Investigating Quadrilaterals as an Ongoing Task

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Abstract:
In this article we discuss an open-ended problem involving quadrilaterals that we continually offer each semester. The task has been posed to undergraduate and graduate students in methods and problem solving classes. The task involves drawing all possible four sided figures with corners at the dots. A four by four array of dots is included in the instructions and students are asked to develop a system for knowing when they have identified all the quadrilaterals. Students are also encouraged to classify them in as many ways as they can and to look at the perimeters and angle measures. The focus of the discussion is on the potential richness of a task and how students engage in non-routine explorations.

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What constitutes a good mathematics task? One way to address this classic question is in terms of mathematics teacher education. Pre-service and in-service teachers stand to gain a variety of new perspectives about mathematics teaching and learning by experiencing rich mathematics tasks themselves. At both the undergraduate and graduate levels, we offer tasks as on-going investigations to encourage critical thinking, meaning making, and connections.

In this article we offer insights into an investigation of quadrilaterals by various groups of undergraduate and graduate students. We will extend our insights through elaborating how elementary students approached the same task. The purpose for sharing our experiences is to highlight both the rich potential of a non-routine task and of student thinking. It is our hope that other mathematics educators will offer the task to their students so that we can continue adding layers to our own thinking and perhaps assist our students in doing the same. In the following sections, we offer a discussion of our philosophical background, what constitutes a good task, an outline of various student solutions of the "quads" problem, and a discussion of the mathematics that has emerged from the quadrilateral investigation.

Philosophical Background
Problem solving takes on a wide range of interpretations and has often been posited as a linear process involving distinct steps such as read, know, plan, solve, and check. The "problem" itself, posed in such a rigid framework, has the tendency to become limited and void of rich mathematical experience for students and teachers. We argue the use of the term "problem" can imply or evoke thoughts of a prescribed process and perhaps hamper student and teacher engagement.

An alternative way to look at problem solving is to see it not just as a skill to be learned but also as a way of learning mathematics. As students work to solve problems they construct mathematical relationships. This approach to problem solving, learning through problem solving, has been the focus of much research and the development of alternative curriculum materials over the last 20 years. Hiebert (1999), in analyzing this research and alternative curriculum approaches, argues "Students learn new concepts and skills while they are solving problems” (p.12, italics in original).

We prefer to use the word “task” in this setting. The teacher selects a task that, based on her/his experience with the learner, has the potential to become problematic for the student; however, it is the student for whom the task becomes a problem – or not, based on that student’s prior mathematics (Wheatley, 1991). If the student has a quick and ready solution to the task then it is not a problem for the student. Only if the task becomes a problem, that is, a task for which no solution is immediately evident, does the task provide a learning opportunity for the student (Glassersfeld, 1990).

These tasks evoke a more open-ended, investigative feeling, distinctly different from traditional problem solving. We argue for a problem-centered (Wheatley, 1991) approach because it is aligned the most closely with our beliefs about learning mathematics through problem solving. In problem-centered learning, students construct their mathematics through solving problems, presenting their solutions, and learning from one another's methods. This is different from traditional methods of teaching mathematics, in which the teacher specifies a procedure from the start, and students practice that procedure until they are comfortable with it. The quadrilaterals task, from our viewpoint, fits well with the problem-centered learning environment. This is not to say, though, that other types of learning environments would not function well with a non-routine task such as the quadrilaterals investigation. However, we do feel environments in which the student and teacher work hand-in-hand at investigating mathematical ideas make the most sense for non-routine problem solving.

What Constitutes a Good Task?

On all levels, not just with teachers, tasks should be potentially meaningful to students, be problem based, encourage students to devise solutions, invite students to make decisions, be replete with patterns, lead somewhere mathematically, promote discussion and communication, and contain an element of surprise (Wheatley, 1991). Davis (1996) adds that they should be rich, open-ended, and allow for variable entry. In addition, students should expect a task to require an investigation and time, to be puzzled, to develop their own methods, to work together and negotiate, and to explain their solutions (Reynolds & Wheatley, 1996).

