Equations on the semidirect product of a finite semilattice by a finite commutative monoid

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Abstract:

Let $\mathbf{Com_{t,q}}$ denote the variety of finite monoids that satisfy the equations $xy = yx$ and $x^t = x^{t+q}$. The variety $\mathbf{Com_{1,1}}$ is the variety of finite semilattices also denoted by $\mathbf{J_1}$. In this paper, we consider the product variety $\mathbf{J_1 * Com_{t,q}}$ generated by all semidirect products of the form $M * N$ with $M \in \mathbf{J_1}$ and $N \in \mathbf{Com_{t,q}}$. We give a complete sequence of equations for $\mathbf{J_1} * \mathbf{Com}_{t,q}$ implying complete sequences of equations for $\mathbf{J_1}$ * ($\mathbf{Com} \cap \mathbf{A}$), $\mathbf{J_1}$ * ($\mathbf{Com} \cap \mathbf{G}$) and $\mathbf{J_1}$ * $\mathbf{Com}$, where $\mathbf{Com}$ denotes the variety of finite commutative monoids, $\mathbf{A}$ the variety of finite aperiodic monoids and $\mathbf{G}$ the variety of finite groups.

Article:

1. Introduction

Let $\mathbf{Com_{t,q}}$ denote the variety of finite monoids that satisfy the equations $xy = yx$ and $x^t = x^{t+q}$. The variety $\mathbf{Com_{1,1}}$ is the variety of finite semilattices also denoted by $\mathbf{J_1}$. In this paper, we give an equational characterization of the product variety $\mathbf{J_1}$ * $\mathbf{Com_{t,q}}$ generated by all semidirect products of the form $M * N$ with $M \in \mathbf{J_1}$ and $N \in \mathbf{Com_{t,q}}$. Our results imply a complete sequence of equations for $\mathbf{J_1}$ * ($\mathbf{Com} \cap \mathbf{A}$), $\mathbf{J_1}$ *($\mathbf{Com} \cap \mathbf{G}$) and $\mathbf{J_1}$ *Com, where $\mathbf{Com}$ denotes the variety of finite commutative monoids, $\mathbf{A}$ the variety of finite aperiodic monoids and $\mathbf{G}$ the variety of finite groups.

Pin [12] has shown that the variety $\mathbf{J_1}$ * $\mathbf{Com_{1,1}}$ is defined by the equations $xux = xux^2$ and $xuyvxy = xuyvyx$. Irastorza [7] has given equations of the particular products $\mathbf{J_1}$ * $\mathbf{Com_{0,q}}$ and has shown that, although the two varieties $\mathbf{J_1}$ and $\mathbf{Com_{0,2}}$ are defined by finite sequences of equations, their product is not. Almeida [1] has given an equational characterization of the variety of finite monoids generated by all semidirect products of i finite semilattices and has shown that it is defined by a finite sequence of equations if and only if $i = 1$ or 2. Ash [2] has shown that the variety $\mathbf{J_1}$ * $\mathbf{G}$ = $\mathbf{Inv}$ is defined by the equation $x^w y^w = y^w x^w$, that is, J_1 * $\mathbf{G}$ is the variety generated by the inverse semigroups.

Our results follow from versions of techniques used in particular by Blanchet-Sadri [3], Brzozowski and Simon [4] and Pin [11, 12].

1.1. Definitions and notations

Let M and N be monoids. We say that M divides N and write $M \prec N$ *if* M is a morphic image of a submonoid of N . Note that the divisibility relation is transitive. An **M**-variety $\mathbf{V}$ is a family of finite monoids that satisfies the following two conditions:

- If $N \in \mathbf{V}$ and $M \prec N$, then $M \in \mathbf{V}$.
- If $M, N \in \mathbf{V}$, then $M \times N \in \mathbf{V}$.

Some examples of **M** -varieties follow.

- The trivial **M**-variety, consisting of the trivial monoid, is denoted by **I**.
- The **M**-variety, consisting of all finite monoids, is denoted by **M**.
- The **M**-variety, consisting of all finite groups, is denoted by **G** and is defined by the equation $x^w = 1$.
- The **M** -variety, consisting of all finite commutative monoids (respectively groups), is denoted by **Com** (respectively $\mathbf{G_{com}}$) and is defined by the equation $xy = yx$ (respectively by the equations $xy = yx$ and $x^w = 1$).

Given **M**-varieties **V** and **W,** we denote by $\mathbf{V} \vee \mathbf{W}$ the least **M**-variety containing both **V** and **W.**

In this paper, we consider the **M**-variety $\mathbf{Com_{t,q}}$ defined by the pair of equations $xy = yx$ and $x^t = x^{t+q}$ where t, q are integers and $t \geq 0$, $q \geq 1$. We get the following **M** -varieties (among others).

- The **M**-variety $\mathbf{Com_{0,1}}$ is the trivial **M**-variety **I.**
- The **M**-variety $\mathbf{Com_{1,1}}$ is the family of finite commutative and idempotent monoids (called semilattices).
- The **M**-variety $\mathbf{Com} \cap \mathbf{A}$ is $\vee_{t\geq 0}\mathbf{Com_{t,1}}$
- The **M**-variety $\mathbf{Com}_{\mathbf{0},q}$ is generated by the cyclic group Z_q of order q and is also denoted by (Z_q).
- The **M**-variety $\mathbf{G_{com}} = \mathbf{Com} \cap \mathbf{G}$ is $\vee_{q\geq 1}\mathbf{Com}_{\mathbf{0},q}$.
- The **M** -variety $\mathbf{Com_{t,q}}$ is generated by the cyclic monoid $Z_{t,q}$ of index t and period q, that is, $Z_{t,q} = \{1, a, a^2 \dots , a^{t+q-1}\}$ with $a^t = a^{t+q}$. The monoid $Z_{t,q}$ is isomorphic with a submonoid of $Z_{t,1} \times Z_q$. Since further $Z_{t,1} \prec Z_{t,q}$ and $Z_q \prec Z_{t,q}$, it follows that $\mathbf{Com_{t,q}} = \mathbf{Com_{t,1}} \vee \mathbf{Com}_{\mathbf{0},q}$
- The **M**-variety **Com** is $\vee_{t,q}\mathbf{Com}_{t,q}$ or $\vee_{t\geq 0}\mathbf{Com_{t,1}} \vee \vee_{q>1}\mathbf{Com_{0,q}}$.

2. Preliminaries

We refer the reader to [5, 8, 10] for terms not explicitly defined here.

*2.1. Varieties V*W*

Let M and N be monoids. It is convenient to write M additively, without however assuming that M is commutative. In particular, we denote by 0 (respectively 1) the unit element in M (respectively N). A left action of N on M is a function $(n, m) \mapsto n \bullet m$ from $N \times M$ into M satisfying the following conditions

$$n \cdot (m + m') = n \cdot m + n \cdot m'$$
$$n \cdot (n' \cdot m) = (nn') \cdot m$$
$$1 \cdot m = m \text{ for all } m \in M$$
$$n \cdot 0 = 0 \text{ for all } n \in N$$

for all $m, m' \in M$ and $n, n' \in N$. Given a left action of N on M, we define the semidirect product $M * N$ as follows. The elements of $M * N$ are pairs (m, n) with $m \in M$ and $n \in N$. Multiplication is given by the formula

$$(m,n)(m', n') = (m + n \bullet m', nn') .$$

The multiplication in $M * N$ is associative. Thus $M * N$ is a monoid with (0,1) as unit element.

