Finding the majority-rule equilibrium under lexicographic comparison of candidates

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Abstract:

This paper considers the well studied problem of the existence of an undominated point, under the assumption of lexicographic preferences of voters, as espoused by Taylor in [24]. We extend Taylor's model to situations were we allow for (i) voters to have different ranings of the issues in n- dimensional issue space and (ii) a candidate to be disregarded by a voter if his stand on any one or more of the issues involved in the election is perceived to be too extreme by the voter and (iii) combinations of (i) and (ii). We extend the results of Taylor by demonstrating the non-existence of an equilibrium point in these models in general and then showing that under special cir- cumstances, specialized variants of the “median” point(s) represent equilib- rium or undominated points in these models too. Thus a model of voting behavior results that is closer approximation of reality in that historically incumbents tend to win. The primary conclusion of the paper is to suggest that incumbents tend to have an advantage when the election process is characterized by a large presence of special interests or as information be- comes more expensive to acquire.

Article:

1 Introduction

Plurality-rule elections are one of the most widely studied problems in voting theory. In this problem, a set of voters elects one of two candidates to office. The candidate which receives the most votes wins the election. Each candidate must therefore determine how to position herself with respect to voters in order to win.

In our model, we assume that there is a two candidate election in a simple plurality-rule system. We further assume that the entire set of voters is concerned with the same issues in the issue space, and that the voters have single-peaked preferences in this space. Finally, we assume the election satisfies the following five conditions [18]:

1. All voters know the candidates' positions on the issues.
2. The candidates' strategic opportunities are identical.
3. Candidates maximize plurality.
4. Candidates know the form of voters' preferences on the issues.

A candidate that cannot be defeated by another candidate in a pairwise election is said to be undominated. If an undominated candidate exists, then a majority-rule equilibrium exists. However, it is not necessarily true that an undominated candidate exists. For this reason, an important problem in voting theory is to define simple conditions for the existence of an undominated point [18].

Many researchers have addressed the problem of determining the conditions for an undominated point under different kinds of voter preferences: Euclidean metric i.e., circular preferences ([1, 4, 5, 9, 10, 14, 19]), city block (L1) norms ([21, 27]) or polyhedral norms ([6]). However, as shown in many studies (see for example, [16], [19], [20]), the probability of existence of such an undominated point is almost always zero, when the number of issues in the election are high enough. In that case, voting cycles are possible and the election may be manipulated [1, 18]. Observing election outcomes however, contradicts this finding in that incumbents usually defeat challengers [26]. This paradox suggests that voting model assumptions do not fully capture actual voting behavior, and the purpose of this paper is to elicit a voting model that attempts to remedy this. However, it must be pointed out that several researchers have attempted to do the same before. For example, it is shown in [25] that if the model of voter behavior is altered only slightly to incorporate any of the following several plausible characteristics of decision making such as, cost of change, cost of uncertainty, bounded rationality etc., the probability of stability coverges to 1 as the population size increases.

Our principal point of departure in this paper is an alternative method of measuring disutility or voter preferences, introduced in the seminal work of Taylor in [24]. This work proposed using lexicographic preferences to rank the candidates in an election. Such voting behavior presupposes that the voters rank the election issues in a hierarchical manner, in the order of their relative importance. For example, a conservative voter may rank deficit control above social welfare expenditure, whereas a liberal may do the reverse. Similarly, an unemployed voter may perceive job creation as the most important issue and place it ahead of both deficit control and social welfare expenditure; or yet another voter may perceive abortion as the leading issue in an election. Notwithstanding the ranking employed by an individual voter, any issue is assumed to dominate the ones that are deemed less important and is in turn, dominated by the ones that are considered more important. Finally, a voter evaluates all the candidates by comparing their stands on the different issues separately and in a fixed sequence; the particular sequence employed by a voter is supposed to mirror his views about the relative significance of the tissues.

The use of lexicographic preferences for measuring disutility in definitely not new and can be traced back almost forty years, when social scientists proposed it to allow a decision maker to choose between alternatives of non-comparable subjective value; a good survey is available in [29]. As far back as 1954, Hausner showed in [13], that if a set of preferences satisfies all axioms for von Neumann and Morgenstern utility, except for the Archimedan axiom, then those preferences can be represented, not by a scalar, but by vec- tors. Further, if these vectors are of finite size, the decision process of choosing the alternative with the least disutility is akin to the lexicographic principle elicited above. The use of lexicographic preferences was further advocated by Georgescu-Roegen [12] in his *Principle of Irreducibility of Human Wants*, which hypothesized that “human needs and wants are hierarchized”, and also by Chipman in [13], who stated that (quote from [24])

> “Utility, in its most general form, is a lexicographic ordering, represented by a finite of infinite dimensional vector with real components ... and these vectors ... are ordered lexicographically”.

In addition to the above, we also argue that the imposition of lexicographic preferences is further justified if one considers rational ignorance or interest group participation in the political process — facts that are particularly true in the context of most present day elections. Voters must expend resources on information to determine where a candidate stands on each issue. A voter may thus find it irrational to acquire information about issues for which they do not feel strongly. Furthermore, interest groups tend to promote single tissues. If one assumes that the goal of any group is to transfer the maximum amount of public wealth to their cause, then group members would always vote for the candidate who had positioned herself closest to their ideal point with respect to that issue.

It is worthwhile to note that similar hierarchical comparisons are well known in other areas such as Decision Theory. For example, in Goal Programming we assume that a decision maker who is faced with optimizing a model with respect to various criteria, i.e., objective functions, first optimizes with respect to the most important one. If multiple optimal solutions exist, then the second most important criteria is optimized within this set of all optimal solutions obtained in the first stage. This process is continued until either all criteria have been optimized hierarchically, or, a unique optimal solution is reached at some intermediate stage and is chosen as the best solution. Another related approach is the well known Analytical Hierarchy Process.