A task that leads somewhere mathematically is intimately linked with the person posing the initial question (in most cases the instructor). It is not just the mathematics that the teacher expects her/his students to “get,” it is also the way s/he imposes for the students to engage in the task. A task that a teacher deems as having rich potential could have little significance for the students if social norms are that of students following specific problem solving steps or imitating procedures. Rather than taking the stance that students should construct their own knowledge, we agree with Lobato, Clark, and Ellis (2005) that students do construct their own knowledge. As is often seen in lesson plan objectives, the wording is usually “students should or will be able to...” For teachers to impose what students are supposed to gain from a task makes sense in some contexts, especially in high stakes testing environments, but what makes a task rich are the unforeseen paths extending through initial mathematical objectives. Challenging students to ask questions, like “what if not” (Brown & Walter, 2005) about a particular problem and perhaps being inspired to go in a direction not anticipated by the instructor is part of a rich task. There is potential for students and teachers to be changed, that is to learn, in such environments. Therefore, it is our stance in this discussion that it is not just the task itself, it is how we engage in the task (Pratt, working paper) that enables change, meaningful mathematics construction, and pattern-relationship exploration.

Sociomathematical Norms and Nonroutine Tasks

Establishing classroom norms that encourage such engagement is crucial for mathematics learning to occur through open-ended tasks. Stein, Grover & Henningsen (1996) propose three phases of task implementation: task as it is presented in a resource or curriculum, the task as envisioned and presented by the teacher, and the task as implemented by students. Thus, as a teacher, choosing and presenting a non-routine problem is not enough. The classroom norms must be such that students feel comfortable enough and know how to share ideas about their thinking and take risks in posing problems so that the mathematics can emerge. If students are used to a traditional mathematics class routine of imitating procedures at the board as the teacher presents them, participation in different mathematics practices such as communication may need to be scaffolded at first as students negotiate new norms for doing mathematics.

Yackel and Cobb (1996) identified sociomathematical norms between students and teachers in reform based mathematical environments while McClain and Cobb (2001) offered an extensive analysis of how sociomathematical norms emerged in a classroom. In both discussions, the types of tasks were deemed as important, but it is carefully noted that, "it was the activities as they were realized in interaction in the classroom that supported the students‟ mathematical development (McClain & Cobb, 2001, p. 244)."

Adapted from McClain and Cobb's ideas, we feel some key elements in developing strong sociomathematical norms between students and teachers include how tasks or problems are posed, how students negotiate a problem, how students engage in a problem, the significant mathematics students bring from or to a task, the ideas and meaning that emerge from a task from both the student and teacher's standpoints, and what type of mathematical objectives can be aligned with a task.

In the following sections we take a look at these elements in relation to a rich non-routine task we have been posing and continue to pose to graduate and undergraduate students.

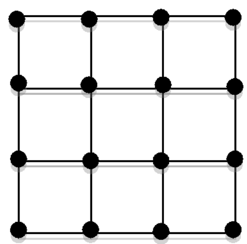

Figure 1. **The Quadrilaterals Task.**

Given the set of sixteen dots shown, draw all possible four sided figures with corners at the dots. Develop a system for knowing when you have identified them all. Classify the quadrilaterals in as many ways as you can. Example: Classify by the number of right angles. What are the possible perimeters? Largest? Smallest? What are the possible angle measures? Largest? Smallest?

The Students and Task

Over the past five years, we have offered the quadrilaterals task to pre-service teachers taking our methods classes, in-service teachers taking our graduate level problem solving classes, and full-time mathematics education doctoral students. Depending on the course, students are assigned various readings throughout the semester and are posed with open-ended questions brought in by both the instructor and students. In addition to the weekly in-class investigations, students are presented with the "quads task" as being the on-going investigation in which they work alone or in groups of two both in and out of class (see Figure 1). At the end of the semester, each group presents their ideas and we focus here on a vast array of student solutions we have collected over time. Students will be unidentifiable and the focus here is to look at the mathematics that has emerged and student approaches to the quadrilaterals task. It is important to note that many of the ideas expressed here also come out of a conference presentation we gave in which fellow educators engaged in the quadrilaterals task and helped add to the growing list of the task's mathematical possibilities (Richardson, Reynolds, & Stein, 2008). In addition, the works of Kennedy (1993) and Kennedy & McDowell (1998) have served as an excellent reference for investigating both triangles and quadrilaterals.

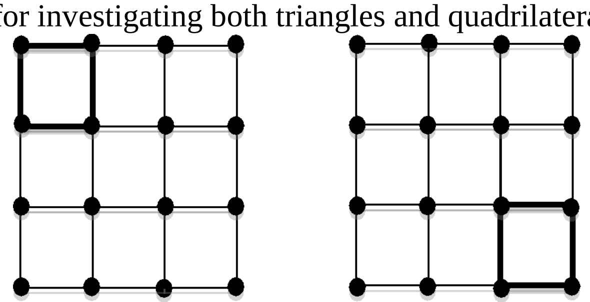

Figure 2. **Are these two solutions the same?**

Messy Beginnings

Yackel and Cobb (1996) suggest that one sociomathematical norm that is negotiated in classrooms is what counts as an acceptable solution. A common occurrence in the initial forays of the students includes a host of questions and attempts ranging from fundamental inquiries about the problem to enacting a random generation of quadrilaterals. For example, does a different location on the array count as a different solution (see Figure 2)? After some negotiation among the class, the consensus is that a different location does not constitute a different quadrilateral. Therefore, a square with the area of one is the same no matter where it is located on the grid.

