Trees, Congruences and Varieties of Finite Semigroups

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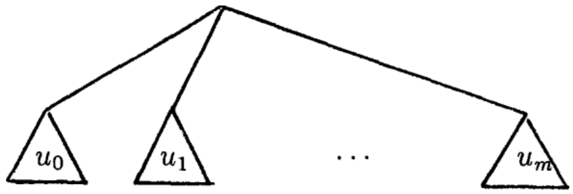
Abstract:
A classification scheme for regular languages or finite semigroups was proposed by Pin through tree hierarchies, a scheme related to the concatenation product, an operation on languages, and to the Schützenberger product, an operation on semigroups. Starting with a variety of finite semigroups (or pseudovariety of semigroups) **V**, a pseudovariety of semigroups $\Diamond_u(\mathbf{V})$ is associated to each tree *u*. In this paper, starting with the congruence γ_A generating a locally finite pseudovariety of semigroups **V** for the finite alphabet *A*, we construct a congruence $\equiv_u(\gamma_A)$ in such a way to generate $\Diamond_u(\mathbf{V})$ for *A*. We give partial results on the problem of comparing the congruences $\equiv_u(\gamma_A)$ or the pseudovarieties $\Diamond_u(\mathbf{V})$. We also propose case studies of associating trees to semidirect or two-sided semidirect products of locally finite pseudovarieties.

Article:
1. Introduction
A result of Kleene [10] shows that the class of recognizable languages (that is, recognized by finite automata) coincides with the class of *regular* or *rational* languages which can be obtained from finite languages by the boolean operations, the concatenation product and the star. *Star-free* languages are those rational languages which can be obtained from finite languages by the boolean operations and the concatenation product only. Several classification schemes for the star-free languages were proposed based on the alternating use of the boolean operations and the concatenation product. This led to the natural notion of dot-depth. However, the first question related to this notion "given a star-free language, is there an algorithm for computing its dot-depth?" appears to be extremely difficult.

A classification scheme for rational languages was proposed by Pin through tree hierarchies [13]. This classification scheme generalizes the above mentioned ones for star-free languages. Tree hierarchies are related to the concatenation product, an operation on languages and to the Schützenberger product, an operation on monoids or semigroups.

In this paper, we give some results on Pin's tree hierarchies. The notion of congruence plays a central role in our approach. For any finite alphabet *A*, denote by A^* the free monoid generated by *A*. We say that a monoid *S* is *A-generated* if there exists a congruence y on A^* such that *S* is isomorphic to A^*/γ. A pseudovariety of monoids **V** is *locally finite* if for any *A*, there are finitely many *A*-generated monoids in **V**. Equivalently, there exists for each *A*, a congruence γ_A such that an *A*-generated monoid *S* is in **V** if and only if *S* is a morphic image of A^*/γ_A. By Eilenberg's one- to-one correspondence between the pseudovariety **V** and a *-variety of languages $\mathcal{V}$, a language *L* of A^* is in A^* if and only if *L* is a union of γ_A-classes.

Starting with the congruence yA, we associate to each tree u a congruence(7,$_4$) in such a way to generate the class $A^* \mathcal{V}_u$ of recognizable languages of A^* defined recursively as follows: If u is the tree reduced to a point, then $A^* \mathcal{V}_u = A^* \mathcal{V};$ if u =



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then $A^* \mathcal{V}_u$, is the boolean algebra generated by the languages $L_{i_0} a_1 L_i, ...a_k L_{i_k}$, where $0 \leq i_0 < i_1 < \bullet \bullet \bullet < i_k \leq m$, $a_1, \ldots, a_k$ are letters of A and for each $0 \leq j \leq k$, L_i, is in $A^* \mathcal{V}_{u_{i_j}}$. Pin showed that the Schützenberger product is perfectly adapted to the operation $(L_0, \ldots, L_k) L_0 a_1 L_1 ... a_k L_k$. This result allows to build, without reference to languages, hierarchies of pseudovarieties of monoids corresponding, via Eilenberg's result, to the above-mentioned hierarchies of *-varieties of languages. In other words, starting with a pseudovariety **V**, a pseudovariety $\Diamond_u(\mathbf{V})$ is associated to each tree u.

We first give partial results on the problem of comparing the congruences $\equiv_u (\gamma_A)$ (Section 3). Our congruence construction shows, in particular, that all the pseudovarieties of the hierarchy built from locally finite pseudovarieties are locally finite (Section 4). Case studies are proposed of associating trees to semidirect or two-sided semidirect products of locally finite pseudovarieties using our congruence construction (Section 5). Definitions and results are given for pseudovarieties of monoids. Up to the obvious changes, they hold also for pseudovarieties of semigroups. Unless otherwise specified, any congruence we discuss has finite index.

2. Preliminaries

This section is devoted to reviewing basic properties of finite monoids and recognizable languages. The reader is referred to the books of Almeida [2], Eilenberg [8] and Pin [12] for further definitions and background.

2.1. Monoids

A *semigroup* is a set S together with an associative binary operation (generally denoted multiplicatively). If there is an element 1 of S such that $1s = s1 = s$ for each s E S, then S is called a *monoid* and 1 is its unit. S is a *group* if S is a monoid and, for each $s \in S$, there exists $s' \in S$ such that $ss' = s's = l$. A subset of S is a *subsemigroup* (respectively *submonoid, subgroup*) of S if the induced binary operation makes it a semigroup (respectively monoid, group).

Let S and T be monoids. *A morphism* $\varphi: S \rightarrow T$ is a mapping such that $\varphi(ss') = \varphi(s) \varphi(s')$ for all $s, s' \in S$ and $\varphi(1) =$ 1. We say that S *divides* T, and write $S < T$, if S is the image by a morphism of a submonoid of T.

If A is a set, we let A^+ be the *free semigroup* on A and A^* be the *free monoid* on A. A^+ is the set of all finite strings $a_1 ... a_i$ of elements of A and $A^* = A+ \ U \{l\}$, where 1 is the empty string (when we write a_j we will always mean a letter in A). The operation in A^* is the concatenation of these strings.

2.1.1. Varieties of finite monoids

A *variety of monoids* is a class of monoids that is closed under division and direct product. An **M**-*variety* is a class of finite monoids that is closed under division and finite direct product. **M**-varieties are also called *pseudovarieties of monoids.* Given a class C of finite monoids, the intersection of all **M**-varieties containing C is still an **M**-variety, called the **M**-*variety generated by C.*

A (monoid) *identity* on a set A is a pair (x, y) of elements of A^*, usually indicated by a formal equality $x = y$. We say that a monoid S *satisfies* an identity $x = y$ (or that the identity $x = y$ *holds* in S) and we write $S \models x = y$ if, for any morphism $\varphi: A^* \to S$, we have $\varphi(x) = \varphi(y)$. For an identity $x = y$ and an **M**-variety **V**, the notation **V** $= x = y$ will abbreviate the fact that each $S \in$ **V** satisfies x y.

Work of Eilenberg and Schiitzenberger [9] showed that **M**-varieties are *ultimately defined* by sequences of identities (that is, a monoid belongs to the given **M**-variety if and only if it satisfies all but finitely many of the identities in the sequence), and that finitely generated **M**-varieties are equational or *defined* by sequences of identities (that is, a monoid belongs to the given **M**-variety if and only if it satisfies all the identities in the sequence).

We now list a few important **M**-varieties that we are going to use:

- A is the **M**-variety of all finite aperiodic monoids (a monoid S is *aperiodic* if all groups in S are trivial).
- I is the trivial **M**-variety consisting only of the 1-element monoid.
- J_1 is the **M**-variety of all finite idempotent and commutative monoids (also called *semilattices*) defined by the identities $x^2 = x$ and $xy = yx$.
- J is the **M**-variety of all finite $\mathfrak{J}$-trivial monoids.
- M is the **M**-variety of all finite monoids.
- R is the **M**-variety of all finite $\mathfrak{R}$-trivial monoids.
- G is the **M**-variety of all finite groups (any **M**-variety contained in **G** will be called a **G**-*variety*).

2.2. Languages

Let A be a finite set. When we deal with languages, A is called an *alphabet* and its elements are called *letters*. The elements of A^* are called *words* on A. A *language* on A is a subset L of A^*. A language L in A^* is said to be *recognizable* if there exists a finite monoid S and a morphism $\varphi: A^* \to S$ such that $L = \varphi^{-1}(\varphi(L))$, that is, if $x \in L$ and $\varphi(x) = \varphi(y)$, then $y \in L$. This is also equivalent to saying that there is a subset X of S such that $L = \varphi^{-1}(X)$. In that case, we say that S (or φ) *recognizes* L. The notions of recognizable sets (by finite monoids and by finite automata) are equivalent. To each language L, we associate a congruence $\sim_L$ defined, for $x, y \in A^*$, by $x \sim_L y$ if and only if uxv and uyv are both in L or both in $A^* \backslash L$, for all u, v in A^*. The congruence $\sim_L$ is called the *syntactic congruence* of L and the monoid $M(L) = A^*/\sim L$ is called the *syntactic monoid* of L. A monoid recognizes L if and only if it is divided by $M(L)$.

2.2.1. Varieties of languages

A *-*variety* $\mathcal{V}$ is a family $A^* \mathcal{V}$ of sets of recognizable languages of A^* defined for all finite alphabets A and satisfying the following three conditions:

1. $A^* \mathcal{V}$ is a boolean algebra, that is, if K and L are in $A^* \mathcal{V}$, then so are $K \cup L$, $K \cap L$ and $A^* \backslash L$.
2. If $\varphi: A^* \to B^*$ is a morphism and $L \in B^* \mathcal{V}$, then $\varphi^{-1}(L) \in A^* \mathcal{V}$.
3. If $L \in A^* \mathcal{V}$ and $a \in A$, then both $\{x \text{ E } A^* \mid ax \in L\}$ and $\{x \in A^* \mid xa \in L\}$ are in $A^* \mathcal{V}$.

Eilenberg [8] proved that **M**-varieties and *-varieties are in one-to-one correspondence. If **V** is an **M**-variety, then $A^* \mathcal{V} = \{L \subseteq A^* \mid M(L) \in \mathbf{V}\}$ defines the corresponding *-variety $\mathcal{V}$. If $\mathcal{V}$ is a *-variety, then the **M**-variety generated by $\{M(L) \mid L \in A^* \mathcal{V}$ for some $A\}$ defines the corresponding **M**-variety **V**.

Let **V** be an **M**-variety generated by the monoids $S_1, \ldots, Sm$. Thus **V** is generated by $S = S_1 \times \cdots \times S_m$. Let $\mathcal{V}$ be the *-variety associated to **V**. Then A^* is the Boolean closure of the sets $\varphi^{-1}(s)$ for all $s \in S$ and all morphisms $\varphi: A^* \to S$. Consequently, $A^* \mathcal{V}$ is finite.

We now list *-varieties of languages associated to some of the **M**-varieties listed previously:

- $A^* \mathfrak{A}$ consists of the star-free languages of A^* [16].
- $A^*, = \{\emptyset, A^*\}$ where $\emptyset$ denotes the empty set.

- $A^*\mathfrak{J}$ consists of the piecewise testable languages of A^* [17].
- $A^*\mathfrak{M}$ consists of the rational languages of A^* [10].

We end this section with a few examples of locally finite **M**-varieties.

