# Partial words and the critical factorization theorem revisited

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**Abstract:**
In this paper, we consider one of the most fundamental results on the periodicity of words, namely the critical factorization theorem. Given a word $w$ and nonempty words $u$, $v$ satisfying $w = uv$, the *minimal local period* associated with the factorization $(u, v)$ is the length of the shortest square at position $|u| - 1$. The critical factorization theorem shows that for any word, there is always a factorization whose minimal local period is equal to the minimal period (or global period) of the word.

Crochemore and Perrin presented a linear time algorithm (in the length of the word) that finds a critical factorization from the computation of the maximal suffixes of the word with respect to two total orderings on words: the lexicographic ordering related to a fixed total ordering on the alphabet, and the lexicographic ordering obtained by reversing the order of letters in the alphabet. Here, by refining Crochemore and Perrin's algorithm, we give a version of the critical factorization theorem for *partial words* (such sequences may contain "do not know" symbols or "holes"). Our proof provides an efficient algorithm which computes a critical factorization when one exists. Our results extend those of Blanchet-Sadri and Duncan for partial words with one hole. A World Wide Web server interface at http://www.uncg.edu/mat/research/cft2/ has been established for automated use of the program.
**Keywords:** Word; Partial word; Period; Weak period; Local period

## Article:

### *1. Introduction*

This paper studies *partial words*, or finite sequences of symbols from a finite alphabet that may have a number of "do not know" symbols or "holes". While a word can be described by a total function, a partial word can be described by a partial function. More precisely, a partial word of length $n$ over a finite alphabet $A$ is a partial function from $\{0, \ldots, n - 1\}$ into $A$. Elements of $\{0, \ldots, n - 1\}$ without an image are called holes (a word is just a partial word without holes). The paper focuses on three important concepts of the *periodicity* of partial words: one is that of *period*, another is that of *weak period*, and the third is that of *local period*, which characterizes a local periodic structure at each position of the partial word.

Results concerning periodicity in the framework of partial words include: First, the well known and basic result of Fine and Wilf [20] intuitively determines how far two periodic events have to match in order to guarantee a common period. This result states that any word having periodicities $p$ and $q$ and length $\geq p + q - \gcd(p, q)$ has periodicity $\gcd(p, q)$, where $\gcd(p, q)$ denotes the greatest common divisor of $p$ and $q$. Moreover, the bound $p + q - \gcd(p, q)$ is optimal, since counterexamples can be provided for words of smaller length. This result was extended to partial words with one hole by Berstel and Boasson [1], to partial words with two or three holes by Blanchet-Sadri and Hegstrom [7], and to partial words with an arbitrary number of holes by Blanchet-Sadri [2].

Second, the well known and unexpected result of Guibas and Odlyzko [22] states that the set of all periods of a word is independent of the alphabet size. In [23], this result was reconsidered through an algorithmic approach that reduces the technical complexity of the proof. Guibas and Odlyzko's result states that for every word $u$, there exists a binary word $v$ that has exactly the same set of periods as $u$. In [4], Blanchet-Sadri and Chriscoe extended Guibas and Odlyzko's result to partial words with one hole. As a consequence, they obtained, for any partial word $u$ with one hole, a binary partial word $v$ with at most one hole that has exactly the same set of periods and the same set of weak periods as $u$. The proof provides a linear time algorithm which, given the partial word $u$, computes the desired binary partial word $v$. And in [6], Blanchet-Sadri, Gafni and Wilson extended Guibas and Odlyzko's result further to partial words with an arbitrary number of holes.

Third, the well known and fundamental critical factorization theorem, of which several versions exist [ 10, 11, 15– 17,26,27], intuitively states that the minimal period (or global period) of a word of length at least two is always locally detectable in at least one position of the word, resulting in a corresponding *critical factorization*. More specifically, given a word $w$ and nonempty words $u$, $v$ satisfying $w = uv$, the *minimal local period* associated with the factorization $(u, v)$ is the length of the shortest square at position $|u|-1$. It is easy to see that no minimal local period is longer than the global period of the word. The critical factorization theorem shows that critical factorizations are unavoidable. Indeed, for any string, there is always a factorization whose minimal local period is equal to the global period of the string. In other words, we consider a string $a_0a_1 \ldots a_{n-1}$ and, for any integer $i$ ($0 \leq i < n - 1$), we look at the shortest repetition (a *square*) *centered* in this position; that is, we look at the shortest (virtual) suffix of $a_0a_1 \ldots a_i$ which is also a (virtual) prefix of $a_{i+1}a_{i+2} \ldots a_{n-1}$. The minimal local period at position $i$ is defined as the length of this shortest square. The critical factorization theorem states, roughly speaking, that the global period of $a_0a_1 \ldots a_{n-1}$ is simply the maximum among all minimal local periods. As an example, consider the word $w$ = *babbaab* with global period 6. The minimal local periods of $w$ are 2, 3, 1, 6, 1, and 3, which means that the factorization (*babb*, *aab*) is critical.

Crochemore and Perrin showed that a critical factorization can be found very efficiently from the computation of the maximal suffixes of the word with respect to two total orderings on words: the lexicographic ordering related to a fixed total ordering on the alphabet $\preccurlyeq_l$, and the lexicographic ordering obtained by reversing the order of letters in the alphabet $\preccurlyeq_r$ [12]. If $v$ denotes the maximal suffix of $w$ with respect to $\preccurlyeq_l$ and $v'$ the maximal suffix of $w$ with respect to $\preccurlyeq_r$, then let $u$, $u'$ be such that $w = uv = u'v'$. The factorization $(u, v)$ turns out to be critical when $|v| \leq |v'|$, and the factorization $(u', v')$ is critical when $|v| > |v'|$. There exist linear time (in the length of $w$) algorithms for such computations [12,13,28] (the latter two use the suffix tree construction).

In [5], Blanchet-Sadri and Duncan extended the critical factorization theorem to partial words with one hole. In this case, they called a factorization *critical* if its minimal local period is equal to the minimal weak period of the partial word. It turned out that for partial words, critical factorizations may be avoidable. They described the class of the so-called *special* partial words with one hole that possibly avoid critical factorizations. They gave a version of the critical factorization theorem for the nonspecial partial words with one hole. By refining the method based on the maximal suffixes with respect to the lexicographic/reverse lexicographic orderings, they gave a version of the critical factorization theorem for the so-called $(k, l)$-*nonspecial* partial words with one hole. Their proof led to an efficient algorithm which, given a partial word with one hole, outputs a critical factorization when one exists or outputs "no such factorization exists".

In this paper, we further investigate the relationship between the local and global periodicity of partial words. We extend the critical factorization theorem to partial words with an arbitrary number of holes. We characterize precisely the class of partial words that do not admit critical factorizations. We then develop an efficient algorithm which computes a critical factorization when one exists.

In [12], a new string matching algorithm was presented, which relies on the critical factorization theorem and which can be viewed as an intermediate between the classical algorithms of Knuth, Morris, and Pratt [25], on the one hand, and Boyer and Moore [8], on the other hand. The algorithm is linear in time and uses constant space as the algorithm of Galil and Seiferas [21]. It presents the advantage of being remarkably simple, which consequently makes its analysis possible. The critical factorization theorem has found other important applications as well, which include the design of efficient approximation algorithms for the shortest superstring problem [9,24,26].

A periodicity theorem on words, which has strong analogies with the critical factorization theorem, and three applications were derived in [29]. There, the authors improved some results motivated by string matching problems [14,21]. In particular, they improved the upper bound on the number of comparisons in the text processing of the Galil and Seiferas' time–space optimal string matching algorithm [21]. For other recent developments on the critical factorization theorem and on the study of properties of local periods, we refer the reader to [17–19].

## 2. Preliminaries

In this section, we fix our terminology on partial words. In particular, we discuss compatibility and conjugacy in Sections 2.1 and 2.2 respectively.

A nonempty finite set, denoted by $A$, is called an *alphabet*. The elements of $A$ are called *letters*. A *word* over $A$ is a finite sequence of letters from $A$. If $u$ is a word over $A$, then the *length* of $u$, denoted by $|u|$, is the number of letters in $u$. The *empty word*, denoted by $\epsilon$, is the unique sequence of length zero over $A$. A word of length $n$ over $A$ can be defined by a total function $u : \{0, \ldots, n-1\} \rightarrow A$ and is usually represented as $u = a_0a_1 \ldots a_{n-1}$ for $a_i \in A$. The $i$ -power of a word $u$, denoted by $u^i$ , is defined inductively by $u^0 = \epsilon$ and $u^i = uu^{i-1}$. We define the *reversal* of a word $u$, denoted by $rev(u)$, as follows: If $u = \epsilon$, then $rev(\epsilon) = \epsilon$, and if $u = a_0a_1 \ldots a_{n-1}$, then $rev(u) = a_{n-1} \ldots a_1a_0$. The set of all words over $A$ (length greater than or equal to zero) is denoted by $A^*$. It is a monoid under the associative operation of concatenation or product of words where $\epsilon$ serves as identity, and is referred to as the *free monoid* generated by $A$. The set of all nonempty words over $A$ is denoted by $A^+$ and it is a semigroup under the concatenation of words and is referred to as the *free semigroup* generated by $A$.

A *partial word* of length $n$ over $A$ is a partial function $u : \{0, \ldots, n-1\} \rightarrow A$. For $0 \leq i < n$, if $u(i)$ is defined, then we say that i belongs to the *domain* of $u$, denoted by $i \in D(u)$; otherwise we say that $i$ belongs to the *set of holes* of $u$, denoted by $i \in H(u)$. A *full* word over $A$ is a partial word over $A$ with an empty set of holes. The length of $u$ will be denoted by $|u|$.

