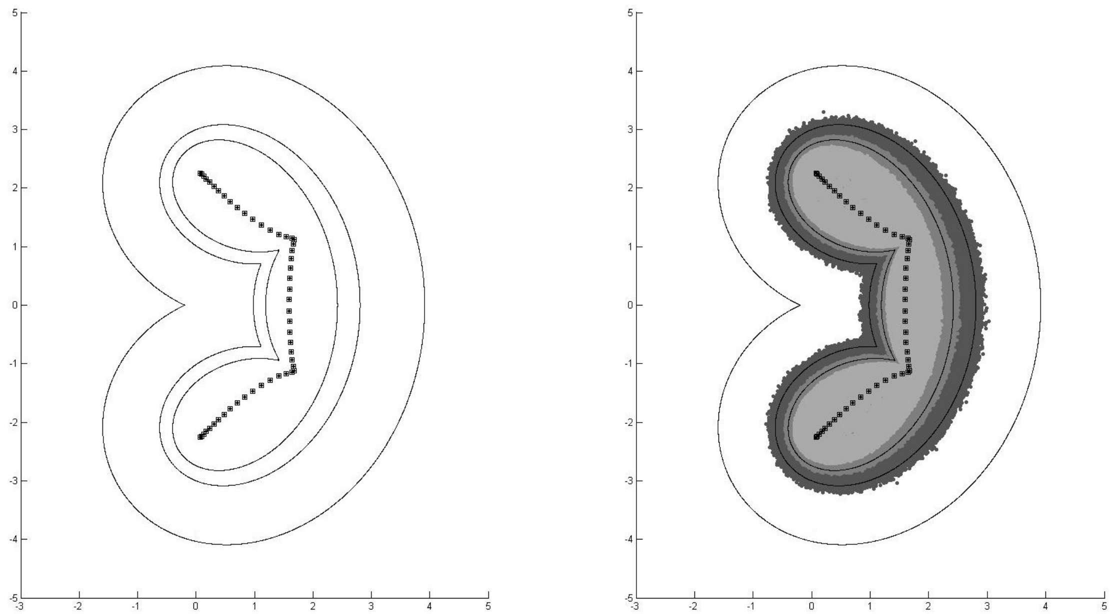

Figure 5. Plotting the contour lines for pseudospectra

It is possible to give a similar equivalent characterization for the structured pseudospectra $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$. In particular, define the set $\mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$ [7, pp.3] by

$$\mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \Lambda(\mathbf{A}) \cup \{\lambda \in \mathbb{C} : \|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\| \geq \frac{1}{\epsilon}\}.^{1} \tag{7}$$

It is known that $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$ [9, pp.813-814], Appendix B. The structured ϵ-pseudospectrum can be approximated as in Figure 5 by using a variations on PSA.m seen in Appendix C.

Matrices $\mathbf{B}$ and $\mathbf{C}$ characterize the types of perturbations in the set $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$ and $\mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$. If we restrict our consideration to the cases in which $\mathbf{B}$ or $\mathbf{C}$

[1] We say the $\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\| = \infty$ for each $\lambda \in \Lambda(\mathbf{A})$. Thus $\Lambda(\mathbf{A}) \subset \mathcal{S}$.

are orthogonal projections, it is possible to define sets of structured pseudospectra based on subspaces of $\mathbb{C}^n$. We do this next. Although the equivalence of the characterizations can be considered a special case of the equivalence $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$, we provide the proof for completeness, and for the explict construction of the relavent rank one perturbation matrices $\mathbf{E}$. For any subspace W of $\mathbb{C}^n$, define the set $\mathcal{T}_1(\mathbf{A};W,\epsilon)$ by $\mathcal{T}_1(\mathbf{A};W,\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{E})$ for some $\mathbf{E}$ such that $\|\mathbf{E}\| \leq \epsilon, \mathcal{R}(\mathbf{E}) \subset W\}$.

Theorem 2. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$ be arbitrary. Let W be a subspace of $\mathbb{C}^n$. Define the set $\mathcal{S}_1(\mathbf{A};W,\epsilon) = \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I}-\mathbf{A})^{-1}|_W\| \geq \frac{1}{\epsilon}\} \cup \Lambda(\mathbf{A})$. Then $\mathcal{T}_1(\mathbf{A};W,\epsilon) = \mathcal{S}_1(\mathbf{A};W,\epsilon)$.*

Proof. Observe that $\|(\lambda\mathbf{I}-\mathbf{A})^{-1}|_W\| \geq \frac{1}{\epsilon}$ if and only if there exist $y \in W$ such that $\|y\| \leq \epsilon\|(\lambda\mathbf{I}-\mathbf{A})^{-1}y\|$.

To prove $\mathcal{T}_1 \subset \mathcal{S}_1$ let $\lambda \in \mathcal{T}_1$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{S}_1$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$, so $(\lambda\mathbf{I}-\mathbf{A})^{-1}$ exists. Since $\lambda \in \mathcal{T}_1$, there exists $\mathbf{E}$ such that $\|\mathbf{E}\| \leq \epsilon$, $\mathcal{R}(\mathbf{E}) \subset W$, and $\lambda \in (\mathbf{A}+\mathbf{E})$. Thus there exists $x \neq 0$ such that $(\mathbf{A}+\mathbf{E})x = \lambda x$, which implies $(\lambda\mathbf{I}-\mathbf{A})x = \mathbf{E}x$. Let $y = \mathbf{E}x$. It follows that $y \in W$, $x = (\lambda\mathbf{I}-\mathbf{A})^{-1}y$, and $\|y\| = \|\mathbf{E}x\| \leq \epsilon\|x\| = \epsilon\|(\lambda\mathbf{I}-\mathbf{A})^{-1}y\|$. Thus $\lambda \in \mathcal{S}_1$.

To prove $\mathcal{S}_1 \subset \mathcal{T}_1$ let $\lambda \in \mathcal{S}_1$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{T}_1$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$. Since $\lambda \in \mathcal{S}_1$, there exist $y \in W$ such that $\|y\| \leq \epsilon\|(\lambda\mathbf{I}-\mathbf{A})^{-1}y\|$. Let $x = (\lambda\mathbf{I}-\mathbf{A})^{-1}y$, which implies $(\lambda\mathbf{I}-\mathbf{A})x = y$. Let $\mathbf{P} : \mathbb{C}^n \to \mathbb{C}^n$ be the orthogonal projection onto the span of x. Thus $\mathbf{P}z = \alpha x$ where $|\alpha| \leq \frac{\|z\|}{\|x\|}$ since $\|\mathbf{P}\| = 1$. In particular, $\mathbf{P} = \frac{1}{\|x\|^2}xx^T$. Define $\mathbf{E} = (\lambda\mathbf{I}-\mathbf{A})\mathbf{P}$. Thus for all $z \in \mathbb{C}^n$,

$\mathbf{E}z = (\lambda\mathbf{I} - \mathbf{A})\mathbf{P}z = \alpha(\lambda\mathbf{I} - \mathbf{A})x = \alpha y$ which is in W. Thus $\mathcal{R}(\mathbf{E}) \subset W$. Also $(\mathbf{A}+\mathbf{E})x = \mathbf{A}x + (\lambda\mathbf{I} - \mathbf{A})\mathbf{P}x = \mathbf{A}x + (\lambda\mathbf{I} - \mathbf{A})x = \lambda x$. Thus $\lambda \in \Lambda(\mathbf{A}+\mathbf{E})$. Finally for all $z \in \mathbb{C}^n$,

$$\begin{aligned} \|\mathbf{E}z\| &= \|(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}z\| = \|\alpha(\lambda\mathbf{I} - \mathbf{A})x\| \\ &= |\alpha|\|y\| \leq \frac{\|z\|}{\|x\|}\epsilon\|(\lambda\mathbf{I} - \mathbf{A})^{-1}y\| = \epsilon\|z\|. \end{aligned}$$

Thus $\|\mathbf{E}\| \leq \epsilon$, and $\lambda \in \mathcal{T}_1$.

Thus $\mathcal{S}_1 = \mathcal{T}_1$. □

For any subspace Z of $\mathbb{C}^n$, define $\mathcal{T}_2(\mathbf{A}; Z, \epsilon)$ by $\mathcal{T}_2(\mathbf{A}; Z, \epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{E})$ for some $\mathbf{E}$ such that $\|\mathbf{E}\| \leq \epsilon, Z^\perp \subset \mathcal{N}(\mathbf{E})\}$.

Theorem 3. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$ be arbitrary. Let Z be a subspace of $\mathbb{C}^n$ and $\mathbf{P}$ be the orthogonal projection onto Z. Define the set $\mathcal{S}_2(\mathbf{A}; Z, \epsilon) = \{\lambda \in \mathbb{C} : \|\mathbf{P}(\lambda\mathbf{I} - \mathbf{A})^{-1}\| \geq \frac{1}{\epsilon}\} \cup \Lambda(\mathbf{A})$. Then $\mathcal{T}_2(\mathbf{A}; Z, \epsilon) = \mathcal{S}_2(\mathbf{A}; Z, \epsilon)$.*

Proof. To prove $\mathcal{T}_2 \subset \mathcal{S}_2$ let $\lambda \in \mathcal{T}_2$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{S}_2$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$, so $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ exists. Since $\lambda \in \mathcal{T}_2$, there exists $\mathbf{E}$ such that $\|\mathbf{E}\| \leq \epsilon$, $Z^\perp \subset \mathcal{N}(\mathbf{E})$, and $\lambda \in \Lambda(\mathbf{A}+\mathbf{E})$. Thus there exists $x \neq 0$ such that $(\mathbf{A}+\mathbf{E})x = \lambda x$, which implies $(\lambda\mathbf{I} - \mathbf{A})x = \mathbf{E}x$. Let $y = \mathbf{E}x$, which implies $y = \mathbf{E}x = \mathbf{E}(\mathbf{P}x + (\mathbf{I} - \mathbf{P})x) = \mathbf{E}\mathbf{P}x$. Thus $\|y\| = \|\mathbf{E}x\| = \|\mathbf{E}\mathbf{P}x\| \leq \epsilon\|\mathbf{P}x\| = \epsilon\|\mathbf{P}(\lambda\mathbf{I} - \mathbf{A})^{-1}y\|$. Thus $\lambda \in \mathcal{S}_2$.