We consider the set $M_N \times N$ where M^N is for the set of all functions $f : N \rightarrow N$. The wreath product is then $M \circ N$ with multiplication defined by the formula

$$(f, n)(g, n') = (h, nn')$$

with $h \in M^N$ given by $n''h = n''f + (n''n)g$. The associativity of the multiplication in $M \circ N$ may be verified by a simple computation. If we define the left action of N on M^N by setting $n''(n \bullet g) = (n''n)g$ for all $g \in M^N$ and all $n'' \in N$, we find that $h = f + n \bullet g$ and thus the wreath product $M \circ N$ is a semidirect product $M^N * N$. Conversely, we can show that any semidirect product $M * N$ is isomorphic to a submonoid of $M \circ N$.

Given **M**-varieties **V** and **W,** we denote by $\mathbf{V} * \mathbf{W}$ the **M**-variety generated by all semidirect products $M * N$ with $M \in \mathbf{V}$, $N \in \mathbf{W}$ and with any left action of N on M. This is equivalent to the **M**-variety generated by all wreath products $M \circ N$ with $M \in \mathbf{V}$ and $N \in \mathbf{W}$. The semidirect product $\mathbf{V} * \mathbf{W}$ is associative.

We end this section with some terminology and well-known results related to equational descriptions of **M**-varieties.

2.2. Varieties defined by equations

Let Σ^* be the free monoid generated by the infinite sequence of letters $x_1, x_2, \ldots$. Given $u, v \in \Sigma^*$ and given a monoid M we say that M satisfies the equation $u = v$ (or that the equation $u = v$ holds in M) if $u\varphi = v\varphi$ for every morphism $\varphi : \Sigma^* \rightarrow M$ of monoids. For a fixed pair (u, v), let $\mathbf{V}(u, v)$ be the family of all monoids satisfying the equation $u = v$. The family $\mathbf{V}(u, v)$ is an **M** -variety.

Given a sequence of pairs $(u_i, v_i) \in \Sigma^* \times \Sigma^*$, $i \geq 1$, we may consider the two **M** –varieties

$$\mathbf{V}' = \bigcap_{i \geq 1} \mathbf{V}(u_i, v_i)$$

$$\mathbf{V}'' = \bigcup_{j \geq 1} \bigcap_{i \geq j} \mathbf{V}(u_i, v_i) .$$

A monoid M is in $\mathbf{V}'$ if it satisfies all the equations $u_i = v_i$. We say that $\mathbf{V}'$ is defined by the equations $u_i = v_i$, $i \geq 1$. A monoid M is in $\mathbf{V}''$ if it satisfies the equations $u_i = v_i$ for all i sufficiently large. We say that $\mathbf{V}''$ is ultimately defined by the equations $u_i = v_i$, $i \geq 1$. Every non-empty **M** -variety **V** is ultimately defined by a sequence of equations and every **M** -variety generated by a single monoid is equational [6].

In this paper, we are interested in the problem of determining equations of **M** -varieties of the form $\mathbf{V} * \mathbf{W}$ knowing equations on **V** and **W.** Very little is known about this problem because it is not possible to adapt the results of varieties of groups [9].

3. A congruence description of J1 * $\mathbf{Com}_{t,q}$

If A is a finite alphabet, then A^+ denotes the free semigroup on A, that is, the set of all strings (or words) made from letters of A. If an empty word (denoted by 1) is adjoined, we obtain A^*, the free monoid on A. A language in a free monoid A^* is any subset of A^*. The syntactic monoid of L, denoted $M(L)$, is the quotient of A^* by the syntactic congruence $\sim_L$ defined by $u \sim_L v$ if and only if for all $x , y \in A^*$, $xuy \in L$ if and only if $xvy \in L$.

We write $|u|_a$ for the number of times the letter a appears in the word $u \in A^*$, and we write ua for the set of letters in u. For any $t \geq 0$, $q \geq 1$, we define on A^* the congruence $\sim_{t,q}$ by $u \sim_{t,q} v$ if and only if for all $a \in A$, $|u|_a = |v|_a$ or $|u|_a , |v|_a \geq t$ and $|u|_a \equiv |v|_a \pmod{q}$. Note the following special cases.

- For all $u, v, \in A^*$, $u \sim_{0,1} v$.
- $u \sim_{1,1} v$ if and only if $ua = va$.

- $u \sim_{t,1} v$ if and only if for all $a \in A$, $|u|_a = |v|_a$ or $|u|_a , |v|_a \geq t$.

- $u \sim_{0,q} v$ if and only if for all $a \in A$, $|u|_a \equiv |v|_a \pmod q$.

Also note that $\sim_{t,q} \subset \sim_{t', q'}$ if and only if $t' \leq t$ and q' divides q.

Unless otherwise specified, any congruence we discuss has finite index and every non-free monoid is finite.

Lemma 3.1 ([10]). *The syntactic monoid of a language $L \subseteq A^*$ belongs to $Com_{t,q}$ if and only if L is in the boolean closure of the languages*

$$\{u \in A^* \mid |u|_a = i\}$$
$$u \in A^* \mid |u|_a \equiv j \pmod q)\}$$

for all $0 \leq i \leq t$, $0 \leq j \leq q$ *and* $a \in A$. *In* terms *of congruences, the syntactic monoid of a language $L \subseteq A^*$ belongs to* $\mathbf{Com_{t,q}}$ *if and only if L is a union of classes modulo* $\sim_{t,q}$.

The previous lemma describes languages L satisfying $M(L) \in \mathbf{Com_{t,q}}$. For languages L satisfying $M(L) \in \mathbf{J_1} * \mathbf{Com_{t,q}}$, we have the following lemma.

Lemma 3.2 ([10]). *The syntactic monoid of a language $L \subseteq A^*$ belongs to* $\mathbf{J_1} * \mathbf{Com_{t,q}}$ *if and only if L is in the boolean closure of the languages of the form K or KaA* satisfying* $M(K) \in \mathbf{Com_{t,q}}$ *and* $a \in$ A.

We now express this last lemma in terms of congruences. Let $A^*/\sim_{t,q}$ be the set of classes modulo $\sim_{t,q}$ and let $\sigma_{t,q}$: $A^* \rightarrow (A^*/\sim_{t,q} \times A)^*$ be the function defined by

$$1\sigma_{t,q} = 1$$
$$(a_1 \dots a_i)\sigma_{t,q} = ([1]_{\sim t,q} a_1)([a_1]_{\sim t,q} , a_2) \dots ([a_1 \dots a_{i-1}]_{\sim t,q} , a_i) .$$

In Eilenberg's terminology, this function is a sequential function realized by the sequential machine $\mathcal{M}_{t,q} = (A^*/\sim_{t,q} , A, \delta_{t,q} , \lambda_{t,q} , [1]_{\sim t,q})$ where A is an alphabet, $A^*/\sim_{t,q}$ is the set of states and $[1]_{\sim t,q}$ the initial state. The transition function $\delta_{t,q}$ and the output function $\lambda_{t,q}$ are pictured in the following diagram, where $w \in A^*$ and $a \in A$.

$$[w]_{\sim t,q} \xrightarrow{a/([w]_{\sim t,q} ,a)} [wa]_{\sim t,q} .$$

Thus, if the machine is in state $[w]_{\sim t,q}$ and reads an a, it moves to state $[wa]_{\sim t,q}$ and prints the letter $([w]\sim_{t,q} a)$.