If $u$ is a partial word of length $n$ over $A$, then the *companion* of $u$, denoted by $u_\Diamond$, is the total function $u_\Diamond : \{0,\ldots, n-1\} \rightarrow A \cup \{\Diamond\}$ defined by

$$u\Diamond(i) = \begin{cases} u(i) & \text{if } i \in D(u), \\ \Diamond & \text{otherise.} \end{cases}$$

The symbol $\Diamond \notin A$ is viewed as a "do not know" symbol. For example, the word $u_\Diamond = ab\Diamond a\Diamond a$ is the companion of the partial word $u$ of length 6 where $D(u) = \{0, 1, 3, 5\}$ and $H(u) = \{2,4\}$. The map $u \longmapsto u_\Diamond$ is a bijection and thus allows us to define for partial words concepts such as concatenation, power, reversal, etc. in a trivial way. We define the concatenation of the partial words $u$ and $v$ by $(uv)_\Diamond = u_\Diamond v_\Diamond$. The $i$-power of the partial word $u$ is defined by $(u^i)_\Diamond = (u_\Diamond)^i$ where $(u_\Diamond)_0 = \epsilon$ and $(u_\Diamond)^i = u_\Diamond(u_\Diamond)^{i-1}$. The reversal of the partial word $u$ is defined by $(rev(u))_\Diamond = rev(u_\Diamond)$. The set of all partial words over $A$ with an arbitrary number of holes will be denoted by $W(A)$. It is a monoid under the operation of concatenation where $\epsilon$ serves as identity.

For partial words $u$ and $v$, we define $u$ is a prefix of $v$, if there exists a partial word $x$ such that $v = ux$; $u$ is a *suffix* of $v$, if there exists a partial word $x$ such that $v = xu$; and $u$ is a *factor* of $v$, if there exist partial words $x$ and $y$ such that $v = xuy$ (the factor $u$ is called proper if $u \neq \epsilon$ and $u \neq v$). The unique *maximal common prefix* of $u$ and $v$ will be denoted by $u \wedge v$. For a subset $X$ of $W(A)$, we denote by $P(X)$ the set of prefixes of elements in $X$ and by $S(X)$ the set of suffixes of elements in $X$. More specifically,

$$P(X) = \{u \mid u \in W(A) \text{and there exists } x \in W(A) \text{ such that } ux \in X\}$$
$$S(X) = \{u \mid u \in W(A) \text{ and there exists } x \in W(A) \text{ such that } xu \in X\}.$$

If $X$ is the singleton $\{u\}$, then $P(X)$ (respectively, $S(X)$) will be abbreviated by $P(u)$ (respectively, $S(u)$).

A *period* of a partial word $u$ is a positive integer $p$ such that $u(i) = u(j)$ whenever $i, j \in D(u)$ and $i \equiv j \bmod p$. In this case, we call $u$ *$p$-periodic*. The smallest period of $u$ is called the *minimal period* of $u$ and will be denoted by

$p(u)$. A *weak period of* $u$ is a positive integer $p$ such that $u(i) = u(i + p)$ whenever $i, i + p \in D(u)$. In this case, we call $u$ *weakly* $p$*-periodic*. The smallest weak period of $u$ is called the *minimal weak period* of $u$, and will be denoted by $p'(u)$. Note that every weakly $p$-periodic full word is $p$-periodic, but this is not necessarily true for partial words. Also even if the length of a partial word $u$ is a multiple of a weak period of $u$, then $u$ is not necessarily a power of a shorter partial word.

For convenience, we will refer to a partial word over $A$ as a word over the enlarged alphabet $A \cup \{\Diamond\}$, where the additional symbol $\Diamond$ plays a special role. This allows us to say, for example, that "the partial word $aba\Diamond aa\Diamond$" instead of "the partial word with companion $aba\Diamond aa\Diamond$".

### 2.1. Compatibility

If $u$ and $v$ are partial words of equal length, then $u$ is said to be contained in $v$, denoted by $u \subset v$, if all elements in $D(u)$ are in $D(v)$ and $u(i) = v(i)$ for all $i \in D(u)$. The notation $u \sqsubset v$ will abbreviate the two conditions $u \subset v$ and $u \neq v$ holding simultaneously.

The partial words $u$ and $v$ are called compatible, denoted by $u \uparrow v$, if there exists a partial word $w$ such that $u \subset w$ and $v \subset w$. The least upper bound of two compatible partial words $u$ and $v$ will be denoted by $u \vee v$. More precisely, $u \vee v$ satisfies the following three conditions: $u \subset u \vee v$ and $v \subset u \vee v$ and $D(u \vee v) = D(u) \cup D(v)$. As an example, $u = aba\Diamond\Diamond a$ and $v = a\Diamond\Diamond b\Diamond a$ are compatible and $u \vee v = abab\Diamond a$. We use $u \Uparrow v$ as an abbreviation for $u \uparrow v$ with $u \not\subset v$ and $v \not\subset u$ holding simultaneously.

For a subset $X$ of $W(A)$, we denote by $C(X)$ the set of all partial words compatible with elements of $X$. More specifically,

$$C(X) = \{u \mid u \in W(A) \text{ and there exists } v \in X \text{ such that } u \uparrow v\}.$$

The following two lemmas, related to the combinatorial property of compatibility, are useful for computing with partial words. For $u, v, w, x, y \in W(A)$, the following hold:

*Multiplication*: If $u \uparrow v$ and $x \uparrow y$, then $ux \uparrow vy$.
*Simplification*: If $ux \uparrow vy$ and $|u| = |v|$, then $u \uparrow v$ and $x \uparrow y$.
*Weakening*: If $u \uparrow v$ and $w \subset u$, then $w \uparrow v$.

Lemma 1 *([1]). Let $u, v, x, y \in W(A)$ be such that $ux \uparrow vy$.*

- *If $|u| \geq |v|$, then there exist $w, z \in W(A)$ such that $u = wz$, $v \uparrow w$, and $y \uparrow zx$.*
- *If $|u| \leq |v|$, then there exist $w, z \in W(A)$ such that $v = wz$, $u \uparrow w$, and $x \uparrow zy$.*

### 2.2. Conjugacy

The following lemma, related to the combinatorial property of conjugacy, is used in particular to prove our main results (Theorems 2 and 3).

Lemma 2 *([3]). Let $u, v \in W(A) \setminus \{\epsilon\}$ and $z \in W(A)$ be such that $|u| = |v|$. Then $uz \uparrow zv$ if and only if $uzv$ is weakly $|u|$-periodic.*

Proof. Let $m$ be defined as $\left\lfloor \frac{|z|}{|u|} \right\rfloor$ and $n$ as $|z| \bmod |u|$. Then let $u = x_0y_0$, $v = y_{m+1}x_{m+2}$ and $z = x_1y_1x_2y_2 \ldots x_my_mx_{m+1}$ where each $x_i$ has length $n$ and each $y_i$ has length $|u| - n$. We may now align $uz$ and $zv$ one above the other in the following way:

$$\begin{array}{ccccccccc} x_0 & y_0 & x_1 & y_1 & \ldots & x_{m-1} & y_{m-1} & x_m & y_m & x_{m+1} \\ x_1 & y_1 & x_2 & y_2 & \ldots & x_m & y_m & x_{m+1} & y_{m+1} & x_{m+2}. \end{array} \quad (1)$$

Assume $uz \uparrow zv$. Then the partial words in any column in (1) are compatible by simplification. Therefore, for all $i$ such that $0 \le i \le m + 1$, $x_i \uparrow x_{i+1}$ and for all $j$ such that $0 \le j \le m$, $y_j \uparrow y_{j+1}$. Thus $uz \uparrow zv$ implies that $uzv$ is weakly $|u|$-periodic. Conversely, assume $uzv$ is weakly $|u|$-periodic. This implies that $x_iy_i \uparrow x_{i+1}y_{i+1}$ for all $i$ such that $0 \le i \le m$. Note that $x_{m+1}y_{m+1}x_{m+2}$ being weakly $|u|$-periodic, as a result $x_{m+1} \uparrow x_{m+2}$. This shows that $uz \uparrow zv$ which completes the proof.

