Proof of a Problem

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This paper is dedicated to my mother and father for always believing in me. It is because of them that I have been able to accomplish everything I have set my mind to.

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Abstract

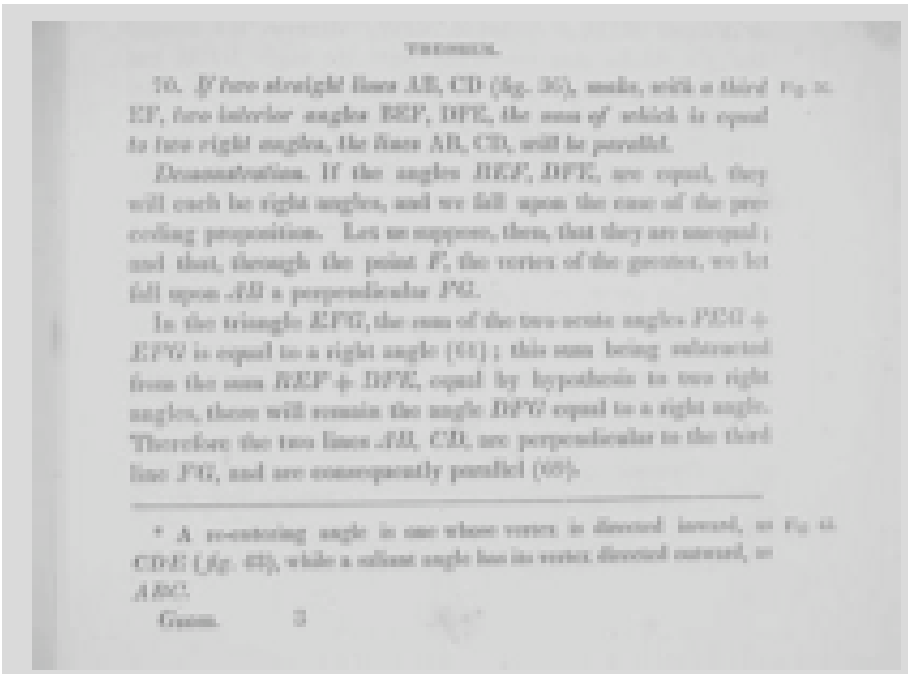
In this paper I have constructed an overview of mathematical proof, which involves several factors. A large part of understanding our current methods of teaching proof is knowing the history of proof. I also take a look at several different methods of proving and why proof is important. The focal point of the paper is the difficulties students have with mathematical proof and the difficulties the students that I did a field study with had. This stems from many different factors, the biggest being definitions. Another problem is lack of emphasis so I have also included an overview of national standards of proof and international comparison for proof. Possible solutions to the difficulties that students have with proof are changing teacher conception of proof, a transitional course, and the modified-Moore method.

Introduction

Transitioning from high school mathematics classes to upper level courses in college has troubled many students throughout the years. One particular area where students have problems is writing proofs. Most of their problems are caused by difficulties with mathematical definitions and notation. The root of their problems is inadequate exposure to proof. Over time there has been a change in how we view and write proofs (Herbst, 2002), but not enough has been done. Students are still having difficulties in this area and it has been found that high school teachers also have trouble with teaching proof. To understand proofs, you should also know the methods of proving and why it is important. After studying the research, I wanted to see first hand the difficulties that high school students have with doing proof so I did field research in this area. With problems in the educational field, there are usually solutions. One way to help with teaching proof is using the modified-Moore method. It is different from the traditional method in that it is not based on lecturing, but allowing students to figure out proofs independently (Weber, 2003). It contrasts drastically with how teaching proof used to be approached.

History of Two-Column Proof

If you look back to almost two centuries ago, you will see a vastly different style of teaching proof than what you see today. The first period of textbooks for proof was deemed the "*Era of Text,"* by Patricio Herbst (2002). In this era, proving in geometry was done by use of extensive text. An example of what a proof would look like is:



[illegible]

10. *If two straight lines AB, CD (fig. 36), make, with a third EF, two interior angles BEF, DFE, the sum of which is equal to two right angles, the lines AB, CD, will be parallel.*

Demonstration. If the angles *BEF*, *DFE*, are equal, they will each be right angles, and we fall upon the case of the preceding proposition. Let us suppose, then, that they are unequal; and that, through the point *F*, the vertex of the greater, we let fall upon *AB* a perpendicular *FG*.