1. For any positive integer q and nonnegative integer m, $\mathbf{Com}_{q,m}$ is the **M**-variety of all finite commutative monoids defined by the identities x^{m+q} x^m and $xy= yx$ (we adopt the convention that $x^0 = 1$). For any word x on A and $a \in A$, we denote by $|x|_a$ the number of occurrences of a in x. We define on A^* the congruence $\beta_{q,m}$ by $x\beta_{q,m}y$, if for all $a \in A$, $|x|_a = |y|_a$ or $x|_a, |y|_a \geq m$ and $x|_a \equiv |y|_a \bmod q$ ($\beta_{1,0}$ will often be abbreviated by ω). An A-generated monoid S is in $\mathbf{Com}_{q,m}$ if and only if S is a morphic image of $A^*/\beta_{q,m}$, (note that $\mathbf{Com}_{1,0} = \mathbf{I}$). The **M**-variety **Com** of all finite commutative monoids (which is the join $\bigvee_{q\geq1,m\geq0} \mathbf{Com}_{q,m}$) is not locally finite; the same is true for $\mathbf{Com} \cap \mathbf{A}$ which is the join $\bigvee_{m\geq0} \mathbf{Com}_{1,m}$ and $\mathbf{Com} \cap \mathbf{G}$ which is the join $\bigvee_{q\geq1} \mathbf{Com}_{q,0}$.

2. A hierarchy was introduced by Straubing [21] for the star-free languages of A^*: the set$\{\emptyset, A^*\}$ constitutes $A^*\mathcal{V}_0$; then, $A^*\mathcal{V}_k$, is the boolean algebra generated by the languages of the form $L_0a_1L_1...a_iL_i$, where I $\geq$ 0, $a_1,...,a_i \in A$ and $L_0,..., L_i \in A^*\ \mathcal{V}_{k-1}$. Straubing's hierarchy induces, by Eilenberg's correspondence, a hierarchy of **M**-varieties : $\mathbf{V}_0 \subseteq \mathbf{V}_1 \subseteq \mathbf{V}_2 \subseteq \cdots$ which is known to be strict [23]. We have $\mathbf{V}_0 = \mathbf{I}$. Simon [17] proved that $\mathbf{V_1} = \mathbf{J}$ and hence $\mathbf{V}_1$ is decidable. The problem remains open as to whether $\mathbf{V}_k$ is decidable for $k \geq 2$.

Straubing's hierarchy can be refined as follows: for each $k \geq 1$, $m \geq 0$, $A^*\mathcal{V}_{k,m}$ is the boolean algebra generated by the languages of the form $L_0a_1L_1...a_iL_i$, where $0 \leq i \leq m$, $a_1,...,a_i \in A$ and $L_0,...,L_i \in A^*\mathcal{V}_{k-1}$. Then, for each positive integer k, $\mathbf{V}_k$ naturally contains a subhierarchy of **M**-varieties : $\mathbf{V}_{k,0} \subseteq \mathbf{V}_{k,1} \subseteq \mathbf{V}_{k,2} \subseteq \cdots \subseteq \mathbf{V}_k$.

A remarkable fact about these hierarchies is their connections with some hierarchies of formal logic [22, 23, 11]. In particular, the congruences $x_{(m_1,...,m_k)}$ defined below are intimately related to Straubing's hierarchy, namely to its kth level.

A word $a_1 ...a$, on A is a subword of a word z on A if there exist words $z_0,...,z_i$ on A such that $z = z_0a_1z_1...a_iz_i$. For any nonnegative integer m and word z on A, we denote by $z_{(m)}(z)$ the set of subwords of z of length less than or equal to m. We define the congruence $\alpha_{(m)}$ on A^* by $x\chi_{(m)}y$ if $\alpha_{(m)}(x)= x_{(m)}(y)$ ($a_{(1)}= \beta_{1,1}$ will often be abbreviated by χ). An A-generated monoid S is in $\mathbf{V}_{1,m}$ or $\mathbf{J}_m$ if and only if S is a morphic image of $A^*/\alpha_{(m)}$

We proceed with a generalization of $\alpha_{(m)}$ related to an Ehrenfeucht-Fraissé game. We identify any word x on A with a word model $x =(\mathcal{U}_x, <^x, (Q_a^x)_{a\in A})$ where the universe $\mathcal{U}_x = \{1,\dots, |x|\}$ represents the set of positions of letters in the word x ($|x|$ denotes the length of x), $<^x$ denotes the usual order relation on $\mathcal{U}_x$, and Q_a^x is a unary relation on $\mathcal{U}_x$, containing the positions with letter a, for each $a \in A$ (we will often write $Q_a^x p$ instead of $p \in Q_a^x$). The game $G_{\overline{m}}(x, y)$, where $\overline{m} = (m_1,...,m_k)$ is a k-tuple of positive integers ($k \geq 0$) and x, y are words on A, is played between two players I and II on the word models x and y. A play of the game consists of k moves. In the ith move, Player I chooses, in x or in y, a sequence of m_i positions; then, Player II chooses, in the remaining word (y or x), also a sequence of m_i positions. Before each move, Player I has to decide whether to choose his next elements from x or from y. After k moves, by concatenating the position sequences chosen from x and from y, two sequences $pi,...,p_n$ from x and $q_1,...,q_n$ from y have been formed where $n = m_i + \cdots + m_k$. Player II has won the play if the following two conditions are satisfied : $p_i <^x p_j$ if and only if $q_i <^y q_j$ for all $1 \leq i, j \leq n$, and $Q_a^x p_i$ if and only if $Q_a^y q_i$ for all $1 \leq i \leq n$ and $a \in A$. Equivalently, the two subwords in x and y given by the position sequences $p_1,\dots,p_n$ and $q_1,\dots,q_n$ should coincide. If there is a winning strategy for Player II in the game to win each play we say that Player II wins $G_{\overline{m}}$(x, y) and write $x\alpha_{\overline{m}}y$. The special case $G_{\overline{1}}(x,y)$ where $\overline{1}$ denotes a k-tuple of l's is the standard Ehrenfeucht-Fraissé game [7]. The relation $\alpha_{\overline{m}}$ naturally defines a finite-index congruence on A^*.

The congruences $\alpha_{\bar{m}}$ can be defined inductively as follows: First, if $x= a_i...a_n$ is a word on A and $1 \leq i \leq j \leq n$ then $x[i,j]$, $x(i,j)$, $x(i,j]$ and $x[i,j)$ denote the factors $a_i\ldots a_j$, $a_{i+1}\ldots a_{j-1}$, $a_{i+1}\ldots a_j$ and $a_i\ldots a_{j-1}$ respectively. Now, we have $x\alpha_{(m,\bar{m})}y$ if and only if

(a) For every $p_i,\ldots, p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$, there exist $q_1,\ldots,q_m \in (q_1 \leq \bullet\bullet\bullet\ q_m)$ such that
- (i) $p_i <^x p_j$ if and only if $q_i <^y q_j$ for all $1 \leq i, j \leq m$,
- (ii) $Q_a^x p_i$ if and only if $Q_a^y q_i$ for all $1 \leq i \leq m$ and $a \in A$,
- (iii) $x[1,p_1)a_{\bar{m}}y[1,q_1)$
- (iv) $x(p_i, p_{i+1})a_{\bar{m}}y(q_i, q_{i+1})$ for all $1 \leq i <$ m,
- (v) $x(p_m,|x|]a_{\bar{m}}y(q_m, |y|]$, and

(b) For every $q_i,\ldots,q_m \in \mathcal{U}_y$ $(q_1 \leq \bullet\bullet\bullet \leq q_m)$, there exist $p_i,\ldots,p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$ such that (i)—(v) hold.

For fixed $\bar{m}$, we define the **M**-variety $\mathbf{V}_{\bar{m}}$ as follows: an A-generated monoid S is in $\mathbf{V}_{\bar{m}}$ if and only if S is a morphic image of $A^* /\alpha_{\bar{m}}$. Note that the equality $\mathbf{V}_{(m)} = \boldsymbol{J}_m$ holds. The **M**-variety $\mathbf{V}_k = \bigvee_{(m_1,\ldots,m_k} \mathbf{V}_{(m_1,\ldots,m_k)}$ is not locally finite.

3. For any words x,z on A with $z = a_1,\ldots,a_i$, the binomial coefficient (z) is defined as the number of distinct factorizations of the form $x = x_0a_1x_1\ldots a_ix_i$ with words $x_0,\ldots,x_i$ on A. For any prime number p and nonnegative integer m, we define on A^* the congruence $\delta_{p,m}$ by $x\delta_{p,m}y$ if $\binom{x}{z} \equiv \binom{y}{z}$ mod p whenever $|z| \leq m$. We define the **M**-variety $\mathbf{H}_{p,m}$ as follows: an A-generated monoid S is in $\mathbf{H}_{p,m}$ if and only if S is a morphic image of $A^*/\delta_{p,m}$. The **M**-variety $\mathbf{G}_p = \bigcup_{m\geq 0} \mathbf{H}_{p,m}$ of all finite p-groups is not locally finite.

3. Congruences associated to trees

We denote by P the set of trees on the alphabet $\{c,\bar{c}\}$. Formally, P is the set of words in $\{c, \bar{c}\}^*$ congruent to 1 in the congruence generated by the relation $c\bar{c}.= 1$. Intuitively, the words of P are obtained as follows: Given a tree, and starting from the root we encode c for going down and $\bar{c}$ for going up. For example,

is encoded by $cc\bar{c}cc\bar{c}c\bar{c}c\bar{c}\bar{c}\bar{c}c\bar{c}$. The number of leaves of a non-empty word u on $\{c, \bar{c}\}$, denoted by $1(u)$, is the number of occurrences of the factor $c\bar{c}$ in u (we define the number of leaves of the empty word, $l(1)$, by 1). The following two properties of trees are satisfied:

- Each non-empty tree u can be written uniquely as $u= cu_0\bar{c}...cu_m\bar{c}$ where $m \geq 0$ and $u_0,\ldots,u_m \in P$. We have $l(u) = \sum_{0\leq i\leq m} l(u_i)$.
- If $u= cu_0\bar{c}\ ...cu_m\bar{c}$ and $u= v_1cv_2\bar{c}v_3$ where $v_2 \in P$, then the tree $cv_2\bar{c}$ is factor of some $cu_i\bar{c}$.

Definition 3.1. Let A be a finite alphabet, u be a tree andbe equivalence relations on A^*. We define an equivalence relation $\equiv_u (\gamma_i,\ldots,\gamma_{l(u)})$ on A^* as follows:

- $\equiv_1 (\gamma) = \gamma$ for each equivalence relation γ on A^*.
- If $u = cu_0\bar{c}$ where $u_0 \in P$, $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)})= \equiv_{u0} (\gamma_1,\ldots,\gamma_{l(u_0)})$.