On A^*, we define an equivalence relation $\approx_{t,q}$ by $u \approx_{t,q} v$ if $u\sigma_{t,q}\alpha = v\sigma_{t,q}\alpha$ and $u \sim_{t,q} v$. The equivalence $\approx_{t,q}$ is in fact a congruence and $\approx_{t,q} \subseteq \sim_{t,q}$.

Lemma 3.3. *The syntactic monoid of a language $L \subseteq A^*$ belongs to* $\mathbf{J_1} * \mathbf{Com_{t,q}}$ if *and only if L is a union of classes modulo* $\approx_{t,q}$. *As a consequence, for any alphabet A, the monoid* $A^*/ \approx_{t,q}$ *belongs to* $\mathbf{J_1} * \mathbf{Com_{t,q}}$.

Proof. Assume that $M(L) \in \mathbf{J_1} * \mathbf{Com_{t,q}}$. By Lemma 3.2, we may assume that $L = K$ or KaA^* with $M(K) \in \mathbf{Com_{t,q}}$ and $a \in A$. If $L = K$, then by Lemma 3.1, K is a union of classes modulo $\sim_{t,q}$ and the result follows since $\approx_{t,q} \subseteq \sim_{t,q}$. If $L = KaA^*$, we show that L is a union of classes modulo $\approx_{t,q}$. Let $u \approx_{t,q} v$ with $u \in L$. Then $u = u_1 a u_2$ for some $u_1 \in K$ and $u_2 \in A^*$. Hence $([u_1]_{\sim t,q}\ a)$ is a letter of $u\sigma_{t,q}$ and also a letter of $v\sigma_{t,q}$ since $u\sigma_{t,q}\alpha = v\sigma_{t,q}\alpha$. Therefore, there exist $v_1, v_2 \in A^*$ such that $v_1 \sim_{t,q} u_1$ *and* $v = v_1 a v_2$. Since K is a union of classes modulo, $\sim_{t,q}$, u1 $\sim_{t,q} v_1$ and $u_1 \in K$ imply $v_1 \in K$ and hence $v \in KaA^*$.

Conversely, the transition monoid $M(\sigma_{t,q})$ of the automaton $(A^*/\sim_{t,q}, A, \delta_{t,q})$ is in $\mathbf{Com}_{t,q}$. To see this, let a, $b \in A$ and $w \in A$. We have $\delta_{t,q}([w]_{\sim t,q}, ab) = \delta_{t,q}([w]_{\sim t,q}, ba)$ and $\delta_{t,q}([w]_{\sim t,q}, a^t) = \delta_{t,q}([w]_{\sim t,q}, a^{t+q})$. Hence $M(\sigma_{t,q})$ satisfies $xy = yx$ and $x^t = x^{t+q}$. Now, let $\alpha : (A^*/\sim_{t,q} \times A)^* \to (A^*/\sim_{t,q} \times A)^*/\sim_{1,1}$ be the canonical morphism. The monoid $(A^*/\sim_{t,q} \times A)^*/\sim_{1,1}$ is in $\mathbf{J_1}$. If L is a union of classes modulo $\approx_{t,q}$, then there exists a subset X of $(A^*/\sim_{t,q} \times A)^*/\sim_{1,1}$ such that $L = X\alpha^{-1}\sigma_{t,q}^{-1}$. It follows that $M(L)$ divides $M(X\alpha^{-1})\circ M(\sigma_{t,q})$ [10]. Since the language $X\alpha^{-1}$ is recognized by $(A^*/\sim_{t,q} \times A)^*/\sim_{1,1}$ the monoid $M(X\alpha^{-1})$ is also in $\mathbf{J_1}$. We therefore conclude that $M(L) \in \mathbf{J_1} * \mathbf{Com}_{t,q}$.

Let $\pi : A^* \to A^*/\approx_{t,q}$ be the canonical morphism. The monoid $A^*/\approx_{t,q}$ divides the direct product of all syntactic monoids of the form $M(([w]_{\approx t,q})\pi^{-1})$ where $w \in A^*$. But $([w]_{\approx t,q})\pi^{-1}$ is a class modulo $\approx_{t,q}$ and hence $M(([w]_{\approx t,q})\pi^{-1})$ belongs to $\mathbf{J_1} * \mathbf{Com}_{t,q}$. Therefore, $A^*/\approx_{t,q} \in \mathbf{J_1} * \mathbf{Com}_{t,q}$

In the next section, we give equations for the product $\mathbf{J_1} * \mathbf{Com}_{t,q}$. If two words u and v form an equation $u = v$ for that product, then $u \approx_{t,q} v$.

4. An equational description of J_1*$Com_{t,q}$

In this section, we give an equational description of $J_1 * Com_{t,q}$. In order to do this, we use a theorem on graphs due to Simon.

A directed graph $\mathcal{G}$ is given by two sets V and E where $E \subseteq V \times V$. The elements of V are called the vertices of $\mathcal{G}$ while the elements of E are called the edges of $\mathcal{G}$. For each edge $e = (v_1, v_2) \in E$, the vertices v_1 and v_2 are called the start and the end vertices of e. The edges e_1 and e_2 are called consecutive if the end vertex of e_1 is the start vertex of e_2. A sequence $p = e_1 \ldots e_i$ is a path if e_j and e_{j+1} are consecutive for all $1 \leq j < i$. The integer i is called the length of the path. Clearly each edge is a path of length 1. The start vertex of p is the start vertex of e_1 and the end vertex of p is the end vertex of e_i. If the start vertex of p, say v, is the end vertex of p, then p is called a loop about v. If $p = e_1 \ldots e_i$ and $p' = e'_1 \ldots e'_j$ are consecutive paths (that is, if e_i and e'_1 are consecutive), then $pp' = e_1 \ldots e_i e'_1 \ldots e'_j$ is a path. An equivalence relation $\equiv$ on the set of all paths in a directed graph is called a *congruence* if it satisfies the following two conditions:

- If $p \equiv p'$, then p and p' are coterminal (that is, the start vertex of p is the start vertex of p', and the end vertex of p is the end vergex of p').
- If $p = p'$, $p'' = p'''$ and p and p'' are consecutive, then $pp'' = pip'''$

Lemma 4.1. (Simon [5]) *Let $\equiv$ be the smallest congruence relation on the set of all paths in a directed graph satisfying $p \equiv p^2$ and $pp' \equiv p'p$ for any two loops p and p' about the same vertex. Then any two coterminal paths traversing the same set of edges (without regard to order and multiplicity) are $\equiv$-equivalent.*

4.1. Equations on $J_1 * Com_{t,1}$

We now define a finite sequence of equations for $\mathbf{J_1} * \mathbf{Com}_{t,1}$.

Theorem 4.1.1. *The* **M** *-variety* $\mathbf{J_1} * \mathbf{Com}_{t,1}$ *is defined by the equation*

(1) $$Xu_1 \ldots xu_t x = xu_1 \ldots xu_t x^2$$

together with all the equations of the form

(2) $$x_1u_1 \ldots x_{2t}u_{2t}xy = x_1u_1 \ldots x_{2t}u_{2t}yx$$

where $x_1, \ldots, x_{2t}$ is a list of t x's and t y's.