The principal result of Taylor in [24] was to show that if all voters had the same ranking of issues in an election, and voted in accordance with the lexicographic process described above, then an undominated point always exists and is given by the set of all medians; a theoretical result that is in agreement with realistic elections. If different voters have different rankings however, he showed that an undominated point may not exist and gave necessary and sufficient conditions for the existence of such a point in the two dimensional issue space. In this paper, we have extended this concept of lexicographic measurement of disutility to more refined models where we allow for (i) voters to have different rankings of the issues in the general *n*-dimensional issue space and (ii) a candidate to be disregarded by a voter if her stand on any one or more of the issues involved in the election is perceived to be too extreme by the voter and (iii) combinations of (i) and (ii). We show that specialized variants of the concept of medians also constitute equilibrium or undominated points in these models.

The remainder of the paper is divided as follows. In the next section we discuss preliminaries and Sect. 3 considers our model under the assumption that different voters use different rankings of the issues in their comparison of candidates. Section 4 then examines the results of the preceding section in light of another assumption, namely, that voters may abstain from voting. Finally the fifth section summarizes the conclusions of our study. The time complexity of all the tests and computations are also discussed at appropriate places in the paper.

2 Preliminaries

We assume that there are n issues involved in the election and that the leftmost and rightmost stands that any voter or candidate can take on any issue are denoted by 0 and 1 respectively; hence, for the purpose of the paper, the term *issue space* will refer to the unit hypercube in $\mathfrak{R}^n$. There are *P* voters, each of whom is assumed to have a unimodal preference function and can therefore be represented by his unique ideal point in the issue space. The voters themselves will be referred to as v^1, v^2,... v^P respectively, with the ideal point of voter v^i being given by the vector $(v_1^i, v_2^i, \ldots v_j^i, \ldots, v_n^i)$ where v_j^i denotes the location of the ideal point of voter v^i on issue *j*. Figure 1 illustrates an example of the configuration of five voters, namely v^1 through v^1 who are located in a two dimensional issue space. Unless otherwise mentioned, we will assume throughout that the voters have equal preferences for all the issues. Where there is no ambiguity, we will refer to a candidate or a voter located at a point $x = (x_1, x_2, \ldots x_n)$ in the issue space as the candidate/voter *x*, with x_i denoting the position of candidate/voter *x* on issue *i*. For any two distinct points *x* and *y* in the issue space, the closed interval $[x,y]$ will be used to denote the entire line segment connecting them, and $(x,y]$ the open ended interval obtained therefrom, by excluding the endpoint x.

We only consider cases where there are two candidates vying for the votes of the P voters, one of whom is an incumbent and the other the challenger. If a voter finds that both candidates are equally "far" from his ideal point, he is assumed to vote for the incumbent. There are several reasons for advocating this incumbency advantage. First, since an incumbent has a voting history, voters feel more comfortable voting for a known commodity (as the old saying goes, "better the devil you know ...."). A second reason for this assumption is the uncertainty/risk faced by voters in voting for a candidate. Voters are always faced with the risk that once in office, a candidate may not vote according to her campaign rhetoric (e.g. Read my lips, no new taxes). Therefore, if two candidates have the same mean on a particular issue, it is reasonable to assume that a voter will perceive the incumbent as the candidate with the lowest probability of deviation from their stated position, and therefore, vote for her. Finally, incumbency advantage can also result from the fact that there is always a "cost" associated with any change in office (see [25] for a good discussion on this issue) and voters will refuse to incur this cost unless the challenger is perceived to have a"substantially" better stand than the incumbent.

In our initial model, we will also assume that every voter is required to vote, relaxing this assumption only later in the paper. With a majority rule for declaring the winner of the election, this would imply that the challenger would have to get at least ($\lfloor LP/2 \rfloor$ + 1) votes to win the election; in other words, the incumbent can win the election if at least $\lceil FP/2 \rceil$ voters vote for her. An *undominated point*, is defined as a point such that if an incumbent were to locate there, no challenger could win an election from her.

In this paper, we often project the voters on their respective stands on any one issue, say *j*. When projected along issue *j*, $v_j^{[1]}$ will be assumed to represent the location of the leftmost voter on this issue, $v_j^{[2]}$ the location of the second voter from left and so forth, with $v_j^{[P]}$ denoting the location of the voter with the rightmost stand on this issue. The location of the "centrist" voter, i.e., the median voter, is therefore anywhere inside the set of all medians along this issue, that we denote by M_j. It is easy to see that M_j is the point $v_j^{[[P/2]]}$ if P odd, or the entire interval $[v_j^{[[P/2]]}, v_j^{[[P/2+1]]}]$ in case P is even. For example, in figure 1, considering the voters only on their stands on issue 2 we see that $v_2^{[1]} = v_2^1$, $v_2^{[2]} = v_2^2$, $v_2^{[3]} = v_2^4$, $v_2^{[4]} = v_2^3$ and $v_2^{[5]} = v_2^5$. Thus $M_2 = v_2^{[3]} = v_2^4$. Given a voter v^i and a point x in the issue space, if $v_j^i \leq x_j$ (respectively, $v_j^i < x_j$) we will say that v^i is to the left (respectively, strictly to the left) of x on issue j. The notion of a voter being to the right, or strictly to the right of a given point is similar. Thus by our definition, in Fig. 1, v^4 is to the left (or right) of y on issue 1 (since $v_1^4 = y^1$), and strictly to the left of y on issue 2.

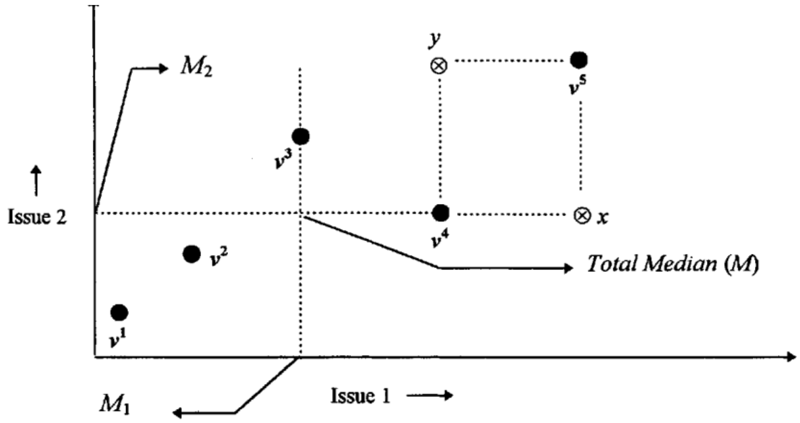

Fig. 1.