- If $u = cu_0\overline{c}\ldots cu_m\overline{c}$ where $m \geq l$ and $u_0\ldots u_m \in P$, $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)})$ is the equivalence relation on A^* where x $\equiv_x$ $(\gamma_1,\ldots,\gamma_{l(u_0)})y$ if and only if

$$x \equiv_{u_i} (\gamma_{l(u_0)+\bullet\bullet\bullet+l(u_{i-1})+1}, \cdots, \gamma_{l(u_0)+\bullet\bullet\bullet+l(u_i)})y \text{ for all } 0 \leq i \leq m,$$

(note that when $i = 0$, this means $x \equiv_{u_0} (\gamma_l,\ldots,\gamma_{l(u_0)})\gamma$) and

1. For every p1,…,$p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$, there exist $q_1,\ldots, q_m \in (q_1 \leq \bullet\bullet\bullet \leq q_m)$ such that

   (a) $p_i <^x p_j$ if and only if $q_i <^y q_j$ for all $1 \leq i, j \leq m$,
   (b) $Q_a^x p_i$ if and only if $Q_a^y q_i$ for all $1 \leq$ i $\leq m$ and $a \in A$,
   (c) $x[1, p_{i+1}) \equiv_{u_i}(\gamma_{l(u_0)+\bullet\bullet\bullet+l(u_{i-1})+1}, \cdots, \gamma_{l(u_0)+\bullet\bullet\bullet+l(u_i)})y[1,q_{i+1})$ for all $0 \leq i < m$,
   (d) $x(p_i, p_{i+1}) \equiv_{u_i}(\gamma_{l(u_0)+\bullet\bullet\bullet+l(u_{i-1})+1}, \cdots, \gamma_{l(u_0)+\bullet\bullet\bullet+l(u_i)})y(q_i,q_{i+1})$ for all $1 \leq i < m$,
   (e) $x(p_i, |x|] \equiv_{u_i}(\gamma_{l(u_0)+\bullet\bullet\bullet+l(u_{i-1})+1}, \cdots, \gamma_{l(u_0)+\bullet\bullet\bullet+l(u_i)})y\ (q_i, |y|)$ for all $0 \leq i < m$,

and

2. For every $q_1,\ldots,q_m \in \mathcal{U}_y$ $(q_1 \leq \bullet\bullet\bullet \leq q_m)$ there exist $p_1,\ldots,p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet\ p_m)$ such that (a)—(e) hold.

If $\gamma_i = \bullet\bullet\bullet = \gamma_j = \gamma$ for $1 \leq i < j \leq l(u)$, then we will abbreviate $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)})$ by

$$\equiv_u(\gamma_1,\ldots,\gamma_{i-1},\gamma^{j-i+1}, \gamma_{j+1},\ldots,\gamma_{l(u)}).$$

We will abbreviate $\equiv_u (\gamma^{1(u)})$ by $\equiv_u (\gamma)$. A consequence of Definition 3.l is that if $u=cu_0\overline{c}\ldots cu_m\overline{c}$ with $u_0,\ldots,u_m \in P$, then we have

$$\equiv_u (\gamma_i,\ldots, \gamma_{l(u)}) = \equiv_{(c,\overline{c})^{m+1}} (\equiv_{u_0}(\gamma_1,\ldots,\gamma_{l(u_o)}),\ldots,$$

$$\equiv_{u_m} (\gamma_{l(u_0)+\bullet\bullet\bullet+l(u_{m-1})+1}, \cdots, \gamma_{l(u_0)+\bullet\bullet\bullet+l(u_m)})).$$

Let $\overline{m} = (m_1,...,m_k)$ be a k-tuple of positive integers ($k \geq 0$). We have that $\equiv_{u_m} (\omega)$ where the tree $\alpha_{\overline{m}}$ is defined, by induction on k, as follows: if $k = 0$, then $u_{\overline{m}}, = $ l; then, for $\overline{m}$ $(m,m_1,\ldots, m_k)$, $u_{\overline{m}} = (cu_{(m_t,\ldots,m_k)}\overline{c})^{m+1}$.

Lemma 3.1. *Let A be a finite alphabet, u be a tree and $\gamma_1,\ldots,\gamma_{l(u)}$ be finite-index congruences on A^*. The equivalence relation $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)})$ is a finite-index congruence on A^*.*

Proof. The proof is by induction on u. If $u = l$, we have $(\gamma) = \gamma$. Otherwise, we factorize u as $u=cu_0\overline{c}\ldots cu_m\overline{c}$ with $u_0,\ldots,u_m \in P$. We have the following two cases: Case 1 ($m = 0$) and Case 2 ($m \geq 1$).

Case 1. We have $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)}) = \equiv_{u_0} (\gamma_1,\ldots,\gamma_{l(u_0)})$ and the result follows by the inductive hypothesis on u_0.

Case 2. Let $x \equiv_u (\gamma_1,\ldots,\gamma_{l(u)})y$ and $x' \equiv_u (\gamma_1,\ldots,\gamma_{l(u)})y'$. We want to show that $xx' \equiv_u (\gamma_1,\ldots,\gamma_{l(u)})yy'$. First, $xx' \equiv_u (\gamma_{l(u_0)+\cdots+l(u_{i-1})+1,\ldots,}\gamma_{l(u_0)+\cdots+l(u_i)})yy'$ for all $0 \leq i \leq m$ by the inductive hypothesis on u_i. Second, let $p_1,\ldots,p_m \in \mathcal{U}_{xx'}$ $(p_i \leq \bullet\bullet\bullet \leq P_m)$ (the proof is similar if starting with $q_1,\ldots,q_m \in \mathcal{U}_{yy'}$). Say $p_i,\ldots, p_n \leq |x|$ and $p_{n+i},\ldots,p_m > |x|$ for some $0 \leq n \leq m$. We treat the case $0 < n < m$ (the other cases are simpler). Put $p'_1 = p_{n+1} - |x|,\ldots,p'_{m-n} = p_m$ — $|x|$. From $x \equiv_u (\gamma_1,\ldots,\gamma_{l(u)})y$, there exist $q_1,\ldots,q_n \in \mathcal{U}_y$ $(q_1 \leq \bullet\bullet\bullet \leq q_n)$ satisfying (a)-(e) (here, we let $p_1,\ldots, p_n$, $p_n,\ldots, p_n \in \mathcal{U}_x$ for a total of m positions), and from $x' \equiv_u (\gamma_1,\ldots,\gamma_{l(u)})y'$, there exist $q'_1,\ldots,q'_{m-n} \in \mathcal{U}_{y'}$ $(q'_1 \leq \bullet\bullet\bullet \leq q'_{m-n})$ satisfying (a)—(e) (here, we let $p'_1,\ldots,p'_1, p'_1,\ldots, p'_{m-n} \in \mathcal{U}_{x'}$ for a total of m positions). Put $q_{n+1} = q'_1 + |y|,\ldots,q_m = q'_{m-n} + |y|$. The positions $q_1,\ldots,q_m \in \mathcal{U}_{yy'}$ are such that $q_1 \leq \bullet\bullet\bullet \leq q_m$ and we have

$$x(p_n,|x|] \equiv_{u_n} (\gamma_{l(u_0)+\cdots+l(u_{n-1})+1},\ldots,\gamma_{l(u_0)+\cdots+l(u_n)})y(q_n, |y|],$$

$$x'[l, p'_1) \equiv_{u_n}(\gamma_{l(u_0)+\cdots+l(u_{n-1})+1},\ldots,\gamma_{l(u_0)+\cdots+l(u_n)})y'[1, q'_1),$$

and by the inductive hypothesis on u_n we get

$$xx'(p_n, p_{n+1}) \equiv_{u_n} (\gamma_{l(u_0)+\cdots+l(u_{n-1})+1},\ldots,\gamma_{l(u_0)+\cdots+l(u_n)})\, yy'(q_n,q_{n+1}).$$

Condition (d) easily follows. Conditions (a)—(c) and (e) are simpler. The relation $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)})$ is hence a congruence on A^*. This obviously is finite-index since $\gamma_1,\ldots,\gamma_{l(u)}$ are. ■

3.1. Inclusion results

This section is concerned with comparing the equivalence relations $\equiv_u \gamma_1,\ldots,\gamma_{l(u)}$ Proposition 3.1, Theorem 3.l, Corollary 3.1 and Theorem 3.2 are adaptations of results of [13].

Proposition 3.1. *Let A be a finite alphabet, u be a tree and $\gamma_1,\ldots,\gamma_{l(u)}$ be congruences on A^*. We have*

$$\equiv_u (\gamma_1,\ldots,\gamma_{l(u)}) = \equiv_{cu\bar{c}} (\gamma_1,\ldots,\gamma_{l(u)}) = \equiv_{c\bar{c}} (\equiv_u (\gamma_1,\ldots,\gamma_{l(u)}))$$

Proof. This is an immediate consequence of Definition 3.1. ■

Theorem 3.1. *Let A be a finite alphabet, $u=v_1cv_2\bar{c}v_3$ be a tree as well as v_2 and $\gamma_1,\ldots,\gamma_{1(u)}$ be congruences on A^*. We have*

$$\equiv_u (\gamma_1,\ldots,\gamma_{l(u)})$$

$$= \equiv_{v_1c\bar{c}v_3} (\gamma_1,\ldots,\gamma_{l(v_1)}, \equiv_{v_2} (\gamma_{l(v_1)+1},\ldots,\gamma_{l(v_1)+l(v_2)}), \gamma_{l(v_1)+l(v_2)+1},\ldots,\gamma_{l(u)})$$

Proof. The proof is by induction on u. If $u= c\bar{c}$, we have $\equiv_{c\bar{c}} (\gamma) = \equiv_{c\bar{c}} (\equiv_1 (\gamma))$. Otherwise, we factorize u as $u = cu_0\bar{c}\ldots cu_m\bar{c}$ with $u_0,\ldots,u_m \in P$. We have the following two cases: Case 1 ($m = 0$) and Case 2 ($m \geq 1$).

Case 1. If $v_1v_3 = 1$, we get $v_2 = u_0$ and by Proposition 3.1, we have $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)}) = \equiv_{c\bar{c}} (\equiv_{v_2}(\gamma_1,\ldots,\gamma_{l(v_2)}))$ Otherwise, we have $v_1 = cv'_1$, $v_3 = v'_3\bar{c}$ and hence $u_0 = v'_1cv_2\,\bar{c}v'_3$. The result follows by Proposition 3.1 and the inductive hypothesis on u_0.

Case 2. Then some $cu_i\bar{c}$ has $cv_2\bar{c}$ as factor. We put $cu_i\bar{c} = v'cv_2\bar{c}v''$ and by using Proposition 3.l and the inductive hypothesis, we get $\equiv_{cu_i\bar{c}} (\gamma_1,\ldots,\gamma_{l(u_i)}) = \equiv_{u_i} \gamma_1,\ldots,\gamma_{l(v')}), \equiv_{v_2}(\gamma_{l(v')+1},\ldots,\gamma_{l(v')+l(v_2)}),$ $(\gamma_{l(v')+l(v_2)+1},\ldots,\gamma_{l(u_1)})$. The result follows from $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)}), = \equiv_{(c\bar{c})^{m+1}} (\equiv_{u_0} (\gamma_1,\ldots,\gamma_{l(u_0)}) ,\ldots, \equiv_{u_m}$ $\gamma_{l(u_0)+\cdots+l(u_{m-1})+1}, \ldots, \gamma_{1(u)}))$. ■

Corollary 3.1. *Let A be a finite alphabet, $u = v_1ccv_2\bar{c}\bar{c}v_3$ be a tree as well as v_2 and $\gamma_1,\ldots,\gamma_{l(u)}$ be congruences on A^*. We have $\equiv_u (\gamma_1,\ldots,\gamma_{l(u)}). = \equiv_{v_1cv_2\bar{c}v_3} (\gamma_1,\ldots,\gamma_{l(u)})$.*

Proof. By Proposition 3.1 and Theorem 3.1. ■

Corollary 3.1 enables us to restrict ourselves to the set P' of trees in which each node is either a leaf or has a number of children greater than 1.

If u is a tree and $u= v_1cv_2\bar{c}v_3$ is a factorization of u, then we say that the occurrences of c and $\bar{c}$ defined by this factorization are *related* if v_2 is a tree. Each occurrence of c in u is related to a unique occurrence of c in u. If u and v are trees, then we say that u is *extracted* from v if u can be obtained from v by removing in v a certain number of related occurrences of c and $\bar{c}$.

Theorem 3.2. *Let A be a finite alphabet, u and v be trees, u be extracted from v and y be a congruence on* A^*. *We have* $\equiv_v(\gamma) \subseteq \equiv_u(\gamma)$.