Another idea that we often see spin off the question posed in the initial problem centers on organization. Students want to know how to organize their quadrilaterals to help them identify new ones as well as duplicates. We provide dot paper and geoboards as a way to support their thinking and many students use the materials as a starting point. Organizing the quadrilaterals in a way that exposes patterns is a route many are interested in because they feel it might lead to the generation of a formula. The discussion about finding a formula ensues because that means an answer could be arrived at in a timely fashion (as often noted in their presentations). As our classes progress throughout the semester, we find that many groups abandon their initial attempts and choose to face a new set of messy beginnings.

Exploring the Mathematical Possibilities

This section is purposefully called "exploring the mathematical possibilities" rather than a "description of solutions." While solutions are part of the possibilities, we find the quadrilaterals task takes on a life of its own because students tend to nestle into a process that almost becomes the problem for them – hence, problems within the quadrilaterals problem. What follows is the listing of mathematical ideas we've seen emerge along with supporting examples.

Systemizing to generate all possibilities

We've seen groups spend the first three classes using the same two points to create as many quadrilaterals as possible containing the segment created by those two points. They often find that by using the same line each time, they are creating too many duplicates. Therefore, they restart their investigation by choosing sections instead of lines, enabling them to eliminate duplicates more efficiently. A way that some have chosen to separate the 4 × 4 array into the sections is by looking at it in terms of a 2 × 2, 2 × 3, 3 × 3, and 3 × 4. So, the 2 × 2 array contains 4 dots and 1 quadrilateral, the 2 × 3 contains 5 dots, and 2 additional quadrilaterals, and so forth (see Figure 3).

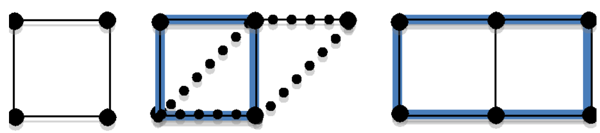

Figure 3. **Four dots, 1 quadrilateral – 5 dots, 2 additional quadrilaterals – 6 dots, 1 additional quadrilateral**

Another approach in systemizing to generate all possibilities is by focusing on the quadrilateral shapes in relation to the rows. For example, drawing all figures that use just the first two rows, then all that used the first three rows and so on. Students have also used a fixed side approach and found as many shapes as possible by moving only one corner around on a geoboard. They've recorded solutions on dot paper and noticed duplicates emerging. It is in the noticing of duplicates for a large number of students that the problem within the problem frequently occurs. The attention then shifts to the elimination of duplicates rather than generating new quadrilaterals. The process for deleting repeats has involved strategies like overlaying clear overhead projection paper over dot paper, tracing it, then rotating the paper to see if it was the same as another one.

Classification and properties of shapes

Another strategy seen is the drawing of squares and rectangles for the purpose of finding all the possible versions of the two polygons. Many note that they want to view the behavior of the quadrilaterals and how seeing a series of trapezoids, for example, would look when created on the four by four array of dots. This image making and image having (Pirie & Kieren, 1994) seems to help the students establish some sort of a focus in where they want to go with the problem.

For group(s) focusing on the shapes themselves, the focus is on parallelograms, squares, rectangles, trapezoids, rhombi, kites, and chevrons. Each week, they focus on finding all the quadrilaterals of each kind. Some will even use slope to eliminate duplicates by identifying irregular shapes. These shapes have sides with a slope other than zero. In addition, the focus can easily turn to a student exploring only chevrons because of the various concave angles.

In *systemizing to generate all possibilities, classification, and focusing on the properties of shapes*, a host of more specific mathematical content has emerged for students. Figure 4 provides an example of mathematical ideas we've collected during the quadrilaterals investigation.