We now define a set of special locations in the issue space, called the set of all *Total Medians* of the voting population. This set, that we denote by M is the location of all the "centrist" candidates who occupy the middle of the road position on every issue, and is characterized as follows. If P is odd then, M is given by the unique point x in the issue space that satisfies:

$$\{x | x_j = v_j^{[[P/2]]} \quad \text{for } j = 1,2,\ldots, n\} \tag{1}$$

On the other hand, when P is even, M is the set of all points x in the issue space that satisfy

$$\{x | v_j^{[[P/2]]} \leq x_j \leq v_j^{[[P/2]]} \text{ for } j = 1, 2,\ldots, n\} \tag{2}$$

It is easy to see that the projection of the set M on any issue j is simply the set Mj (see Fig. 1 where the points representing M_1, M_2 and the Total Median M, of the five voters in the figure are shown).[1]

As mentioned in the introduction, our model assumes that voters vote in accordance with the lexicographic choice rule, as elicited by Taylor in [24]. For any voter v^i in a homogenous population, this process can be summarized as the following algorithm.

Algorithm *Lexico-Comparison* (Homogenous Population).

1. Choose the most important issue at the Present Issue.
2. Compare both candidates according to their stand on the Present Issue.
3. Vote for the candidate whose position on the Present Issue has the least disutility and stop. If both candidates are found to be equal and all issues have been considered, vote for the incumbent and stop. Else go to Step 4.
4. Choose the next most important issue as the Present Issue and go back to Step 2.

By this method of measuring disutility, it can be verified that if x and y are the two candidates in Fig. 1, then voter v^4 would vote for y but voter v^5 would vote for x instead. Based on this voting behavior, Taylor stated that:

Lemma 1 (from Taylor [24]): *Given a homogeneous population of voters that votes in accordance with Algorithm Lexico-Comparison, a point is undominated iff it is a Total Median of the voting population.*[2]

Since a Total Median of a voting population is always guaranteed to exist, Taylor's result therefore guarantees that in a homogenous population of voters, an undominated point will always exist. Further, it also allows a simple characterization of the *core* of the voter distribution, which is defined as the set of all undominated points. By this lemma, it can be asserted the core is given by the set of all Total Medians, i.e. M, and hence characterized by (1) and (2).

3 Heterogeneous voting populations

As mentioned before, the primary result in [24] is about a population of homogeneous voters, i.e., every voter agrees that issue 1 is most important, followed by issue 2, issue 3 etc., in that order. However, in more realistic situations, this assumption may not be true. It is only reasonable to expect that in most real world elections, the different voters involved have different priorities and hence, different perceptions about the relative importance of the issues involved. For this purpose, Taylor also studied the case where different voters have different rankings of the issues involved in the election. We refer to such a population as being *heterogeneous* and its study is the focus of this section.

When the voting population is heterogeneous, different voters may have different perceptions about the relative importance of the n different issues. Hence different voters may compare the two candidates by evaluating their respective stands on the n issues in different sequences. Thus the entire voting population would compare the candidates involved in accordance with *Algorithm Lexico-Comparison,* with the modification that voters may differ in the ranking of the issues used in the algorithm, and may hence compare the candidates' stands on the issues in different orders. For example, in Fig. 1 , if we assume that voters v^1, v^2 and v^3 rank issue 1 as more important than issue 2 but voters v^4 and v^5 do the reverse, then it can be verified that the voting pattern is as follows: v^1, v^2 and v^3 vote for the candidate at y; v^4 votes for x and v^5 votes for y. This is in contrast to the homogenous case where v^4 voted for y and v^5 for x.

In [24], Taylor considered the special case of a heterogeneous voting population when there are only two issues involved, i.e., the issue space is two dimensional. He showed that an undominated point may not exist and gave necessary and sufficient conditions for its existence. We will now study the general case where the number of issues, i.e., dimensions, is arbitrary.

For that purpose, the following definitions are needed first. An issue will be referred to as a *dominant issue* if there is at least one voter who ranks it as the most important issue; without loss of generality, we will assume for the rest of this section that the dominant issues in the election are issues 1 through k where $k < n$. Then the set of all voters that rank issue j, where $1 < j < k$, as the most important issue, will be referred to as Gj, and the number of voters in this set will be denoted by $|Gj|$ — it is worthwhile to note that in the two dimensional case, this reduces to what Taylor refers to as salience groups in [18]. It is easy to see that for any two different issues, say 1 and 2, the two groups $G1$ and $G2$ do not have any voter in common; in other words, the set $\{G1 \cap G2\}$ is empty. Given a dominant issue j, we often project the voters in Gj along this issue. When this is done, the set of all medians of the voters in Gj when they are projected along the issue that they consider most important, namely, issue j, will be denoted by $M(Gj)$.