To prove $\mathcal{S}_2 \subset \mathcal{T}_2$ let $\lambda \in \mathcal{S}_2$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{T}_2$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$. Since $\lambda \in \mathcal{S}_2$, there exists $y \neq 0$ such that $\|y\| \leq \epsilon\|\mathbf{P}(\lambda\mathbf{I} - \mathbf{A})^{-1}y\|$.

Let $x = (\lambda\mathbf{I} - \mathbf{A})^{-1}y$, which implies $x \neq 0$ and $\mathbf{P}x \neq 0$. Define $\mathbf{E} = \frac{1}{\|\mathbf{P}x\|^2}y(\mathbf{P}x)^T$. Since $\mathbf{E}z = \frac{1}{\|\mathbf{P}x\|^2}y(\mathbf{P}x)^Tz = \frac{1}{\|\mathbf{P}x\|^2}yx^T\mathbf{P}z$, it follows that $Z^\perp \subset \mathcal{N}(\mathbf{E})$. Also since $(\mathbf{P}x)^Tx = x^T\mathbf{P}x = x^T\mathbf{P}\mathbf{P}x = (\mathbf{P}x, \mathbf{P}^Tx) = (\mathbf{P}x, \mathbf{P}x) = \|\mathbf{P}x\|^2$, we have

$$\begin{aligned}
(\mathbf{A} + \mathbf{E})x &= \mathbf{A}x + \mathbf{E}x \\
&= \mathbf{A}x + \frac{1}{\|\mathbf{P}x\|^2}y(\mathbf{P}x)^Tx \\
&= \mathbf{A}x + \frac{1}{\|\mathbf{P}x\|^2}(\lambda\mathbf{I} - \mathbf{A})x\|\mathbf{P}x\|^2 \\
&= \mathbf{A}x + (\lambda\mathbf{I} - \mathbf{A})x = \lambda x,
\end{aligned}$$

which shows $\lambda \in \Lambda(\mathbf{A} + \mathbf{E})$. Finally for all z,

$$\begin{aligned}
\|\mathbf{E}z\| &= \frac{1}{\|\mathbf{P}x\|^2}\|y(\mathbf{P}x)^Tz\| \\
&= \frac{\|y\|\|z\|}{\|\mathbf{P}x\|} \\
&\leq \epsilon\frac{\|\mathbf{P}(\lambda\mathbf{I} - \mathbf{A})^{-1}y\|\|z\|}{\|\mathbf{P}x\|} \\
&= \epsilon\|z\|,
\end{aligned}$$

so $\|\mathbf{E}\| \leq \epsilon$. Thus $\lambda \in \mathcal{T}_2$.

Thus $\mathcal{S}_2 = \mathcal{T}_2$. □

II.II Normal Matrices

Definition 6. A matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is diagonalizable if and only if there exists a matrix $\mathbf{V} \in \mathbb{C}^{n \times n}$ such that $\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$, where $\mathbf{D} \in \mathbb{C}^{n \times n}$ is a diagonal matrix with all entries equal to 0 except for the main diagonal entries which are the eigenvalues of $\mathbf{A}$ [5, pp.338].

Definition 7. A matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ is unitary if and only if $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$, where $\mathbf{U}^*$ is the conjugate transpose of $\mathbf{U}$ [12, pp.47].

Definition 8. A matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is normal if and only if it is diagonalizable by a unitary matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$; $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$ where $\mathbf{D}$ is a diagonal $n \times n$ matrix where the diagonal entries are the eigenvalues of $\mathbf{A}$, and the columns of the unitary matrix $\mathbf{U}$ are the eigenvectors of $\mathbf{A}$ [10, pp.258].

This unitary diagonalization of a matrix plays an important role in analyzing the pseudospectra of a matrix. In the matrix 2-norm, unitary matrices preserve the norm; $\|\mathbf{U}x\|_2 = \|x\|_2$ for any unitary matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ and vector $x \in \mathbb{C}^n$ [10, pp.257]. Thus the 2-norm of a unitary matrix $\mathbf{U}$ is $\|\mathbf{U}\|_2 = \max\limits_{x \neq 0} \frac{\|\mathbf{U}x\|_2}{\|x\|_2} = \max\limits_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} = 1$; thus $\|\mathbf{U}^*\|_2 = 1$ [10, pp.54]. Therefore for a normal matrix $\mathbf{A}$, the norm of $\mathbf{A}$ is equal to that of the diagonal matrix that exists by the unitary diagonalization of $\mathbf{A}$; $\|\mathbf{A}\|_2 = \|\mathbf{U}\mathbf{D}\mathbf{U}^*\|_2 = \|\mathbf{D}\|_2$ [8].

Our main interest is understanding when small perturbations can cause large changes to the eigenvalues of a matrix. First we answer the question of what is the best case scenario. That is, is there a 'minimum' configuration of the set $\mathcal{T}(\mathbf{A};\epsilon)$ in $\mathbb{C}$? The answer is yes, and it is based on the observation that $\lambda \in \Lambda(\mathbf{A})$ if and only if $\lambda + \omega \in \Lambda(\mathbf{A} + \omega\mathbf{I})$ for all $\omega \in \mathbb{C}$. Observe that if $\lambda \in \Lambda(\mathbf{A})$ and $|\gamma - \lambda| \leq \epsilon$, then $\gamma \in \Lambda(\mathbf{A} + (\gamma - \lambda)\mathbf{I})$ and $\|(\gamma - \lambda)\mathbf{I}\| = |\gamma - \lambda| \leq \epsilon$. Thus at a minimum, the ϵ-pseudospectrum of a matrix $\mathbf{A}$ contains the set $\Omega(\mathbf{A};\epsilon)$ defined by

$$\Omega(\mathbf{A};\epsilon) = \{\gamma \in \mathbb{C} : |\gamma - \lambda| \leq \epsilon \text{ for some } \lambda \in \Lambda(\mathbf{A})\}. \tag{8}$$

That is, we have $\Omega(\mathbf{A};\epsilon) \subset \mathcal{T}(\mathbf{A};\epsilon)$, so the ϵ-pseudospectrum always contains the union of the set of ϵ-balls centered at the eigenvalues of $\mathbf{A}$. The next result shows that this is all of $\mathcal{T}(\mathbf{A};\epsilon)$ when $\mathbf{A}$ is a normal matrix. Thus this is the best case scenario.

Theorem 4. *If a matrix $\mathbf{A}$ is a normal matrix, then the pseudospectrum for a given $\epsilon > 0$ in the 2-norm is the set $\mathcal{T}(\mathbf{A};\epsilon) = \mathcal{S}(\mathbf{A};\epsilon) = \Omega(\mathbf{A};\epsilon)$ [10, pp.94].*

Proof. [18, pp.19] Let ϵ be arbitrary. For a normal matrix $\mathbf{A}$ the ϵ-pseudospectrum of $\mathbf{A}$ is $\mathcal{S}(\mathbf{A};\epsilon) = \Lambda(\mathbf{A}) \cup \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_2 \geq \frac{1}{\epsilon}\}$. If $\lambda \in \Lambda(\mathbf{A})$ then trivially $\lambda \in \mathcal{S}(\mathbf{A};\epsilon)$. Thus if we let $\lambda \in \mathcal{S}(\mathbf{A};\epsilon) \setminus \Lambda(\mathbf{A})$ we get

$$\begin{aligned} \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_2 &= \|(\lambda\mathbf{I} - \mathbf{U}\mathbf{D}\mathbf{U}^*)^{-1}\|_2 \\ &= \|\mathbf{U}(\lambda\mathbf{I} - \mathbf{D})^{-1}\mathbf{U}^*\|_2 \\ &= \|(\lambda\mathbf{I} - \mathbf{D})^{-1}\|_2 \\ &= \frac{1}{\displaystyle\min_{\gamma \in \Lambda(\mathbf{A})} |\lambda - \gamma|} \geq \frac{1}{\epsilon}. \end{aligned} \tag{9}$$

Thus the ϵ-pseudospectrum of a $\mathbf{A}$ in the 2-norm is $\Omega(\mathbf{A};\epsilon)$. □

From the perspective of stability analysis, the pseudospectrum of a normal matrix is 'boring' i.e. small perturbations to a matrix $\mathbf{A}$ can only cause small movements to the eigenvalues of $\mathbf{A}$. This can be seen in Figure 6 where the pseudospectrum is the union of the ϵ-balls centered around the eigenvalues of the randomly generated normal matrix limited to the precision of PSA.m, Appendix C.

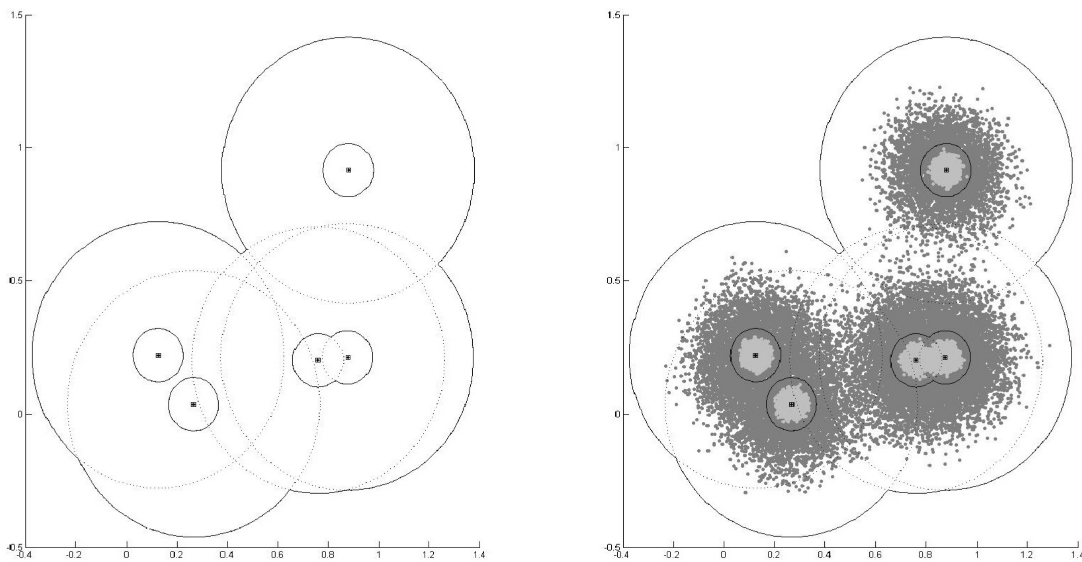

Figure 6. Pseudospectra of a randomly generated normal matrix

Definition 9. The <u>condition number</u> of a nonsingular matrix $\mathbf{B}$ is $\kappa(\mathbf{B}) = \|\mathbf{B}\|\,\|\mathbf{B}^{-1}\|$ [12, pp.385].