Proof. If $M \in \mathbf{J_1}$ and $N \in \mathbf{Com}_{t,1}$, then $M * N$ satisfies Equation (1) and all the equations of the form (2). To see this, if we replace x by $(m, n) \in M * N$, y by $(m', n') \in M * N$ and u_i by $(m_i, n_i) \in M * N$ for $1 \le i \le 2t$, then for some $m'' \in M$,

$$\begin{aligned} xu_1 \ldots xu_t x &= (m'', n^t n_1 \ldots n_t)(m, n) \\ &= (m'' + (n^t n_1 \ldots n_t) \cdot m, n^t n_1 \ldots n_t) \\ &= (m'' + (n^t n_1 \ldots n_t) \cdot m + (n^t n_1 \ldots n_t) \cdot m, n^t n_1 \ldots n_t) \\ &= (m'', n^t n_1 \ldots n_t)(m + n \cdot m, n^2) \\ &= xu_1 \ldots xu_t x^2 \end{aligned}$$

and moreover, if $x_1, \ldots, x_{2t}$ is a list of t x's and t y's, then for some $m''' \in M$,

$$\begin{aligned} x_1 u_1 \ldots x_{2t} u_{2t} xy &= (m''', n^t (n')^t n_1 \ldots n_{2t})(m + n \cdot m', nn') \\ &= (m''', n^t (n')^t n_1 \ldots n_{2t})(m' + n' \cdot m, n'n) \\ &= x_1 u_1 \ldots x_{2t} u_{2t} yx \; . \end{aligned}$$

Conversely, let $: A^* \rightarrow M$ be a surjective morphism. We also denote by φ the (nuclear) congruence on A^* associated with φ and defined by $u\varphi v$ if and only if $u\varphi = v\varphi$. We prove that if the monoid M satisfies all the equations in the statement of the theorem, then $M \in \mathbf{J_1} * \mathbf{Com}_{t,1}$. We do this by showing that $\approx_{t,1} \subseteq \varphi$ which implies $M = A^*/\varphi \prec A^*/\approx_{t,1}$. But since $A^*/\approx_{t,1}$ belongs to $\mathbf{J_1} * \mathbf{Com}_{t,1}$ by Lemma 3.3, the membership of M in $\mathbf{J_1} * \mathbf{Com}_{t,1}$ follows.

In order to show that $\approx_{t,1} \subseteq \varphi$, we define a directed graph $\mathcal{G}$ as follows: the set of vertices of $\mathcal{G}$ is $A^*/\sim_{t,1}$ and the set of edges is the set of 3-tuples of the form $([w]_{\sim t,1}, a, [wa]_{\sim t,1})$ where $a \in A$. Thus $\mathcal{G}$ is just the graph associated to the sequential machine $\mathcal{M}_{t,1}$. To any path

$$P = ([w_0]_{\sim t,1}, a_1, [w_1]_{\sim t,1}) \ldots ([w_{i-1}]_{\sim t,1}, a_i, [w_i]_{\sim t,1})$$

of $\mathcal{G}$, we associate the word $\bar{p} = a_1 \ldots a_i$ of A^*. Define a congruence relation $\equiv$ on the set of paths in $\mathcal{G}$ by $p \equiv p'$ if

- p and p' are coterminal.
- For all paths p'' from the vertex $[1]_{\sim t,1}$ to the start vertex of p and p', $(\bar{p}''\bar{p})\varphi = (\bar{p}''\bar{p}')\varphi$.

Let p and p' be two loops about the same vertex $[w]_{\sim t,1}$, or

$p = ([w]_{\sim t,1}, a_1, [wa_1]_{\sim t,1}) \ldots ([wa_1 \ldots a_{i-1}]_{\sim t,1}, a_i, [wa_1 \ldots a_i]_{\sim t,1})$

$p' = ([w]_{\sim t,1}, b_1, [wb_1]_{\sim t,1}) \ldots ([wb_1 \ldots b_{j-1}]_{\sim t,1}, b_j, [wb_1 \ldots b_j]_{\sim t,1})$

where $wa_1 \ldots a_i \sim_{t,1} w \sim_{t,1} wb_1 \ldots b_j$. We show the following two claims:

Claim (1): $p \equiv p^2$, and

Claim (2): $pp' \equiv p'p$.

Lemma 4.1 implies that any two coterminal paths traversing the same set of edges are $\equiv$-equivalent.

To any word $u = a_1 \ldots a_i$ of A^*, we can associate the path

$$p_u = ([1]_{\sim t,1}, a_1, [a_1]_{\sim t,1})([a_1]_{\sim t,1}, a_2, [a_1 a_2]_{\sim t,1}) \ldots ([a_1 \ldots a_{i-1}]_{\sim t,1}, a_i, [a_1 \ldots a_i]_{\sim t,1}) \; .$$

Intuitively, p_u is the path obtained by reading u in the sequential machine $\mathcal{M}_{t,1}$. Now if $u \approx_{t,1} v$ (and hence $u \sim_{t,1} v$), then $u\sigma_{t,1}\alpha = v\sigma_{t,1}\alpha$. Hence p_u and p_v are coterminal paths (with start vertex $[1]_{\sim t,1}$ and end vertex $[u]_{\sim t,1} = [v]_{\sim t,1}$) traversing the same set of edges. Hence, by Lemma 4.1, $p_u = p_v$ and $\bar{p}_u\varphi = \bar{p}_v\varphi$ Therefore, $u\varphi = v\varphi$ and hence $\approx_{t,1} \subseteq \varphi$.

Let us now prove *Claim* (1) and *Claim* (2). It is easy to see that $u \sim_{t,1} uv$ if and only if for all $a \in A$, $|v|_a = 0$ or $|u|_a \geq t$, $|uv|_a > t$. It then follows that for all $a \in A$, $|a_1 \dots a_i|_a = 0$ or $|w|_a \geq t$, $|wa_1 \dots a_i|_a > t$, and also for all $a \in A$, $|b_1 \dots b_j|_a = 0$ or $|w|_a \geq t$, $|wb_1 \dots b_j|_a > t$.

Proof of Claim (1). The condition $p \equiv p^2$ follows by showing that $(w\bar{p})\varphi = (w\bar{p}^2)\varphi$. More precisely,

$$\begin{aligned}
(w\bar{p}^2)\varphi &= (wa_1 \dots a_i a_1 \dots a_i)\varphi \\
&= (wa_1 \dots a_{i-1} a_i^2 a_1 \dots a_{i-1})\varphi \quad \text{(using instances of (2))} \\
&= (wa_1 \dots a_i a_1 \dots a_{i-1})\varphi \quad \text{(using (1))} \\
&= (wa_1 \dots a_{i-2} a_{i-1}^2 a_i a_1 \dots a_{i-2})\varphi \quad \text{(using instances of (2))} \\
&= (wa_1 \dots a_i a_1 \dots a_{i-2})\varphi \quad \text{(using (1))} \\
&= \dots = \\
&= (wa_1 \dots a_i)\varphi \quad \text{(using (1) and instances of (2))} \\
&= (w\bar{p})\varphi \,.
\end{aligned}$$

Proof of *Claim* (2). The condition $pp' \equiv p'p$ follows by showing that $(w\bar{p}\,\bar{p}')\varphi = (w\bar{p}'\bar{p})\varphi$. More precisely,

$$\begin{aligned}
(w\bar{p}\,\bar{p}')\varphi &= (wa_1 \dots a_i b_1 \dots b_j)\varphi \\
&= (wb_1 a_1 \dots a_i b_2 \dots b_j)\varphi \quad \text{(using instances of (2))} \\
&= (wb_1 b_2 a_1 \dots a_i b_3 \dots b_j)\varphi \quad \text{(using instances of (2))} \\
&= \dots = \\
&= (wb_1 \dots b_j a_1 \dots a_i)\varphi \quad \text{(using instances of (2))} \\
&= (w\bar{p}'\bar{p})\varphi \,.
\end{aligned}$$

Theorem 4.1 implies a complete sequence of equations for the **M** -variety $\mathbf{J_1} * (\mathbf{Com} \cap \mathbf{A})$. Our preceding result generalizes a result of Pin.