The following lemma is used to prove Theorems 4 and 5. It relates to the compatibility relations $x \uparrow y$ and $ux \uparrow yv$ holding simultaneously. Note that when $x = y = z$, this reduces to $uz \uparrow zv$. Let $m$ be defined as $\left\lfloor \frac{|x|}{|u|} \right\rfloor$. Then let $u = x_0y_0$, $v = y_{m+1}x_{m+2}$, $x = x_1y_1x_2y_2 \ldots x_my_mx_{m+1}$, and $y = x'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}$ where each $x_i$, $x'_i$ has length $|x| (\mathrm{mod}\ |u|)$ and each $y_i$, $y'_i$ has length $|u| - |x| (\mathrm{mod}\ |u|)$. Denoting $x_iy_i$ by $\alpha_i$ and $x'_iy'_1$ by $\alpha'_i$ for every $1 \le i \le m$, we have $x = \alpha_1\alpha_2 \ldots \alpha_mx_{m+1}$ and $y = \alpha'_1\alpha'_2 \ldots \alpha'_mx'_{m+1}$. The $|u|$-*pshuffle* and $|u|$-*sshuffle* of $ux$ and $yv$ are defined as

$$\mathrm{pshuffle}_{|u|}(ux, yv) = u\alpha'_1\alpha_1\alpha'_2 \ldots \alpha_{m-1}\alpha'_m\alpha_mx'_{m+1}y_{m+1}x_{m+1},$$
$$\mathrm{sshuffle}_{|u|}(ux, yv) = x_{m+1}x_{m+2}.$$

Lemma 3 *([3]). Let $u, v, x, y \in W(A) \setminus \{\epsilon\}$ be such that $|x| = |y|$ and $|u| = |v|$. Then $x \uparrow y$ and $ux \uparrow yv$ if and only if pshuffle$_{|u|}(ux, yv)$ is weakly $|u|$-periodic and sshuffle$_{|u|}(ux, yv)$ is $|x|$ (mod $|u|$) -periodic.*

Proof. We may align $x$ and $y$ (respectively, $ux$ and $yv$) one above the other in the following way:

$$\begin{matrix} x_1 & y_1 & x_2 & y_2 & \ldots & x_{m-1} & y_{m-1} & x_m & y_m & x_{m+1} \\ x'_1 & y'_1 & x'_2 & y'_2 & \ldots & x'_{m-1} & y'_{m-1} & x'_m & y'_m & x'_{m+1} \end{matrix} \qquad (2)$$

$$\begin{matrix} x_0 & y_0 & x_1 & y_1 & \ldots & x_{m-1} & y_{m-1} & x_m & y_m & x_{m+1} \\ x'_1 & y'_1 & x'_2 & y'_2 & \ldots & x'_m & y'_m & x'_{m+1} & y_{m+1} & x_{m+2}. \end{matrix} \qquad (3)$$

Assume $x \uparrow y$ and $ux \uparrow yv$. Then the partial words in any column in (2) (respectively, (3)) are compatible using the simplification rule. Therefore for all $0 \le i < m$, $x_iy_i \uparrow x'_{i+1}y'_{i+1}$ and $x'_{i+1}y'_{i+1} \uparrow x_{i+1}y_{i+1}$. Also, we have $y_m \uparrow y_{m+1}$ and the following sequence of compatibility relations: $x_m \uparrow x'_{m+1}$, $x'_{m+1} \uparrow x_{m+1}$, and $x_{m+1} \uparrow x_{m+2}$. Thus, pshuffle$_{|u|}(ux, yv)$ is weakly $|u|$-periodic and sshuffle$_{|u|}(ux, yv)$ is ($|x|$ mod $|u|$)-periodic. The converse follows symmetrically.

Throughout the rest of this paper, A denotes a fixed alphabet.

## 3. Orderings

In this section, we define two total orderings on partial words, $\preccurlyeq_l$ and $\preccurlyeq_r$, and state two lemmas related to them that will be used to prove our main results.

First, let the alphabet $A$ be totally ordered by $\prec$ and let $\Diamond \prec a$ for all $a \in A$. The first total ordering of $W(A)$, denoted by $\prec_l$, is simply the lexicographic ordering related to a fixed total ordering on $A$ and is defined as follows: $u \prec_l v$, if either $u$ is a proper prefix of $v$, or $u = (u \wedge v)ax$, $v = (u \wedge v)by$ with $a, b \in A \cup \{\Diamond\}$ satisfying $a \prec_l b$. The second total ordering of $W(A)$, denoted by $\prec_r$, is obtained from $\prec_l$ by reversing the order of letters in the alphabet; that is, for $a, b \in A$, $a \prec_l b$ if and only if $b \prec_r a$. Note that $\Diamond \prec_l a$ as well as $\Diamond \prec_r a$ for every $a \in A$.

Now, if $u \in W(A)$ and $0 \le i < j \le |u|$, then $(u[i..j))_\Diamond$ denotes the factor of $u_\Diamond$ satisfying $(u[i..j))_\Diamond = u_\Diamond(i) \ldots u_\Diamond(j - 1)$. The *maximal suffix* of $u$ with respect to $\preccurlyeq_l$ (respectively, $\preccurlyeq_r$) is defined as $u[i..|u|)$ where $0 \le i < |u|$ and where $u[j..|u|) \preccurlyeq_l u[i..|u|)$ (respectively, $u[j..|u|) \preccurlyeq_r u[i..|u|)$) for all $0 \le j < |u|$. For example, if $a \prec_l b \prec_l c$, then the maximal suffix of $a\Diamond cbac$ with respect to $\preccurlyeq_l$ is $cbac$, and with respect to $\preccurlyeq_r$ is $ac$.

Lemma 4 *([5]). Let $\prec$ be a total ordering of A extended to the total ordering $\prec'$ of $W(A)$ by setting $\Diamond \prec a$ for all $a \in A$. Let $u, v, w \in W(A)$ be such that $v$ is the maximal suffix of $w = uv$ with respect to $\preccurlyeq'$. Then*

1. *No nonemptypartial words $x, y$ are such that $y \subset x$, $u = rx$ and $v = ys$ for some $r, s \in W(A)$.*
2. *No nonemptypartial words $x, y, s$ are such that $y \subset x$, $u = rx$ and $y = vs$ for some $r \in W(A)$.*

Lemma 5. *Let $u, v \in W(A) \setminus \{\epsilon\}$. Then both $u \preccurlyeq_l v$ and $u \preccurlyeq_r v$ if and only if $u \in P(v)$ or there exist $x, y \in W(A)$*

*and $a \in A$ such that $u = (u \wedge v)\Diamond x$ and $v = (u \wedge v)ay$.*

Proof. If $u \preccurlyeq_l v$ and $u \preccurlyeq_r v$, then either $u \in P(v)$, or $u = (u \wedge v)bx$ and $v = (u \wedge v)cy$ where $x, y \in W(A)$ and where $b, c \in A \cup \{\Diamond\}$ satisfy $b \prec_l c$ and $b \prec_r c$. The latter leads to $b = \Diamond$. Conversely, if $u \in P(v)$, then $u \preccurlyeq_l v$ and $u \preccurlyeq_r v$ by definition. And if there exist $x, y \in W(A)$ and $b \in A$ such that $u = (u \wedge v)\Diamond x$ and $v = (u \wedge v)by$, then $u \preccurlyeq_l v$ and $u \preccurlyeq_r v$ since $\Diamond \prec_l b$ and $\Diamond \prec_r b$ for all $b \in A$.

## 4. Critical factorization theorem on partial words with an arbitrary number of holes

In this section, we discuss our first version of the critical factorization theorem on partial words with an arbitrary number of holes. Intuitively, our theorem states that the minimal weak period of a *nonspecial* partial word $w$ of length at least two can be locally determined in at least one position of $w$. More specifically, if $w$ is nonspecial according to Definition 2, then there exists a *critical* factorization $(u, v)$ of $w$ with $u, v \neq \epsilon$ such that the minimal local period of $w$ at position $|u| - 1$ (as defined below) equals the minimal weak period of $w$.

Definition 1 ([5]). Let $w \in W(A) \setminus \{\epsilon\}$. A positive integer $p$ is called *a local period of w at position i* if there exist $u, v, x, y \in W(A) \setminus \{\epsilon\}$ such that $w = uv$, $|u| = i + 1$, $|x| = p$, $x \uparrow y$, and such that one of the following conditions holds for some partial words $r, s$:

1. $u = rx$ and $v = ys$ (internal square),
2. $x = ru$ and $v = ys$ (left-external square if $r \neq \epsilon$),
3. $u = rx$ and $y = vs$ (right-external square if $s \neq \epsilon$),
4. $x = ru$ and $y = vs$ (left- and right-external square if $r, s \neq \epsilon$).

*The minimal local period of w at position i is denoted by $p(w, i)$. Clearly, $1 \leq p(w, i) \leq p'(w) \leq |w|$.*

A partial word being special is defined as follows.

Definition 2. Let $w \in W(A) \setminus \{\epsilon\}$ be such that $p'(w) > 1$. Let $v$ (respectively, $v'$) be the maximal suffix of $w$ with respect to $\preccurlyeq_l$ (respectively, $\preccurlyeq_r$). Let $u, u'$ be partial words such that $w = uv = u'v'$.

- If $|v| \leq |v'|$, then $w$ is called *special* if one of the following holds:
  1. $p(w, |u| - 1) < |u|$ and $r \notin C(S(u))$ (as computed according to Definition 1).
  2. $p(w, |u| - 1) < |v|$ and $s \notin C(P(v))$ (as computed according to Definition 1).

- If $|v| \geq |v'|$, then $w$ is called *special* if one of the above holds when referring to Definition 1, where $u$ is replaced by $u'$ and $v$ by $v'$.

The partial word $w$ is called *nonspecial* otherwise.

To illustrate Definition 2, first consider $w = aa\Diamond\Diamond ba\Diamond\Diamond bb$ together with $a \prec_l b$. The maximal suffixes of $w$ with respect to $\preccurlyeq_l$ and $\preccurlyeq_r$ are $v = bb$ and $v' = aa\Diamond\Diamond ba\Diamond\Diamond bb$ respectively. Here, $|v| \leq |v'|$ and $u = aa\Diamond\Diamond ba\Diamond\Diamond$. We get that $w$ is special since $1 = p(w, |u| - 1) < |u| = 8$ and $r = aa\Diamond\Diamond ba\Diamond \notin C(S(u))$. Now, consider $w = ab\Diamond\Diamond a$ with maximal suffixes $v = b\Diamond\Diamond a$ and $v' = ab\Diamond\Diamond a$. Again, $|v| \leq |v'|$. We have $2 = p(w, |u| - 1) < |v| = 4$ but $s = \Diamond a \in C(P(v))$, and so $w$ is nonspecial.