We will now show that for a heterogeneous population of voters, the non- existence of an undominated point, as demonstrated in [24], is entirely due to the fact that in our original model, the candidates are allowed to assume similar stands on the dominant issues in the election. This can be demonstrated by focusing on a voting game where the admissible strategy set disallows such coincident locations in the issue space; in other words, a situation where the challenger is prohibited from assuming the same stand as the incumbent on any dominant issue. Restrictive though it may be, this can occur in elections where the process of entry by the candidates into the issue space, i.e., announcing their respective stands on different issues, or at least the dominant ones, is *sequential* rather than *simultaneous*. In other words, these are situations where the incumbent gets the opportunity to announce her election platform before the challenger gets to do so[3]. An example in point is the case of parliamentary elections in Commonwealth countries such as Canada, where the ruling party gets to choose and announce the election date, without having to consult the opposition parties. Unlike the United States, successive parliamentary elections in Canada do not have to be at equal and therefore, predetermined intervals (see [28], pg. 139, 251); these intervals are allowed to differ by several months, giving the ruling party considerable leverage in deciding when to hold an election and when to announce it to the public. In turn, this provides the governing party an advantage over the opposition, by (i) allowing it to arrange for an adequate amount of lead time to formulate its own election platform and (ii) being the first to announce this to the voters, well in advance of the opposition. With such a sequential process, where the challenger is relegated to being the follower in terms of informing the voters, our assumption about the challenger never assuming an identical stand as the incumbent on any issue, is then a simple logical consequence of our previous assumption of incumbency advantage. In other words, given our earlier assumption of incumbency advantage (a voter votes for the incumbent if he finds both the challenger and incumbent similar), it would be unreasonable for the challenger to assume the same position as the incumbent on any issue, particularly the dominant ones.

In addition to the above, another observation about most real life elections also lends further credence to our assumption of disallowing coincident locations between the incumbent and the challenger on dominant issues. Given the polarization amongst political parties in most real elections (conservative vs. liberal or right-leaning vs. left-leaning or Republican vs. Democrat) it is rarely the case that candidates from the two main parties have *absolutely identical* views on any important subject, even in the case where their views are similar. A good example of this would be in the context of USA and Canada, where a major issue in most recent elections has been deficit and debt reduction. On this issue however, although all major parties have favored deficit cutting, they have consistently disagreed, often substantially, on the extent and nature of the cuts.

Notwithstanding the justification for this rule elicited above, we now define the *Dominant Median* of a heterogeneous voting population, denoted by M^D. Given the dominant issues 1 through k, the j^{th} coordinate of the Dominant Median of the voting population is given by:

$$M_j^D = \begin{cases} \text{Any point inside the set } M(Gj) \text{for } 1 \le j \le k \\ \text{Any point inside the set } [0,1] \text{ for } (k+1) \le j \le n \end{cases} \tag{3}$$

Thus M_j^D, the j^{th} coordinate of the Dominant Median, is determined as follows: if issue j is a dominant issue then M_j^D is any point that is a median of all voters in Gj, when they are projected along issue j. If issue j is non-dominant, then the exact value of M_j^D is irrelevant; any point in the set [0, 1] can serve as the j^{th} coordinate of the Dominant Median[4]. As an illustration, consider the situation shown in Fig. 2 with $G1 = \{v^1, v^2\}$ and $G2 = \{v^3\}$. In this case, one Dominant Median exists as shown in the figure. If however, all voters ranked issue 1 as more important than issue 2, then a Dominant Median in the figure would be given by the point v_1^2 on the abscissa (in fact, any point on the vertical line at v_1^2 whose height is at most 1, would suffice). On the other hand, if all voters ranked issue 2 ahead of issue 1, the point v_2^1 on the ordinate would be a Dominant Median in that figure.

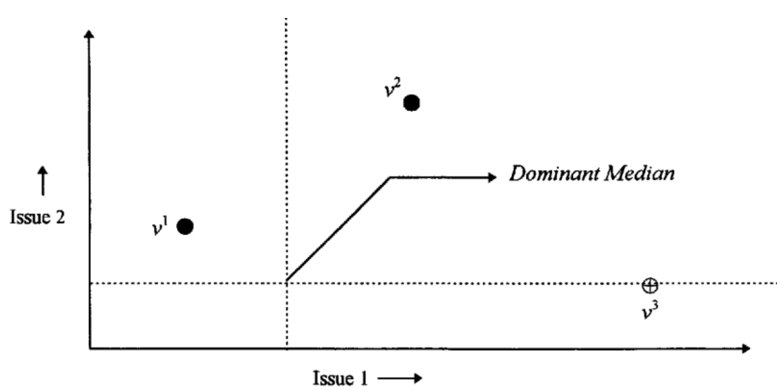

Fig. 2.

It is important to note that *only the dominant issues are critical in deciding the location of the Dominant Median* of a heterogeneous population. Further, similar to a Total Median of a homogeneous voting population, a Dominant Median of a heterogeneous voting population is also guaranteed to exist. With these definitions it can be stated that:

Lemma 2. *Given a heterogeneous voting population that votes in accordance with Algorithm Lexico-Comparison, if the incumbent and challenger always assume different stands on the dominant issues, then any Dominant Median will be an undominated point.*

Proof. Since M^D is purported to be the undominated point, the incumbent is located there and the challenger is located at some point in the issue space, say y, with the property that $y_i \neq M_j^D$ for every issue j from 1 to k. Consider the group $G1$. All voters in this group consider issue 1 as the most important issue, and it is known that $y_1 \neq M_1^D$ by our choice of M_1^D as per (3). This ensures that exactly $\lceil |G1|/2 \rceil$ voters in this group will find themselves closer to the incumbent than the challenger on issue 1 and hence, will vote for him. Therefore, the incumbent is assured of at least $\lceil |G1|/2 \rceil$ votes from the group $G1$. By using the same argument, it is possible to see that the incumbent is guaranteed at least $\lceil |Gj|/2 \rceil$ votes from each of the groups $G1$ through Gk. Taken together with the fact that there is no voter common to any two or more of these groups and that $\sum_{j=1,\ldots,k} |Gj| = P$, it can be seen that this implies that the incumbent is assured a minimum of

$$\sum_{j=1}^{k} \left\lceil \frac{|Gj|}{2} \right\rceil \geq \left\lceil \frac{P}{2} \right\rceil$$

votes, thereby ensuring that the challenger cannot defeat him in an election. ■

Lemma 2 suggests that given its assumptions, an undominated point always exists, and its location is determined solely by the dominant issues of the election. If the assumptions of the lemma hold, then the incumbent should identify the dominant issues of the election and choose a position that is dictated by the location of the voters in each of the different groups in this heterogeneous voting population. Alternatively stated, given that the incumbent has an established reputation, it will be impossible for a challenger to locate in the same position as the challenger. Voters will always select the incumbent since there is less risk that he will deviate from his stated position in such a scenario.