In the triangle *EFG*, the sum of the two acute angles *FEG* + *EFG* is equal to a right angle ([illegible]); this sum being subtracted from the sum *BEF* + *DFE*, equal by hypothesis to two right angles, there will remain the angle *DFG* equal to a right angle. Therefore the two lines *AB*, *CD*, are perpendicular to the third line *FG*, and are consequently parallel ([illegible]).

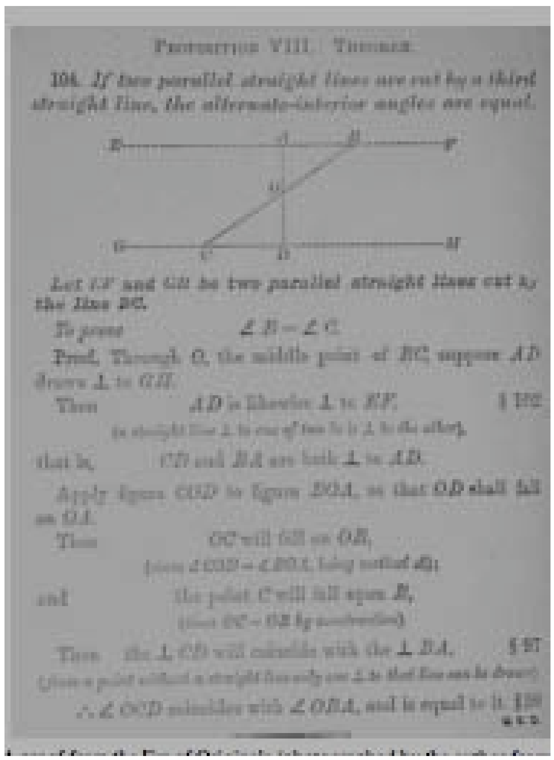
* A re-entering angle is one whose vertex is directed inward, as *CDE* (*fig.* [illegible]), while a salient angle has its vertex directed outward, as *ABC*.

Geom. 3

Figure 1. A proof from Herbst's "*Era of Text*" (2002, p. 289)

He states that the purpose of geometry was to show relationships between geometrical objects, but there were no demonstrations of these relationships or geometrical objects. The textbooks that were presented to students were nothing like the picture and example filled textbooks of today. There were no generalities for proofs or given methods of proving, just written paragraphs that students were expected to memorize.

Because of these difficulties in memorizing huge blocks of texts, there had to be a change in how proofs were to be taught to make it more meaningful to students. This was especially more needed due to the increase of geometry courses being taught in high schools. Authors like Greenleaf and Chauvenet wrote texts that not only provided contextual content, but also provided students with work to do at the end of each section. This era of proofs was deemed as the "*Era of Originals"* because of the original propositions students had to prove. An example of a proof problem for this era is:


PROPOSITION VIII. THEOREM.

104. *If two parallel straight lines are cut by a third straight line, the alternate-interior angles are equal.*

Let EF and GH be two parallel straight lines cut by the line BC.

To prove ∠ B = ∠ C.

Proof. Through O, the middle point of BC, suppose AD drawn ⊥ to GH.

Then AD is likewise ⊥ to EF. § 102

that is, CD and BA are both ⊥ to AD.

Apply figure COD to figure BOA, so that OD shall fall on OA.

Then OC will fall on OB,

and the point C will fall upon B,

Then the ⊥ CD will coincide with the ⊥ BA. § 97

∴ ∠ OCD coincides with ∠ OBA, and is equal to it.

Figure 2. A proof from Herbst's "*Era of Originals*" (2002, p. 294).

These so-called originals were not exercises as we know them today. They allowed students to gain more knowledge, however, by doing an "original" proof. These originals proved to be just as difficult as proving the theorems that authors gave in the text. There was a need for something different and that was found in diagrams that were given. The diagrams were becoming more detailed and gave visual aid to what relationships should be considered. You can see an example of the diagrams in the above figure. The diagrams are important to the history of proof because they showed the educational system was changing in order to *help* the students prove. It also increased the involvement teachers had. They were now checking students' improvement (or checking to see what needed to be done to move forward).