The condition number is in the range $1 \leq \kappa(\cdot) < \infty$,[2] though the condition number of a matrix is equal to 1 if the matrix is unitary [18, pp.19]. Therefore the following results from the Bauer-Fike theorem [1].

[2]We define $\kappa(\mathbf{B}) = \infty$ for a singular matrix $\mathbf{B}$.

Theorem 5. *[18, pp.20] Suppose* $\mathbf{A}$ *is diagonalizable as above. Then for each* $\epsilon > 0$, $\Omega(\mathbf{A};\epsilon) \subseteq \mathcal{T}(\mathbf{A};\epsilon) \subseteq \Omega(\mathbf{A};\epsilon \times \kappa(\mathbf{V}))$.

II.III Change of Norm

The idea of normal matrices producing ϵ-balls for their respective pseudospectra is a valuable tool, but limiting matrices to being normal is not always conducive to the intended results of a particular analysis. Therefore it is reasonable to ask if there is a way the idea of a ϵ-ball-pseudospectrum of a normal matrix can be applied to non-normal matrices. Since the normality of a matrix is dependent upon the underlying norm and inner product, we shall explore the effect on the pseudospectra caused by changing the underlying norm and inner product. A standard idea is that any invertible matrix $\mathbf{B}$ can be used to define a new inner product by $(x,y)_{\mathbf{B}} \equiv y^*\mathbf{B}^*\mathbf{B}x$. This gives us the vector norm $\|x\|_{\mathbf{B}} = \|\mathbf{B}x\|$, and thus we can define a matrix norm induced by this inner product. Thus for any $\mathbf{A} \in \mathbb{C}^{n\times n}$, the matrix $\mathbf{B}$-norm of $\mathbf{A}$ is equal to the matrix 2-norm of $\mathbf{BAB}^{-1}$; $\|\mathbf{A}\|_{\mathbf{B}} = \|\mathbf{BAB}^{-1}\|_2$ [18, pp.379]. To see this, observe that

$$\begin{aligned}\|\mathbf{A}\|_{\mathbf{B}} &= \max_{x\in\mathbb{C}^n\setminus 0}\left(\frac{x^*\mathbf{A}^*\mathbf{B}^*\mathbf{B}\mathbf{A}x}{x^*\mathbf{B}^*\mathbf{B}x}\right)^{1/2}\\ &= \max_{y\in\mathbb{C}^n\setminus 0}\left(\frac{y^*\mathbf{B}^{-*}\mathbf{A}^*\mathbf{B}^*\mathbf{B}\mathbf{A}\mathbf{B}^{-1}y}{y^*y}\right)^{1/2}\\ &= \|\mathbf{BAB}^{-1}\|_2. \end{aligned} \tag{10}$$

Therefore the ϵ-pseudospectrum of $\mathbf{A}$ in the matrix $\mathbf{B}$-norm can be represented in the Euclidean 2-norm, since the resolvent satisfies

$$\|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_{\mathbf{B}} = \|\mathbf{B}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}^{-1}\|_2. \tag{11}$$

Let us introduce the notation $\mathcal{S}_{\mathbf{B}}(\mathbf{A};\epsilon)$ and $\mathcal{T}_{\mathbf{B}}(\mathbf{A};\epsilon)$ to denote the ϵ-pseudospectrum of $\mathbf{A}$ with respect to the $\|\cdot\|_{\mathbf{B}}$ norm. Thus

$$\mathcal{T}_{\mathbf{B}}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \Lambda(\mathbf{A} + \mathbf{E}) \text{ for some } \mathbf{E} \text{ such that } \|\mathbf{E}\|_{\mathbf{B}} \leq \epsilon\} \tag{12}$$

$$\text{and } \mathcal{S}_{\mathbf{B}}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_{\mathbf{B}} \geq \frac{1}{\epsilon}\} \cup \Lambda(\mathbf{A}). \tag{13}$$

It turns out that such pseudospectra are equivalent to certain structured pseudospectra in the 2-norm. In particular, for a given matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, and arbitrary $\epsilon > 0$,

$$\begin{aligned}
\mathcal{S}_{\mathbf{B}}(\mathbf{A};\epsilon) &= \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_{\mathbf{B}} \geq \epsilon^{-1}\} \\
&= \{\lambda \in \mathbb{C} : \|\mathbf{B}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}^{-1}\|_2 \geq \epsilon^{-1}\} \\
&= \mathcal{S}(\mathbf{A};\mathbf{B^{-1}},\mathbf{B},\epsilon).
\end{aligned} \tag{14}$$

Thus $\mathcal{T}_{\mathbf{B}}(\mathbf{A};\epsilon) = \mathcal{T}(\mathbf{A};\mathbf{B^{-1}},\mathbf{B},\epsilon)$.

Recall that for a normal matrix $\mathbf{A}$, we have $\mathcal{T}(\mathbf{A};\epsilon) = \Omega(\mathbf{A};\epsilon)$ for all ϵ. If $\mathbf{A}$ is not normal in the 2-norm, but is diagonalizable, it is known that there is an equivalent norm in which $\mathbf{A}$ is normal. The following result shows that in this equivalent norm, the pseudospectrum behaves as expected. In the next chapter, we will use this observation as a basis for a conjecture regarding which perturbations cause the greatest movement of the eigenvalues of $\mathbf{A}$.

Theorem 6. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be diagonalizable by a matrix $\mathbf{V}$ such that $\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is a diagonal matrix of eigenvalues of $\mathbf{A}$. Then for all $\epsilon > 0$, $\mathcal{S}_{\mathbf{V}^{-1}}(\mathbf{A};\epsilon) = \Omega(\mathbf{A};\epsilon)$.*

Proof. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be diagonalizable by a matrix $\mathbf{V}$ such that $\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is a diagonal matrix of eigenvalues of $\mathbf{A}$. Let $\epsilon > 0$ be given.

$$\begin{aligned}
\mathcal{S}_{\mathbf{V}^{-1}}(\mathbf{A};\epsilon) &= \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_{\mathbf{V}^{-1}} \geq \epsilon^{-1}\} \\
&= \{\lambda \in \mathbb{C} : \|\mathbf{V}^{-1}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{V}\|_2 \geq \epsilon^{-1}\} \\
&= \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{V}^{-1}\mathbf{A}\mathbf{V})^{-1}\|_2 \geq \epsilon^{-1}\} \\
&= \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{D})^{-1}\|_2 \geq \epsilon^{-1}\} \\
&= \{\lambda \in \mathbb{C} : \min_{\gamma\in\Lambda(\mathbf{A})} |\lambda - \gamma| \leq \epsilon\} \\
&= \Omega(\mathbf{A};\epsilon) \qquad (15)
\end{aligned}$$

□

In other words, $\mathbf{A}$ is normal in the $\mathbf{V^{-1}}$-norm. It is possible to define other equivalent norms for diagonalizable matrices which give this result.

Theorem 7. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be diagonalizable by a matrix $\mathbf{V}$ such that $\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is a diagonal matrix of eigenvalues of $\mathbf{A}$. Then there exist a positive definite matrix $\mathbf{Q}$ such that for all $\epsilon > 0$, $\mathcal{S}_{\mathbf{Q}}(\mathbf{A};\epsilon) = \Omega(\mathbf{A};\epsilon)$.*

Proof. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be diagonalizable by a matrix $\mathbf{V}$ such that $\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is a diagonal matrix of eigenvalues of $\mathbf{A}$. Let $\mathbf{Q} = \sqrt{\mathbf{V}^{-*}\mathbf{V}^{-1}}$. Since $\mathbf{V}$ is nonsingular, $\mathbf{Q}$ is positive definite. Let $\lambda \in \mathcal{S}_{\mathbf{Q}}(\mathbf{A}; \epsilon)$. Thus

$$\begin{aligned}
\frac{1}{\epsilon} \leq \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_{\mathbf{Q}} &= \max_{x\in\mathbb{C}^n\setminus 0} \left(\frac{x^*(\lambda\mathbf{I} - \mathbf{A})^{-*}\mathbf{Q}^*\mathbf{Q}(\lambda\mathbf{I} - \mathbf{A})^{-1}x}{x^*\mathbf{Q}^*\mathbf{Q}x} \right)^{1/2} \\
&= \max_{x\in\mathbb{C}^n\setminus 0} \left(\frac{x^*(\lambda\mathbf{I} - \mathbf{A})^{-*}\mathbf{Q}\mathbf{Q}(\lambda\mathbf{I} - \mathbf{A})^{-1}x}{x^*\mathbf{Q}\mathbf{Q}x} \right)^{1/2} \\
&= \max_{x\in\mathbb{C}^n\setminus 0} \left(\frac{x^*(\lambda\mathbf{I} - \mathbf{A})^{-*}\mathbf{V}^{-*}\mathbf{V}^{-1}(\lambda\mathbf{I} - \mathbf{A})^{-1}x}{x^*\mathbf{V}^{-*}\mathbf{V}^{-1}x} \right)^{1/2} \\
&= \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|_{\mathbf{V}^{-1}} \\
&= \|(\lambda\mathbf{I} - \mathbf{V}^{-1}\mathbf{A}\mathbf{V})^{-1}\|_2 \\
&= \|(\lambda\mathbf{I} - \mathbf{D})^{-1}\|_2. \qquad (16)
\end{aligned}$$

Thus $\lambda \in \Omega(\mathbf{A}; \epsilon)$. ☐

This leads to some interesting ideas surrounding the movement of the eigenvalues of a matrix in each of its pseudospectra. In the next chapter we will discuss how these matrix induced norms can help define what perturbation matrices move the eigenvalues of the given matrix the most in each pseudospectrum.