4 Admissible sets and abstentions

One of the shortcomings of the lexicographic choice rule espoused in the previous discussion is that when comparing the candidates sequentially on each issue, the first time that a voter finds a candidate closer to his ideal point on an issue, he immediately votes for the candidate regardless of how extreme he finds the candidate's position. It may be that the voter desires both candidates to be within an acceptable range on all relevant issues. We use two examples to illustrate this concept. Voters may rank defense issues above other issues only if defense expenditures fall within some given range. Defense expenditures below some minimum level may be too low to affect any kind of change in a country's preparedness to protect itself from invasion. Increasing expenditures above some maximum level may be viewed as wasteful in that once a certain defense threshold has been reached, all further expenditures would be superfluous. If abortion is plotted in the issue space from the position of absolutely legal to absolutely illegal, we may again find voter averse to extreme positions. Most voters would probably fall somewhere between the positions of allowing abortion in the case of rape or incest to disallowing abortion in the third trimester. Hence, in an attempt to address this shortcoming, we now introduce the concept of the *Admissible Set* of a voter. To keep the analysis tractable, we begin by assuming that the voting population is homogenous, relaxing it only towards the end of the section.

Consider a voter v^i and his ideal location along issue j, which is given by v_j^i. We will now assume that in addition, every voter v^i also requires that in order for a candidate to be even acceptable to him to vote for, the candidate's stand on issue j must be between two limits $v^i_{j-\min}$ and $v^i_{j-\max}$, where $0 \le v^i_{j-\min} \le v^i_j \le v^i_{j-\max} \le 1$. Considering all the issues involved in the election, v^i would thus require that any candidate that is outside his *Admissible Set* A^i, where

$$A^i = \{x | v^i_{j-min} \le x_j \le v^i_{j-max} \text{ for } j = 1,2,\dots,n\} \qquad (4)$$

will not even be eligible for his vote. In general, the Admissible Set of each voter v^i would be defined by $2n$ inequalities. Thus in two dimensions, the set A^i would define a rectangle around the location of v^i (see Fig. 3), in three dimensions, a cube and so forth. Further, if both the challenger and the incumbent assume a position that is outside A^i, then v^i will not vote in the election at all. This concept of abstention is similar to that of *reservation price* in economics, where a consumer wishing to purchase a nonessential good will not do so if the price of the good exceeds a certain limit (called the reservation price). If the act of voting imposes costs on voters in terms of time, etc. we could further argue that the marginal cost of voting exceeds the marginal benefits when all candidates position themselves outside of the admissible set. It is also interesting to note that such ideas of imposing limits on the possible stands that the candidates can assume is not entirely new — see [15] for a similar voting model where the set of all feasible positions in a two dimensional issue space is restricted by a line. Finally, note that the result of Lemma 1 can now be considered a special case of the results of the present section, one where every voter's Admissible Set in the entire issue space itself. Taken together, this implies that the voting behaviour of a voter v^i can now be described by the following modified version of the previous such algorithm.

Algorithm *Admissible* Lexico-Comparison

1. Define the Admissible Set A^i. Discard from consideration any candidate that is outside A^i. Thus a candidate is eligible only if she is within A^i. Examine the eligible candidates.
   a) If no eligible candidates exists, stop and abstain from voting.
   b) If only one eligible candidate is found, vote for her and stop.
   c) If both candidates are found eligible, go to Step 2.
2. Choose the most important issue as the Present Issue.
3. Compare both candidates according to their stand on the Present Issue. Vote for that candidate whose position on the Present Issue has the least disutility. If both candidates are found to be equal on the Present Issue and all issues have been considered, vote for the incumbent and stop. If not, go to Step 4.
4. Choose the next most important issue as the Present Issue and go back to Step 2. ■

Thus with this voting behavior, if x and y in Fig. 3 represent the two candidates standing for election, then voter v^1 will vote for the candidate at x, voter v^3 for y and v^2 will abstain from voting for any one of these two candidates. Since we now allow for an Admissible Set around each voter's ideal point, it is possible to have situations where the distribution of the voter's ideal points and their Admissible Sets are such that there is no location in the issue space that is within the Admissible Set of all the voters. Consider the following example shown in Fig. 4, where five voters v^1 through v^5 are assumed to be in the simplest space possible, namely, a one dimensional issue space.

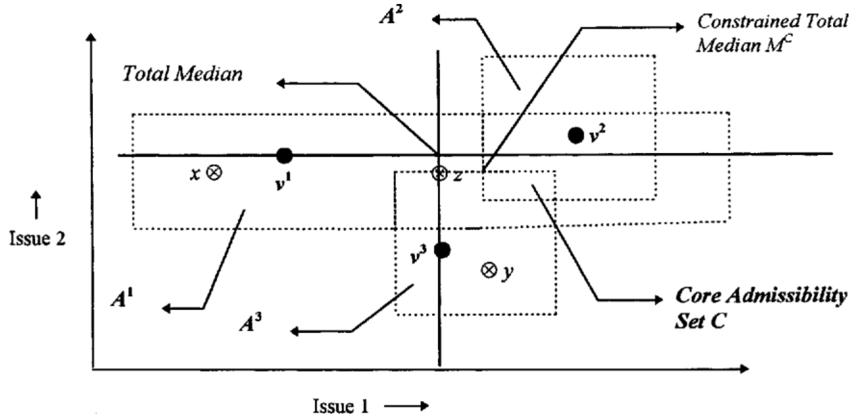

Fig. 3.

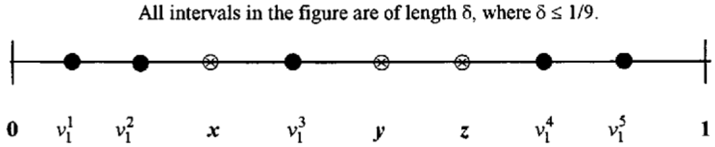

Fig. 4.