Proof. We treat the case where $v = v_1cv_2\bar{c}v_3$ with $v_2 \in P$ and $u = v_1v_2v_3$. The proof is by induction on v. If $v = c\bar{c}$, then $u = 1$ and the result is obvious. Otherwise, we factorize v as $v = cw_0\bar{c}...cw_m\bar{c}$ with $w_0,\ldots,w_m \in P$. We have the following two cases: Case 1 ($m = 0$) and Case 2 ($m \geq 1$).

Case l. If $v_1v_3 = 1$, we get $v_2 = w_0 = u$ and the result follows. Otherwise, we have $v_1 = cv_1'$ and $v_3 = v_3'\bar{c}$ and the equality $w_0\ v_1'cv_2\bar{c}v_3'$ results. By using the inductive hypothesis on w_0, we deduce

$$\equiv_v(\gamma) = \equiv_{w_0}(\gamma) \subseteq \equiv_{v_1'v_2v_3'}(\gamma) = \equiv_{cv_1'v_2v_3'\bar{c}}(\gamma) = \equiv_u(\gamma).$$

Case 2. Then some $cw_i\bar{c}$ has $cv_2\bar{c}$ as factor. We put $cw_i\bar{c} = v'cv_2\bar{c}v''$ and $cw_t'\bar{c} = v'v_2v''$. By using the inductive hypothesis $\equiv_{w_i}(\gamma) \subseteq \equiv_{w_i'}(\gamma)$, we get

$$\equiv_v(\gamma) = \equiv_{(c\bar{c})^{m+1}}(\equiv_{w_0}(\gamma),\ldots,\equiv_{w_m}(\gamma))$$
$$\subseteq \equiv_{(c\bar{c})^{m+1}}(\equiv_{w_0}(\gamma),\ldots,\equiv_{w_i'}(\gamma),\ldots,\equiv_{w_m}(\gamma)) = \equiv_u(\gamma). \qquad \blacksquare$$

Let m be a positive integer. We now define the (m) positions in a word x that will lead to an inclusion result useful for our purposes. These positions were defined in some of our earlier papers (like [4]) but they are needed to understand the proofs of our new results. So we repeat their definition for the sake of completeness.

Let x be a word on a finite alphabet A. To find the positions that spell the first occurrences of every subword of length $\leq m$ of x (or the (m) *first* positions in x), proceed inductively as follows:

- Let x_1 denote the smallest prefix of x such that $\alpha(x_1)= \alpha(x)$ (call p_1 the last position of x_1),
- Let x_{i+1} denote the smallest prefix of $x(p_i,|x|]$ such that $\alpha(x_{i+1})= \alpha(x(p_i,|x|])$ (call p_{i+1} the last position of x_{i+1}) for $1 \leq i < m$.

If $|\alpha(x)| = 1$ ($|\alpha(x)|$ denotes the cardinality of $\alpha(x)$), the positions $p_1,\ldots,p_m$ are the ones we are looking for and the procedure terminates. If $|\alpha(x)| > 1$, the positions $p_1,\ldots,p_m$ are among the ones we are looking for. To find the others, repeat the process to find the (m) first positions in $x[1, p_1)$ and the (m — i) first positions in $x(p_i, p_{i+1})$ for $1 \leq i < m$.

We can define similarly the positions that spell the last occurrences of every subword of length $\leq m$ of x (or the (m) *last* positions in x). The (m) first and the (m) last positions in x are called the (m) positions in x.

Consider the following example: Let A = {a, b} and

$$x = \underline{aaa}aaa\underline{babab}bbbbababaaabbab\overline{b}aaa\overline{a}\overline{a}\overline{b}\overline{b}\overline{a}\overline{a}.$$

The underlined (respectively overlined) positions of x are the (3) first (respectively last) positions in x.

The following lemmas give necessary and sufficient conditions for $\equiv_{(c\overline{c})^{n+1}}$ ($\alpha_{(m)}$, ω^{n-1}, $\alpha_{(m)}$)-equivalence, as well as $\equiv_{(c\overline{c})^2}$ ($\alpha_{(m)}$, γ)- and $\equiv_{(c\overline{c})^2}$ (γ, $\alpha_{(m)}$)-equivalences.

Lemma 3.2. *Let A be a finite alphabet, x and y be words on A and m n be positive integers. Let $p_1,\ldots,p_s \in \mathcal{U}_x$ ($p_1 < \bullet\bullet\bullet < p_s$) (respectively $q_1,\ldots,q_t \in \mathcal{U}_y$ ($q_1 < \bullet\bullet\bullet < g_t$)), be the (m) positions in x (respectively y). We have $x \equiv_{(c\overline{c})^{n+1}}$ ($\alpha_{(m)}$, ω^{n-1} $\alpha_{(m)}$)y if and only if the following three conditions are satisfied:*

1. *$s = t$.*
2. *$Q_a^x\, p_i$ if and only if $Q_a^y q_i$ for all $1 \leq i \leq s$ and $a \in A$.*
3. *$x(p_i, p_{i+1})\alpha_{(n)}y(q_i,q_{i+1})$ for all $1 \leq i < s$.*

Proof. Assume that Conditions (1)—(3) hold. First, the $\alpha_{(m)}$-equivalence of x and y follows from (1) and (2). Second, let $p'_1,\ldots,p'_n \in \mathcal{U}_x$ ($p'_1 \leq \bullet\bullet\bullet \leq p'_n$) (the proof is similar when starting with positions in $\mathcal{U}_y$).

Case l. If some of the p'_j's are among $p_i,\ldots,p_s$, then for each such p'_j, there exists $1 \leq i_j \leq s$ such that $p'_j = p_{i_j}$. Since (1) holds, we may consider $q'_j = q_{i_j}$. Condition (2) implies that $Q_a^x p'_j{}'_i$ if and only if $Q_a^y p'_j$ for $a \in A$.

Case 2. If $p'_j,\ldots, p'_{j'} \in \mathfrak{U}_{x(p_i,p_{i+1})}$ for some $1 \leq i < s$, $1 \leq j \leq \bullet\bullet\bullet \leq j'$ then from (3), there exist $q'_j,\ldots, q'_{j'} \in \mathfrak{U}_{x(q_i,q_{i+1})}$ ($q'_j \leq \bullet\bullet\bullet \leq q'_{j'}$) such that $p'_k <^x p'_j$ if and only if $q'_k <^y q'_\ell$ for all $j \leq k$, $\ell \leq j'$, and $Q_a^x p'_\ell$ if and only if $Q_a^x q'_\ell$ for all $j \leq \ell \leq j'$ and $a \in A$.

The positions $q'_l,\ldots, q'_n \in \mathcal{U}_y$ are such that $q'_j \leq \bullet\bullet\bullet \leq q'_n$ and satisfy

- $p'_i <^x p'_j$ if and only if $q'_i <^y q'_j$ for all $1 \leq i,j \leq n$,
- $Q_a^x p'_i$ if and only if $Q_a^y q'_i$ for all $1 \leq i \leq n$ and $a \in A$,
- $x[1, p'_1)\alpha_{(m)}y[1, q'_l)$,
- $x(p'_n,|x|]\alpha_{(m)}y(q'_n, |y|]$

Conversely, assume $x \equiv_{(c\overline{c})^{n+1}}(\alpha_{(m)}, \omega^{n-1}, \alpha_{(m)})y$. Conditions (1) and (2) hold by considering each of the (m) positions in turn. To see that Condition (3) holds, let $p'_1,\ldots, p'_n \in \mathfrak{U}_{x(p_i,p_{i+1})}$ ($p'_1 \leq \bullet\bullet\bullet \leq p'_n$) (the proof is similar when starting with positions in $\mathfrak{U}_{y(q_i,q_{i+1})}$). There exist suitable positions $q'_1,\ldots, q'_n \in \mathcal{U}_y$ ($q'_1 \leq \bullet\bullet\bullet \leq q'_n$) The facts that $x[1, p'_1)\alpha_{(m)}\, y[1, q'_1)$ and $x(q'_n, |x|]\alpha_{(m)}y(q'_n, |y|]$ guarantee the membership of $q'_1,\ldots, q'_n$ in $\mathfrak{U}_{y(q_i,q_{i+1})}$. ■

Lemma 3.13. *Let A be a finite alphabet, x and y be words on A, γ be a congruence on A* and m be a positive integer. Let $p_1,\ldots,p_s \in \mathcal{U}_x$ ($p_1 < \bullet\bullet\bullet < p_s$) (respectively $q_1,\ldots, q_t \in \mathcal{U}_y$ ($q_1 < \bullet\bullet\bullet < q_t$)) be the (m) first positions in x (respectively y). We have $x \equiv_{(c\overline{c})^2}$ ($\alpha_{(m)}$, γ)y if and only if the following five conditions are satisfied:*

1. *$s = t$.*
2. *$Q_a^x p_i$ if and only if $Q_a^y q_i$ for all $1 \leq i \leq s$ and $a \in A$.*
3. *$x(p_i,|x|]\gamma y(q_i,|y|]$ for all $1 \leq i \leq s$.*
4. *For all $1 \leq i < s$ and for every $p \in \mathfrak{U}_{x(p_1,p_{i+1})}$ (respectively $q \in \mathfrak{U}_{y(q_1,q_{i+1})}$), there exists $q \in \mathfrak{U}_{y(q_1,q_{i-1})}$ (respectively $p \in \mathfrak{U}_{x(p_1,p_{i+1})}$) such that*
    1. *$Q_a^x p$ if and only if $Q_a^x q$ for $a \in A$,*
    2. *$x(p,|x|]\gamma y(q,|y|]$.*
5. *For every $p \in \mathfrak{U}_{x(p_s,|x|)}$ (respectively $q \in \mathfrak{U}_{y(q_s,|y|)}$), there exists $q \in \mathfrak{U}_{y(q_s,|y|)}$ $p \in \mathfrak{U}_{x(p_s,|x|)}$ such that (a)—(b) hold.*

A similar statement is valid for the (m) last positions and $\equiv_{(c\overline{c})^2}$ (γ,$\alpha_{(m)}$)-equivalence.

Proof. Assume that Conditions (1)—(5) hold. First, the $\alpha_{(m)}$-equivalence of x and y follows from (l) and (2), and their γ-equivalence from (2) and (3) (with i = 1) and the fact that $p_1 = q_i$ = l. Second, let p be a position in $\mathcal{U}_x$ (the proof is similar when starting with a position in $\mathcal{U}_y$). Assume $Q_a^x p$.

Case 1. $p = p_i$ for some $1 \le i < s$. Since (1) holds, we may consider $q = q_i$. Condition (2) implies that $Q_a^y q$.

Case 2. $p \in \mathfrak{U}_{x(p_1,p_{i+1})}$ for for some $1 \le i < s$. From (4), there exists $q \in \mathfrak{U}_{y(q_1,q_{i+1})}$ such that $Q_a^y q$.

Case 3. $p \in \mathfrak{U}_{x(p_s,|x|]}$. From (5), there exists $p \in \mathfrak{U}_{y(q_s,|y|]}$, such that $Q_a^y q$.

In all cases, (1)—(5) and the choice of q imply that $x[l, p)\alpha_{(m)}y[1,q)$ and $x(p, |x|)\gamma,y(q,|y|]$.