CHAPTER III

NUMERICAL INVESTIGATIONS

In this chapter we use our previous results about pseudospectra to motivate some numerical explorations. First we make an observation which is important for numerical computations of pseudospectra. We recall the definition

$$\mathcal{T}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{E}) \text{ for some } \mathbf{E} \in \mathbb{C}^{n\times n},\ \|\mathbf{E}\| \leq \epsilon\},$$

and define the following set:

$$\tilde{\mathcal{T}}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\tilde{\mathbf{E}}) \text{ for some } \tilde{\mathbf{E}} \in \mathbb{C}^{n\times n},\ \|\tilde{\mathbf{E}}\| \leq \epsilon, \text{ and } \operatorname{rank}(\tilde{\mathbf{E}}) = 1\}. \tag{17}$$

Clearly $\tilde{\mathcal{T}}(\mathbf{A};\epsilon) \subset \mathcal{T}(\mathbf{A};\epsilon)$, but upon consideration of the proof in Appendix B, we see that $\tilde{\mathcal{T}}(\mathbf{A};\epsilon) = \mathcal{T}(\mathbf{A};\epsilon)$. In fact, for $\lambda \in \mathcal{T}(\mathbf{A};\epsilon)$ the proof indicates how to construct a rank one perturbation matrix $\tilde{\mathbf{E}}$ with $\|\tilde{\mathbf{E}}\| = \epsilon$ and $\lambda \in \Lambda(\mathbf{A}+\tilde{\mathbf{E}})$. From a computational perspective this observation is significant since it means we can considerably restrict the class of perturbation matrices to be considered. Also, we have a recipe for constructing a rank one perturbation matrix for any $\lambda \in \mathcal{T}(\mathbf{A};\epsilon)$, indicated in the following three lines of MATLAB code.

Code 8.

```
n=length(A);
[U,S,V]=svd(A-lambda*eye(n));
E=-S(n,n)*U(:,n)*V(:,n)';
```

III.I Constructing Perturbation Matrices

For a given matrix $\mathbf{A}$, the pseudospectrum $\mathcal{T}(\mathbf{A};\epsilon)$ gives a reasonable picture of the behavior of the eigenvalues, and the stability, of $\mathbf{A}$ under perturbations. Therefore a reasonable question to explore would be given $\mathbf{A}$ and $\epsilon > 0$, which perturbation matrices cause the greatest movement in the eigenvalues of $\mathbf{A}$. Because of the observations above, we may restrict our attention to rank one perturbation matrices $\tilde{\mathbf{E}}$. Furthermore, because $\mathcal{T}(\mathbf{A};\epsilon) = \mathcal{S}(\mathbf{A};\epsilon)$, we know that $\mathcal{T}(\mathbf{A};\epsilon)$ is a closed subset of $\mathbb{C}$ whose boundary $\mathcal{C}(\mathbf{A};\epsilon)$ is a level set of $\|(\lambda\mathbf{I} - \mathbf{A})^{-1}\|$. If $\lambda \in \mathcal{T}(\mathbf{A};\epsilon)$ then there exists a $v \in \mathbb{C}^n$ and a matrix $\mathbf{E}_\lambda$ satisfying $\|\mathbf{E}_\lambda\| \leq \epsilon$ such that $(\mathbf{A} + \mathbf{E}_\lambda)v = \lambda v$. Therefore $(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{E}_\lambda v = v$, and thus

$$\|(\lambda\mathbf{I} - \mathbf{A})^{-1}\| \|\mathbf{E}_\lambda\| \geq= 1. \tag{18}$$

Thus for all $\lambda \in \mathcal{T}(\mathbf{A};\epsilon)$, there exists a matrix $\mathbf{E}_\lambda$ such that $\|\mathbf{E}_\lambda\| \geq \frac{1}{\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\|}$. Therefore if we apply the $\mathbf{V}^{-1}$-norm to Equation 18, we get for all $\lambda \in \mathcal{T}_{\mathbf{V}^{-1}}(\mathbf{A};\epsilon)$, there exists a matrix $\mathbf{E}_\lambda$ such that

$$\|\mathbf{E}_\lambda\|_{\mathbf{V}^{-1}} \geq \frac{1}{\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\|_{\mathbf{V}^{-1}}} = \min_{\gamma\in\Lambda(\mathbf{A})}|\lambda-\gamma|. \tag{19}$$

Therefore the distance an eigenvalue of a perturbed matrix moves must be less than or equal to the $\mathbf{V}^{-1}$-norm of the perturbing matrix that produces the eigenvalue. Let us define

$$\mathrm{d}(\mathbf{A};\lambda) = \min_{\gamma\in\Lambda(\mathbf{A})}|\lambda-\gamma|, \tag{20}$$

and

$$\mathrm{D}(\mathbf{A};\epsilon) = \max_{\lambda\in\mathcal{T}(\mathbf{A};\epsilon)}\mathrm{d}(\mathbf{A};\lambda). \tag{21}$$

Thus we have parameters on the maximizing perturbation matrices $\tilde{\mathbf{E}}$.

Lemma 9. *Let* $\mathbf{A}$ *be a diagonalizable matrix such that* $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{D}$*. For each* $\lambda \in \tilde{\mathcal{T}}(\mathbf{A};\epsilon)$ *such that* $d(\mathbf{A};\lambda) = D(\mathbf{A};\epsilon)$*, all* $\tilde{\mathbf{E}}$ *such* $\lambda\in\Lambda(\mathbf{A}+\tilde{\mathbf{E}})\subset\tilde{\mathcal{T}}(\mathbf{A};\epsilon)$ *must satisfy*

$$\begin{aligned}&(1)\ \|\tilde{\mathbf{E}}\| = \epsilon\\ &(2)\ \mathrm{D}(\mathbf{A};\epsilon) \leq \|\tilde{\mathbf{E}}\|_{\mathbf{V}^{-1}} \leq \epsilon\times\kappa(\mathbf{V}).\end{aligned} \tag{22}$$

Therefore we can pose our question as, given $\mathbf{A}$ and $\epsilon > 0$, can we construct a $\tilde{\mathbf{E}}'$ such that there exists $\lambda' \in \Lambda(\mathbf{A}+\tilde{\mathbf{E}}')$ where $\mathrm{d}(\mathbf{A};\lambda')=\mathrm{D}(\mathbf{A};\epsilon)$?

Of course this question is trivial when $\mathbf{A}$ is normal. In that case $\mathrm{D}(\mathbf{A};\epsilon) = \epsilon$ and for any $\lambda \in \mathcal{C}(\mathbf{A};\epsilon)$ we can construct an $\tilde{\mathbf{E}}$ as above, and we get the desired property. Therefore let us restrict our attention to matrices $\mathbf{A}$ which are diagonalizable, say $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{D}$. Thus even if $\mathbf{A}$ is nonnormal in the 2-norm we know it is normal in the $\mathbf{V}^{-1}$-norm. Further, the eigenvalue λ' of the $\tilde{\mathbf{E}}'$ we are seeking will in fact lie on the boundary $\mathcal{C}(\mathbf{A};\epsilon)$. Thus the $\tilde{\mathbf{E}}'$ we are seeking can be found by searching the set of all rank one perturbation matrices $\tilde{\mathbf{E}}$ such that $\|\tilde{\mathbf{E}}\| = \epsilon$ and $\Lambda(\mathbf{A}+\tilde{\mathbf{E}}) \cap \mathcal{C}(\mathbf{A};\epsilon) \neq \emptyset$.

III.II Conjecture and Investigations

We shall restrict our search to those rank one matrices $\tilde{\mathbf{E}}$ constructed as in the proof in Appendix B, and we shall numerically explore the following conjecture:

Conjecture 10. *For each $\lambda \in \mathcal{C}(\mathbf{A};\epsilon)$, let $\tilde{\mathbf{E}}_\lambda$ be the rank one matrix as constructed in the proof of $\mathcal{T} = \mathcal{S}$ in Appendix* B*, satisfying $\|\tilde{\mathbf{E}}_\lambda\| = \epsilon$ and $\lambda \in \Lambda(\mathbf{A}+\tilde{\mathbf{E}}_\lambda)$. If $d(\mathbf{A};\lambda') = D(\mathbf{A};\epsilon)$, then $\|\tilde{\mathbf{E}}_{\lambda'}\|_{\mathbf{V}^{-1}} \geq \|\tilde{\mathbf{E}}_\lambda\|_{\mathbf{V}^{-1}}$ for all $\lambda \in \mathcal{C}(\mathbf{A};\epsilon)$.*

In other words, from the class of rank one perturbations which put an eigenvalue on the boundary, the one which moves the eigenvalues of $\mathbf{A}$ the furthest is the one with the largest $\mathbf{V}^{-1}$-norm. Our reasoning for this conjecture is as follows. For a fixed matrix $\mathbf{A}$ and $\epsilon > 0$, let λ' be a location on the curve $\mathcal{C}(\mathbf{A};\epsilon)$ which is the furthest from its closest eigenvalue of $\mathbf{A}$. Therefore $\mathrm{d}(\mathbf{A};\lambda') = \mathrm{D}(\mathbf{A};\epsilon)$. This is a pseudospectrum value which represents an eigenvalue that has been moved furthest under the ϵ-perturbations in the 2-norm. Now suppose we select $\hat{\epsilon}$ so that the set $\Omega(\mathbf{A};\hat{\epsilon})$ encloses $\mathcal{C}(\mathbf{A};\epsilon)$. The set $\Omega(\mathbf{A};\hat{\epsilon})$ is the $\hat{\epsilon}$-pseudospectra in the $\mathbf{V}^{-1}$-norm.

Now visualize shrinking $\hat{\epsilon}$ until the boundary of $\Omega(\mathbf{A};\hat{\epsilon})$ just intersects $\mathcal{C}(\mathbf{A};\epsilon)$. From geometric considerations, this intersection should occur at λ', and the remainder of $\mathcal{C}(\mathbf{A};\epsilon)$ should be in the interior of $\Omega(\mathbf{A};\hat{\epsilon})$. Thus the λ values on the remainder of $\mathcal{C}(\mathbf{A};\epsilon)$ correspond to smaller $\hat{\epsilon}$ values in the $\mathbf{V}^{-1}$-norm. We investigate this numerically by giving a discretization to the curve $\mathcal{C}(\mathbf{A};\epsilon)$. For each λ in the discretization, we calculate the values $\mathrm{d}(\mathbf{A};\lambda)$ and $\|\tilde{\mathbf{E}}_\lambda\|_{\mathbf{V}^{-1}}$, where $\tilde{\mathbf{E}}_\lambda$ is constructed as in Code 8. We then plot both values for each λ along the curve $\mathcal{C}(\mathbf{A};\epsilon)$.