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Assume, as shown in the figure, that the voters are located at $v_1^1 = \delta$; $v_1^2 = 2\delta$; $v_1^3 = 4\delta$; $v_1^4 = 7\delta$ and $v_1^5 = 8\delta$, where δ is a positive number less than 1/9. Let *x*, *y* and *z* be three additional points at 3δ, 5δ and 6δ respectively. In addition, define the Admissible Sets of the voters as follows $v_{1-\min}^1 = v_{1-\min}^2 = 0$; $v_{1-\max}^1 = v_{1-\max}^2 = v_1^3$; $v_{1-\min}^3 = v_1^3$; $v_{1-\max}^3 = z$; $v_{1-\min}^4 = v_{1-\min}^5 = y$ and $v_{1-\max}^4 = v_{1-\max}^5 = 1$. Now suppose that: (**i**) the incumbent is located on the median, which is given by v_1^3 in this case. Then the challenger can locate on v_1^2 and can win two votes, namely those of v^1 and v^2. 1 The incumbent would only get the vote of v^3 and thus lose the election. Consider next case (**ii**), where the incumbent is located strictly to the right of the median. Then the challenger can locate on the median and get the three votes of v^1, v^2 and v^3, thus defeating the incumbent. This leaves the following two cases: (iii) either the incumbent locates somewhere in the interval $[x, v_1^3)$ or (iv) or in the interval [0, *x*). In case (**iii**) the challenger can locate on v_1^2 thereby getting the votes of v^1 and v^2 . The incumbent would not get any votes at all, and would thus lose the election. Which brings us to the final case (**iv**), where the incumbent is located in the interval [0,*x*). The challenger can now locate at y and win the three votes of v^3, v^4 and v^5, thereby defeating the incumbent. This example thus allows us to conclude that in a homogeneous population of voters that votes in accordance with Algorithm *Admissible Lexico-Comparison*, there may not exist an undominated point.

Having demonstrated the possibility of non-existence of an undominated point in the case of voters who abstain, we now restrict our attention to special cases where an undominated point is guaranteed to exist. To that end define the *Core Admissibility Set*, C, as the set of all locations in the issue space which are within the Admissible Set of all the voters, as shown in Fig. 3. In other words,

$$C = \bigcap_{i=1}^{n} A^i \qquad (5)$$

Note that the set *C* may not exist, as in the example shown in Fig. 4. But if it does, then it would consist of all points *x* in the issue space that satisfy

$$v_{j-\min}^i \le x_j \le v_{j-\max}^i \text{ for } i = 1,2,\dots,P \text{ and } j = 1,2,\dots,n \qquad (6)$$

Given this assumption, there is an easy way of characterizing *C*. Consider the voters on any given issue *j*; if there exists a pair of voters *i* and *k* such that $v_{j-\max}^i < v_{j-\min}^k$, then the set *C* can not exist. Conversely, if there does not exist any such pair of voters, and we define

$$C_{j-\min} = \underset{i}{\mathrm{Min}}\{v_{j-\max}^i\} \text{ and } C_{j-\max} = \underset{i}{\mathrm{Max}}\{v_{j-\min}^i\} \qquad (7)$$

as the two endpoints of the interval set *C* on issue *j*, then it is guaranteed that $C_{j\text{—}\min} \le C_{j\text{—}\max}$.

Proposition 3. *Given that each voter has an Admissible Set around his ideal point in the issue space, the Core Admissibility Set C exists iff* $C_{j\text{—}\min} \le C_{j\text{—}\max}$ *for each issue j, where Cj*—min *and Cj*—max *are defined by* (7). *Further, if C exists, then it is characterized by the inequalities:*

$$C = \{x | C_{j\text{—}\min} \le x_j \le C_{j\text{—}\max} \; for\; j = 1,2,\dots n\} \qquad (8)$$

If the Core Admissibility Set C exists, then any candidate that locates inside it would find herself to be universally acceptable to all the voters. For the rest of the discussion, we will only focus on the cases where the set C exists and examine two special cases where an undominated point does exist. The first of these, and the simplest, is when the set C exists and there is at least one Total Median contained in it. In that case, it is readily shown that Lemma 1 still holds. This enables us to restate the lemma as:

Lemma 4. *Given a homogeneous population of voters that votes in accordance with Algorithm Admissible Lexico-Comparison, if the Core Admissibility Set exists and at least one Total Median of the voter distribution is contained in it, then any Total Median that is inside C is an undominated point.*

Lemma 4 establishes one set of sufficient conditions for an undominated point to exist. Although the condition described in the lemma may seem overly restrictive, there is one interesting instance where it may hold. To illustrate this example, note that the condition that M be contained inside C implies that along every issue j, there exists a point x_j in the interval $[C_{j\text{-min}}, C_{j\text{-max}}]$ with the property that an equal number of the voters are to its left and right. One of the conditions under which this is true is shown in Fig. 5, where the entire voting population is polarized into three distinct groups: the "leftists" as given by v^1 and v^2 in the figure, that assume a left leaning position on every issue and are therefore to the left of $C_{j\text{-min}}$ on every issue; an equal number of "rightists", as given by v^4 and v^5, who are to the right of $C_{j\text{-max}}$ on every issue and finally, some "centrists", as given by v^{3}', that are inside C on every issue. It can be verified that whenever the voting population is polarized in this manner, with an equal number of "leftists" and "rightists", it is guaranteed that there will exist a Total Median that is contained in C, thus satisfying the condition of Lemma 4.