The following theorem holds.

Theorem 1. *If $w \in W(A)$ is nonspecial and satisfies $|w| \geq 2$, then w has at least one critical factorization.*
More specifically, the proof of the following theorem not only shows the existence of a critical factorization for a given nonspecial partial word of length at least two as claimed in Theorem 1, but also gives an algorithm to compute such a factorization explicitly.

Theorem 2. *Let $\prec$ be any total ordering of A, and let $w \in W(A)$ satisfy $|w| \geq 2$. If $p'(w) > 1$, then let v denote the maximal suffix of w with respect to $\preccurlyeq_l$ and v' the maximal suffix of w with respect to $\preccurlyeq_r$. Let u, u' be partial words such that $w = uv = u'v'$. Then w is nonspecial if and only if $|v| \leq |v'|$ and the factorization (u, v) is critical, or $|v| > |v'|$ and the factorization (u', v') is critical.*

Proof. If $p'(w) = 1$, then $w = a_0^{m_0} \diamond a_1^{m_1} \diamond \ldots a_{n-1}^{m_{n-1}} \diamond a_n^{m_n}$ for some $a_0, a_1, \ldots, a_n \in A$ and integers $m_0, m_1, \ldots, m_n \geq 0$. The result trivially holds in this case. We now assume that $p'(w) > 1$ and that $|v| \leq |v'|$ (the case where $p'(w) > 1$ and $|v| > |v'|$ is proved similarly, but requires that the orderings $\preccurlyeq_l$ and $\preccurlyeq_r$ be interchanged). Assume that $u = \epsilon$, and thus $w = v$. Since $|v| \leq |v'|$, we also have $w = v'$. Setting $w = az$ for some $a \in A$ and $z \in W(A)$ , we argue as follows. If $b \in A$ is a letter in $z$, then $b \preccurlyeq_l a$ and $b \preccurlyeq_r a$. Thus, $b = a$ and $w$ is unary. We get $p'(w) = 1$, contradicting our assumption, and therefore $u \neq \epsilon$. Now let us denote $p(w, |u| - 1)$ by $p$. We consider the following four cases:

Case 1. $p \geq |u|$ and $p \geq |v|$
If $p \geq |u|$ and $p \geq |v|$, then Definition 1(4) is satisfied. There exist $x, y, r, s \in W(A)$ such that $|x| = p$, $x \uparrow y$, $x = ru$, and $y = vs$. First, if $|r| > |v|$, then $p = |x| = |ru| > |uv| = |w|$, which leads to a contradiction. Similarly, we see that $|s| \leq |u|$. Now, if $|r| \leq |v|$, then we may choose $r, s, z, z' \in W(A)$ such that $v = rz$, $u = z's$, and $z \uparrow z'$. There exists $z'' \in W(A)$ such that $z \subset z''$ and $z' \subset z''$. Thus, $uv \subset z''srz''$, showing that $p = |z''sr|$ is a weak period of $uv$, and so $p'(w) \leq p$. On the other hand, $p'(w) \geq p$. Therefore, $p'(w) = p$ which shows that the factorization $(u, v)$ is critical.

Case 2. $p < |u|$ and $p > |v|$
If $p < |u|$ and $p > |v|$, then Definition 1(3) is satisfied. There exist $x, y, r, s, \gamma \in W(A)$ such that $|x| = p$, $x \uparrow y$, $u = rx = r\gamma s$, and $y = vs$. If $v \subset \gamma$, then $y \subset x$, and $v$ being the maximal suffix of $w$ with respect to $\preccurlyeq_l$, we get a contradiction with Lemma 4(2). If $\gamma \sqsubset v$ or $\gamma \not\uparrow v$, then we consider whether or not $r \in C(S(u))$. If $r \notin C(S(u))$, then $w$ is special by Definition 2(1). If $r \in C(S(u))$, then $x'r \uparrow rx$ for some $x'$. By Lemma 2, $u = rx$ is weakly $|x|$-periodic, and so $rxy = rxvs$ is weakly $|x|$-periodic since $x \uparrow y$. Therefore, $p = |x|$ is a weak period of $uv = rxv$.

Case 3. $p < |u|$ and $p \leq |v|$
If $p < |u|$ and $p \leq |v|$, then Definition 1 (1) is satisfied. There exist $x, y, r, s \in W(A)$ such that $|x| = p$, $x \uparrow y$, $u = rx$, and $v = ys$. If $y \subset x$, then v being the maximal suffix of $w$ with respect to $\preccurlyeq_l$, we get a contradiction with Lemma 4(1). If $x \sqsubset y$ or $x \not\uparrow y$, then we argue as follows. If $r \notin C(S(u))$ or $s \notin C(P(v))$, then $w$ is special by Definition 2(1) or Definition 2(2). If $r \in C(S(u))$ and $s \in C(P(v))$, then $x'r \uparrow rx$ and $ys \uparrow sy'$ for some $x'$, $y'$. By Lemma 2, $u = rx$ is weakly $|x|$ -periodic and $v = ys$ is weakly $|y|$ -periodic. Therefore, $p = |x| = |y|$ is a weak period of $uv = rxys$ since $x \uparrow y$.

Case 4. $p \geq |u|$ and $p < |v|$
If $p \geq |u|$ and $p < |v|$, then Definition 1(2) is satisfied. There exist $x, y, r, s \in W(A)$ such that $|x| = p$, $x \uparrow y$, $x = ru$, and $v = ys$. Then $w$ is special by Definition 2(2) unless $s \in C(P(v))$. If $s \in C(P(v))$, then $ys \uparrow sy'$ for some $y'$. By Lemma 2, $v = ys$ is weakly $|y|$-periodic, and so $xys = ruys$ is weakly $|y|$-periodic since $x \uparrow y$. Therefore, $p = |y|$ is a weak period of $uv = uys$.

Referring to Definition 2, the following table, where it is assumed that $a \prec_l b$ and $b \prec_r a$, provides special partial words $w$ with no position $i$ satisfying $p'(w) = p(w, i)$ (these examples show why Theorem 2 excludes the special partial words):

| $w$ | **Def2** | $r \in C(S(u))$ | $s \in C(P(v))$ | $u$ | $v$ | $x$ | $y$ | $r$ | $s$ |
|---|---|---|---|---|---|---|---|---|---|
| $aa\diamond\diamond b\diamond\diamond\diamond\diamond bba$ | 2 | *yes* | *no* | $rx$ | $ys$ | $\diamond$ | $b$ | $aa\diamond\diamond b\diamond\diamond\diamond\diamond$ | $ba$ |
| $baa\diamond bb\diamond$ | 1 | *no* | *yes* | $rx$ | $ys$ | $\diamond$ | $b$ | $baa$ | $b\diamond$ |
| $ab\diamond a\diamond a$ | 2 | | *no* | $a$ | $ys$ | $ru$ | $b\diamond$ | $b$ | $a\diamond a$ |
| $\diamond b\diamond bbabbb$ | 1 | *no* | | $rx$ | $bbb$ | $\diamond bba$ | $vs$ | $\diamond b$ | $a$ |

From the proof of Theorem 2, we can obtain an algorithm that outputs a critical factorization for a given partial word $w$ with $p'(w) > 1$ and with an arbitrary number of holes of length at least two when $w$ is nonspecial, and that outputs "special" otherwise. The algorithm computes the maximal suffix $v$ of $w$ with respect to $\preccurlyeq_l$ and the maximal suffix $v'$ of $w$ with respect to $\preccurlyeq_r$. The algorithm finds partial words $u, u'$ such that $w = uv = u'v'$. If $| | \leq |v'|$, then it computes $p = p(w, |u| - 1)$ and does the following:

1. If $p < |u|$, then it finds partial words $x, y, r, s$ satisfying Definition 1. If $r \notin C(S(u))$, then it outputs "special".

2. If $p < |v|$, then it finds partial words x, y, r, s satisfying Definition 1. If $s \notin C(P(v))$, then it outputs "special".

3. Otherwise, it outputs $(u, v)$.

If $|v| > |v'|$, then the algorithm computes $p = p(w, |u'| - 1)$ and does the above where $u$ is replaced by $u'$ and $v$ by $v'$. As an example, consider $w = aaab\Diamond babb$. Its maximal suffix with respect to $\preccurlyeq_l$ (where $a \prec b$) is $v = bb$ and with respect to $\preccurlyeq_r$ (where $b \prec a$) is $v' = aaab\Diamond babb$. Here $|v| < |v'|$ and the factorization $(aaab\Diamond ba, bb)$ is not critical since $w$ is special. Now, if we consider $rev(w) = bbab\Diamond baaa$, its maximal suffix with respect to $\preccurlyeq_l$ is $v = bbab\Diamond baaa$, and with respect to $\preccurlyeq_r$ is $v' = aaa$. Here $|v| > |v'|$ and $rev(w)$ is nonspecial, and so the factorization $(bbab\Diamond b, aaa)$ of $rev(w)$ (which corresponds to the factorization $(aaa, b\Diamond babb)$ of $w$) is critical. This observation leads us to improve our algorithm by considering both $w$ and $rev(w)$.

Algorithm 1. Step 1: Compute the maximal suffix $v_0$ of $w$ with respect to $\preccurlyeq_l$ and the maximal suffix $v_0'$ of $w$ with respect to $\preccurlyeq_r$. Also compute the maximal suffix $v_1$ of $rev(w)$ with respect to $\preccurlyeq_l$ and the maximal suffix $v_1'$ of $rev(w)$ with respect to $\preccurlyeq_r$.