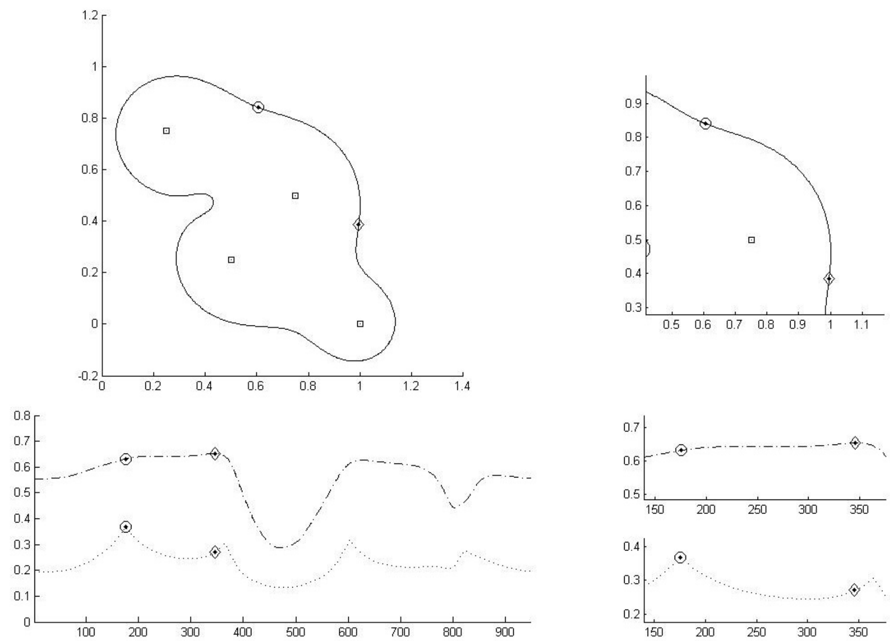

Figure 7. Pseudospectrum of random matrix for $\epsilon = 0.1$

In Figure 7, the top left picture is the pseudospectrum of the matrix which we will denote as $\mathbf{A}$ with the top right being its magnification. The bottom left is a plot of $\mathrm{d}(\mathbf{A};\lambda)$ for each $\lambda \in \mathcal{C}(\mathbf{A};0.1)$, the lower line, and $\mathbf{V}^{-1}$-norm of the $\tilde{\mathbf{E}}$ constructed from Code 8 for each $\lambda \in \mathcal{C}(\mathbf{A};0.1)$, the upper line. The two plots in the bottom right are the respective magnifications of the two lines in the bottom left. In figure 7, $\odot$ represents the $\lambda' \in \mathcal{C}(\mathbf{A}; 0.1)$ such that $\mathrm{d}(\mathbf{A};\lambda') = \mathrm{D}(\mathbf{A}; 0.1)$, and $\diamondsuit$ represents $\lambda \in \mathcal{C}(\mathbf{A};\epsilon)$ that gives the maximum of the $\|\tilde{\mathbf{E}}\|_{\mathbf{V}^{-1}}$ constructed above, $\|\tilde{\mathbf{E}}'\|_{\mathbf{V}^{-1}}$. The eigenvalues of $\mathbf{A}$ are represented by $\boxdot$. We can see that for each λ, its respectively constructed $\tilde{\mathbf{E}}$ has $\mathbf{V}^{-1}$-norm greater than one over the $\mathbf{V}^{-1}$-norm of the resolvent. We can also see that, though they are different, the eigenvalue λ that gives the maximum of the $\mathbf{V}^{-1}$-norms of the $\tilde{\mathbf{E}}$'s, $\|\tilde{\mathbf{E}}'\|_{\mathbf{V}^{-1}}$, is close to the eigenvalue that gives the maximum of the distance the λ's move from their closest eigenvalue of $\mathbf{A}$, λ'.

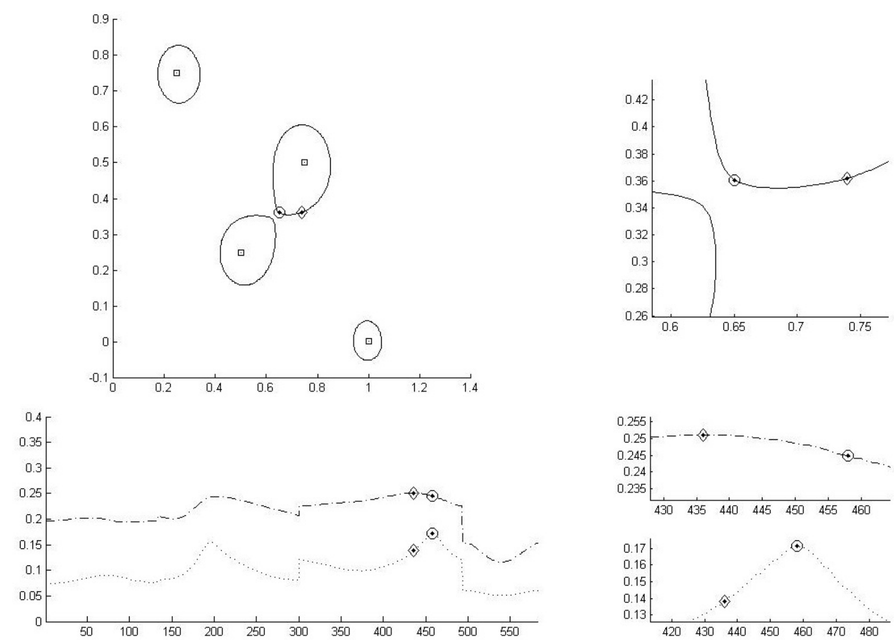

Figure 8. Pseudospectrum of random matrix for $\epsilon = 0.035$

In other words, this example gives numerical evidence that our conjecture is not true. However, further consideration of the geometric motivation of the conjecture indicates it is likely only valid if ϵ is sufficiently small enough so that the set $\mathcal{C}(\mathbf{A};\epsilon)$

separates into distinct closed curves about each of the eigenvalues of $\mathbf{A}$. Since this was not the case in the example, we perform the calculations again with the smaller value $\epsilon = 0.035$ and show the results in Figure 8. Here the numerical evidence is much more supportive of the conjecture, although there is a small but noticeable difference in the λ that maximize $\mathrm{d}(\mathbf{A};\lambda)$ and $\|\tilde{\mathbf{E}}_\lambda\|_{\mathbf{V}^{-1}}$, respectively (our conjecture is that the same λ maximizes both of these values). We can explain this behavior by observing that the matrix $\mathbf{A}$ is not too nonnormal (the condition number of the diagonalizing matrix is 2.78). Thus the closed curves which comprise $\mathcal{C}(\mathbf{A};\epsilon)$ are nearly ϵ-balls which manifests itself in the relatively flat nature of the graph of $\|\tilde{\mathbf{E}}_\lambda\|_{\mathbf{V}^{-1}}$. Thus it is difficult numerically to use $\|\tilde{\mathbf{E}}_\lambda\|_{\mathbf{V}^{-1}}$ to distinguish values of λ.

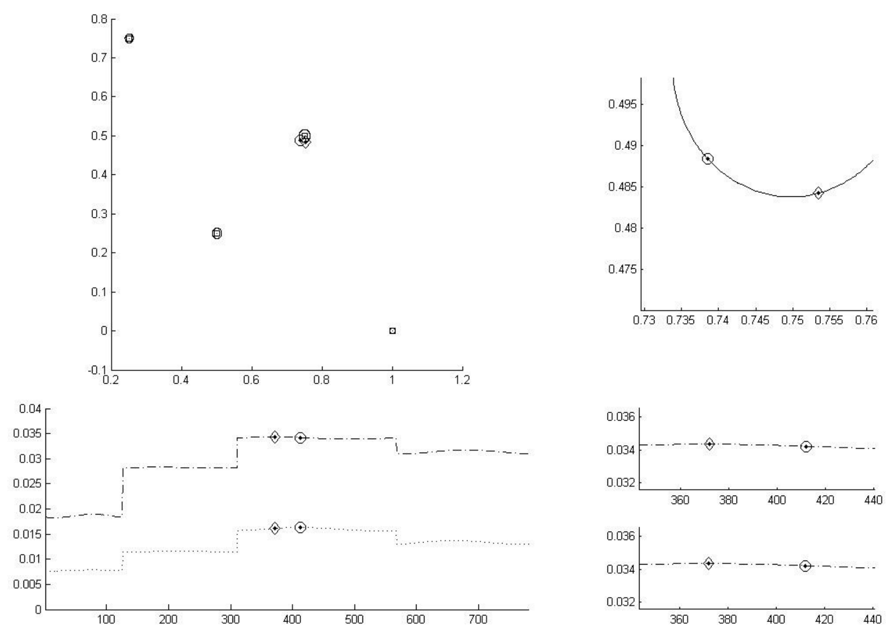

Figure 9. Pseudospectrum of random matrix for $\epsilon = 0.005$

The problem is only made worse with a smaller value of ϵ, as illustrated in Figure 9. Based upon these observations, in order to see more compelling numerical evidence in support of the conjecture, we likely need to consider a somewhat nonnormal matrix with a sufficiently small ϵ. The results for such an example are presented next for a matrix $\mathbf{B}$ where the condition number of the diagonalizing matrix is 28.2.

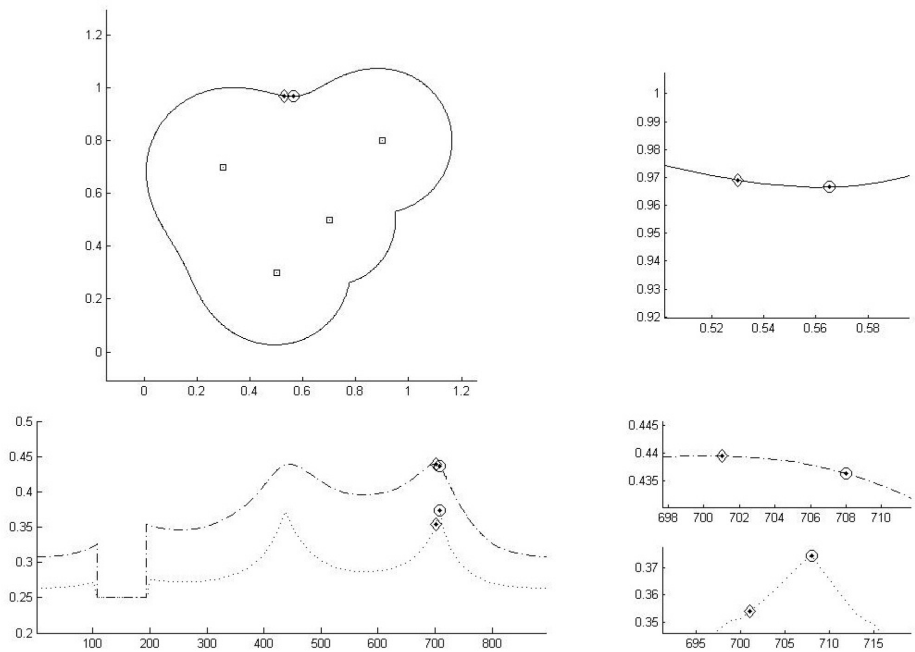

Figure 10. Pseudospectrum of random matrix for $\epsilon = 0.25$

In Figure 10, we see again that for an ϵ value which does not give distinct closed curves about each of the eigenvalues of the matrix, the numerical evidence does not support our conjecture. Therefore in Figure 11 we reduce the ϵ, and we see that our conjecture has merit to be true.