Theorem 4.1.2. ([12]) *The M-variety* $\mathbf{J_1} * \mathbf{J_1}$ *is defined by the equations* $xux = xux^2$ *and* $xuyvxy = xuyvyx$.

Proof. By Theorem 4.1.1 and the fact that $\mathbf{J_1} * \mathbf{J_1} = \mathbf{J_1} * \mathbf{Com_{1,1}}$

4.2. Equations on $J_1 * Com_{0,q}$

In this subsection, we define a sequence of equations for $\mathbf{J_1} * \mathbf{Com_{0,q}}$.

Let $q \geq 1$ and $r \geq 1$. Consider a circular list of at least 1 and at most q^r distinct strings of r q-ary digits such that consecutive strings $d_1 \dots d_r$ and $d_1' \dots d_r'$ are so that there exists $1 \leq i \leq r$ satisfying $d_i' \equiv d_i + 1 \pmod q$ and $d_j' = d_j$ for $j \neq i$, and such that the last string differs from the first string in the same manner. For example, 000, 001, 011, 111, 101, 100, 110, 010 is such a circular list for $q = 2$ and $r = 3$ and is also called a *Gray code of length* 3. Such lists can be relabeled as follows: a string of r q-ary digits $d_1 \dots d_r$ is relabeled by x_i if the following string in the list is $d_1 \dots d_{i-1} d_i' d_{i+1} \dots d_r$ where $d_i' \equiv d_i + 1 \pmod q$. In the example above, 000, 001, 011, 111, 101, 100, 110, 010 can be relabeled as $x_3, x_2, x_1, x_2, x_3, x_2, x_1, x_2$. Let χ_q^r denote the finite set of such relabeled lists. In the example above, $x_3, x_2, x_1, x_2, x_3, x_2, x_1, x_2$ is a list in χ_2^3. Note that every x_j in a list in χ_q^r occurs a multiple of q times in the list. We have $\chi_q^1 \subseteq \chi_q^2 \subseteq \chi_q^3 \subseteq \dots$.

We can view the construction of a circular list of length q^r in χ_q^r as a graph-theoretic problem. Let $V(\mathcal{G})$ be the set $\{0,1, \dots, q-1\}^r$ of q-ary r -strings, and put an edge from v to v' if $v = d_1 \dots d_r$ and $v' = d_r'$ are so that there exists $1 \leq i \leq r$ satisfying $d_i' \equiv d_i + 1 \quad d,+1$ (mod q) and $d_j' = d_j$ for $j \neq i$. A circular list of length q^r in χ_q^r is, in effect, a Hamilton circuit of the graph $\mathcal{G}$.

Circular lists in χ_q^r of length q^r always exist. To see this, we fix q and we use induction on r and consider the graph $\mathcal{G}_r$ in which a Hamilton circuit corresponds to a circular list of length q^r in χ_q^r, as described above. If $r = 1$, the list $0, 1, \ldots, q - 1$ is a circular list of length q which is relabeled as $x_1, \ldots, x_1$ (q times) in χ_q^1 and corresponds to the Hamilton circuit $0,1, \ldots, q - 1,0$ of the graph $\mathcal{G}_1$. Define the function *pred* from $\{0,1,\ldots, q - 1\}$ into $\{0,1,\ldots, q - 1\}$ by $pred(0) = q - 1$ and $pred(i) = i - 1$ for $i \geq 1$. Call $pred(i)$ the predecessor of i. Let $V(\mathcal{H}_i)$ consist of (r +1)-strings with i in the 1st digit and let $E(\mathcal{H}_i)$ consist of the edges of $\mathcal{G}_{r+1}$ connecting vertices in $V(\mathcal{H}_i)$. The function from $\mathcal{G}_{r+1}$ to $\mathcal{G}_r$ which simply leaves off the first coordinate determines an isomorphism from $\mathcal{H}_i$ onto $\mathcal{G}_r$ From a Hamilton circuit for $\mathcal{G}_r$ which looks like $0\bar{0}, \ldots, (q - 1)\bar{0}, 0\bar{0}$ where $\bar{0}$ denotes the $(r - 1)$-string $0 \ldots 0$, form a Hamilton circuit for $\mathcal{G}_{r+1}$ as follows. First, make q copies of the circular list $0\bar{0}, \ldots, (q-1)\bar{0}$ of length q^r called *Copy*(0), *Copy*(1), , *Copy*(q - 1). *Copy*(0) is just $0\bar{0}, \ldots, (q - 1)\bar{0}$ and *Copy*(i) is just *Copy*(i - 1) where the 1st digit of each r-string in the list has been replaced by its predecessor. So *Copy*(1) looks like $(q-1)\bar{0}, \ldots, (q-2)\bar{0}, \ldots,$ and *Copy*(q -1) like $1\bar{0}, \ldots, 0\bar{0}$. Now, starting with the $(r + 1)$-string $00\bar{0}$ in $\mathcal{H}_0$ follow *Copy*(0) in $\mathcal{H}_0$ until you reach $0(q - 1)\bar{0}$. Then take the edge from $0(q-1)\bar{0}$ to $1(q - 1)\bar{0}$ which exists. Then follow *Copy*(1) in $\mathcal{H}_1$ until you reach $1(q - 2)\bar{0}$. Then take the edge from $1(q - 2)\bar{0}$ to $2(q - 2)\bar{0}, \ldots$. After following *Copy*(q -1) in $\mathcal{H}_{q-1}$ until you reach $(q - 1)0\bar{0}$, take the edge from $(q - 1)0\bar{0}$ to $00\bar{0}$, the starting point. Every vertex in $\mathcal{H}_i$ will have been visited exactly once.

Since $V(\mathcal{G}_{r+1}) = \bigcup_{i \geq 0} V(\mathcal{H}_i)$, the path

$00\bar{0}, \ldots, 0(q-1)\bar{0}, 1(q-1)\bar{0}, \ldots, 1(q-2)\bar{0}, \ldots, (q-1)1\bar{0}, \ldots, (q-1)\, 0\bar{0}, 00\bar{0}$

is a Hamilton circuit of $\mathcal{G}_{r+1}$ and gives a circular list of length q^{r+1}.