Step 2: Find partial words $u_0, u_0'$ such that $w = u_0v_0 = u_0'v_0'$ . Also find partial words $u_1, u_1'$ such that $rev(w) = u_1v_1 = u_1'v_1'$.

Step 3: If $|v_0| \leq |v_0'|$ and $|v_1| \leq |v_1'|$, then compute $p_0 = p(w, |u_0| - 1)$ and $p_1 = p(rev(w), |u_1| - 1)$.

Step 4: If $p_0 \geq p_1$ , then do the following:

1. If $p_0 < |u_0|$, then find partial words $x, y, r, s$ satisfying Definition 1. If $r \notin C(S(u_0))$, then output "special".
2. If $p_0 < |v_0|$, then find partial words $x, y, r, s$ satisfying Definition 1. If $s \notin C(P(v_0))$, then output "special".
3. Otherwise, output $(u_0, v_0)$.

Step 5: If $p_0 < p_1$, then do the work of Step 4 with $p_1$, $u_1$ and $v_1$ instead of $p_0$, $u_0$ and $v_0$.

Step 6: If $|v_0| > |v_0'|$ (or $|v_1| > |v_1'|$), then do the work of Step 3 with $u_0'$ and $v_0'$ instead of $u_0$ and $v_0$ (or do the work of Step 3 with $u_1'$ and $v_1'$ instead of $u_1$ and $v_1$). The algorithm may produce $(u_0', v_0')$ unless $w$ is special (or may produce $(u_1', v_1')$ unless $rev(w)$ is special) (in those cases, output "special").

## 5. A class of special partial words

In this section, the nonempty suffixes of a given partial word $w$ are ordered as follows according to $\preccurlyeq_l$:

$$v_{0,|w|-1} \prec_l v_{0,|w|-2} \prec_l \cdots \prec_l v_{0,0}.$$

The factorizations $(u_{0,0}, v_{0,0})$, $(u_{0,1}, v_{0,1})$ ,... of $w$ result. Similarly, the nonempty suffixes of $w$ are ordered as follows according to $\preccurlyeq_r$:

$$v'_{0,|w|-1} \prec_r v'_{0,|w|-2} \prec_r \cdots \prec_r v'_{0,0}.$$

The factorizations $(u'_{0,0}, v'_{,00})$, $(u'_{0,1}, v'_{0,1})$, . . . of $w$ result. The nonempty suffixes of $rev(w)$ are ordered as follows:

$$v_{1,|w|-1} \prec_l v_{1,|w|-2} \prec_l \cdots \prec_l v_{1,0}$$
$$v'_{1,|w|-1} \prec_r v'_{1,|w|-2} \prec_r \cdots \prec_r v'_{1,0}.$$

The factorizations $(u_{1,0} , v_{1,0})$, $(u_{1,1} , v_{1,1})$ , ... , $(u'_{1,0}, v'_{1,0})$, $(u'_{1,1}, v'_{1,1})$, . . . of $rev(w)$ result.

Referring to Definition 2, the following table provides examples of special partial words w whose reversals are also special and for which there exists a position $i$ such that $p'(w) = p(w, i)$ or $p'(w) = p(rev(w), i)$, resulting in a critical factorization (it is assumed that $a \prec_l b$ and $b \prec_r a$):

| $w$ | Fact | Critical | Fact | Critical | Fact | Critical |
|---|---|---|---|---|---|---|
| $aaa\diamond\diamond ba$ | $(u_{0,0}, v_{0,0})$ | no | $(u'_{1,0}, v'_{1,0})$ | no | $(u_{1,0}, v_{1,0})$ | yes |
| $abba\diamond abb$ | $(u_{0,0}, v_{0,0})$ | no | $(u'_{1,0}, v'_{1,0})$ | no | $(u_{0,1}, v_{0,1})$ | yes |
| $a\diamond abb\diamond bbbaa$ | $(u'_{0,0}, v'_{0,0})$ | no | $(u_{1,0}, v_{1,0})$ | no | $(u_{0,2}, v_{0,2})$ | yes |
| $a\diamond cbac$ | $(u'_{0,0}, v'_{0,0})$ | no | $(u'_{1,0}, v'_{1,0})$ | no | $(u_{0,2}, v_{0,2})$ | yes |

For instance, if we consider $w = aaa\Diamond\Diamond ba$, then the factorization $(u_{0,0}, v_{0,0})$ is not critical since $w$ is special. If we consider $rev(w) = ab\Diamond\Diamond aaa$, then the factorization $(u'_{1,0}, v'_{1,0})$ is not critical either, since $rev(w)$ is special. However, $w$ has a critical factorization (the factorization $(u_{1,0}, v_{1,0})$ of $rev(w)$ is critical implying a corresponding critical factorization of $w$).

The above examples lead us to refine Theorem 2. First, we define the concept of an $(k,l)$-*special* partial word (note that the concept of *special* in Definition 2 is equivalent to the concept of (0, 0)-special in Definition 3).

Definition 3. Let $w \in W(A) \setminus \{e\}$ be such that $p'(w) > 1$, and let k, l be a pair of integers satisfying $0 \leq k, l < |w|$.

- If $|v_{0,k}| \leq |v'_{0,l}|$, then $w$ is called $(k, l)$-*special* if one of the following holds:
  1. $p(w, |u_{0,k}| - 1) < |u_{0,k}|$ and $r \notin C(S(u_{0,k}))$ (as computed according to Definition 1).
  2. $p(w, |u_{0,k}| - 1) < |v_{0,k}|$ and $s \notin C(P(v_{0,k}))$ (as computed according to Definition 1).

- If $|v_{0,k}| \geq |v'_{0,l}|$, then $w$ is called $(k, l)$-*special* if one of the above holds when referring to Definition 1 where $u_{0,k}$ is replaced by $u'_{0,l}$ and $v_{0,k}$ by $v'_{0,l}$.

The partial word $w$ is called $(k, l)$-*nonspecial* otherwise.

We now describe our algorithm (based on Theorem 3) that outputs a critical factorization for a given partial word $w$ with $p'(w) > 1$, with an arbitrary number of holes of length at least two when such a factorization exists, and that outputs "no critical factorization exists" otherwise.

Algorithm 2. Step 1: Compute the nonempty suffixes of w with respect to $\preccurlyeq_l$ (say $v_{0,|w|-1} \prec_l \cdots \prec_l v_{0,0}$) and the nonempty suffixes of w with respect to $\preccurlyeq_r$ (say $v'_{0,|w|-1} \prec_r \cdots \prec_r v'_{0,0}$). Also compute the nonempty suffixes of $rev(w)$ with respect to $\preccurlyeq_l$ (say $v_{1,|w|-1} \prec_l \cdots \prec_l v_{1,0}$) and the nonempty suffixes of $rev(w)$ with respect to $\preccurlyeq_r$ (say $v'_{1,|w|-1} \prec_r \cdots \prec_r v'_{1,0}$).

Step 2: Set $k_0 = 0$, $l_0 = 0$, $k_1 = 0$, $l_1 = 0$, and $mwp = 0$.

Step 3: If $k_0 \geq |w| - \|H(w)\|$ or $l_0 \geq |w| - \|H(w)\|$ or $k_1 \geq |w| - \|H(w)\|$ or $l_1 \geq |w| - \|H(w)\|$, then output "no critical factorization exists."

Step 4: If $v_{0,k_0} \prec_l v'_{0,l_0}$, then update $l_0$ with $l_0 + 1$ and go to Step 3. If $v'_{0,l_0} \prec_r v_{0,k_0}$, then update $k_0$ with $k_0 + 1$ and go to Step 3. If $v_{1,k_1} \prec_l v'_{1,l_1}$, then update $l_1$ with $l_1 + 1$ and go to Step 3. If $v'_{1,l_1} \prec_r v_{1,k_1}$, then update $k_1$ with $k_1 + 1$ and go to Step 3.

Step 5: If $k_0 > 0$ and $v'_{0,l_0} = w$, then update $l_0$ with $l_0 + 1$ and go to Step 3. If $l_0 > 0$ and $v_{k0_0} = w$, then update $k_0$ with $k_0 + 1$ and go to Step 3. If $k_1 > 0$ and $v'_{l1_1} = rev(w)$, then update $l_1$ with $l_1 + 1$ and go to Step 3. If $l_1 > 0$ and $v_{1,k_1} = rev(w)$, then update $k_1$ with $k_1 + 1$ and go to Step 3.

Step 6: Find partial words $u_{0,k_o}$, $u'_{0,l_o}0$ such that $w = u_{0,k_o}\ v_{0,k_0} = u'_{0,l_0}v'_{0,l_0}$. Also find partial words $u_{1,k_1}$, $u'_{1,l_1}$ such that $rev(w) = u_{1,k_1}\ v_{1,k_1} = u'_{1,l_1}v'_{1,l_1}$ .

Step 7: If $|v_{0,k_0}| \leq |v'_{0,l_0}|$ and $|v_{1,k_1}| \leq |v'_{1,l_1}|$, then compute $p_{0,k_0} = p(w, |u_{0,k_0}| - 1)$ and $p_{1,k_1} = p(rev(w), |u_{1,k_1}| - 1)$.

Step 8: If $p_{0,k_0} \leq mwp$, then *move up* which means to update $k_0$ with $k_0 + 1$ and to go to Step 3. If $p_{1,k_1} \leq mwp$, then *move up* which means one needs to update $k_1$ with $k_1 + 1$ and to go to Step 3.