Conversely, assume $x \equiv_{(c\bar{c})^2} (\alpha_{(m)}\gamma)y$. Conditions (l)—(3) hold by considering each of the (m) first positions in turn. To see that Condition (4) holds, let p be in $\mathfrak{U}_{x(p_1,p_{i+1})}$ (the proof is similar when starting with q in $q \in \mathfrak{U}_{y(q_1,q_{i+1})}$). Assume $Q_a^x p$. Hence there exists q in $\mathcal{U}_y$, such that $Q_a^y q$, $x[1, p)\alpha_{(m)}y[1,q)$ and $x(p,|x|]\gamma y(q,|y|]$. Assume that $q \notin \mathfrak{U}_{y(q_1,q_{i+1})}$. Hence $q \in \mathfrak{U}_{y[1,q_i]}$ or $q \in \mathfrak{U}_{y[q_{i+1},|y|]}$. From the choice of the p_j's and the q_j's, we get a contradiction with either $q \in Q_a^y$; or $x[l, p)\alpha_{(m)}\, y[1,q)$. Condition (5) follows similarly. ■

Note that in the case where $y = \omega$, Conditions (3)—(5) can be replaced by

$$x(p_s, |x|]\alpha y(q_s, |y|] \text{ and } x(p_i, p_{i+1})\alpha y(q_i, q_{i+1}) \text{ for all } 1\le i < s.$$

Theorem 3.3. *Let A be a finite alphabet, γ be a congruence on A^* and m be a positive integer. We have*

$$\equiv_{c(c\bar{c})^{m+1}\bar{c}c\bar{c}} (\omega^{m+1}, \gamma)= \equiv_{c^{m+1}(\bar{c}c\bar{c})^{m+1}} (\omega^{m+1}, \gamma)$$

and

$$\equiv_{c\bar{c}c(c\bar{c})^{m+1}\bar{c}} (\gamma, \omega^{m+1}) = \equiv_{(c\bar{c}c)^{m+1}\bar{c}^{m+1}} (\gamma,\omega^{m+1}).$$

Proof. The inclusion, $\equiv_{c^m(\bar{c}c\bar{c})^m}(\omega) \subseteq \equiv_{(c\bar{c})^{m+1}} (\omega)$ is clear from Theorem 3.2. So $\equiv_{c^{m+1}(\bar{c}c\bar{c})^{m+1}} (\omega^{m+1}, \gamma) = \equiv_{(c\bar{c})^2} (\equiv_{c^m(\bar{c}c\bar{c})^m}(\omega), \gamma) \subseteq \equiv_{(c\bar{c})^2} (\equiv_{(c\bar{c})^{m+1}} (\omega), y) = \equiv_{c(c\bar{c})^{m+1}\bar{c}c\bar{c}} (\omega^{m+1},\gamma)$ by Theorem 3.l. For the reverse inclusion, let us assume that x, y are such that $x \equiv_{c(c\bar{c})^{m+1}\bar{c}c\bar{c}} (\omega^{m+1}, \gamma)y$ or $x \equiv_{(c\bar{c})^2} (\alpha_{(m)},\gamma)y$. We want to show that $x \equiv_{c^{m+1}(\bar{c}c\bar{c})^{m+1}} (\omega^{m+1}\ \gamma)y$ or $x \equiv_{c(c^m(\bar{c}c\bar{c})^m\bar{c}c\bar{c}} (\omega^{m+1}, \gamma)y$. By Definition 3.l, we need to show that $x \equiv_{c^m(\bar{c}c\bar{c})^m} (\omega)y$, $x\gamma y$ and

- For every $p \in \mathcal{U}_x$, there exists $q \in \mathcal{U}_y$ such that
  (a) $Q_a^x p$ if and only if $Q_a^y q$ for $a \in A$,
  (b) $x[1, p) \equiv_{c^m(\bar{c}c\bar{c})^m} (\omega)y[1,q)$,
  (c) $x(p, |x|]\gamma y\ (q,|y|]$, and

- For every $q \in \mathcal{U}_y$, there exists $p \in \mathcal{U}_x$ such that (a)—(c) hold.

Under our assumption, this is equivalent to showing that $x\gamma y$ and

- For every $p \in \mathcal{U}_x$, there exists $q \in \mathcal{U}_y$ such that (a)—(c) hold, and
- For every $q \in \mathcal{U}_y$, there exists $p \in \mathcal{U}_x$ such that (a)—(c) hold.

To see this, we proceed by induction on m. We have $x \equiv_{c^m(\bar{c}c\bar{c})^m} (\omega)y$ if and only if $x \equiv_{c(c^{m-1}(\bar{c}c\bar{c})^{m-1})\bar{c}c\bar{c}} (\omega)y$ if and only if $x \equiv_{cc^{m-1}(\bar{c}c\bar{c})^{m-1}} (\omega)y$ and

- For every $p \in \mathcal{U}_x$, there exists $q \in \mathcal{U}_y$ such that
  (d) $Q_a^x p$ if and only if $Q_a^y q$ for $a \in A$,
  (e) $x[1, p \equiv_{c^{m-1}(\bar{c}c\bar{c})^m}(\omega)y[1,q)$, and
- For every $q \in \mathcal{U}_y$, there exists $p \in \mathcal{U}_x$ such that (d)—(e) hold.

For m = 1,(w)y if and only if $x\alpha y$ (which is part of our assumption). The result follows since $\alpha_{(m)} \subseteq \alpha_{(m\text{-}1)}$ and $\equiv_{c^m(\bar{c}c\bar{c})^m} (\omega) \subseteq \equiv_{c^{m-1}(\bar{c}c\bar{c})^{m-1}} (\omega)$.

Now, the γ-equivalence of x and y is part of our assumption. Next, since $x \equiv_{(c\bar{c})^2} (\alpha_{(m)},\gamma)y$, the ($m$) first positions in x and y satisfy (l)—(5) of Lemma 3.13. So let $p \in \mathcal{U}_x$, (the proof is similar if starting with $q \in \mathcal{U}_y$). Assume $Q_a^x p$.

Case 1. $p = p_i$ for some $1 \leq i \leq s$. Since (l) holds, we may consider $q = q_i$. Conditions (2) and (3) imply that $Q_a^y q$ and $x(p,|x|]\gamma y(q,|y|]$.

Case 2. $p \in \mathfrak{U}_{x(p_i,p_{i+1})}$ for some $1 \leq i < s$. From (4), there exists $q \in \mathfrak{U}_{y(p_i,p_{i+1})}$ such that $Q_a^y q$ and $x(p,|x|]\gamma y(q, |y|]$

Case 3. $p \in \mathfrak{U}_{x(p_v,|x|]}$. From (5), there exists $q \in \mathfrak{U}_{y(p_s,|y|]}$ such that $Q_a^y q$ and $x(p,|x|]\gamma y(q,|y|]$.

In all cases, (l)—(5) and the choice of q imply that $x[l, p) \equiv_{c^m(\bar{c}c\bar{c})^m} (\omega)y[l,q)$. This is done by induction on m. For m= l, x[l, p) $\equiv_{c^m(\bar{c}c\bar{c})^m} (\omega)y[1,q)$ if and only if x[l, p)xy[1,q). For m > l, we will show that x [1, p) $\equiv_{c^m(\bar{c}c\bar{c})^m} (\omega)y[1,q)$ by showing that x[l, p) $\equiv_{c(c\bar{c})^m\bar{c}c\bar{c}} (\omega)y[1,q)$ or x[I, p) $\equiv_{(c\bar{c})^2} (x_{(m\text{-}1)}, \omega)$ y[1,q) (using the inductive hypothesis). We treat Case 2 (Case 1 and Case 3 are handled similarly).

We need to show that x[l, p)$\alpha_{(m\text{-}1)}$$y$[1, q) (which is obvious) and

- For every $p' \in \mathcal{U}_{x[1,p)}$, there exists $q' \in \mathcal{U}_{y[1,q)}$ such that
  (f) $Q_b^x p'$ if and only if $Q_b^y q'$ for $b \in A$,
  (g) $x[1, p')x_{(m\text{-}1)}\, y[1,q')$, and
- For every $p' \in \mathcal{U}_{y,[1,p)}$, there exists $p' \in \mathcal{U}_{x[1,p)}$ such that (f)—(g) hold.

So let $p' \in \mathcal{U}_{x[1,p)}$ (the proof is similar if starting with $q' \in \mathcal{U}_{y[1,q)}$). Assume $Q_b^x p'$.

Case 2.1. $p' \in \mathfrak{U}_{x[1,p_i)}$. If $p' = p_j$ for some $1 \leq j < i$, consider $q' = q_j$ which satisfies $Q_b^y q'$. If $p' \in \mathfrak{U}_{x(p_j,p_{i+1})}$ for some $1 \leq j < i$, then from (4), consider $q' \in \mathfrak{U}_{y(q_j,q_{i+1})}$ satisfying $Q_b^y q'$.

Case 2.2. $p' = p_i$. Consider $q' = q_i$ satisfying $Q_b^y q'$.

Case 2.3. $p' \in \mathfrak{U}_{x(p_i,p)}$. Here, let p_i be the last of the (m — l) first positions in x[1, p_i) (p_i exists, otherwise x(p_i p_{i+1}) = 1). Consider q' to be the first occurrence of b *in* $\mathfrak{U}_{x(p_j,q_i]}$

In Cases 2.l-2.3, we see that x[1, p')$\alpha_{(m\text{-}1)}$ y[1, q'). ■

We end this section with a lemma similar to Lemma 3.2 involving the congruence $\beta_{1,m}$ instead of $\alpha_{(m)}$.

Lemma 3.4. *Let A be a finite alphabet, x and y be words on A and m,n be positive integers. Let* $p_1,\ldots,p_s \in \mathcal{U}_x$ *(*$p_1 < \bullet\bullet\bullet < p_s$*) (respectively* $q_1,\ldots,q_t \in (q_1 < \bullet\bullet\bullet < q_t)$*) be the positions that spell the first m and the last m occurrences of every letter of x (respectively y). We have* $x \equiv_{(c\overline{c})^{n-1}} (\beta_{1,m},\omega^{n-1},\beta_{1,m})y$ *if and only if the following three conditions are satisfied:*

1. $S = t$.
2. $Q_a^x p_i$ *if and only if* $Q_a^y q_i$ *for all* $1 \le i \le s$ *and* $a \in A$.
3. $x(p_i\, p_{i+1})\alpha_{(n)}y(q_i, g_{i+1})$ *for all* $1 \le i < s$.

Proof. The proof is similar to that of Lemma 3.2. ■

4. Pseudovarieties associated to trees

We are now going to review a few facts about the *Schützenberger product.* A first version 0f this product was introduced in [16], and it was generalized in [20].

Let m be a positive integer and $S_1,\ldots,S_m$ be finite monoids. We define the Schiitzenberger product of $S_1,\ldots,S_m$, denoted by $\Diamond_m(Si,\ldots,S_m)$, to be the submonoid of $m \times m$ matrices with the usual multiplication of matrices, of the form x = (x_{ij}), $1 \le i, j \le m$, in which the (i, j)-entry is a subset of $S_1 \times \bullet\bullet\bullet \times S_m$ and satisfying the following three conditions:

1. If $i > j$, then $x_{ij} = \emptyset$
2. If $i = j$, then $x_{ii} = \{(1,\ldots,1,s_i,l,\ldots,1)\}$ for some $s_i \in S_i$ (here, s_i is the ith component in the m-tuple).
3. If $i < j$, then $x_{ij} \subseteq \{(s_1,\ldots,s_m) \in S_1 \times \bullet\bullet\bullet \times S_m \mid s_1 = \bullet\bullet\bullet = s_{i-1} = l = s_{j+1} = \bullet\bullet\bullet = s_m\}$ (here, 1 is the unit of $S_1,\ldots,S_m$).