Definition 4.2.1. Let $q \geq 1$ and $r \geq 1$. $\mathcal{E}_{0,q}^r$ is the finite sequence of all the equations of the form

$$y_1 z_1^q \ldots y_i z_i^q y_{i+1} = (y_1 z_i^q y_{i+1})^2$$

where $y_1, \ldots, y_i, y_{i+1}$ is a list in $\mathcal{X}_q^r$.

For example, the equation $x = x^2$ where

$$x = x_3 z_1^2 x_2 z_2^2 x_1 z_3^2 x_2 z_4^2 x_3 z_5^2 x_2 z_6^2 x_1 z_7^2 x_2$$

belongs to the sequence $\mathcal{E}_{0,2}^2$.

The sequence $\mathcal{E}_{0,1}^r$ is equivalent to the equation $x = x^2$. The sequence $\mathcal{E}_{0,q}^1$ is equivalent to the equation $xz_1^q \ldots xz_{q-1}^q x = (xz_1^q \ldots xz_{q-1}^q x)^2$ which has as a particular instance the equation $x^q = x^{2q}$. Every equation in the sequence $\mathcal{E}_{0,q}^r$ is also in the sequence $\mathcal{E}_{0,q}^{r+1}$ for $r \geq 1$.

Lemma 4.2.2. *Let A be an alphabet of r letters where $r \geq 1$. A monoid M generated by A belongs to $J_1 * Com_{0,q}$ if and only if M satisfies the equation*

$$x^q y^q = y^q x^q$$

together with all the equations

$$\mathcal{E}_{0,q}^r$$

Proof. If $M \in J_1$ and $N \in Com_{0,q}$ are monoids generated by an alphabet A of r letters where $r \geq 1$, then $M * N$ satisfies $x^q y^q = y^q x^q$ and $\mathcal{E}_{0,q}^r$. To see this, if we replace x by $(m_1, n_1) \in M * N$ and y by $(m_2, n_2) \in M * N$, then $x^q =$

$= (m_1, n_1)^q = (m,1)$ and $y^q = (m_2, n_2)^q = (m',1)$ for some $m, m' \in M$ since $1 = x^q$ holds in N. Therefore, $M * N$ satisfies the equation $x^q y^q = y^q x^q$, since $x^q y^q = (m, 1)(m', 1) = (m + m', 1) = (m' + m, 1) = (m', 1)(m, 1) = y^q x^q$. Now, let $y_1, \ldots, y_i, y_{i+1}$ be a list in $\mathcal{X}_q^r$. If we replace y_j by $(m_j, n_i) \in M * N$ for $1 \leq j \leq i+1$ and z_j^q by $(m_j',1) \in M * N$ for $1 \leq j \leq I$, then for some $m \in M$

$$(y_1 z_1^q \ldots y_i z_i^q y_{i+1})^2 =$$
$$((m_1, n_1)(m_1', 1) \ldots (m_i, n_i)(m_i', 1)(m_{i+1}, n_{i+1}))^2 = (m, n_1 \ldots n_i n_{i+1})^2 =$$
$$(m, 1)^2 \text{ (since every } n_j \text{ in } n_1 \ldots n_i n_{i+1} \text{ occurs a multiple of } q \text{ times)} =$$
$$(m, 1) =$$
$$y_1 z_1^q \ldots y_i z_i^q y_{i+1} .$$

The proof of the converse is similar to that of Theorem 4.1.1 except that it deals with the congruences $\sim_{0,q}$ and $\approx_{0,q}$ instead of the congruences $\sim_{t,1}$ and $\approx_{t,1}$. Let A be an alphabet of r letters where $r \geq 1$. It is easy to see that $u \sim_{0,q} uv$ if and only if for all $a \in A$, $|v|_a \equiv$ (mod q). Using the same notation as in the proof of Theorem 4.1.1, it follows that $|a_1 \ldots a_i|_a \equiv 0$ (mod q) and $|b_1 \ldots b_j|_a \equiv 0$ (mod q) for all $a \in A$. If $A = \{c_1, \ldots, c_r\}$, strings over A like $a_1 \ldots a_i$ and $b_1 \ldots bj$ can be viewed as loops in the graph $\mathcal{G}_r$ ($\mathcal{G}_r$ is explained at the beginning of this subsection). For instance, if $q = 2$, the string $c_3c_2c_1c_2c_3c_2c_1c_2$ over $A = \{c_1, c_2, c_3\}$ can be viewed as the loop 000, 001, 011, 111, 101, 100, 110, 010, 000 in $\mathcal{G}_3$. A string u over A satisfying $|u|_a \equiv 0$ (mod q) for all $a \in A$, can be viewed as a loop about the r-string 0 ... 0 where the i^{th} digit in the r -strings is used to record the number (modulo q) of c_i's in the string u.

Claim 1.
The condition $p = p^2$ follows by showing that $(w\bar{p})\varphi = (w\bar{p}^2)\varphi$. Here, we can show that $(\bar{p})\varphi = (\bar{p}^2)\varphi$ (and therefore $(\bar{p})\varphi = (\bar{p}^q)\varphi)$. The string $\bar{p}$ has the property P that "each of its letters occurs a multiple of q times". A string x over A with the property P can be factorized as follows: let x_1 be the smallest nonempty prefix of x with the property P, let x_2 be the smallest nonempty prefix of $u \backslash u_1$ with the property P,... . So x is the concatenation of factors with the property P. Factors in x are either of type (1), that is, a^q for some $a \in A$, or of type (2), that is, $y_1z_1 \ldots y_kz_ky_{k+1}$ where $y_1, \ldots, y_k, y_{k+1} \in \chi_q^r$ and where the z's have the property P. Since the z's have the property P, they can be factorized as above and the process can be repeated. The most elementary factors of type (2) look like $y_1z_1^q \ldots y_kz_k^q y_{k+1}$ where $y_1, \ldots, y_k, y_{k+1} \in \chi_q^r$ and where the z's are either empty or of type (1). In such situations,

$$(y1z_1^q \ldots ykz_k^q yk+1)\varphi =$$
$$((y_1z_1^q \ldots y_kz_k^q y_{k+1})^2)\varphi \text{ (using an instance of } \varepsilon_{0,q}^r)$$

and therefore $(y_1z_1^q \ldots y_kz_k^q y_{k+1})\varphi = ((y_1z_1^q \ldots. y_kz_k^q y_{k+1})^q)\varphi$. The string $(\bar{p})\varphi$ can have subfactors of the form $(x_1^q \ldots x_1^q)\varphi$ and in such situations,

$$(x_1^q \ldots x_1^q)\varphi =$$
$$(x_1^{2q} \ldots x_1^{2q})\varphi \text{ (using } x^q = x^{2q} \text{ which is an instance of } \varepsilon_{0,q}^1) =$$
$$((x_1^q \ldots x_1^q)^2)\varphi \text{ (using } x^q y^q = y^q x^q \text{ several times)}$$

and therefore, $(x_1^q \ldots x_1^q)\varphi = ((x_1^q \ldots x_1^q)^q)\varphi$. It is then easy to see that $(\bar{p})\varphi$ isof the form $x\varphi$ where x is the concatenation of factors of the form y^q. And as above, $(\bar{p})\varphi = (\bar{p}^2)\varphi$.