However, it is possible that no Total Median of the voter distribution is inside C. For example, in Fig. 3, the unique Total Median is outside the set C shown in the figure. Therefore we now turn to situations where such a condition may be true and discuss another special case where an undominated point always exists. In this version of our model we stipulate that one of the rules of the elections is that any candidate standing for the election must be universally acceptable, thus requiring all candidates to locate within the Core Admissibility Set C. Such a requirement could be implemented in practice by holding a two stage election where in the first stage, each voter nominates all the candidates that are acceptable to him, or conversely, the ones that are unacceptable to him. In the second stage the contest is between those candidates that have been found to be acceptable to all voters in the first stage. One example of such an election process could be in the selection of a candidate for an important senior management job, by the members of an interview board. Given this requirement of universal acceptability however, we now define a point within C that is referred to as the *Constrained Total Median* of the voting population.

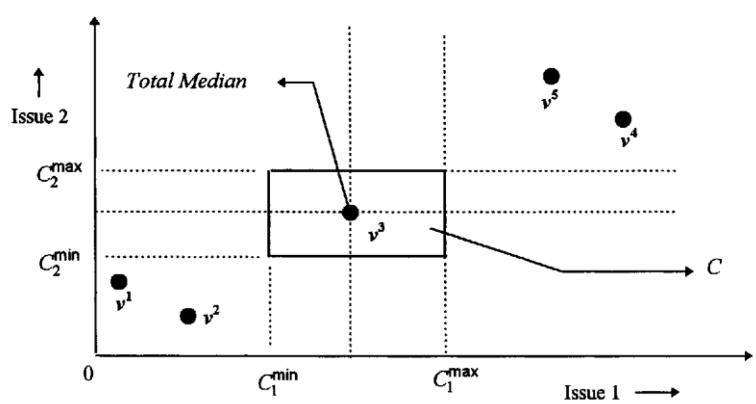

Fig. 5.

The Constrained Total Median, that we denote by M^C, is obtained on the basis of the location of the median of the voters along each issue. To illustrate this, consider all the voters along issue j and the set of all medians along this issue, i.e. M_j. If the two intervals $[C_{j\text{—min}}, C_{j\text{—max}}]$ and M_j intersect, then the j^{th} coordinate of M^C, denoted by M_j^C, is given by any point, (and therefore, as we will assume, the midpoint) of this intersection. If not, it is given by either one of the two endpoints $C_{j\text{—}\min}$ or $C_{j\text{—}\max}$, depending on which of them is closer to M_j. In other words, M^C is characterized as

$$M_j^C = \begin{cases} \text{Midpoint of} & \\ M_j \bigcap [C_{j-\min}, C_{j-\max}] & \text{if} \cap [C_{j-\min}, C_{j-\max}] \neq \Phi \\ C_{j-\min} & \text{if } M_j \in [0, C_{j-\min}) \\ C_{j-\max} & \text{if } M_j \in [C_{j-\max}, 1] \end{cases} \quad (9)$$

The Constrained Total Median of the voting population in Fig. 3 is shown therein. Thus M^C is obtained by trying to "graviate" towards the nearest median along every issue, while being constrained to remain inside C; hence the name. The definition of M^C as per (9) also ensures that if the set C exists, then the Constrained Total Median is also guaranteed to exist. Given this definition, we can state that:

Lemma 5. *Given a homogeneous population of voters that votes in accordance with Algorithm Admissible Lexico-Comparison, if the Core Admissibility Set exists and it is required that any candidate standing for the election be universally acceptable to all voters, i.e., locate within the Core Admissibility Set, then the Constrained Total Median of the voting population, as obtained in* (9)*, is an undominated point.*

Proof. Suppose that the statement is not true and hence, there is a challenger located at a point x inside C that can take at least ($\lfloor P/2 \rfloor + 1$) votes from an incumbent at M^C. Assume that x and M^C are coincident on issues 1 though k and on issue (k + 1), $x_{k+1} \neq M_{k+1}^C$. However, since both candidates are required to be universally acceptable to all voters, it is guaranteed that x_{k+1} is contained in the interval $[C_{(k+1)\text{—}\min}, C_{(k+1)\text{—}\max}]$. Nevertheless, regardless of where x_{k+1} is inside the interval $[c_{(k+1)\text{—}\min}, C_{(k+1)\text{—}\max}]$, by virtue of the fact that $x_{k+1} \neq M_{k+1}^C$, at least $\lceil P/2 \rceil$ voters will find M^C closer to themselves then x on issue (k + 1). Since M^C is universally acceptable to the entire voting population, these voters will therefore vote for the incumbent, thereby making it impossible for the challenger to get any more than $\lfloor P/2 \rfloor$ votes. ■

Finally, note that the sufficient conditions outlined in Lemma 5 above are based on the crucial assumption that any candidate standing for the election be universally acceptable to all voters. If this does not happen, then the point M^C is no longer an undominated point. To see this, consider the situation shown in figure 3 and assume that the incumbent is located at the Constrained Total Median shown therein. Then an incumbent at the point z, who is not acceptable to voter v^2, can take away the votes of v^1 and v^3, thereby defeating the incumbent.[5]

As a final consideration, we examine our basic model under the two assumptions of a heterogeneous voting population and that each voter has an Admissibility Set defined around his ideal point, which may not be the entire issue space itself. As before, the primary aim of the discussion will be to investigate conditions that guarantee the existence of an undominated point. Since this is the most complex version of our basic model, we restrict our discussion to one special case, namely, the heterogeneous counterpart of Lemma 4.

To that end, we stipulate that the Core Admissibility Set of all voters, C, exist. Then, by combining the arguments used in Lemma 1 and Lemma 2, we can show an analogous Lemma.