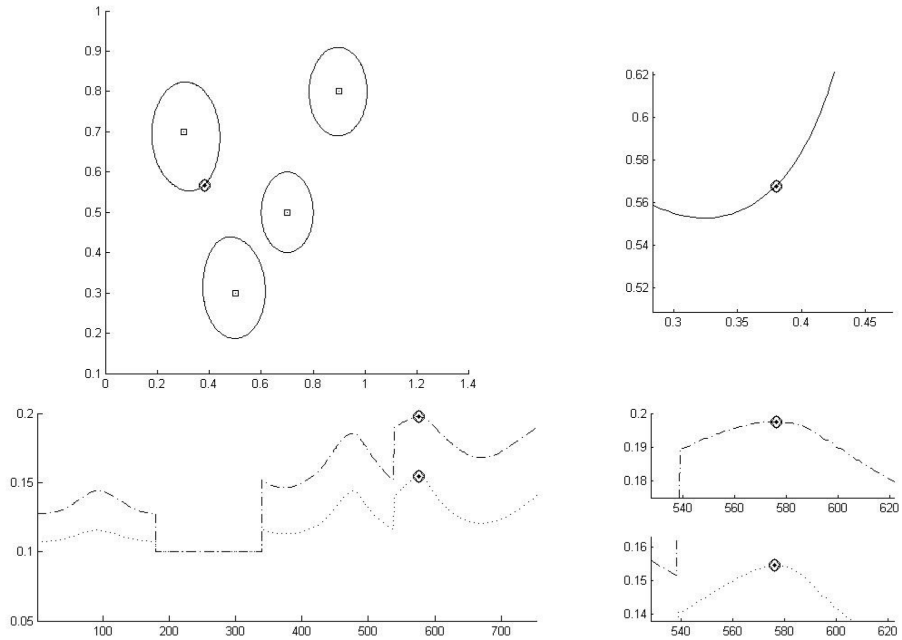

Figure 11. Pseudospectrum of random matrix for $\epsilon = 0.1$

Our final observation that we believe is relevant to our conjecture is on the subject of stability. In [9, pp.811], the stability radius of a system represented by a matrix $\mathbf{A}$ is defined to be the supremum of ϵ such that $\mathcal{T}(\mathbf{A};\epsilon)$ is a subset of the stability region. We know that if $\lambda \in \mathcal{C}(\mathbf{A};\epsilon)$, all $\mathbf{E}$ such that $\lambda \in \Lambda(\mathbf{A}+\mathbf{E}) \subset \mathcal{T}(\mathbf{A};\epsilon)$ must have $\|\mathbf{E}\| = \epsilon$. Thus for every $\lambda \in \mathbb{C} \setminus \Lambda(\mathbf{A})$, there exists $\epsilon > 0$ such that $\epsilon = \frac{1}{\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\|}$ and $\lambda \in \mathcal{C}(\mathbf{A};\epsilon)$. Therefore for a given matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ where there is a defined stability region $\mathbb{C}_s \subset \mathbb{C}$, if $\Lambda(\mathbf{A}) \cap (\mathbb{C}\setminus\mathbb{C}_s) = \emptyset$, then for all $\epsilon > 0$ such that $\min\limits_{\gamma\in\overline{\mathbb{C}\setminus\mathbb{C}_s}} \left[\frac{1}{\|(\gamma\mathbf{I}-\mathbf{A})^{-1}\|}\right] > \epsilon > 0$, $\mathcal{T}(\mathbf{A};\epsilon) \cap (\mathbb{C}\setminus\mathbb{C}_s) = \emptyset$. In particular, if $\mathbb{C}_s$ is closed, then for all $\epsilon > 0$ such that $\min\limits_{\gamma\in\overline{\mathbb{C}\setminus\mathbb{C}_s}} \left[\frac{1}{\|(\gamma\mathbf{I}-\mathbf{A})^{-1}\|}\right] \geq \epsilon > 0$, $\mathcal{T}(\mathbf{A};\epsilon) \cap (\mathbb{C}\setminus\mathbb{C}_s) = \emptyset$. We give two examples to numerically confirm this observation. Our first example is a matrix defined by a system with the stability region taken to be the left half complex plane.

We first take the matrix $\mathbf{A}$ which is similar to the `grcar(10)` matrix, and find the minimum of one over the norm of the resolvent as defined above for a finite number of points in the closure of the instability region $\mathbb{C}\backslash\mathbb{C}_s$ which is the complement of the left half complex plane. In this case we see that $\mathbb{C}\backslash\mathbb{C}_s$ is closed. For this example we compute $\min_{\gamma\in\overline{\mathbb{C}\backslash\mathbb{C}_s}}\left[\frac{1}{\|(\gamma\mathbf{I}-\mathbf{A})^{-1}\|}\right] = 0.220$. The left plot in Figure 12 is $\mathcal{T}(\mathbf{A}; 0.2204)$, and the right plot is a magnification of the left. $\boxdot$ represents eigenvalues of $\mathbf{A}$. $\odot$ represents the points of intersection. We compute $\mathcal{T}(\mathbf{A}; 0.2204)\cap(\mathbb{C}\backslash\mathbb{C}_s) = \{0.3274i, -0.3274i\}$. Therefore for all ϵ such that $0.2204 > \epsilon > 0$, $\mathcal{T}(\mathbf{A}; \epsilon) \cap (\mathbb{C} \setminus \mathbb{C}_s) = \emptyset$.

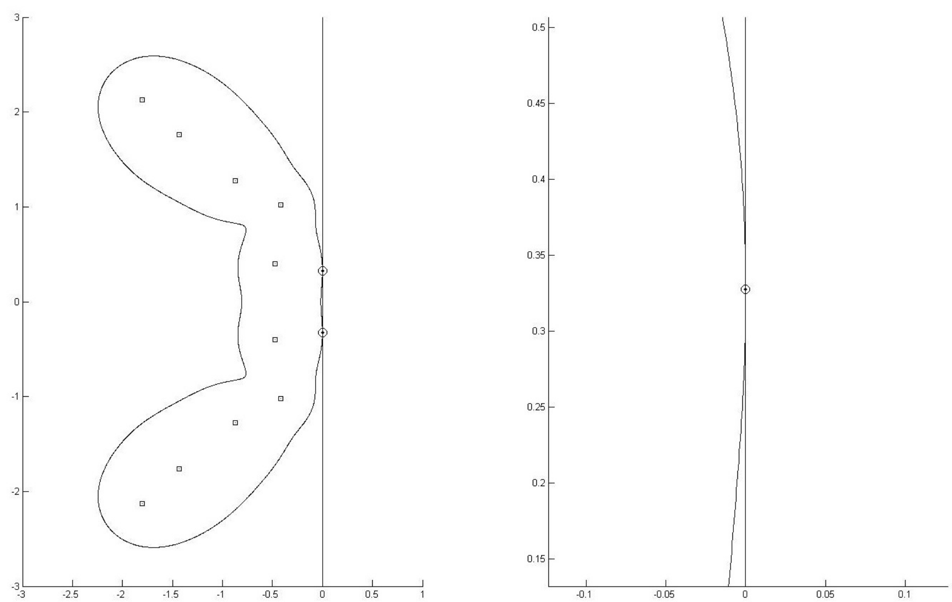

Figure 12. Pseudospectrum - negative eigenvalues

Our second example is for a matrix $\mathbf{B}$ defined for a system with the stability region taken to be the interior of the unit circle. We can numerically calculate the minimum of one over the norm of the resolvent for a finite number of points in the closure of the instability region $\mathbb{C} \setminus \mathbb{C}_s$ which is the complement of the interior of the unit circle. In this case we see that $\mathbb{C} \setminus \mathbb{C}_s$ is closed. For this example we compute $\min_{\gamma \in \overline{\mathbb{C} \setminus \mathbb{C}_s}} \left[\frac{1}{\|(\gamma \mathbf{I} - \mathbf{B})^{-1}\|} \right] = 0.0424$. The left plot in Figure 13 is $\mathcal{T}(\mathbf{B}; 0.0424)$, and the right plot is a magnification of the left. $\boxdot$ represents eigenvalues of $\mathbf{B}$. $\odot$ represents the points of intersection. We compute $\mathcal{T}(\mathbf{B}; 0.0424) \cap (\mathbb{C} \setminus \mathbb{C}_s) = \{-0.0125 - 0.9999i\}$. Therefore for all ϵ such that $0.0424 > \epsilon > 0$, $\mathcal{T}(\mathbf{B}; \epsilon) \cap (\mathbb{C} \setminus \mathbb{C}_s) = \emptyset$. Both examples numerically confirm the expected observation pertaining to stability radius.

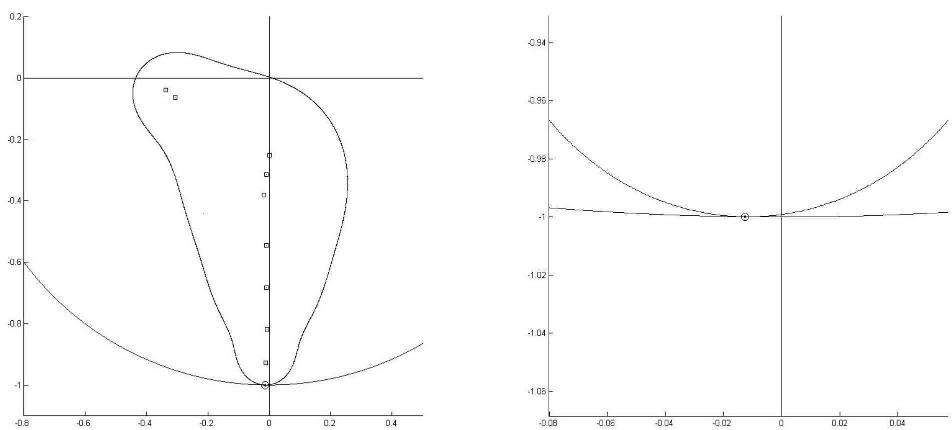

Figure 13. Pseudospectrum - eigenvalues inside the unit circle

III.III Conclusion

We have surveyed definitions and characterizations of pseudospectra and structured pseudospectra. By considering how the normality of a matrix can be affected by changing the underlying norm, we posed a possible method for calculating those perturbations which have the greatest effect on the eigenvalues of a matrix. This idea was examined numerically and confirmed for certain cases.