Note that these matrices are exactly the upper-triangular matrices whose ith diagonal entry corresponds to a singleton of S_i and whose (i,j)-entry (if $i < j$) to a subset of $S_i \times \bullet\bullet\bullet \times S_j$ If $\overline{s} = (s_i,\ldots,s_j) \in S_i \times \bullet\bullet\bullet \times S_j$ and $= \overline{s}'$, $(s'_{i'},\ldots,s'_{j'})$, then $\overline{s}\overline{s}' = (s_i,\ldots, s_{j-1},s_js'_{i'}, s'_{i'+1},\ldots, s'_{j'})$ if $j= i'$, and is undefined otherwise. This multiplication is extended to sets in the usual fashion; addition is given by set union. It is easy to check that $\Diamond_m(S_1,\ldots,S_m)$ is a monoid.

If **W**, $\mathbf{W}_1,\ldots,\mathbf{W}_m$ are **M**-varieties, $\Diamond_m(\mathbf{W}_1,\ldots,\mathbf{W}_m)$ denotes the **M**-variety generated by the products of the form $\Diamond_m(S_1,\ldots,S_m)$ with $S_i \in W$ for all $1 \le i \le m$. Also, we write $\Diamond_m(\mathbf{W})$ for $\Diamond_m(\mathbf{W},\ldots,\mathbf{W})$ and $\Diamond(\mathbf{W}) = \bigcup_{m\ge1}\Diamond_m(\mathbf{W})$ It is not difficult to see that $\Diamond_m(\mathbf{W}) \subseteq \Diamond_{m+1}(\mathbf{W})$ and that $\Diamond(\mathbf{W})$ is an **M**-variety.

The algebraic operation on monoids that corresponds to the concatenation of languages was identified to be the Schiitzenberger product.

Proposition 4.1 (Pin [13], Reutenauer [14], Straubing [20]). *Let m be a positive integer. Let* $\mathcal{W}_0,\ldots,\mathcal{W}_m$ *be *-varieties and* $\mathbf{W}_0,\ldots,\mathbf{W}_m$ *be the associated* **M**-*varieties. If* $\mathcal{W}$ *is the *-variety associated to* $\Diamond_{m+1}(\mathbf{W}_0,\ldots,\mathbf{W}_m)$*, then for each finite alphabet A,* $A^*\mathcal{W}$ *is the Boolean algebra generated by the languages of the form* $L_{i_0}a_1L_{i_1}\ldots a_kL_{i_k}$*, where* $0 \le i_0 < i_1 < \bullet\bullet\bullet < i_k \le a_1,\ldots,a_k \in A$ *and* $L_{i_1} \in A^*\mathcal{W}_{i_j}$*, for all* $0 \le j \le k$.

The following definition associates pseudovarieties to trees.

Definition 4.1 (Pin [13]). Let u be a tree and $\mathbf{W}_1,\ldots,\mathbf{W}_{l(u)}$ be **M**-varieties. We define an **M**-variety $\Diamond_u(\mathbf{W}_1,\ldots,\mathbf{W}_{l(u)})$ as follows:

- $\Diamond_1(\mathbf{W}) = \mathbf{W}$ for each **M**-variety **W**.
- If $u = cu_0\overline{c}$, where $u_0 \in P$, $\Diamond_u(\mathbf{W}_1,\ldots,\mathbf{W}_{l(u)}) = \Diamond_{u_0}(\mathbf{W}_1,\ldots,\mathbf{W}_{l(u_0)})$.

- If $u = cu_0\bar{c}...cu_m\bar{c}$ where $m \geq 1$ and $u_0, \ldots, u_m \in P$, $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u_0)})$ is the M- variety generated by the Schiitzenberger products of the form $\lozenge_{m+1}(S_0, \ldots, S_m)$, where

$$S_0 \in \lozenge_{u_0}(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u_0)}), \ldots, S_m \in \lozenge_{u_m}(\mathbf{W}_{l(u_0)+\cdots+l(u_{m-1})+1}, \ldots, \mathbf{W}_{l(u_0)+\cdots+l(u_m)}).$$

If $\mathbf{W}_i = \bullet\bullet\bullet = \mathbf{W}_j = \mathbf{W}$ for $1 \leq i < j \leq l(u)$, then we will abbreviate $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)})$ by

$$\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{i\text{-}1}, \mathbf{W}^{j-i+1}, \mathbf{W}_{j+1}, \ldots, \mathbf{W}_{l(u)}).$$

We will abbreviate $\lozenge_u(\mathbf{W}^{l(u)})$ by $\lozenge_u(\mathbf{W})$. More generally, if $L \subseteq P$, we denote by $\lozenge_L(\mathbf{W})$ the join $\bigvee_{u \in L} \lozenge_u(\mathbf{W})$. A consequence of Definition 4.1 is that if $u = cu_0\bar{c}\ldots cu_m\bar{c}$ with $u_0, \ldots, u_m \in P$, then we have

$$\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)}) = \lozenge_{(c\bar{c})^{m+1}}(\lozenge_{u_0}(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u_0)}),$$

$$\ldots, \lozenge_{u_m}(\boldsymbol{W}_{l(u_0)+\cdots+l(u_{m-1})+1}, \ldots, \mathbf{W}_{l(u_0)+\cdots+l(u_m)})).$$

The following theorem together with Proposition 4.1 describe, for each tree u, the *-variety of languages associated to the **M**-variety $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)})$.

Theorem 4.1 (Pin [13]). *If m is a positive integer and* $\mathbf{W}_0, \ldots, \mathbf{W}_m$ *are* **M**-*varieties, then*

$$\lozenge_{(c\bar{c})^{m+1}}(\mathbf{W}_0, \ldots, \mathbf{W}_m) = \lozenge_{m+1}(\mathbf{W}_0, \ldots, \mathbf{W}_m).$$

Now, let u be a tree and $\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)})$ be locally finite **M**-varieties. The following proposition shows that $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)})$ is also locally finite.

Proposition 4.2. *Let A be a finite alphabet, u be a tree and* $\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)}$ *be locally finite* **M**-*varieties. For* $1 \leq i \leq 1(u)$, *let* γ_i, *be the congruence generating* $\mathbf{W}_i$ *for A. Then, an A-generated monoid S belongs to* $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)})$ *if and only if S is a morphic image of* $A^* / \equiv_u (\gamma_1, \bullet\bullet\bullet \gamma_{l(u)})$.

Proof. Let $\mathcal{V}_u$ be the *-variety of languages associated to $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)})$. We want to show that $A^*\mathcal{V}_u = \mathcal{L}_{\equiv_u(\gamma_1, \ldots, \gamma_{l(u)})}$ where $\mathcal{L}_{\equiv_u(\gamma_1, \ldots, \gamma_{l(u)})}$ denotes the set of languages on A that are unions of classes of $\equiv_u(\gamma_1, \ldots, \gamma_{l(u)})$. The proof is by induction on u. If $u = 1$ and γ is the congruence generating $\mathbf{W}$ for A, then $\lozenge_1(\mathbf{W}) = \mathbf{W}$ and $\equiv_1(\gamma) = \gamma$. Otherwise, we factorize u as $u = cu_0\bar{c}...cu_m\bar{c}$ with $u_0, \ldots, u_m \in P$. If $m = 0$, then $\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)}) = \lozenge_{u_0}(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u_0)})$, $\equiv_u(\gamma_1, \ldots, \gamma_{l(u)}) = \equiv_{u_0}(\gamma_1, \ldots, \gamma_{l(u_0)})$ and the result follows by the inductive hypothesis on u_0. If $m \geq 1$, then from

$$\lozenge_u(\mathbf{W}_1, \ldots, \mathbf{W}_{l(u)}) = \lozenge_{(c\bar{c})^{m+1}}(\lozenge_{u_0}(\mathbf{W}_1, \ldots, \boldsymbol{W}_{l(u_0)}),$$

$$\ldots, \lozenge_{u_m}(\mathbf{W}_{l(u_0)+\cdots+l(u_{m-1})+1}, \ldots, \mathbf{W}_{l(u_0)+\cdots+l(u_m)}))$$

using the inductive hypothesis, Proposition 4.1 and Theorem 4.1, we can conclude that $A^*\mathcal{V}_u$ is the boolean algebra generated by the languages of the form $L_{i_0}a_1L_{i_1}...a_kL_{i_k}$, where $0 \leq i_0 < i_1 < \bullet\bullet\bullet < i_k \leq m$, $a_1, \ldots, a_k \in A$ and $L_{i_j} \in \mathcal{L}_{\equiv_{u_{i_j}}(\gamma_{l(u_0)+\cdots+l(u_{i_{j-1}})+1}, \ldots, \gamma_{l(u_0)+\cdots+l(u_{i_j})})}$ for all $0 \leq j \leq k$. The result follows since each $\equiv_u(\gamma_1, \ldots, \gamma_{l(u)})$-class is a boolean combination of sets of the form $L_{i_0}a_1L_{i_1}...a_kL_{i_k}$, where $0 \leq i_0 < i_1 < \bullet\bullet\bullet < i_k \leq m$, $a_1, \ldots, a_k \in A$ and each L_{i_j} is a $\equiv_{u_{i_j}}(\gamma_{l(u_0)+\cdots+l(u_{i_{j-1}})+1}, \ldots, \gamma_{l(u_0)+\cdots+l(u_{i_j})})$-class (this comes directly from Definition 3.l where the sets $L_{i_0}a_1L_{i_1}...a_kL_{i_k}$ are induced by the corresponding positions $p_1, \ldots, p_m$ ($p_1 \leq \bullet\bullet\bullet \leq p_m$) (a total of k different positions) and $q_1, \ldots, q_m$ ($q_1 \leq \bullet\bullet\bullet \leq q_m$) (a total of k different positions)). ■

5. Semidirect products

We are now going to review a few facts about semidirect products.

Let S and T be monoids. For the sake of clarity, when semidirect products are considered, we will usually express the operation of S additively (without assuming commutativity) and T multiplicatively. We will let 0 denote the unit of S and 1 the unit of T. A *left unitary action* of T on S is a map $(t,s) \mapsto ts$ from $T \times S$ into S satisfying $(tt')s = t(t's)$, $t(s + s') = ts + ts'$, $t0 = 0$ and $1s = s$ for all $s,s' \in S$ and $t,t' \in T$; a *right unitary action* of T on S is a map $(t,s) \mapsto st$ from $T \times S$ into S satisfying $s(tt')$-= $(st)t'$, $(s + s')t = st + s't$, $0t = 0$ and $s1 = s$ for all $s,s' \in S$ and $t, t' \in T$. If a left unitary action of T on S is given, the *semidirect product* $S *T$ is the set S x T with operation $(s,t)(s', t') = (s + ts', tt')$. If commuting left and right unitary actions of Ton S are given (that is, $t(st') = (ts)t'$ for all $s \in S$ and $t,t' \in T$), the *two-sided semidirect product* $S * * T$ is the set $S \times T$ with operation $(s,t)(s',0=(st' + ts',tt')$. Properties of the semidirect product are studied in [8] and properties of the two-sided semidirect product are found in [15]. Semidirect products are special cases of two-sided semidirect products.

Two-sided semidirect products induce an operation on **M**-varieties. Let **V** and **W** be **M**-varieties. We define **V * * W** to be the **M**-variety generated by the products $S ** T$ with $S \in$ V and $T \in$ W. We have $S \in$ V * * W if and only if S divides some product $S * * T$ with $S \in$ V and $T \in$ **W.** The definition of the **M**-variety **V** * **W** is similar. Note that * is associative on **M**-varieties and that ** is not. Neither * nor ** is associative on monoids. The operation * behaves well with respect to directed unions [8, 15].

