Traveling Representations in a Fifth Grade Classroom: An Exploration of Algebraic Reasoning

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Abstract:

In this three-day teaching experiment along with follow up interviews, algebraic concepts related to pattern-finding tasks were examined with 25 fifth grade students. The specific focus centered on representations from a realistic mathematics education perspective, meaning a model “of” a situation toward a model “for” a situation. Within this context, certain situational models were found that seemed to travel and permeate throughout the entire class. Students were able to generalize and justify based on the models developed during whole class discussions. Several weeks after the teaching experiment, follow up interviews indicated that the representations generated were still prevalent in students’ descriptions of the activities. Findings, analysis of findings, and implications of the study will be discussed.

Article:

OBJECTIVE

The early algebra movement in the United States is gaining momentum throughout all areas of the elementary curriculum (Kaput, Carraher, & Blanton, 2008; NCTM, 2000). Recent research focuses on the need for preservice teachers to develop a better understanding of their own early algebra concepts (Richardson, Berenson, & Staley, 2009). Representations often play a vital role in helping students to reason algebraically (Presmeg, 2005; Smith, 2008; NCTM, 2000). The purpose of this report is to take recent research (Richardson, Berenson, & Staley, 2009) and extend it further through the eyes of fifth grade students using similar tasks. The research here is focused on fifth grade representations, namely the models generated within them, and the effects of a teaching experiment designed to make notable improvements in their algebraic reasoning (Lesh & Kelly, 2000).

THEORETICAL FRAMEWORK

For 30 plus years, the Dutch Realistic Mathematics Education (RME) movement has provided a framework for a host of studies. The term *realistic* refers to problems having a context of ‘real-world’ or simply imagined (Presmeg, 2003). Centered on the idea of mathematics as human activity, Freudenthal (1977) insisted that context must play an important role in the teaching and learning of mathematics. Progressive formalization or mathematization is a key process within the RME philosophy and is comprised of students exploring mathematical ideas informally and then making gradual progress to more formal, higher level thinking. A variety of mathematical ideas are defined within progressive formalization and here the focus is on models. Models, in the context of this study, are defined as representations of problem situations that contain a realistic or imaginable context, possess flexibility, and can be re-invented by students on their own (Van den Heuvel-Panhuizen, 2003). As discussed by Presmeg (2003), Van den Heuvel-Panhuizen (2003) notes form-function shifts in the types of models students generate during their mathematical activity. Van den Heuvel-Panhuizen also draws from Streefland's (1985) work where he described a model *of* a situation to a model *for* a situation. Meaning, a student uses a model to investigate a particular problem but then later transforms the model to relate to other situations and/or to provide a way to better understand the situation at hand.

The work described here offers an adaptation of Van den Hevel-Panhuizen's and Streefland's research. For this study, *Context* accounts for how the student initially engaged in the problem, both through verbalizing ideas and modeling those ideas. *Flexibility* refers to how the student took the context of the problem and started finding patterns, hence working flexibly within the context. *Reinvention* indicates how the student re-conceptualized the problem. Flexibility and reinvention are closely related and difficult to separate. A unique contribution from the authors of this study is the idea of *traveling*, which means to capture the permeation of an idea within a class. A teaching experiment (Lesh & Kelly, 2000) is utilized along with task-based interviews (Goldin, 2000) to explore traveling and other moments of student investigations. Thus, the purpose of the research is to focus on specific representations that fifth graders used within a teaching experiment to solve algebraic pattern-finding tasks. Our specific research question is, how do representations of early algebra ideas travel over space and time in a fifth grade classroom?

METHODOLOGY

Design and Subjects

This teaching experiment focused on a fifth grade class at a primarily white, rural, science/mathematics focused elementary school in the southeastern part of the United States. A whole class teaching experiment was used because the researchers felt it was the most well suited setting to get at children's mathematical thinking (Lesh & Kelly, 2000). The researchers used task-based interviews because it was agreed that they were the most powerful way to focus on the individual student (Goldin, 2000).

In terms of mathematical ability, the class being observed had 25 average to above average students. During three consecutive days of instruction, video cameras focused on three student dyads recommended by the classroom teacher. Audio data were collected for all 12 dyads. Six weeks after the original three-day instructional period (about one and a half to two hours) the researchers returned to the school and did follow up interviews with students. In these interviews the researchers then asked them to extend their understanding of the original tasks.

In a larger study (Richardson, 2010), the focus is on 25 students but for this preliminary study, the focus is on both Dan's work and the work of other students related to his work. Part of Dan's work was whole-class and small group and another part was from his work from the one-hour follow up interview. Dan was confident and wrote clear explanations on his paper. He also stood out in the group because he was able to generalize rules, even on the first day of the teaching experiment.