Claim 2.
The condition $pp' \equiv p'p$ follows by $(\bar{p}\bar{p}')\varphi = (\bar{p})\varphi(\bar{p}')\varphi = (\bar{p}^q)\varphi(\bar{p}'^q)\varphi = (\bar{p}^q\bar{p}'^q)\varphi = (\bar{p}'^q\bar{p})\varphi = (\bar{p}'\bar{p})\varphi$ (using $x^q y^q = y^q x^q$).

Theorem 4.2.3. *The* **M**-*variety* $\mathbf{J_1} * \mathbf{Com_{0,q}}$ *is defined by the equation*

$$x^q y^q = y^q x^q$$

together with all the equations

$$\varepsilon^{r}_{0,q},\ r \geq 1\ .$$

Proof. By Lemma 4.2.2.
Using Schützenberger's notation as explained in [10], we get the following theorem.

Theorem 4.2.4. *The* **M** -variety $\mathbf{J_1} * (\mathbf{Com} \cap \mathbf{G})$ *is defined by the equation*

$$x^{w}y^{w} = y^{w}x^{w}$$

together with all the equations

$$\varepsilon^{r}_{0,w},\ r \geq 1.$$

Proof. By Theorem 4.2.3.

4.3. Equations on $J_1 * Com_{t,q}$

In this subsection, we define a sequence of equations for $\mathbf{J_1} * \mathbf{Com_{t,q}}$. Let us first define recursively what we mean by "x is of the form (*)".

Definition 4.3.1. Let $q \geq 1$ and $r \geq 1$ be fixed.

Basis. If there exists a list $y_1, \ldots, y_i, y_{i+1}$ in χ^{r}_{q} and $z_1, \ldots, z_i$ (that may be empty) such that $x = y_1 z_1^{q} \ldots y_i z_i^{q} y_{i+1}$, then we say that x is of the form (*).

Recursive step. If there exists a list $y_1, \ldots, y_i, y_{i+1}$ in χ^{r}_{q} and $z_1, \ldots, z_i$ (that may be empty or of the form (*)) such that $x = y_1 z_1^{q} \ldots y_i z_i^{q} y_{i+1}$, then we say that x is of the form (*).

Closure. x is of the form (*) only if it can be obtained from the basis by a finite number of applications of the recursive step.

Note that if x is of the form (*), it is built only from $x_1, \ldots, x_r$, the variables that build the lists in χ^{r}_{q}. Note also that if $q = 1$, x is of the form (*) if and only if x is one of $x_1, \ldots, x_r$.

Definition 4.3.2. Let $t \geq 0$, $q \geq 1$ and $r \geq 1$. $\mathcal{F}^{r}_{t,q}$ is the sequence of all the equations of the form

$$(3) \qquad u_1 v_1 \ldots u_{rt} v_{rt} x = u_1 v_1 \ldots u_{rt} v_{rt} x^2$$

where x is of the form (*) and where $u_1, \ldots, u_{rt}$ is a list of t x_1 's, … , t x_r's, together with all the equations of the form

$$(4) \qquad u_1 v_1 \ldots u_{rt} v_{rt} xy = u_1 v_1 \ldots u_{rt} v_{rt} yx$$

where x and y are of the form (*) and where $u_1, \ldots u_{rt}$ is a list of t x_1's, ... , t x_r's.

Note that every equation in the sequence $\mathcal{F}^{r}_{t,q}$ is also in the sequence $\mathcal{F}^{r+1}_{t,q}$ for $r \geq 1$.

Theorem 4.3.3. *The* **M**-*variety* $\mathbf{J_1} * \mathbf{Com_{t,1}}$ *is defined by all the equations*

$$\mathcal{F}^{r}_{t,1},\ r \geq 1\ .$$

Proof. $\mathcal{F}^{r}_{t,1}$ is the finite sequence of all the equations of the form

$$(5) \qquad u_1 v_1 \ldots u_{rt} v_{rt} x = u_1 v_1 \ldots u_{rt} v_{rt} x^2$$

where x is one of $x_1,...,x_r$ and where $u_1, \ldots, u_{rt}$ is a list of t x_1's,...,t x_r's, together with all the equations of the form

(6) $u1v_1 \ldots urtv_{rt}xy = u_1v_1 \ldots u_{rt}v_{rt}yx$

where x and y are among $x_1,...,x_r$ and where $u_1,...,u_{rt}$ is a list of t x_1 's, ... ,t x_r 's. We prove that the sequence $\mathcal{F}^r_{t,1}$, $r \geq 1$, is equivalent to the finite sequence of equations (described in Theorem 4.1) that define $\mathbf{J_1} * \mathbf{Com_{t,1}}$.

It is easy to see that Equation (1) is of the form (5) for r = 1 and all the equations of the form (2) are of the form (6) for r = 2.

So there remains to show that all the equations of the form (5) and (6) are deducible from Equation (1) and all the equations of the form (2). To see this, let $u_1, \ldots u_{rt}$ be a list of t x_1 's, ... , t x_r's and assume $x = x_i$ for some $1 \leq i \leq$ r. Then

$$u_1v_1 \ldots u_{rt}v_{rt}x_i = u_1v_1 \ldots u_{rt}v_{rt}x_i^2$$

(using Equation (1)) since x_i occurs t times in $u_1v_1 \ldots u_{rt}v_{rt}$. Now, assume $x = x_i$ and $y = x_j$ for some $1 \leq i, j \leq r$. Then

$$u_1v_1 \ldots u_{rt}v_{rt}x_ix_j = u_1v_1 \ldots u_{rt}v_{rt}x_jx_i$$

(using an instance of (2)) since x_i and x_j occur t times in $u_1v_1 \ldots u_{rt}v_{rt}$.

Theorem 4.3.4. *The* **M** -variety $\mathbf{J_1} * \mathbf{Com_{o,q}}$ *is defined by all the equations*

$$\mathcal{F}^r_{0,q}, \ r \geq 1$$

Proof. $\mathcal{F}^r_{0,q}$ is the sequence of all the equations of the form

(7) $x = x^2$

where x is of the form (*), together with all the equations of the form

(8) $xy = yx$

where x and y are of the form (*). We prove that the sequence $\mathcal{F}^r_{0,q}$, $r \geq 1$ is equivalent to the equation $x^qy^q = y^qx^q$ together with the sequence $\mathcal{E}^r_{0,q}$, $r \geq 1$.

It is easy to see that the equation $xq = x^{2q}$ is of the form (7) for r = 1 and hence deducible from $\mathcal{F}^1_{0,q}$ (take $x = (x_1)^q$ to get $(x_1)^q = x = x^2 = ((x_1)^q)^2 = (x_1)^{2q}$). The equation $x^qy^q = y^qx^q$ is of the form (8) for r = 2 and hence deducible from $\mathcal{F}^2_{0,q}$ (take $x = (x_1)^q$ and $y = (x_2)^q$ to get $(x_1)q(x_2)^q = xy = yx = (x_2)^q(x_1)^q$). Now, let $y_1,\ldots, y_i, y_{i+1}$ be a list in $\mathcal{X}^r_q$. We have

$$\begin{aligned} y_1z_1^q \ldots y_iz_i^qy_{i+1} &= y_1z_1^{2q} \ldots y_iz_i^{2q}y_{i+1} \text{ (using } x^q = x^{2q}) \\ &= y_1(z_1^q)^q \ldots y_i(z_i^q)^qy_{i+1} \text{ (using } x^q = x^{2q} \text{ several times)} \\ &= (y_1(z_1^q)^q \ldots y_i(z_i^q)^qy_{i+1})^2 \text{ (using the equation} \end{aligned}$$
$$y_1((x_{r+1})^q)^q \ldots y_i((x_{r+i})^q)^qy_{i+1} = (y_1((x_{r+1})^q)^q \ldots y_i((x_{r+i})^q)^qy_{i+1})^2$$
$$\text{in } \mathcal{F}^{r+i}_{0,q}) = (y_1z_1^q \ldots y_iz_i^qy_{i+1})^2 .$$

So the equation $y_1z_1^q \ldots y_iz_i^qy_{i+1} = (y_1z_1^q \ldots y_iz_i^qy_{i+1})^2$ in $\mathcal{E}^r_{0,q}$ is deducible from $\mathcal{F}^r_{0,q}$, $r \geq 1$.