Step 9: If $p_{0,k_0} \geq p_{1,k_1}$, then update *mwp* with $p_{0,k_0}$. Do the following:

1. If $p_{0,k_0} < |u_{0,k_0}|$, then find partial words *x*, *y*, *r*, *s* satisfying Definition 1. If $r \notin C(S(u_{0,k_0}))$, then move up, which means update $k_0$ with $k_0 + 1$ and go to Step 3.
2. If $p_{0,k_0} < |v_{0,k_0}|$, then find partial words *x*, *y*, *r*, *s* satisfying Definition 1. If $s \notin C(P(v_{0,k_0}))$, then move up which means update $k_0$ with $k_0 + 1$ and go to Step 3.
3. Otherwise, output $(u_{0,k_0}, v_{0,k_0})$.

Step 10: If $p_{0,k_0} < p_{1,k_1}$, then update *mwp* with $p_{1,k_1}$ and do the work of Step 9 with $p_{1,k_1}$, $u_{1,k_1}$ and $v_{1,k_1}$ instead of $p_{0,k_0}$, $u_{0,k_0}$ and $v_{0,k_0}$.

Step 11: If $|v_{0,k_0}| > |v'_{0,l_0}0|$ (or $|v_{1,k_1}| > |v'_{1,l_1}|$), then compute $p_{0,l_0} = p(w, |u'_{0,l_0}| - 1)$ and do the work of Step 8 with $p_{0,l_0}$, $u'_{0,l_0}$ and $v'_{0,l_0}$ instead of $p_{0,k_0}$, $u_{0,k_0}$ and $v_{0,k_0}$ (*move up* here means update $l_0$ with $l_0 + 1$ and go to Step 3) (or compute $p_{1,l_1} = p(rev(w), |u'_{1,l_1}| - 1)$ and do the work of Step 8 with $p_{1,l_1}$, $u'_{1,l_1}$ and $v'_{1,l_1}$ instead of $p_{1,k_1}$, $u_{1,k_1}$ and $v_{1,k_1}$ (*move up* here means update $l_1$ with $l_1 + 1$ and go to Step 3)). The algorithm may produce $(u'_{0,l_0}, v'_{0,l_0})$ unless *w* is $(k_0, l_0)$-special (or may produce $(u'_{1,l_1}, v'_{1,l_1})$ unless *rev*(*w*) is $(k_1, l_1)$-special) (in those cases, move up).

We illustrate Algorithm 2 with the following example.

Example 1. Below are tables for the nonempty suffixes of the partial word $w = a\Diamond cbac$ and its reversal $rev(w) = cabc\Diamond a$. These suffixes are ordered in two different ways: The first ordering is on the left and is an $\prec_l$-ordering according to the order $\Diamond \prec a \prec b \prec c$, and the second is on the right and is an $\prec_r$ -ordering where $\Diamond \prec c \prec b \prec a$. The tables also contain the indices used by the algorithm, $k_0$, $l_0$, $k_1$, $l_1$, and the local periods that needed to be calculated in order to compute the critical factorization $(a\Diamond c, bac)$. The minimal weak period of *w* turns out to be equal to 4.

| $k_0$ | $p_{0,k_0}$ | $v_{0,k_0}$ | $v'_{0,l_0}$ | $p_{0,l_0}$ | $l_0$ |
|---|---|---|---|---|---|
| 5 | | $\Diamond cbac$ | $\Diamond cbac$ | | 5 |
| 4 | | $a\Diamond cbac$ | $c$ | | 4 |
| 3 | | $ac$ | $cbac$ | | 3 |
| 2 | 4 | $bac$ | $bac$ | | 2 |
| 1 | 3 | $c$ | $a\Diamond cbac$ | | 1 |
| 0 | 1 | $cbac$ | $ac$ | 3 | 0 |

| $k_1$ | $p_{1,k_1}$ | $v_{1,k_1}$ | $v'_{1,l_1}$ | $p_{1,l_1}$ | $l_1$ |
|---|---|---|---|---|---|
| 5 | | $\Diamond a$ | $\Diamond a$ | | 5 |
| 4 | | $a$ | $c\Diamond a$ | | 4 |
| 3 | | $abc\Diamond a$ | $cabc\Diamond a$ | | 3 |
| 2 | | $bc\Diamond a$ | $bc\Diamond a$ | | 2 |
| 1 | 4 | $c\Diamond a$ | $a$ | 1 | 1 |
| 0 | | $cabc\Diamond a$ | $abc\Diamond a$ | 3 | 0 |

Algorithm 2 starts with the pairs $(v_{0,0}, v'_{0,0}) = (cbac, ac)$, $(v_{1,0}, v'_{1,0}) = (cabc\Diamond a, abc\Diamond a)$ and selects the shortest component of each pair, that is, $v'_{0,0}$ and $v'_{1,0}$. In Step 11, $p_{0,0}$ is computed as 3 and $p_{1,0}$ as 3. Since $p_{0,0} \geq p_{1,0} >$

$mwp$ = 0, the factorization ($u'_{0,0}$, $v'_{0,0}$) = ($a\Diamond cb$, $ac$) is chosen and the algorithm discovers that $w$ is (0,0)-special according to Definition 3. The variable $l_0$ is then updated to 1 and the pairs ($v_{0,0}$, $v'_{0,1}$) = ($cbac$, $a\Diamond cbac$), ($v_{1,0}$, $v'_{1,0}$) = ($cabc\Diamond a$, $abc\Diamond a$) are treated with shortest components $v_{0,0}$, $v'_{1,0}$ respectively. Now, $p_{0,0}$ is computed as 1 and $p_{1,0}$ as 3. Since $p_{0,0} < p_{1,0} \leq mwp = 3$, $k_0$ gets updated to 1 and $l_1$ to 1. Now, the pairs ($v_{0,1}$, $v'_{0,1}$) = ($c$, $a\Diamond cbac$), ($v_{1,0}$, $v'_{1,1}$) = ($cabc\Diamond a$, $a$) are considered and in Step 5, $l_0$ is updated to 2 since $k_0 = 1 > 0$ and $v'_{0,l_0} = v'_{0,1} = w$. The pairs ($v_{0,1}$, $v'_{0,2}$) = ($c$, $bac$), ($v_{1,0}$, $v'_{1,1}$) = ($cabc\Diamond a$, $a$) are treated and in Step 5, $k_1$ is updated to 1 since $l_1 = 1 > 0$ and $v_{1,k_1} = v_{1,0} = rev(w)$. Comes the turn of ($v_{0,1}$, $v'_{0,2}$) = ($c$, $bac$), ($v_{1,1}$, $v'_{1,1}$) = ($c\Diamond a$, $a$) with shortest components $v_{0,1}$ and $v'_{1,1}$. The algorithm computes $p_{0,1} = 3$ and $p_{1,1} = 1$. Since $p_{1,1} < p_{0,1} \leq mwp = 3$, the indices $k_0$ and $l_1$ get updated to 2 and the pairs ($v_{0,2}$, $v'_{0,2}$) = ($bac$, $bac$), ($v_{1,1}$, $v'_{1,2}$) = ($c\Diamond a$, $bc\Diamond a$) are then considered with shortest components $v_{0,2}$, $v_{1,1}$ and with $p_{0,2} = 4$, $p_{1,1} = 4$ calculated in Step 7. Since $p_{0,2} \geq p_{1,1} > mwp = 3$ leads to an improvement of the number $mwp$, the algorithm outputs ($u_{0,2}$, $v_{0,2}$) in Step 9 with $mwp = p_{0,2} = 4$ (here $w$ is (2, 2)-nonspecial).

We now prove Theorem 3.

Theorem 3. 1. *Let ($k_0$, $l_0$) be a pair of nonnegative integers being considered at Step 9 (when $p_{0,k_0} > mwp$ or when $p_{0,l_0} > mwp$). If $w \in W(A)$ is ($k_0$, $l_0$)-nonspecial satisfying $|w| \geq 2$ and $p'(w) > 1$, then w has at least one critical factorization. More specifically, the factorization ($u_{0,k_0}$ , $v_{0,k_0}$) is critical when $|v_{0,k_0}| \leq |v'_{0,l_0}|$, and the factorization ($u'_{0,l_0}$, $v'_{0,l_0}$) is critical when $|v_{0,k_0}| > |v'_{0,l_0}|$. Moreover, if $|v_{0,k_0}| \leq |v'_{0,l_0}|$ and the factorization ($u_{0,k_0}$ , $v_{0,k_0}$) is critical, then w is ($k_0$ , $l_0$) -nonspecial, and if $|v_{0,k_0}| > |v'_{0,l_0}|$ and the factorization ($u'_{0,l_0}$, $v'_{0,l_0}$) is critical, then w is ($k_0$, $l_0$)-nonspecial.*

2. *Let ($k_1$, $l_1$) be a pair of nonnegative integers being considered at Step 10 (when $p_{1,k_1} > mwp$ or when $p_{1,l_1} > mwp$). If rev(w) $\in W(A)$ is ($k_1$, $l_1$)-nonspecial satisfying $|w| \geq 2$ and $p'(w) > 1$, then rev(w) has at least one criticalfactorization. More specifically, thefactorization ($u_{1,k_1}$ , $v_{1,k_1}$) is critical when $|v_{1,k_1}| \leq |v'_{1,l_1}|$, and thefactorization ($u'_{1,l_1}$, $v'_{1,l_1}$) is critical when $|v_{1,k_1}| > |v'_{1,l_1}|$. Moreover, if $|v_{1,k_1}| \leq |v'_{1,l_1}|$ and the factorization ($u_{1,k_1}$ , $v_{1,k_1}$) is critical, then rev(w) is ($k_1$ , $l_1$)-nonspecial, and if $|v_{1,k_1}| > |v'_{1,l_1}|$ and thefactorization ($u'_{1,l_1}$, $v'_{1,l_1}$) is critical, then rev(w) is ($k_1$ , $l_1$) -nonspecial.*