Task and Instruction

The first task was called square tables (see Figure 1 for an abbreviated version). In this task students were asked to determine how many people could sit around a square table, if one person could sit on each side. They then determined how many people could sit around two contiguous square tables. The primary objective was for the students to be able to answer how many people could sit around *n t*ables, where *n* was an arbitrarily large number. The same question was asked on the second day (see Figure 2) but used triangles and the third day involved hexagons. Students were asked each day to organize their data in a table, using pattern blocks to build the larger models as needed, and to continue to describe any patterns they discovered during the teaching experiment. While this set of train tasks usually begins with triangle pattern blocks, earlier results of these perimeter tasks led us to change the hypothetical trajectory of the experiment to include the square tables in the first week, and then the triangle tables in the second week (Berenson, Wilson, P.H., Mojica, G., Lambertus, A., & Smith, R., 2007). The triangle task is also difficult for students to re-invent and give context to since one does not generally sit at a triangle shaped table.

During the follow up interviews the tasks were similar to the ones described above except that students were asked to consider tables that had two people sitting on each side instead of one. If they showed an ability to quickly grasp and generalize the new problem, they were then asked to examine tables with two people on each side of a table shaped like a pentagon, a shape that had not been part of the original set of three tasks. The researchers were interested in what insights they might construct during this new pattern finding activity.

Figure 1. First task investigated by fifth grade students in an algebraic reasoning teaching experiment.

Day 1 - If you have one square table, how many chairs will fit around the table if you have one chair on each side of the square? Two square tables? Three square tables? Do you see a pattern yet? If yes, write down a description of your number pattern.

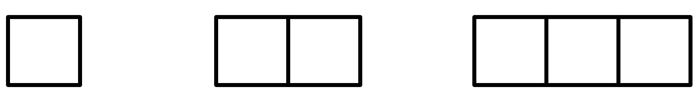
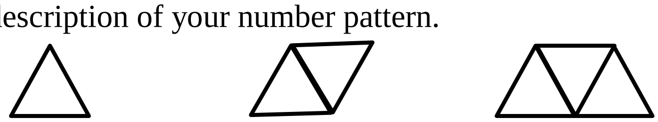


Figure 2. Second task investigated by fifth grade students in an algebraic reasoning teaching experiment.

Day 2 - If you have one triangular table, how many chairs will fit around the table if you have one chair on each side of the triangle? Two triangular tables? Three triangular tables? Do you see a pattern yet? If yes, write down a description of your number pattern.



Evidence and Analysis

Sources of data included video, audio, and written work of fifth grade students. The conversations were filmed using digital video cameras and conversations were also captured using digital audio recorders. The videotape interviews were transcribed and the transcripts were analyzed. Pseudonyms are used in all descriptions. Our overall analysis looks at modeling *of* the problem to modeling *for* understanding, which is demonstrated in Figure 3. It is what happens within modeling *for* understanding that is analyzed, so three areas of modeling were coded for: context, flexibility, and reinvention (Van den Hevel-Panhuizen, 2003). Table 1 lists our coding of Dan's work.

Figure 3. Dan's progression on the square tables task from 4, 6, and 100. Adapted from Streefland (1985). **NOT AVAILABLE**

Table 1. Dan's progression throughout the entire teaching experiment.

Task	***Contextualizing the Problem***	***Demonstrating Flexibility***	***Reinventing Model***
Day 1 – Square Task	Drew squares and labeled them to represent the tables. Also made T chart.	Stopped drawing tables and wrote rule "For every table added 2 is added to the chairs."	Line drawn with 100 on bottom, 100 on top, and 1 on each end.
Day 2 – Triangle Task	Drew triangles and labeled them to represent the tables. Also made T chart.	Wrote rule, "every table added, one chair is added"	Did not draw reinvented model.
Day 3 – Hexagon, etc. Task	Drew no shapes. Simply filled in worksheets w/ rules.	Wrote rules in written and symbolic form.	Did not draw reinvented model.
Follow-up Interview Task (pentagons)	Drew T chart, pentagons and labeled some parts	Verbally expressed rules.	Line drawn with 200 on top, 100 on bottom, and 1 on each end.

RESULTS

There were two major results from the analysis of Dan's data. First, he reinvented the initial model of squares into a new model during day one and the follow up interview as a means to generalize algebraic patterns. Second, Dan's model had an impact on other student models in the classroom and six students utilized his model or a form of his model to describe their solutions to the algebraic patterning tasks. The results of each major finding are listed and the results detailed.

Dan's use of his reinvented model

Figures 3 and 4 demonstrate Dan's modeling during day 1 and during his final interview.

Observations from the video data show Dan meticulously drawing the squares, labeling them, and then drawing a T table (not shown) next to his work. When asked how many people could sit around 100 tables, he started his work on a new sheet of paper, at which time his re-invented model was drawn. Other students kept filling in their T tables until they reached 100; others tried to find a pattern on their T tables, while some tried to multiply and add – all of which were notable occurrences for each student. However, Dan's re-invented model seemed to enable him to generalize a rule and justify that rule. For example, on day one, he wrote, "For every table added, 2 is added to the chairs." He went on to note 100 tables = 200 chairs + 2 = 202. Although on days two and three, he did not draw his reinvented model, he was easily able to express a generalization and justify his answers. On the follow up interview, though, he revisited his reinvented model and even used it to express his patterns for the pentagon task. This was surprising to the researchers because the students had difficulty in giving context to the other patterns that were not squares.