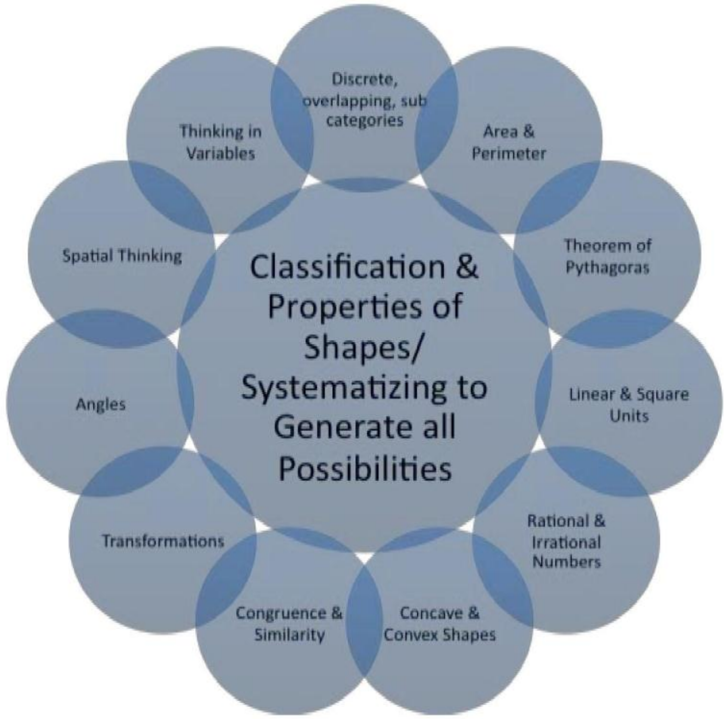

***Figure 4*. Some of the Mathematics in the Quadrilaterals Problem**

On two separate occasions we have been surprised in unique ways by students' approaches to the task. In one class a pair of students who had generated a large number of quadrilaterals was having difficulty sorting through them to identify and eliminate duplicates. None of the methods they tried proved satisfactory. One of the pair had, in discussion with a friend who was an engineer, heard him talk about something called 'vectors' The pair knew that vectors had something to do with direction of line segments and felt it might somehow be a way to identify duplicates. While they were never able to actually make vectors work for them in their analysis of the various quadrilaterals, over the subsequent weeks they learned a considerable amount about something that was for them a whole new field of mathematics, vector analysis.

In another class two students took on another noteworthy approach inspired by a statistics course they were taking together at the time. They connected their experiences in the class with the degrees of freedom concept. For example, imagine you are given the average of four numbers. The first three numbers could be any value, but once they are determined there is only one number that can serve as the fourth value if you want to keep the same average. Similarly, they thought that if two of the sides of a quadrilateral were set, once you drew the third side, the fourth side would be a given. Therefore, their strategy was to draw all the possible quadrilaterals when two adjacent sides were fixed (see Figure 4).

Sharing the Quadrilaterals Task with Elementary Students

We have also had our students offer the task to elementary students. They have reported that children jump right into the problem but need coaching about what a quadrilateral is because they tend to make figures with three or five sides. They are also quick to find the regular quadrilaterals (squares and rectangles) but fewer think of irregular ones. For students in fourth through sixth grade the quadrilaterals task connects well with so much mathematics they are just making sense of particularly area, square numbers, and the Theorem of Pythagoras. In trying to find the area of some of their quadrilaterals they have to identify a "unit" as none is specified. Some define a unit length as the distance between two adjacent points on the grid (horizontal or vertical). We have seen this lead to an interesting dilemma as they often also label the distance between two adjacent points along a diagonal as "one." Others go straight to labeling a square formed by joining four adjacent points as a "square unit." When they connect adjacent points diagonally to make a square they frequently claim it is also a square unit (see Figure 6). It is only when they draw their shapes on the square grid paper and cut out their various quadrilaterals in an attempt to classify them and check for duplicates by laying shapes on top of each other that they notice that these are not the same length or area. It is often at that time that students begin to question how they can find the length of these diagonal line segments. This becomes an opportune time to introduce the Theorem of Pythagoras. Students have searched websites to explore a variety of visual animations of this theorem.