Proof. We prove Statement 1 (Statement 2 is proved similarly). The pair ($k_0$, $l_0$) = (0, 0) was treated in Theorem 2. So, we may assume that ($k_0$, $l_0$) $\neq$ (0, 0). We consider the case where $|v_{0,k_0}| \leq |v'_{0,l_0}|$ (the case where $|v_{0,k_0}| > |v'_{0,l_0}|$ is handled similarly, but requires that the orderings $\preccurlyeq_l$ and $\preccurlyeq_r$ be interchanged). Here, $u_{0,k_0} \neq \epsilon$ unless $v_{0,k_0} = v'_{0,l_0} = w$. In such cases where $v_{0,k_0} = v'_{0,l_0} = w$, if $w$ begins with $\Diamond$, then the algorithm will discover in Step 3 that $w$ has no critical factorization. And if $w$ begins with $a$ for some $a \in A$, then $k_0 < |w| - ||H(w)||$ and $l_0 < |w| - ||H(w)||$. In such case, we have ($k_0 > 0$ and $v'_{0,l_0} = w$) or ($l_0 > 0$ and $v_{0,k_0} = w$). In the former case, Step 5 will update $l_0$ with $l_0 + 1$ resulting in the pair ($k_0$, $l_0 + 1$) being considered in Step 3; in the latter case, Step 5 will update $k_0$ with $k_0 + 1$, and ($k_0 + 1$, $l_0$) will be considered in Step 3.

We now consider the following cases where $p_{0,k_0}$ denotes $p$(w, $|u_{0,k_0}| - 1$).

Case 1. $p_{0,k_0} \geq |u_{0,k_0}|$ and $p_{0,k_0} \geq |v_{0,k_0}|$

Here Definition 1(4) is satisfied, and there exist $x, y, r, s \in W(A)$ such that $|x| = p_{0,k_0}$, $x \uparrow y$, $x = ru_{0,k_0}$ and $y = v_{0,k_0}s$. First, if $|r| > |v_{0,k_0}|$, then $p_{0,k_0} = |x| = |ru_{0,k_0}| > |u_{0,k_0}v_{0,k_0}| = |w| \geq p'(w)$, which leads to a contradiction. Now, if $|r| \leq |v_{0,k_0}|$, then by Lemma 1, there exist $r'$, $z \in W(A)$ such that $v_{0,k_0} = r'z$, $r \uparrow r'$, and $u_{0,k_0} \uparrow zs$. There exists $r''$ such that $r \subset r'$ and $r' \subset r''$, and there exist $z'$, $s'$ such that $u_{0,k_0} \subset z's'$, $z \subset z'$ and $s \subset s'$. Thus, $u_{0,k_0}v_{0,k_0} \subset z's'r'z'$, showing that $p_{0,k_0} = |z's'r'|$ is a weak period of $u_{0,k_0}v_{0,k_0}$, and $p'(w) \leq p_{0,k_0}$. On the other hand, $p'(w) \geq p_{0,k_0}$. Therefore, $p'(w) = p_{0,k_0}$, which shows that the factorization ($u_{0,k_0}$, $v_{0,k_0}$) is critical.

Case 2. $p_{0,k_0} < |u_{0,k_0}|$ and $p_{0,k_0} > |v_{0,k_0}|$
Here, Definition 1(3) is satisfied and there exist $x, y, r, s, \gamma \in W(A)$ such that $|x| = p_{0,k_0}$, $\gamma \uparrow v_{0,k_0}$, $u_{0,k_0} = rx = r\gamma s$, and $y = v_{0,k_0}s$. Note that if $k_0 = 0$ and $v_{0,k_0} \subset \gamma$, then $y \subset x$, and we get a contradiction with Lemma 4(2). If $r \in C(S(u_{0,k_0}))$, then $w$ is $(k_0, l_0)$-special by Definition 3(1). If $r \in C(S(u_{0,k_0}))$, then there exists $x'$ such that $x'r \uparrow rx$. The result follows as in Case 2.

Case 3. $p_{0,k_0} < |u_{0,k_0}|$ and $p_{0,k_0} \leq |v_{0,k_0}|$
Here Definition 1(1) is satisfied, and there exist $x, y, r, s \in W(A)$ such that $|x| = p_{0,k_0}$, $x \uparrow y$, $u_{0,k_0} = rx$, and $v_{0,k_0} = ys$. Note that if $k_0 = 0$ and $y \subset x$, then we get a contradiction with Lemma 4(1). Here $w$ is $(k_0, l_0)$-special by Definition 3, unless $r \in C(S(u_{0,k_0}))$ and $s \in C(P(v_{0,k_0}))$. If the two conditions hold, then $x'r \uparrow rx$ and $ys \uparrow sy'$ for some $x'$, $y'$. The result follows as in Case 3.

Case 4. $p_{0,k_0} \geq |u_{0,k_0}|$ and $p_{0,k_0} < |v_{0,k_0}|$
Here Definition 1(2) is satisfied, and there exist $x, y, r, s \in W(A)$ such that $|x| = p_{0,k_0}$, $x \uparrow y$, $x = ru_{0,k_0}$ and $v_{0,k_0} = ys$. Note that if $k_0 = 0$ and $r = \epsilon$ and $y \subset x$, then we get a contradiction with Lemma 4(1). Here $w$ is $(k_0, l_0)$-special by Definition 3(2) unless $s \in C(P(v_{0,k_0}))$. If $s \in C(P(v_{0,k_0}))$, then $ys \uparrow sy'$ for some $y'$ and the result follows as in Case 4.

We conclude this section by characterizing the special partial words that admit critical factorizations. If $w$ is such a special partial word satisfying $|v_{0,0}| \leq |v'_{0,0}|$, then $p_{0,0} = p(w, |u_{0,0}| - 1) < p'(w)$. The following theorems give a bound of how far $p_{0,0}$ is from $p'(w)$ and explain why Algorithm 2 is faster in average than a trivial algorithm where every position would be tested for critical factorization.

Theorem 4. *Let $w \in W(A)$ be a special partial word that admits a critical factorization, and let $v_{0,0}$ (respectively, $v'_{0,0}$) be the maximal suffix of $w$ with respect to $\preccurlyeq_l$ (respectively, $\preccurlyeq_r$). Let $u_{0,0}$, $u'_{0,0}$ be partial words such that $w = u_{0,0}v_{0,0} = u'_{0,0}v'_{0,0}$. If $w$ is special according to Definition 2(1), then the following hold: 0 0*

- *If $|v_{0,0}| \leq |v'_{0,0}|$, then the following hold:*
  1. *If $p_{0,0} \leq |v_{0,0}|$, then there exist for nonnegative integers $m$, $n$, partial words*

     $$x_0, \ldots, x_{m+2}, x'_1, \ldots, x'_{m+1}$$

     *of length $n$, and partial words $y_0, \ldots, y_{m+1}, y'_1, \ldots, y'_m$ of length $p'(w) - p_{0,0} - n$ such that*
     - *$x_0y_0x'_1y'_1x_1y_1x'_2y'_2 \ldots x_{m-1}y_{m-1}x'_my'_mx_my_mx'_{m+1}y_{m+1}x_{m+1}$ has a weak period of $p'(w) - p_{0,0}$,*
     - *$x_{m+1} \uparrow x_{m+2}$,*
     - *$p_{0,0} = |x_1y_1x_2y_2 \ldots x_my_mx_{m+1}| < p_{0,0} + |x_0y_0| = p'(w)$,*
     - *$u_{0,0}$ is a suffix of a weakly $p'(w)$-periodic partial word ending with*

       $$x_0y_0x_1y_1x_2y_2 \ldots x_my_mx_{m+1},$$

     - *$v_{0,0}$ is a prefix of a weakly $p'(w)$-periodic partial word starting with*

       $$x'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}y_{m+1}x_{m+2}.$$

  2. *If $p_{0,0} > |v_{0,0}|$, then let $s$ denote the nonempty suffix of length $p_{0,0} - |v_{0,0}|$ of $u_{0,0}$. Then there exist nonnegative integers $m$, $n$ and partial words as above except that*
     - *$p_{0,0} = |x_1y_1x_2y_2 \ldots x_my_mx_{m+1}s|$,*
     - *$u_{0,0}$ is a suffix of a weakly $p'(w)$-periodic partial word ending with*

       $$x_0y_0x_1y_1x_2y_2 \ldots x_my_mx_{m+1}s,$$

     - *$v_{0,0} = x'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}$.*
- *If $|v_{0,0}| \geq |v'_{0,0}|$, then the above hold when replacing $u_{0,0}$, $v_{0,0}$ by $u'_{0,0}$, $v'_{0,0}$ respectively.*

Proof. Let $x, y, r \in W(A) \setminus \{\epsilon\}$ and $s \in W(A)$ be such that $|x| = p_{0,0}$, $x \uparrow y$, $u_{0,0} = rx$, and either $v_{0,0} = ys$ or $y = v_{0,0}s$. We first assume that $v_{0,0} = ys$ (this case is related to Statement 1). Since $w$ admits a critical factorization, there exists $(k_0, l_0) \neq (0, 0)$ such that $w$ is $(k_0, l_0)$-nonspecial and either $(u_{0,k_0}, v_{0,k_0})$ (if $|v_{0,k_0}| \leq |v'_{0,l_0}|$) or $(u'_{0,l_0}, v'_{0,l_0})$