Figure 4. Dan's reinvented model appearing again during the final interview questions. **NOT AVAILABLE**

The traveling of Dan's re-invented model

During day one of the teaching experiment, students were asked by the researchers to present their work to the entire class. Some students explained that the answer to the 100 table question was to find how many people could sit around 10 tables which was 22 and then multiply that by 10 to get 220. While pointing to his reinvented model, Dan presented his work stating that it was 100 x 2 + 2 = 202 but did a poor job in verbalizing why he had gotten this. Anna soon came up and re-drew his model in the same way and attempted to explain it but got stuck. It was at this point that the researcher asked, "Where does the plus two come from? Do you know where the plus two comes from in that drawing?" Anna was unable to elaborate more and looked to someone else to come up and continue. At which time Kevin came up and drew a form of Dan's model. It was a long rectangle, instead of just a line, with sloppy marks drawn inside of it to indicate the individual tables. He emulated Dan's labeling by writing a 100 across the top and a 100 across the bottom, with a 1 on each end. He stated, "These – there are 100 people on each side, but there's another person on each side – so there's two right here. So that's what it comes from."

It was after this explanation that students abandoned the 220 solution and researchers observed additional students drawing some form of Dan's model to illustrate the accurate 202 answer. In total, six students explicitly drew a form of Dan's model and wrote in words a generalization of the patterns they found. For example, Brenda wrote "Multiply tables by 2 and then add 2 to find the number of chairs." Another student, Kim, wrote, "100 people on each side of the tables so 100 + 100 = 200 then count the people on the end = 202." Two other students used Dan's model to verbally express their generalization but did not explicitly draw it. For example, Melanie wrote, "2 sides on a table (long) then multiply 100 x 2 for the 2 sides then add 2 end sides." Like Melanie, Stephen wrote out his rule in words as opposed to drawing it explicitly. He wrote, "You would take 100 and x it by 2 and get 200 + 2 gives you the answer." Upon close look at the video data and written work, it is fairly clear here that both Melanie and Stephen are using a version of the model that Dan had first come up with to state their own rules in word form even though they were not necessarily drawing this model on their papers.

Researchers Perceptions

The researchers perceived that each day within the teaching experiment built on the previous day since the students generally were more detailed in their descriptions as each new task was introduced. The follow up interviews, which were conducted six weeks after the teaching experiment, surprised the researchers because the students were able to easily engage in the tasks posed and remembered how they had worked their initial problems.

DISCUSSION

The analysis of Dan's modeling, algebraic generalizations created from his modeling, and the impact of his work on other fifth graders, indicate both individual and whole class growth of algebraic reasoning. The researchers are reminded that learning is dynamic and additional time is needed to find out more about fifth graders' development of early algebra concepts and how modeling enhances those concepts. An examination of the modeling categories informs researchers and instructors of how a student can take a common model (e.g., squares, triangles, etc.) and re-imagine the model in a way that enables the student to make key algebraic generalizations. The ability to contextualize the square tables task, meaning students could imagine people sitting at tables, greatly helped the students re-invent within the problem. On days two and three, where the tasks were less contextual, meaning the same questions posed utilized triangles and hexagons, students were easily able to make a generalization by simply drawing the shapes and recording the patterns in a T table thus generating a rule. Posing a realistic question first, in this case the square tables task, enabled the students to later think about less contextual problems with ease, as indicated by the data.

Spending day 1 with only the square tables task was an important decision made by the researchers. Although the students were actively engaged in the task, they were reserved in expressing their findings. Their eagerness to share and their energy levels were much higher on days two and three and their ability to work more efficiently and less recursively were evident. However, their attention to detail and determination to work through several issues that arose on day one were instrumental in their successes for days two and three.

Putting the analysis of this teaching experiment, and Dan's work in particular, within the framework of realistic mathematics gives researchers valuable insight into how students conduct mathematical investigations and how they construct models. Examining Dan's work and seeing how much impact it had upon the understanding of the rest of the class demonstrates this clearly. His ability to work flexibly and reinvent problems enabled him to come up with creative and unique solutions that in turn benefited his peers. This permeation of an idea within a class of students or group of people is what the researchers have defined as *traveling.* It was quite clear through the course of this teaching experiment that Dan's idea permeated the entire class and helped his peers solve the problem in a similar manner as he did. Further more the researchers saw that this idea had longevity in that some of his classmates used the same strategy six weeks later in the follow up interviews. It is for this reason that the researchers consider the notion of traveling, and how the idea that traveled came into being as an important concept.

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