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APPENDIX A

NOTATIONS

$\mathbb{C}$: The set of all complex numbers 2

$\mathbb{C}^n$: The set of all vectors of length n with complex entries 2

$\mathbb{C}^{m\times n}$: The set of all matrices of size $m \times n$ with complex entries 2

$\mathbf{I}$: The identity matrix in $\mathbb{C}^{n\times n}$ 2

$\mathcal{R}(\mathbf{A})$: The range of the matrix $\mathbf{A}$ 2

$\mathcal{N}(\mathbf{A})$: The null space of the matrix $\mathbf{A}$ 2

$\|\cdot\|$: The norm of a vector in $\mathbb{C}^n$ or a matrix in $\mathbb{C}^{m\times n}$ 2

$\Lambda(\mathbf{A})$: The spectrum of the matrix $\mathbf{A}$ 2

(x, y) : The Euclidean inner product of $x, y \in \mathbb{C}^n$ 2

y^* : The conjugate transpose of y 2

$\|x\|_2$: The vector 2-norm of the vector $x \in \mathbb{C}^n$ 2

$\|\mathbf{A}\|_2$: The induced matrix 2-norm of the matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ 2

$\mathcal{T}(\mathbf{A};\epsilon)$: The set $\mathcal{T}(\mathbf{A};\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{E})$ for some $\mathbf{E}$ with $\|\mathbf{E}\| \le \epsilon\}$ 6

$\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$: The set $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \{\lambda \in \mathbb{C} : \lambda \in \Lambda(\mathbf{A}+\mathbf{BEC})$ for some $\mathbf{E}$ with $\|\mathbf{E}\| \le \epsilon\}$ 9

$(\lambda\mathbf{I}-\mathbf{A})^{-1}$: The matrix defining the resolvent of $\mathbf{A}$ 12

$\mathcal{S}(\mathbf{A};\epsilon)$: The set $\Lambda(\mathbf{A}) \cup \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I}-\mathbf{A})^{-1}\| \ge \epsilon^{-1}\}$ 12

$\mathcal{C}(\mathbf{A};\epsilon)$: The boundary of the ϵ-pseudospectrum of $\mathbf{A}$ 13

$\mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$: The set $\Lambda(\mathbf{A}) \cup \{\lambda \in \mathbb{C} : \|\mathbb{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\| \ge \epsilon^{-1}\}$ 15

APPENDIX B

PROOFS

Proof $\mathcal{S}(\mathbf{A};\epsilon) = \mathcal{T}(\mathbf{A};\epsilon)$

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$ be arbitrary.

Proof. [17, `http://www.cs.ox.ac.uk/pseudospectra/thms/thm1.pdf`]

To prove $\mathcal{T} \subset \mathcal{S}$ let $\lambda \in \mathcal{T}$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{S}$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$. Thus $\lambda \in \Lambda(\mathbf{A}+\mathbf{E})$ for some $\mathbf{E} \in \mathbb{C}^{n\times n}$ with $0 < \|\mathbf{E}\| \le \epsilon$. Thus there exists $v \in \mathbb{C}^n$ with $\|v\| = 1$ such that $(\mathbf{A}+\mathbf{E})v = \lambda v$. Thus $\mathbf{E}v = (\lambda\mathbf{I}-\mathbf{A})v$ which implies $(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{E}v = v$. Now by taking the norm of both sides we get the following: $1 = \|v\| = \|(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{E}v\| \le \|(\lambda\mathbf{I}-\mathbf{A})^{-1}\|\|\mathbf{E}\|\|v\| \le \epsilon\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\|$. Thus $\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\| \ge \frac{1}{\epsilon}$. Thus $\lambda \in \mathcal{S}$.

To prove $\mathcal{S} \subset \mathcal{T}$ let $\lambda \in \mathcal{S}$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{T}$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$. Thus $\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\| \ge \frac{1}{\epsilon}$. Thus there exists $u \in \mathbb{C}^n$ where $u \neq 0$ such that $\|(\lambda\mathbf{I}-\mathbf{A})^{-1}u\| \ge \frac{1}{\epsilon}\|u\|$. Define $v = (\lambda\mathbf{I}-\mathbf{A})^{-1}u$ which implies that $(\lambda\mathbf{I}-\mathbf{A})v = u$. Thus $\|u\| = \|(\lambda\mathbf{I}-\mathbf{A})v\| \le \epsilon\|v\|$. Define $\mathbf{E}x = (\lambda\mathbf{I}-\mathbf{A})\frac{(x,v)}{\|v\|^2}v$. Thus $\|\mathbf{E}x\| \le \|(\lambda\mathbf{I}-\mathbf{A})v\|\frac{|(x,v)|}{\|v\|^2} \le \epsilon\|v\|\frac{\|x\|\|v\|}{\|v\|^2} = \epsilon\|x\|$. Thus $\|\mathbf{E}\| \le \epsilon$. Therefore $(\mathbf{A}+\mathbf{E})v = \mathbf{A}v + (\lambda\mathbf{I}-\mathbf{A})\frac{(v,v)}{\|v\|^2}v = \mathbf{A}v + (\lambda\mathbf{I}-\mathbf{A})v = \lambda v$. Thus $\lambda \in \mathcal{T}$.

Thus $\mathcal{S} = \mathcal{T}$. □

Proof $\mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times s}$, $\mathbf{C} \in \mathbb{C}^{t\times n}$ and $\epsilon > 0$ be arbitrary, $\mathbf{B}$ and $\mathbf{C} \neq 0$.

Proof. [8]

To prove $\mathcal{T} \subset \mathcal{S}$ let $\lambda \in \mathcal{T}$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{S}$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$. Thus there exists $\mathbf{E}$ such that $\lambda \in \Lambda(\mathbf{A} + \mathbf{BEC})$. Thus there exists $v \neq 0$, $v \in \mathbb{C}^n$ such that $(\mathbf{A} + \mathbf{BEC})v = \lambda v$. Define $u = \mathbf{EC}v$. Thus $u \neq 0$ and

$$\begin{aligned} u &= \mathbf{EC}v \\ &= \mathbf{EC}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{BEC}v \\ &= \mathbf{EC}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u. \end{aligned}$$

This implies that $\|u\| = \|\mathbf{EC}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u\|$; which implies

$$\begin{aligned} 1 &= \|\mathbf{EC}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u\|\,\|u\|^{-1} \\ &\leq \|\mathbf{EC}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\|\,\|u\|\,\|u\|^{-1} \\ &\leq \|\mathbf{E}\|\,\|\mathbf{C}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\|. \end{aligned}$$

Thus $\|\mathbf{C}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\| \geq \|\mathbf{E}\|^{-1} \geq \frac{1}{\epsilon}$. Thus $\lambda \in \mathcal{S}$.

To prove $\mathcal{S} \subset \mathcal{T}$ let $\lambda \in \mathcal{S}$. If $\lambda \in \Lambda(\mathbf{A})$ then $\lambda \in \mathcal{T}$. Therefore assume $\lambda \notin \Lambda(\mathbf{A})$. Thus there exist a $u \in \mathbb{C}^s$, $\|u\| = 1$ such that $\|\mathbf{C}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u\| = \|\mathbf{C}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\|$. Define $y = \left(\frac{1}{\|\mathbf{C}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u\|}\right)\mathbf{C}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u$.

Thus

$$
\begin{aligned}
y^*\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u &= \left(\frac{1}{\|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\|}\right) u^*(\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B})^*\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u \\
&= \frac{\|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u\|^2}{\|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u\|} \\
&= \|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u\|.
\end{aligned}
$$

Define $\hat{\epsilon} = \frac{1}{\|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\|} \leq \epsilon$. Thus

$$
\begin{aligned}
u &= \hat{\epsilon}\|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\|u \\
&= \hat{\epsilon}u\|\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u\| \\
&= \hat{\epsilon}uy^*\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u.
\end{aligned}
$$

Define $\mathbf{E} = \hat{\epsilon}uy^*$. Thus $\mathbf{E}$ is an $s \times t$ matrix and $\|\mathbf{E}\| = \|\hat{\epsilon}uy^*\| \leq \hat{\epsilon} \leq \epsilon$. Define $x = (\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u$. Thus

$$
\begin{aligned}
\mathbf{EC}x &= \mathbf{EC}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u \\
&= \hat{\epsilon}uy^*\mathbf{C}(\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}u \\
&= u.
\end{aligned}
$$

Thus $x \neq 0$ and $x = (\lambda\mathbf{I}-\mathbf{A})^{-1}\mathbf{BEC}x$ which implies $\lambda x = (\mathbf{A}+\mathbf{BEC})x$. Thus $\lambda \in \mathcal{T}$. Thus $\mathcal{S}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon) = \mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$. □