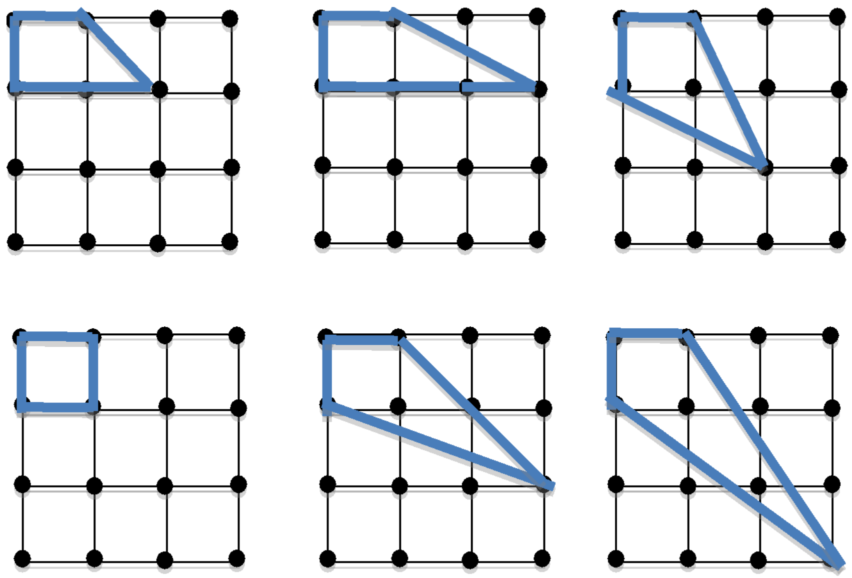

***Figure 5*. Degrees of Freedom**

Discussion

As instructors, we harbor a certain sense of anticipation when posing the quadrilaterals investigation, which is perhaps why we continue to challenge students and ourselves with the task. The anticipation we experience is informed, similar to Davis' (1996) ideas on curriculum anticipation, and we continually add to our list and add to the layers of our thinking about the problem. We don't expect the students to engage in the task in a certain way.

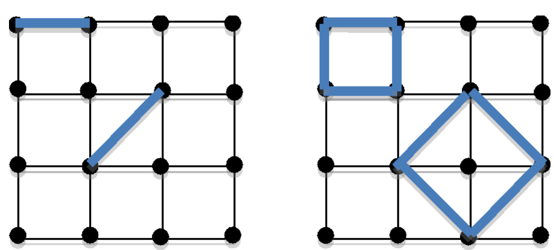

***Figure 6*. Horizontal unit, diagonal unit, square unit, square unit on diagonal**

One interesting aspect of this particular task is that more often than not, students do not actually reach a consensus that they have found all the quadrilaterals over the extended time they spend immersed in the task. For some, this can become frustrating; they realize there are many possibilities and wonder how they are going to find them all and know that they have done so. It is at that point that, as teachers, we need to encourage them in their explorations and not worry about "getting an answer." In a sense, these students need to 'let go' of their focus on a number and concentrate on making sense of relationships they are seeing, categorizing shapes, and developing some system for generating the various shapes and eliminating duplicates. It is only then that they really begin to develop some important mathematics. In using this task with teachers this becomes a time when we can discuss the importance of the investigation rather than the answer. One example we use is the centuries that were spent by mathematicians attempting to prove Fermat's last theorem. Whole new branches of mathematics were developed during that time as a direct result of the chase for a proof. When Andrew Wiles announced his proof in 1993/4 some mathematicians expressed a sense of sadness. Mathematician, James Conway, captured this sense in the Nova program (Lynch, 1997) devoted to Wiles' proof of Fermat's last theorem: "I'm relieved that this result is now settled. But I'm sad in some ways, because Fermat's last theorem has been responsible for so much. What will we find to take its place?"

While students may not be inventing new branches of mathematics as they explore the quadrilaterals task, when given the chance, they are acting mathematically in new ways. As they become immersed in the challenge of systematizing their search for all possible quadrilaterals they have opportunities to explore a rich set of mathematical ideas, some of which are new ideas for them.

The quadrilaterals task and tasks like it provide an opportunity for prospective and practicing teachers to experience firsthand the potential for significant mathematics learning through problem solving (Hiebert, 1999; Wheatley, 1991). They can begin to conceptualize the mathematics curriculum as a coherent whole where a variety of standards and objectives are integrated. This is distinct from their previous more traditional approaches to state and district standards and objectives as a "laundry list" of items to be checked off as they are taught one after another. The Principles and Standards for School Mathematics process standards (NCTM, 2000) are evident in the classroom interactions as students negotiate their way through the task. They find themselves solving a variety of problems as they make sense of the task (Problem Solving), communicating and defending their mathematical ideas (Communication; Reasoning & Proof), symbolizing those ideas in various ways (Representation), and making connections with several areas of mathematics (Connections). In many cases our students have not previously had opportunities to experience mathematics learning in ways they are or will be expected to teach their students. They now have a better sense of the potential learning opportunities in designing their mathematics classrooms so that they are more in line with reform efforts in mathematics education.

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