Straubing has given a general construction to describe the languages recognized by the semidirect product of two finite monoids ("principle of the semidirect product") [19]. Weil has given such a construction for two-sided products [24]. The following results are consequences of their constructions and the equality $\mathbf{R} = \bigcup_{m\geq 0} \mathbf{J}_1^m$ where $\mathbf{J}_l^m$ denotes $\mathbf{J}_1 * \cdots * \mathbf{J}_1$ ($\mathbf{J}_1$ appears m times) [18].

Proposition 5.1 (Pin [13]). *Let $\mathcal{V}$ be a *-variety and* **V** *be the associated* **M**-*variety. If $\mathcal{W}$ is the *-variety associated to* $\mathbf{J}_1$, * **V,** *then for each finite alphabet A, $A^*\mathcal{W}$ is the boolean algebra generated by the languages of the form L or LaA^*, where $a \in A$ and $L \in A^*\mathcal{V}$. In other words,* $\mathbf{J}_1 * \mathbf{V} = \lozenge_{(c\bar{c})^2}$ **(V, I).** *If $\mathcal{W}'$ is the *-variety associated to* **R * V,** *then for each finite alphabet A, A^* $\mathcal{W}'$ is the smallest boolean algebra containing A^* $\mathcal{V}$ and closed for the operations $L \mapsto LaA^*$ where $a \in A$.*

Proposition 5.2 (Weil [24]). *Let $\mathcal{V}$ be a *-variety and* **V** *be the associated* **M**-*variety. If $\mathcal{W}$ is the *-variety associated to* $\mathbf{J}_1$ *** * V**, *then for each finite alphabet A, A^* $\mathcal{W}$ is the boolean algebra generated by the languages of the form L or LaL', where $a \in A$ and $L,L' \in A^*\mathcal{V}$. In other words,* $\mathbf{J}_1 * * \mathbf{V} = \lozenge_{(c\bar{c})^2}$ (V).

The following representations of free objects for **V** * **W** and **V** * * **W** were obtained by Almeida and Weil. The free object on the alphabet A in the variety generated by a pseudovariety **V** is represented by $F_v(A)$. In general, $Fv(A)$ does not lie in **V**. We have $F_v(A) \in$ **V** if and only if $F_v(A)$ is finite. In case **V** is **M**, $F_v(A)$ is A^*.

Proposition 5.3 (Almeida and Weil **[l]).** *Let* **V** *and* **W** *be* **M**-*varieties such that $F_v(A) \in$* **V** *and $F_w(A) \in W$ for all finite alphabets A. Then so is* **V** * **W**.

Moreover, for a finite alphabet A, let $T = F_w(A)$ and $S= F_v(T \times A)$. Consider:

1. *The left unitary action of T on S defined by $t(t_1,a) = (tt_1,a)$ for all $t,t_1 \in T$ and $a \in A$.*
2. *The associated semidirect product $S * T$.*

Then there exists a one-to-one morphism from $F_{v*w}(A)$ into $S * T$ that maps a into $((1,a),a)$.

Proposition 5.4 (Almeida and Weil [3]). *Let* **V** *and* **W** *be* **M**-*varieties such that $Fv(A) \in$* **V** *and $F_w(A) \in$* **W** *for all finite alphabets A. Then so is* **V** ** **W**.

Moreover, for a finite alphabet A, let $T = F_w(A)$ and $S = F_v(T \times A \times T)$. Consider:

1. *The left unitary action of T on S defined by $t(t_1,a,t_2) = (tt_1,a,t_2)$ for all $t,t_1,t_2 \in T$ and $a \in A$.*
2. *The right unitary action of T on S defined by $(t_1, a, t_2)t = (t_1, a, t_2t)$ for all $t, t_1, t_2 \in T$ and $a \in A$.*
3. *The associated two-sided semidirect product $S ** T$.*

Then there exists a one-to-one morphism from $F_{v**w}(A)$ into $S ** T$ that maps a into $((1,a,1),a)$.

5.1. Congruences associated to semidirect products

In this section, we associate congruences to semidirect and two-sided semidirect products of locally finite **M**-varieties.

Let A be a finite alphabet. Let **W** be a locally finite **M**-variety and γ_A be the finite-index congruence on A^* such that an A-generated monoid S belongs to **W** if and only if S is a morphic image of A^*/γ_A. The free object $F_w(A)$ is isomorphic to A^*/γ_A. The pseudovariety **W** is such that $F_w(A) \in$ **W**. We denote by π_{γ_A} the projection from A^* into $F_w(A)$ that maps a to the generator a of $F_w(A)$. If $x, y \in A^*$, then $\pi_{\gamma_A}(x) = \pi_{\gamma_A}(y)$ if and only if $\pi_{\gamma_A} y$.

Definition 5.1. Let $B = F_w(A) \times A$ and z be a word on A. Let $\sigma^z_{\gamma_A}: A^* \to B^*$ be defined by $\sigma\sigma^z_{\gamma_A}(1) = 1$ and

$$\sigma^z_{\gamma_A}(a_1 \ldots a_i) = (\pi_{\gamma_A}(z),a_1)(\pi_{\gamma_A}(za_1), a_2)\ldots(\pi_{\gamma_A}(za_1\ldots a_{i-1}),a_i).$$

We often denote $\sigma^z_{\gamma_A}(x)$ simply by $\sigma_{\gamma_A}(x)$.

Definition 5.2. Let $B = F_w(A) \times A \times F_w(A)$ and z,z' be words on A. Let $\tau^{z,z'}_{\gamma_A}: A^* \to B^*$ be defined by $\tau^{z,z'}_{\gamma_A}(1)= 1$ and

$$\tau^{z,z'}_{\gamma_A}(a_1\ldots a_i) = (\pi_{\gamma_A}(z), a_1, \pi_{\gamma_A}(a_2\ldots a_i z'))(^1r),(\pi_{\gamma_A}(za_1), a_2, \pi_{\gamma_A}(a_3\ldots a,z'))$$
$$\ldots(\pi_{\gamma_A}(za_1\ldots a_{i-1}), a_i, \pi_{\gamma_A}(z'))$$

We often denote $\tau^{1,1}_{\gamma_A}(x)$ simply by $\tau_{\gamma_A}(x)$.

Fix two locally finite M-varieties **V** and **W.** Let β_A (respectively γ_A) be the finite-index congruence generating **V** (respectively **W)** for the finite alphabet A.

5.1.1. The case V * W

Let A be a finite alphabet and $B = F_w(A) \times A$. We define an equivalence relation $\sim_{\beta_B,\gamma_A}$ on A^* as follows:

$$x \sim_{\beta_B,\gamma_A} y \text{ if and only if } \sigma_{\gamma_A}(x)\beta_B\sigma_{\gamma_A}(y) \text{ and } x\gamma_A y.$$

Proposition 5.5. *The equivalence relation $\sim_{\beta_B,\gamma_A}$ is a finite-index congruence on A^*.*

Proof. We will abbreviate β_B by β and γ_A by γ throughout the proof. Assume x $\sim_{\beta,\gamma} y$ and $x' \sim_{\beta,\gamma}$ y'. We have

$$\sigma_\gamma(x)\beta\sigma_\gamma(y) \text{ and } x\gamma y$$

and similarly with x and y replaced by x' and y'. Since y is a congruence we have $xx'yyy'$. The above, the fact that $\pi_\gamma(x) = \pi_\gamma(y)$, and the fact that β is a congruence imply that

$$\sigma_\gamma(xx') = \sigma_\gamma(x)\sigma^x_\gamma(x') = \sigma_\gamma(x)\, \sigma^y_\gamma(x')\beta\sigma_\gamma(y)\, \sigma^y_\gamma(y')=\sigma_\gamma(yy').$$

Thus xx' $\sim_{\beta,\gamma}$ yy' showing that $\sim_{\beta,\gamma}$ is a congruence. This obviously is a finite-index congruence since β and y are. ■

Proposition 5.6. *Let* **V** *and* **W** *be locally finite* **M***-varieties. Let* γ_A (*respectively* β_B) *be the finite-index congruence generating* **W** (*respectively* **V**) *for the finite alphabet* A (*respectively,* $B = F_w(A) \times A$). *Then, an A-generated monoid S belongs to* **V** * **W** *if and only if S is a morphic image of* $A^*/{\sim}{\sim}_{\beta_B,\gamma_A}$.

Proof. We will abbreviate β_B by β and γ_A by γ throughout the proof. Let $x = y$ be an identity on A, say $x = a_1\ldots a_i$ and $y = b_1\ldots b_j$. Then $x = y$ holds in **V** * **W** if and only if x and y represent the same element of $F_{\mathbf{v}*\mathbf{w}}(A)$. By Proposition 5.3, this is equivalent to x and y having the same image under the one-to-one morphism from $F_{\mathbf{v}*\mathbf{w}}$ (A) into $F_{\mathbf{v}}(B)*F_w(A)$ defined by $a \mapsto ((1,a),a)$ and where the left unitary action of $F_w(A)$ on $F_v(B)$ is given by $t(t_1,a)=(tt_1,a)$. The above morphism maps x to

$$((1, a_1) + (a_1,a_2) + \bullet\bullet\bullet + (a_1\ldots a_{i-1}, a_i),a_1\ldots a_i), \qquad (1)$$

and y to

$$((1, b_1) + (b_1,b_2) + \bullet\bullet\bullet + (b_1\ldots b_{j-1}, b_j),a_1\ldots b_j), \qquad (2)$$

(here, $F_{\mathbf{v}}(B)$ is written additively). The identity $x = y$ holds in $F\mathbf{v}_*\mathbf{w}(A)$ if and only if corresponding components of the pairs (1) and (2) coincide. Denote by x' (respectively y') the first component of (1) (respectively (2)). Then, $F_{v*w}(A) = x = y$ is equivalent to the two conditions $F_v(B)$ |= $x' = y'$ and $F_w(A)= x =$ y, or $\sigma_\gamma(x)\beta\sigma_\gamma(y)$ and $x\gamma y$. ■

5.1.2. The case V * * W

Let A be a finite alphabet and $B = F_w(A) \times A \times F_w(A)$. We define an equivalence relation $\approx_{\beta_B,\gamma_A}$ on A^* as follows:

$x \approx_{\beta_B,\gamma_A}$ if and only if $\tau_{\gamma_A}(x)\beta_B\,\tau_{\gamma_A}(y)$ and $x\gamma_A y$.

Proposition 5.7. *The equivalence relationis a finite-index congruence on* A^*.

Proof. Using the notation in the proof of Proposition 5.5, assuming $x \approx_{\beta,\gamma} y$ and $x' \approx_{\beta,\gamma} y'$, the result follows from

$$\tau_\gamma(xx') = \tau_\gamma^{1,x'}(x)\,\tau_\gamma^{x,1}(x') = \tau_\gamma^{1,y'}(x)\,\tau_\gamma^{y,1}(x')\,\beta\tau_\gamma^{1,y'}(y)\,\tau_\gamma^{y,1}(y') = \tau_\gamma(yy'). \quad ■$$

Proposition 5.8. *Let* **V** *and* **W** *be locally finite* **M***-varieties. Let* y_A (*respectively* β_B) *be the finite-index congruence generating W* (*respectively* **V**) *for the finite alphabet A* (*respectively* $B = F_w(A) \times A$ x $F_w(A)$)*. Then, an A-generated monoid S belongs to* **V*** * **W** *if and only if S is a morphic image of* $A^*/{\approx}_{\beta_B,\gamma_A}$.

Proof. The proof is similar to that of Proposition 5.6 using Proposition 5.4 instead of Proposition 5.3. ■

5.2. Trees associated to semidirect products

In this section, we associate trees to some semidirect and two-sided semidirect products of locally finite **M**-varieties.