So there remains to show that all the equations of the form (7) and (8) are deducible from the equation $x^q y^q = y^q x^q$ and all the equations $\mathcal{E}^r_{0,q}$, $r \geq 1$. To see this, let x be of the form (*) for some r. Then $x = x^2$ (using an instance of $\mathcal{E}^r_{0,q}$). Now, let x and y be of the form (*) for some r. Then

$$xy = (y_1 z_1^q \ldots y_i z_i^q y_{i+1})(y'_1 (z'_1)^q \ldots y'_j (z'_j)^q y'_{j+1})$$

for some y1, … , yi, yi+1 and $y'_1, \ldots, y'_j, y'_{j+1}$ in $\mathcal{X}^r_q$ and where the z's and the z's are either empty or of the form (*). But the latter is equal to

$$(y_i z_1^q \ldots y_i z_i^q y_{i+1})^2 (y'_1 (z'_1)^q \ldots y'_j (z'_j)^q y'_{j+1})^2$$

using instances of $\mathcal{E}^r_{0,q}$ and hence to

$$(y_1 z_1^q \ldots y_i z_i^q y_{i+1})^2 (y'_1 (z'_1)^q \ldots y'_j (z'_1)^q \ldots y'_j (z'_j)^q y'_{j+1})^q .$$

But using the equation $x^q y^q = y^q x^q$, the latter becomes

$$(y'_1 (z'_1)^q \ldots y'_j (z'_j)^q y'_{i+1})^q = (y_1 z_1^q \ldots \mathrm{yi} z_i^q y_{j+1})^q$$

which is equal to yx.

Theorem 4.3.5. *The* **M** -variety $\mathbf{J_1} * \mathbf{Com_{t,q}}$ *is defined* by *all the equations*

$$\mathcal{F}^r_{t,q} \ , r \geq 1 \ .$$

Proof.Let A be an alphabet of r letters where $r \geq 1$. We show that a monoid generated by A belongs to $\mathbf{J_1} * \mathbf{Com_{t,q}}$ if and only if it satisfies all the equations of the form (3) and all the equations of the form (4) in $\mathcal{F}^r_{t,q}$.

If $M\ \mathcal{E}\ \mathbf{J_1}$ and $N \in \mathbf{Com_{t,q}}$ are monoids generated by A, then $M * N$ satisfies all the equations of the form (3) and all the equations of the form (4) in $\mathcal{F}^r_{t,q}$. We show that $M * N$ satisfies all the equations of the form (3) in $\mathcal{F}^r_{t,q}$ (the proof is similar for the equations of the form (4) in $\mathcal{F}^r_{t,q}$).

Consider

$$(9) \qquad u_1 v_1 \ldots u_{rt} v_{rt} (y_1 z_1^q \ldots y_i z_i^q y_{i+1}) = u_{rt} v_{rt} (y_1 z_1^q \ldots y_i z_i^q y_{i+1})^2$$

where $y_i, \ldots y_i, y_{i+1}$ is a list in $\mathcal{X}^r_q$, where the z 's are either empty or of the form (*) and where $u_1, \ldots, u_{rt}$ is a list of t x_1's,...,t x_r's. By replacing x_j by $n_i \in N$ for $1 \leq j \leq r$ and v_j by $n'_j \in N$ for $1 \leq j \leq rt$ in both sides of (9), we get $(n_1)^t \ldots (n_r)^t n'_1 \ldots n'_{rt}$ since $xy = yx$ and $x^t = x^{t+q}$ hold in N, every variable in the list $u_1, \ldots, u_{rt}$, is one of $x_1,\ldots,x_r$ and occurs t times in $u_1, \ldots, u_{rt}$, every variable in the list $y_1, \ldots, y_i, y_{i+1}$ is one of $x_1,\ldots, x_r$ and occurs a multiple of q times in $y_1, \ldots, y_i, y_{i+1}$, and also every variable in z_j^q is one of $x_1,\ldots, x_r$ and occurs a multiple of q times in z_j^q. Now, by replacing $u_1 v_1 \ldots u_{rt} v_{rt}$ by (m, n) and $y_1 z_1^q \ldots y_i z_i^q y_{i+1}$ by (m', n') for some $m, m' \in M$ and $n, n' \in N$, we get

$$\begin{aligned}
& u_1 v_1 \ldots u_{rt} v_{rt} (y_1 z_1^q \ldots y_i z_i^q y_{i+1})^2 \\
&\quad = (m,n)(m',n')^2 \\
&\quad = (m + n \cdot m' + (nn') \cdot m', n(n')^2) \\
&\quad = (m + n \cdot m' + n \cdot m', nn') \text{ (since } n = nn' \text{ holds in } N) \\
&\quad = (m + n \cdot m', nn') \text{ (since } x = x^2 \text{ holds in } M) \\
&\quad = (m,n)(m',n') \\
&\quad = u_1 v_1 \ldots u_{rt} v_{rt} (y_1 z_1^q \ldots y_i z_i^q y_{i+1}) \ .
\end{aligned}$$

It follows that $M * N$ satisfies the equation

$$u_1v_1 \ldots u_{rt}v_{rt}(y_1z_1^q \ldots y_iz_i^q y_{i+1}) = u_1v_1 \ldots u_{rt}v_{rt}(y_1z_1^q \ldots y_iz_i^q y_{i+1})^2 .$$

The proof of the converse is similar to that of Lemma 4.2.2 except that it deals with the congruences $\sim_{t,q}$ and $\approx_{t,q}$ instead of the congruences $\sim_{t,1}$ and $\approx_{t,1}$. It is easy to see that $u \sim_{t,q} uv$ if and only if, for all $a \in A$, $|v|_a = 0$ or $|u|_a \geq t$ and $|v|_a \equiv 0 \pmod q$. It then follows that for all $a \in A$, $|a_1 \ldots a_i|_a = 0$ or $|w|_a \geq t$, $|a_1 \ldots a_i|_a \equiv 0 \pmod q$, and also for all $a \in A$, $|b_1 \ldots b_j|_a = 0$ or $|w|_a \geq t$, $|b_1 \ldots b_j|_a \equiv 0 \pmod q$. The rest of the proof imitates that of Lemma 4.2.2.

Theorem 4.3.6. *The* **M**-*variety* $\mathbf{J_1} * \mathbf{Com}$ *is defined by all the equations*

$$\mathcal{F}_{t,w}^r , \ r \geq 1 .$$

Proof. By Theorem 4.3.5.

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