Lemma 6. *Given a heterogeneous voting population that votes in accordance with Algorithm Admissible Lexico-Comparison, the following three conditions are sufficientfor the existence of an undominated point.*

(i) The Core Admissibility Set exists.
(ii) *For every dominant issue j, the intervals M*(*Gj*) *and* $[C_{j\text{-}\min}, C_{j\text{-}\max}]$ *intersect and,*
(iii) *Both candidates always assume different stands on each of the dominant issues,*

Further, when these three conditions hold, the location of an undominatedpoint is given by any point x whose coordinates satisfy:

$$x_j = \begin{cases} Any\ point\ inside\ the \\ set\ M(Gj) \cap [C_{j-\min}, C_{j-\max}] & for\ 1 \le j \le k \\ Any\ point\ inside\ the & (k+1) \le j \le n \\ set [C_{j-\min}, C_{j-\max}] \end{cases} \qquad (11)$$

Note that Lemma 6 requires that $M(Gj)$ and $[C_{j\text{-min}}, C_{j\text{-max}}]$ intersect for each dominant issue j. This is less stringent than the corresponding requirement of Lemma 5, that the entire sets M and C intersect. This is due to the fact the former result also assumes that candidates are disallowed from taking similar stands on any of the dominant issues.[6]

5 Conclusions, limitations and future work

In this paper, we have reconsidered the problem of the existence of an undominated point, under the assumption of lexicographic preferences of voters, that was introduced by Taylor in [18]. We argue that the assumption of lexicographic preferences may be fairly realistic in that: i) information of a candidate's position may be costly to obtain and process, and ii) special interest politics will always rank their interest above those of any other group. In our paper we have extended Taylor's model to situations where we allow for (i) voters to have different rankings of the issues in the general *n*-dimensional issue space and (ii) a candidate to be disregarded by a voter if his stand on any one or more of the issues involved in the election is perceived to be too extreme by the voter and (iii) combinations of (i) and (ii). We show that specialized variants of the concept of medians also constitute equilibrium or undominated points in these models. Thus a model of voting behavior results that is closer approximation of reality in that historically incumbents tend to win. Our results therefore suggest that incumbents tend to have an advantage when the election process is characterized by a large presence of special interests or as information becomes more expensive to acquire.

Any model that attempts to explain human behavior almost always has limitations, and of course, ours are no exception. Hence, we will devote the final part of the paper to discussing the limitations of the models considered in this paper and how future research can address some of them.

In our opinion, one of the biggest limitations of lexicographic preferences is that it does not allow for trade-off between different issues. That many not always be realistic — for example, it is perfectly reasonable to assume that a voter may be willing to compromise on a moral issue, such as prayer in schools, for an economic issue, such as tax cuts. We believe that the best way to incorporate this phenomenon would be by researching hybrid models that synthesize some commonly used preferences, such as circular preferences with a lexicographic one; and this is our first suggested strand of future research on this topic.

A second important shortcoming of our analysis is that we have not performed any"sensitivity analysis" of the results in this paper. For example, how would the existence results change, if one were to allow for a perturbation in the lexicographic preference of the voters - most importantly, this refers to a perturbation of the *ranking* of the issues by each voter. Do the existence results still hold? Our conjecture is that they may not. In that case, what other assumptions and/or restrictions are necessary for an undominated point to exist?

Our final suggested avenue of future research is to focus solely on the case when an undominated point does not exist. Given that, it will be interesting to investigate and develop elegant characterizations of another solution concept, namely, the Simpson point, and efficient ways for computing it.

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Notes:

1. To calculate the time it would take to compute the set M, note that along each issue j, the set of all medians M_j can be obtained in $O(P)$ time by using the algorithm in [2]. Since M is obtained by repeating this for all the issues, given the distribution of the voters in the issue space, M can be computed in $O(nP)$ time.
2. Interestingly, although Taylor was the first to enunciate the use of lexicographic preferences, similar ideas, based on the idea of restricting the decision making by the voter to one issue at a time, was also proposed elsewhere in the literature — a good discussion of this is available in [10] — see for example, Theorems 2 and 3 in that paper.
3. It is interesting to note that such sequential entry models are well studied in the context of competitive location models in geographical, rather than issue, space. In competitive location literature, they are referred to as Stackelberg [23] location models/games — see [T] for a recent survey of these models.

4. Using similar arguments as before, it can be easily concluded that that MD can be computed in $O(kP)$ time.
5. A comment now on the time complexity of all the computations and checks involved. It can be seen that the two endpoints of C on each issue j, namely, $C_{j\text{-min}}$ and $C_{j\text{-max}}$ can be computed $O(P)$ time by taking the minimum and maximum of the numbers $v^i_{j-\max}$ and $v^i_{j-\min}$ respectively. Checking whether $C_{j\text{-min}} < C_{j\text{-max}}$ can then be performed in $O(1)$ time. Since finding C entails repeating this for all issues, the existence of C can be verified in $O(nP)$ time, and if C exists, it can be characterized as per (8) in the same time as well. As mentioned before, M, the set of all Total Medians, can be computed in $O(nP)$ time. Given the two intervals Mj and $[C_{j\text{-min}}, C_{j\text{-max}}]$, checking if they intersect can be done in $O(1)$ time. This also implies that the j^{th} coordinate of the Constrained Total Median, namely M^C_j, can be computed in $O(1)$ time too, and hence, given M and C, M^C can be found in $O(n)$ time. By repeating the check for the intersection of M_j and $[C_{j\text{-min}}, C_{j\text{-max}}]$ for all the issues, it can also be verified that, given M and C, checking if these two sets intersect, can be done in $O(n)$ time, and if they do intersect, a Total Median that is contained inside C can be produced in $O(n)$ time as well. Taken together, these statements imply that all the computations and checks required by Lemma 4 and Lemma 5, can be performed in the same time as it takes to compute M, i.e., $O(nP)$ time.
6. Finally, the issue of time complexity. As discussed previously, $M(Gj)$ can be computed in $O(P)$ time for any dominant issue j and the Core Admissibility Set can be computed in $O(nP)$ time. Given C and $M(Gj)$ for any dominant issue j, checking if the two intervals $M(Gj)$ and $[C_{j\text{-min}}, C_{j\text{-max}}]$ intersect can be accomplished in $O(1)$ time. Since the computation of the undominated point in Lemma 6 entails calculating $M(Gj)$ and performing this check for every dominant issue j, it can be concluded that given C, this computation can be accomplished in $O(kP)$, and hence, $O(nP)$ time. Therefore, as in Lemmas 4 and 5, all the computations and tests of Lemma 6 can also be performed in $O(nP)$ time.