The following theorem provides equalities which relate with Propositions 5.l and 5.2. Let γ be the finite-index congruence generating a locally finite **M**-variety **V** for the finite alphabet A. By Proposition 5.6 (respectively 5.8), the congruence (respectively generates $\mathbf{J}_1$ * **V** (respectively $\mathbf{J}_1$* * **V**) for A. By Proposition 4.2, $\equiv_{(c\bar{c})^2}(\gamma,$ $\omega)$ (respectively $\equiv_{(c\bar{c})^2}(\gamma)$) generates $\lozenge_{(c\bar{c})^2}$(**V**, **I**) (respectively $\lozenge_{(c\bar{c})^2}$(**V**)) for A.

Theorem 5.1. *Let A be a finite alphabet and γ be a congruence on A^*. We have $\sim_{x,y} = \equiv_{(c\overline{c})^2}(\gamma, \omega)$ and $\approx_{\alpha,\gamma} = \equiv_{(c\overline{c})^2}(\gamma)$.*

Proof. We have $x \equiv_{(c\overline{c})^2}(\gamma, \omega)y$ if and only if $x\gamma y$ and

1. For every $p \in \mathcal{U}_x$, there exists $q \in \mathcal{U}_y$ such that
   (a) $Q_a^x p$ if and only if $Q_a^y q$ for $a \in A$,
   (b) $x[1, p)\gamma\, y[1,q)$, and
2. For every $q \in \mathcal{U}_y$ there exists $p \in \mathcal{U}_x$ such that (a) and (b) hold.

It is easy to see that $x \equiv_{(c\overline{c})^2}(y, \omega)y$ if and only if $\sigma_y(x)\alpha\sigma_y(y)$ and xyy.

We have $x \equiv_{(c\overline{c})^2}(\gamma)y$ if and only if xyy and

1. For every $p \in \mathcal{U}_x$ there exists $q \in \mathcal{U}_y$ such that
   (a) $Q_a^x p$ if and only if $Q_a^y q$ for $a \in A$,
   (b) $x[1, p)\gamma y[1,q)$,
   (c) $x(p, |x|]\gamma y(q, |y|]$, and
2. For every $q \in \mathcal{U}_y$, there exists $p \in \mathcal{U}_x$ such that (a)—(c) hold.

It is easy to see that $x \equiv_{(c\overline{c})^2}(\gamma)y$ if and only if $\tau_y(x)\alpha\tau_y(y)$ and $x\gamma y$. ■

Corollary 5.1. *Let y_i be the sequence of congruences defined by $y_i = \alpha$ and $y_{i+1} = \equiv_{(c\overline{c})^2}(y_i)$. The equality $y_i = \equiv_{u_{\overline{1}_i}}(\omega) = \alpha_{\overline{1}_i}$ holds where $\overline{1}_i$ is a sequence of i 1's.*

Theorem 5.2. *Let m be a positive integer,* **H** *be a locally finite* **G***-variety and y be the congruence generating* **H** *for the finite alphabet A. Then $\sim_{a_{(m)},\gamma} = \approx_{\alpha_{(m)},\gamma} = \equiv_{(c\overline{c})^{m+1}}(y)$.*

Proof. We have $x \equiv_{(c\overline{c})^{m+1}}(\gamma)y$ if and only if $x\gamma y$ and

1. For every $p_1,\dots,p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$, there exist $q_1,\dots,q_m \in (q_1 \leq \bullet\bullet\bullet \leq q_m)$ such that
   (a) $p_i <^x p_j$ if and only if $q_i <^y q_j$ for all $1 \leq i,j \leq m$,
   (b) $Q_a^x p_i$ if and only if $Q_a^y q_i$ for all $1 \leq i \leq m$ and $a \in A$,
   (c) $x[1, p_{i+1})\gamma y[1,q_{i+1})$ for all $0 \leq i < m$,
   (d) $x(p_i, p_{i+1})\gamma y(q_i,q_{i+1})$ for all $1 \leq i < m$,
   (e) $x(p_i, |x|]\gamma y(q_i, |y|]$ for all $0 < i < m$, and
2. For every $q_i,\dots,q_m \in \mathcal{U}_y$ $(q_i \leq \bullet\bullet\bullet \leq q_m)$, there exist $p_1,\dots,p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$ such that (a)—(e) hold.

We have $x \sim_{a_{(m)},\gamma} y$ if and only if $\sigma_y(x)\alpha_{(m)}\sigma_y(y)$ and $x\gamma y$. This is equivalent to saying that $x \sim_{a_{(m)},\gamma} y$ if and only if $x\gamma y$ and

1. For every $p_i,\dots,p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$, there exist $q_1,\dots,q_m$ $\mathcal{U}_y$ $(q_1 \leq \bullet\bullet\bullet \leq q_m)$ such that (a)—(c) hold, and
2. For every $q_i,\dots,q_m \in \mathcal{U}_y$ $(q_1 \leq \bullet\bullet\bullet \leq q_m)$, there exist $p_1,\dots,p_m \in \mathcal{U}_x$ $(p_1 \leq \bullet\bullet\bullet \leq p_m)$ such that (a)—(c) hold.

We have $x \approx_{a_{(m)},\gamma} y$ if and only if $\tau_\gamma(x)\alpha_{(m)}\tau_\gamma(y)$ and $x\gamma y$. This is equivalent to saying that $x \approx_{a_{(m)},\gamma} y$ if and only if $x\gamma y$ and

1. For every $p_1,\ldots, p_m \in \mathcal{U}_x$ ($p_1 \leq \cdots \leq p_m$), there exist $q_1,..,q_m \in \mathcal{U}_y$ ($q_1 \leq \cdots \leq q_m$) such that (a)—(c) and (e) hold, and
2. For every $q_i,\ldots,q_m \in \mathcal{U}_y$ ($q_1 \leq \cdots \leq q_m$), there exist $p_1,\ldots,p_m \in \mathcal{U}_x$ ($p_1 \leq \cdots \leq p_m$) such that (a)—(c) and (e) hold.

Since **H** is a **G**-variety and γ generates **H** for A (γ is a group congruence), the conditions $x\gamma y$ and (a)—(c) imply (d) and (e). We conclude that x $\equiv_{(c\bar{c})^{m+1}} (\gamma)y$ if and only if $x \sim_{\alpha_{(m)},\gamma} y$ if and only if $x \approx_{\alpha_{(m)},\gamma} y$. ■

Theorem 5.3. *Let A be a finite alphabet and m, n be positive integers. We have*

$$\approx_{\alpha_{(n)}\alpha_{(m)}} = \equiv_{(c\bar{c})^{n+1}} (\alpha_{(m)}, \omega^{n-1}, \alpha_{(m)}).$$

Consequently, $\mathbf{J}_n ** \mathbf{J}_m = \Diamond_{(c\bar{c})^{n+1}} (\mathbf{J}_m)\mathbf{I}^{n-1}, \mathbf{J}_m) \Diamond_{c(c\bar{c})^{m+1}(c\bar{c})^{n-1}(c\bar{c})^{m+1}\bar{c}} (\mathbf{I})$.

Proof. By Lemma 3.2, we have x $\equiv_{(c\bar{c})^{n+1}} (\alpha_{(m)}, \omega^{n-1}, \alpha_{(m)})y$ if and only if $\tau_{\alpha_{(m)}}(x)\alpha_{(n)}\, \tau_{\alpha_{(m)}}(y)$ and x$\alpha_{(m)}$y. ■

Theorem 5.4. *Let A be a finite alphabet and m, n be positive integers. We have*

$$\approx_{\alpha_{(n)}\beta_{1,m}} = \equiv_{(c\bar{c})^{n+1}} (\beta_{1,m}, \omega^{n-1}, \beta_{1,m}).$$

Consequently, $\mathbf{J}_n ** \mathbf{Com}_{1,m} = \Diamond_{(c\bar{c})^{n+1}} (\mathbf{Com}_{1,m}, \mathbf{I}^{n-1}, \mathbf{Com}_{1,m})$.

Proof. By Lemma 3.4, we have $x \equiv_{(c\bar{c})^{n+1}} (\beta_{1,m}\, \omega^{n-1}, \beta_{1,m})y$ if and only if $\tau_{\beta_{1,m}}(x)\alpha_{(n)}\, \tau_{\beta_{1,m}}(y)$ and $x\beta_{1,m}y$. ■

We end this section with a few results on a conjecture of Pin. It was conjectured in [13] that if $u, v \in P'$, then $\Diamond_u(\mathbf{I}) \subseteq \Diamond_v (\mathbf{I})$ (in other words, $\equiv_v (\omega) \subseteq \equiv_u (\omega)$) if and only if u is extracted from v. The following two results give counterexamples.

Theorem 5.5 (Blanchet-Sadri [5]). *If m > 1, then*

$$\mathbf{J}_1^{m+1} = \Diamond_{c^{m+1}(\bar{c}c\bar{c})^{m+1}} (\mathbf{I}) = \Diamond_{c(c\bar{c})^{m+1}\bar{c}c\bar{c}} (\mathbf{I}) = \mathbf{J}_1 * \mathbf{J}_m$$

Proof. By Theorem 5.l, if γ_i is the sequence of congruences defined by $\gamma_i = \sim_{\alpha,\alpha}$ and $\gamma_{i+1} = \equiv_{(c\bar{c})^2} (\gamma_i, \omega)$, then the equality $\gamma_i = \equiv_{c^{i+1}(\bar{c}c\bar{c})^{i+1}} (\omega)$ holds. Also, we have the equality $\sim_{\alpha,\alpha_{(m)}} = \equiv_{(c\bar{c})^2} (\alpha_{(m)}, \omega)$ by Theorem 5.1. The result then follows from Theorem 3.3 with $\gamma = \omega$. ■

Theorem 5.6. *If m> 1, then*

$$\Diamond_{c^{m+1}(\bar{c}c\bar{c})^{m+1}} (\mathbf{I}) \subseteq \Diamond_{(cc\bar{c}c\bar{c}\bar{c})^{m+1}} (\mathbf{I}).$$

Proof. The equality $\Diamond_{c^{m+1}(\bar{c}c\bar{c})^{m+1}} (\mathbf{I}) = \Diamond_{c(c\bar{c})^{m+1}\bar{c}c\bar{c}} (\mathbf{I})$ holds by Theorem 5.5. But the latter is included in $\Diamond_{(c(c\bar{c})^{m+1}\bar{c})^2} (\mathbf{I})$ since $\equiv_{(c(c\bar{c})^{m+1}\bar{c})^2} (\omega) \subseteq \equiv_{c(c\bar{c})^{m+1}\bar{c}c\bar{c}} (\omega)$ by Theorem 3.2. We have $(c(c\bar{c})^{m+1}\,\bar{c})^2 = u_{(m,1)}$ and $(cc\bar{c}c\bar{c}\bar{c})^{m+1} = u_{(m,1)}$. The result then follows from the inclusion $\equiv_{u_{(m,1)}} (\omega) = \alpha_{(m,1)} \subseteq \alpha_{(l,m)} = \equiv_{u_{(1,m)}} (\omega)$ from [6]. ■

Theorem 5.6 answers a statement at the end of Section 3 of [13]. But Pin's conjecture was shown to be true in an important special case.

Theorem 5.7 (Blanchet-Sadri [6]). *Let P" be the set of trees* $u_{\overline{m}}$ *where* $\overline{m}$ *is a tuple of positive integers either of length* 1 *or of the form* $(m_1,\ldots,m_k, 1)$ *for some* $m_1,\ldots,m_k$. *If u, v ∈ P", then* $\Diamond_u(\mathbf{I}) \subseteq \Diamond_v(\mathbf{I})$ *if and only if u is extracted from v.*

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