(if $|v_{0,k_0}| > |v'_{0,l_0}|$) is critical with minimal local period $q$ (here $p_{0,0} < q = p'(w)$). Let $\alpha, \beta \in W(A)\setminus\{\epsilon\}$ be such that $\alpha x \uparrow y\beta$, $|\alpha x| = |y\beta| = q$, either $u_{0,0}$ is a suffix of $\alpha x$ or $\alpha x$ is a suffix of $u_{0,0}$, and either $y\beta$ is a prefix of $v_{0,0}$ or $v_{0,0}$ is a prefix of $y\beta$. Let $m$ be defined as $\left\lfloor \frac{x}{\alpha} \right\rfloor$ and $n$ as $|x|(\text{mod } |\alpha|)$. Then let $\alpha = x_0y_0$, $\beta = y_{m+1}x_{m+2}$, $x = x_1y_1x_2y_2 \ldots x_my_mx_{m+1}$, and $y = x'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}$ where each $x_i$, $x'_i$ has length $n$ and each $y_i$, $y'_i$ has length $|\alpha| - n$. By Lemma 3, $\text{pshuffle}_{|\alpha|}(\alpha x, y\beta) = x_0y_0x'_1y'_1x_1y_1x'_2y'_2 \ldots x_{m-1}y_{m-1}x'_my'_mx_my_mx'_{m+1}y_{m+1}x_{m+1}$ is weakly $|\alpha|$ -periodic and $\text{sshuffle}_{|\alpha|}(\alpha x, y\beta) = x_{m+1}x_{m+2}$ is $|x|$ (mod $|\alpha|$) -periodic (which means that $x_{m+1} \uparrow x_{m+2}$) and the result follows. We now assume that $y = v_{0,0}s$ with $s \neq e$ (this case is related to Statement 2). Set $x = \gamma s$. Here $\alpha x \uparrow v_{0,0}\beta s$ for some $\alpha$, $\beta \in W(A) \setminus \{\epsilon\}$. By simplification, $\alpha\gamma \uparrow v_{0,0}\beta$, and we also have $\gamma \uparrow v_{0,0}$. The result follows similarly as above.

Theorem 5. *Let $w \in W(A)$ be a special partial word that admits a criticalfactorization, and let $v_{0,0}$ (respectively, $v'_{0,0}$) be the maximal suffix of $w$ with respect to $\preccurlyeq_l$ (respectively, $\preccurlyeq_r$). Let $u_{0,0}$, $u'_{0,0}$ be partial words such that $w=u_{0,0}v_{0,0}= u'_{0,0}v'_{0,0}$. If $w$ is special according to Definition 2(2), then the following hold:*

- *If $|v_{0,0}| \leq |v'_{0,0}|$, then the following hold:*
  1. *If $p_{0,0} \leq |u_{0,0}|$, then there exist for nonnegative integers $m, n$, partial words*

     $$x_0, \ldots, x_{m+2}, x'_1, \ldots, x'_{m+1}$$

     *of length n, and partial words $y_0, \ldots, y_{m+1}, y'_1, \ldots, y'_m$ of length $p'(w) - p_{0,0} - n$ such that*
     - *$x_0y_0x'_1y'_1x_1y_1x'_2y'_2 \ldots x_{m-1}y_{m-1}x'_my'_mx_my_mx'_{m+1}y_{m+1}x_{m+1}$ has a weak period of $p'(w) - p_{0,0}$,*
     - *$x_{m+1} \uparrow x_{m+2}$,*
     - *$p_{0,0} = |x'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}| < p_{0,0} + |y_{m+1}x_{m+2}| = p'(w)$,*
     - *$u_{0,0}$ is a suffix of a weakly $p'(w)$-periodic partial word ending with*

       $$x_0y_0x_1y_1x_2y_2 \ldots x_my_mx_{m+1},$$

     - *$v_{0,0}$ is a prefix of a weakly $p'(w)$-periodic partial word starting with*

       $$x'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}y_{m+1}x_{m+2}.$$

  2. *If $p_{0,0} > |u_{0,0}|$, then let $r$ denote the nonempty prefix of length $p_{0,0} - |u_{0,0}|$ of $v_{0,0}$. Then there exist nonnegative integers $m, n$ and partial words as above, except that*
     - *$p_{0,0} = |rx'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}|$,*
     - *$u_{0,0} = x_1y_1x_2y_2 \ldots x_my_mx_{m+1}$,*
     - *$v_{0,0}$ is a prefix of a weakly $p'(w)$-periodic partial word starting with*
       $rx'_1y'_1x'_2y'_2 \ldots x'_my'_mx'_{m+1}y_{m+1}x_{m+2}$.
- *If $|v_{0,0}| \geq |v'_{0,0}|$, then the above hold when replacing $u_{0,0}$, $v_{0,0}$ by $u'_{0,0}$, $v'_{0,0}$ respectively.*

Proof. Let $x, y, s \in W(A) \setminus \{\epsilon\}$ and $r \in W(A)$ be such that $|x| = p_{0,0}$, $x \uparrow y$, either $u_{0,0} = rx$ or $x = ru_{0,0}$, $v_{0,0} = ys$, and let $(k_0, l_0)$ and $q$ be as in the proof of Theorem 4. Statement 1 is similar to Statement 1 of Theorem 4. For Statement 2, let $\alpha, \beta, \gamma \in W(A) \setminus \{\epsilon\}$ be such that $y = r\gamma$, $r\alpha u_{0,0} \uparrow y\beta$, $|\alpha x| = |y\beta| = q$, and either $y\beta$ is a prefix of $v_{0,0}$ or $v_{0,0}$ is a prefix of $y\beta$. By simplification, $\alpha u_{0,0} \uparrow \gamma\beta$, and we also have $u_{0,0} \uparrow \gamma$. The result follows from Lemma 3.

## 6. Conclusion

In this paper, we considered one of the most fundamental results on the periodicity of words, namely the critical factorization theorem, and extended it to partial words (such sequences may contain “do not know symbols” or “holes”). While the critical factorization theorem on words shows that *critical factorizations* are unavoidable, Theorem 2 shows that such factorizations can be possibly avoidable for the so-called *special* partial words. Then, Theorem 3 refines the class of special partial words to the class of the so-called $(k,l)$-*special* partial words. Theorem 3’s proof leads to an efficient algorithm which, given a partial word with an arbitrary number of holes, outputs “no critical factorization exists” or outputs a critical factorization that gets computed from the lexicographic/reverse lexicographic orderings of the nonempty suffixes of the partial word and its reversal. Finally, Theorems 4 and 5 characterize the (0, 0)-special partial words that admit critical factorizations.
In our testing of the algorithm, we felt it important to make a distinction between partial words that have a critical factorization and partial words for which no critical factorization exists. In the table below, we provide data concerning partial words without critical factorizations. Tests were run on all partial words with an

arbitrary number of holes over a three letter alphabet from sizes two to twelve.

| Size | Number of partial words without CFs | Number of partial words | % |
|---|---|---|---|
| 2 | 0 | 16 | 0.0 |
| 3 | 0 | 64 | 0.0 |
| 4 | 24 | 256 | 9.375 |
| 5 | 144 | 1024 | 14.063 |
| 6 | 816 | 4096 | 19.922 |
| 7 | 3852 | 16384 | 23.511 |
| 8 | 17376 | 65536 | 26.514 |
| 9 | 73962 | 262144 | 28.214 |
| 10 | 311460 | 1048576 | 29.703 |
| 11 | 1269606 | 4194304 | 30.270 |
| 12 | 5115750 | 16777216 | 30.492 |

In the case where a partial word has no critical factorization, we exhaustively search $|w| - \|H(w)\|$ positions for a factorization. Now we show the average values for our indices $k_0$, $l_0$, $k_1$, $l_1$ after the algorithm completes over the same data set. Also, we show the average values for these indices when partial words without critical factorizations are ignored.

| Size | All partial words | | | | Partial words with CFs | | | |
|---|---|---|---|---|---|---|---|---|
| | $k_0$ | $l_0$ | $k_1$ | $l_1$ | $k_0$ | $l_0$ | $k_1$ | $l_1$ |
| 2 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 4 | 0.137 | 0.180 | 0.105 | 0.102 | 0.0 | 0.0 | 0.0 | 0.0 |
| 5 | 0.352 | 0.377 | 0.233 | 0.212 | 0.017 | 0.017 | 0.010 | 0.010 |
| 6 | 0.617 | 0.657 | 0.453 | 0.394 | 0.049 | 0.049 | 0.033 | 0.033 |
| 7 | 0.848 | 0.910 | 0.651 | 0.568 | 0.083 | 0.081 | 0.058 | 0.058 |
| 8 | 1.093 | 1.181 | 0.862 | 0.763 | 0.123 | 0.121 | 0.091 | 0.090 |
| 9 | 1.297 | 1.413 | 1.050 | 0.945 | 0.160 | 0.158 | 0.121 | 0.120 |
| 10 | 1.505 | 1.650 | 1.242 | 1.134 | 0.196 | 0.194 | 0.151 | 0.150 |
| 11 | 1.676 | 1.848 | 1.407 | 1.301 | 0.229 | 0.228 | 0.180 | 0.179 |
| 12 | 1.834 | 2.030 | 1.562 | 1.460 | 0.262 | 0.261 | 0.209 | 0.209 |

From this data, we see that if a partial word has a critical factorization, then the algorithm discovers it extremely quickly.

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