APPENDIX C

MATLAB CODES

PM.m – Poor Man's Method to Construction Pseudospectra

```
% This function is based upon the 'Poor Man's' method presented in
% [17, pp.378−379], and developed from ps.m [6].
%%
function PM(matrix,epsilon,number)
% 'matrix' is the input matrix that the pseudospectra will be found
% for.
% 'epsilon' is the vector of epsilon values to be found.
% 'epsilon' must be a positive real number.
% 'number' is the number of perturbation per epsilon.
%%
% Check dimensions, and get basic values for computation.
[rows cols]=size(matrix);
if rows≤1 || cols≤1 || isempty(matrix)
   error('Please choose a square matrix with minimum dimension ...
       of two.')
end
if abs(rows−cols)>0
   error('Please choose a square matrix')
end
n=length(matrix); lambda=eig(matrix); times=length(epsilon);
%%
% Other error checks.
if nargin<2 || isempty(epsilon)
```

```
    epsilon=[1E-1 1E-2 1E-3];
end
if nargin<3 || isempty(number)
    number=250;
end
for i=1:times
    if real(epsilon(i))≤0 || abs(imag(epsilon(i)))>0
        error('Please make sure all epsilon values are positive ...
            real numbers.')
    end
end
if number<1 || abs(fix(number)-number)>0
    error('Please provide a positive non-zero integer for the ...
        number of perturbations.')
end
%%
% Here store the eigenvalues of the ('times'*'number')
% perturbed given matrices which are perturbed by random
% matrices with norm equal to the respective epsilon.
epsilon=sort(epsilon,'descend'); X=zeros(number*n,times);
del=zeros(n,times*n);
for i=1:times;
    for j=1:number;
        del1=randn(n)+randn(n)*1i;
        del(1:n,((i-1)*n+1):i*n)=epsilon(i)*del1/(norm(del1));
        X((j-1)*n+1:j*n,i)=eig(matrix+del(1:n,((i-1)*n+1):i*n));
    end;
```

```
end;
%%
% Plot the eigenvalues of the perturbations.
figure
hold on
% if length(epsilon)==1
% title('Epsilon Pseudospectrum');
% else title('Epsilon Pseudospectra');
% end
COLORS=gray(2*times);
for i=1:times
   plot(real(X(1:number*n,i)),imag(X(1:number*n,i)),'o', ...
      'color',COLORS(i,:),'markersize',4,'MarkerFaceColor', ...
         COLORS(i,:));
end
% Plot eigenvalues of matrix
plot(lambda,'k+','MarkerSize',18,'LineWidth',4);
hold off;
%%
% Clean up the visual data
fig = gcf;
axis tight
axis equal
xlim=get(gca,'Xlim'); ylim=get(gca,'Ylim');
xtick=(xlim(2)-xlim(1))/100; ytick=(ylim(2)-ylim(1))/100;
axis([xlim(1)-xtick xlim(2)+xtick ylim(1)-ytick ylim(2)+ytick]);
axes = gca;
```

```
set (get (axes, 'XLabel'), 'FontSize', 20);
set (get (axes, 'YLabel'), 'FontSize', 20);
set (get (axes, 'Title'), 'FontSize', 24);
set (axes, 'Box', 'off', 'TickDir', 'out', 'XMinorTick', 'on', ...
    'YMinorTick', 'on', 'FontSize', 16);
screen_size = get(0, 'ScreenSize');
set(fig, 'Position', [0 0 screen_size(3) screen_size(4)]);
end
```

PSA.m − Construction of Contour Lines of Pseudospectra

```
% This code is developed from the eigtool.m present by
% Trefethen and others.
% It utilizes the base algorithm to find the contour plot of the
% pseudospectra of a given matrix for given epsilons.
% Lines 92—113 are from the eigtool.m algorithm lines 368—390.
%%
function PSA(matrix,epsilon,d,maxit)
time=tic;
% 'matrix' is the input matrix that the pseudospectra will be found
% for.
% 'epsilon' is the vector of epsilon values to be found.
% 'epsilon' must be a positive real number.
% 'd' is the value of the distance between points in the meshgrid.
% The smaller the value of 'd', the more precision the contour plot
% will have.
% 'd' must be a positive real number.
% 'maxit' is a value used by the eigtool.m to max the true
```

```
% value of the norm of the resolvent at each point in the meshgrid;
% it reduces the difference in the calculated value and the true
% value due to calculation errors such as round error.
% 'maxit' must be a positive integer since it is being used as a
% counter.
% The user should normally not input a number in for maxit; 10 is
% sufficient for visual data, but for computational data with
% very small epsilon values or very non-normal matrices, a larger
% value should be used.
%%
% Check dimensions, and get basic values for computation.
[rows cols]=size(matrix);
if rows≤1 || cols≤1 || isempty(matrix)
   error('Please choose a square matrix with minimum dimension ...
       of two.')
end
if abs(rows-cols)>0
   error('Please choose a square matrix')
end
if nargin<2 || isempty(epsilon)
   epsilon=[1E-1 1E-2 1E-3];
end
n=length(matrix); lambda=eig(matrix); number=length(epsilon);
%%
% Other error checks.
if nargin<3 || isempty(d)
   d=0.05;
```

```
end
if nargin<4 || isempty(maxit)
   maxit=10;
end
if number<1
   error('Please provide at least one positive real number ...
       epsilon.')
end
for i=1:number
   if real(epsilon(i))≤0 || abs(imag(epsilon(i)))>eps
      error('Please make sure all epsilon values are positive ...
          real numbers.')
   end
   if epsilon(i)<abs(d) || d≤0
      error('Please choose a smaller postive value d>0 for the ...
          mesh grid.')
   end
end
if maxit<0 || abs(fix(maxit)−maxit)>eps
   error('Please choose a positive integer for the computation ...
       error.')
end
%%
% Set 'E' to be the reciprocal values of the 'epsilon' entry to set
% the contour values to be plotted.
% This is done for as in the S set, the norm of the resolvent is
% compared in the inequallity to the reciprocal of epsilon.
```

```
E=1./epsilon;
%%
% Here we are computing the area for and creating the meshgrid to
% make sure that we include enough points to include the
% contour plots for all epsilons.
% Find midpoint value between the largest and smallest real and
% imagenary values of the eigenvalues.
q=(max(real(lambda))+min(real(lambda)))/2;
p=(max(imag(lambda))+min(imag(lambda)))/2;
% Set the distance away from the midpoint values to be included in
% the meshgrid
r=norm(matrix)*norm(inv(matrix))+1.5*max(epsilon);
% Make all values integers to eliminate errors in the next step
q=fix(round(q)); p=fix(round(p)); r=fix(round(r));
% Set the intervals for the meshgrid
x=q-r:d:q+r; y=p-r:d:p+r;
% Meshgrid
[X,Y]=meshgrid(x,y);
%%
% This is the eigtool.m algorithm. The algorithm computes the value
% of the norm of the resolvent at each point in the mesh grid, but
% uses the Schur decompostion of the given matrix instead of the
% given matrix to speed up the function.
Z=zeros(length(y),length(x)); sigmin=Z; T=schur(matrix,'complex');
for k=1:length(x)
   for j=1:length(y)
      T1=(x(k)+y(j)*1i)*eye(length(matrix))-T; T2=T1';
```

```
        sigold=0; qold=zeros(length(matrix),1); beta=0; H=[];
        q=randn(length(matrix),1)+1i*randn(length(matrix),1);
        q=q/norm(q);
        for p=1:maxit
            v=T1\(T2\q)-beta*qold; alpha=real(q'*v); v=v-alpha*q;
            beta=norm(v); qold=q; q=v/beta; H(p+1,p)=beta;
            H(p,p+1)=beta; H(p,p)=alpha; sig=max(eig(H(1:p,1:p)));
            if abs(sigold/sig-1)<1e-3, break, end
            sigold=sig;
        end
        sigmin(j,k)=sqrt(sig); Z(j,k)=sigmin(j,k);
    end
end
%%
% Here we are going to develop the contour plot figure.
figure
hold on;
% if length(epsilon)==1
% title('Epsilon Pseudospectrum');
% else title('Epsilon Pseudospectra');
% end
% If the matrix is normal, input the epsilon balls around each
% eigenvalue of 'matrix' for each value in 'epsilon'.
% This is done first with black dotted lines so that they will be
% under the contour lines thus not interfering with the visual data
% of the contour lines.
if (norm(matrix*matrix'-matrix'*matrix)≤0.001)
```

```
    ang=0:0.01:2*pi;
    for e=1:n;
       x1=real(lambda(e)); y1=imag(lambda(e));
       for i=1:number;
          x2=epsilon(i)*cos(ang)+x1; y2=epsilon(i)*sin(ang)+y1;
          plot(x2,y2,'k--','LineWidth',0.5);
       end;
    end;
end;
% Plot the contour lines
% The 'if else' is used due to the exclusive input needed when only
% one epsilon value is used.
if number==1
    [¬,h]=contour(X,Y,Z,[E E]);
else
    [¬,h]=contour(X,Y,Z,E);
end
set(h,'LineWidth',2)
colormap(gray(number))
% Plot the eigenvalues of 'matrix' for visual interpretation of the
% movement of said eigenvalues.
plot(lambda,'k+','MarkerSize',4);
hold off
% Clean up the visual data
fig = gcf;
axis tight
axis equal
```

```
148 xlim=get(gca,'Xlim'); ylim=get(gca,'Ylim');
149 xtick=(xlim(2)-xlim(1))/100; ytick=(ylim(2)-ylim(1))/100;
150 axis([xlim(1)-xtick xlim(2)+xtick ylim(1)-ytick ylim(2)+ytick]);
151 axes = gca;
152 set (get (axes, 'XLabel'), 'FontSize', 20);
153 set (get (axes, 'YLabel'), 'FontSize', 20);
154 set (get (axes, 'Title'), 'FontSize', 24);
155 set (axes, 'Box', 'off', 'TickDir', 'out', 'XMinorTick', 'on', ...
        'YMinorTick', 'on', 'FontSize', 16);
156 screen_size = get(0, 'ScreenSize');
157 set(fig, 'Position', [0 0 screen_size(3) screen_size(4)]);
158 Time=toc(time)
159 end
```

Construction of Contour Lines of Structured Pseudospectra

For the computation of the contour line of a structured pseudospectrum $\mathcal{T}(\mathbf{A};\mathbf{B},\mathbf{C},\epsilon)$, replace lines 85-106 in PSA.m with the following:

```
%%
% Here we will calculate the norm of the resolvent with
% respect to the structure defined by B and C.
Z=zeros(length(y),length(x));
for k=1:length(x)
   for j=1:length(y)
      Z(j,k)=norm(C*inv((X(j,k)+Y(j,k)*1i)*eye(n)-matrix)*B);
   end
end
```

GenNormal.m [20] − Construction of Random Normal Matrix

```
% GENNORMAL Generate any type of normal matrix
% N=GENNORMAL(n)
% Generates an arbitrary complex normal matrix of size n.
%
% Raf Vandebril
% raf.vandebril − a − cs.kleuven.be
% Revision Date: 1/9/2008
function N=GenNormal(n)
% Define I to be the square root of −1
I=sqrt(−1);
D=diag(rand(n,1)+I*rand(n,1));
A=rand(n,n)+I*rand(n,n);
[Q,R]=qr(A);
N=Q'*D*Q;
end
```