In addition to the changes in diagrams, there was also a change in how proofs were presented. Wentworth began this change by writing in between each step the reason for the step. The reasons were spelled out for the students. This new style of text laid the foundation of how future authors would write proofs. Students were then expected to be able to show they understood by demonstrating how to prove the theorems and doing original proofs.

Soon after these changes, the Committee of Ten (a committee in the late 1800's dedicated to studying the problems related to college requirements) made proofs the center of geometry. In the early 1900's, descriptions of what a proof should *look* like began to appear along with strategies and methods for proofs. It was made important that the students should arrange their statements in a logical order. The proofs should be in two columns: one side should be the "statements" and the other side "reasons" (Herbst 2002).

The originals became exercise in which students were "drilled" until they understood the theories. The exercises purpose also changed in that they were now to reinforce the ideas after they were taught instead of being a preparation to what was going to be taught.

With all of the new exercises there was need for a change in the foundation of proofs. Axioms were made shorter by leaving out propositions that were deemed "self-evident." Herbst gives the example that "all right angles are equal." This axiom was not needed because if you were given a right angle it is a rule that it would equal 90 degrees. It can be said that it is self evident that another right angle would also equal 90 degrees therefore you do not need an axiom that says all right angles are equal to each other. The proofs were expected to be logical so there was a need for a logical language to support it. The new notation gave stability in the written proofs. Diagrams changed once again and showed more clearly what was to be proved (Herbst, 2002).

Almost everything about writing proofs changed in the early 20^{th} century. Students were given instruction on theorems and expected to be able to do examples using the learned material. Writing a proof became like a simple algebraic equation in which you solve for a variable. In those types of problems you were proving why the variable is the number you find it to be by using given rules. Whenever the problems got more complicated there was a need for *methods* of proving.

Methods of Writing Proof

There are many methods of proving. The first method that was actually *taught* to me was the "forward-backward" method. This method goes forward from your *if* statement and backward from your *then* statement. It allows the student to use what they know about both statements to link the two together. Another method that is straight forward is just the direct method in which you simply move forward from your given statements. Other methods[1] are contrapositive[2], construction, choose, specialization, and proof by elimination.

While those methods of proving are fairly common, there are three methods that are most commonly used, more especially in Real Analysis. These methods are induction, proof by cases, and proof by contradiction[3]. Proof by contradiction allows you to assume the *if* statement and the negative (opposite) of the *then* statement. If you have done your proof correctly then you will arrive at the end of the proof with a *contradiction* of what you assumed, thus proving the original *then* statement had to be true. Proof by mathematical induction has two main steps. First, show that $P(n_0)$ is true. The second step is to show that what you are proving is true for any *n*, so you substitute (n + 1) for *n* and prove that it works as well. The last type of proving to be mentioned is proof by cases. This is when your *if* statement can be in the form of “C OR D” (p. 137). This method of proving is also important due to the fact that you often use cases within other methods of proving (Solow, 2004).

Importance of Mathematical Proof

There are a lot of texts that have been developed over the history of mathematics in the subject area of proofs and might leave some wondering why it is so important. One answer given by Herbst is that “proof is essential in mathematics education not only as a valuable process for students to engage in…but, more importantly, as a necessary aspect of knowledge construction” (2002, p. 308). I think that it is important for students to know *why* they are able to certain things in mathematics. Another reason for the

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[1] A complete list of methods of proving given by Solow (2004) can be found in Appendix A.
[2] An example of contrapositive is given in Appendix B.
[3] Examples of proof by induction, cases, and contradiction are give in Appendix B.

importance of proofs being taught in high school is because it will help to prepare students who will be entering college.

Proof is also a matter of logic. Mathematically, logic is done in geometry using “truth tables.” These tables[4] help organize truth-values (the truth or falsity of a statement). Students use logic in proofs by using statements with the same truth-value. For example, you are given an “if-then” (conditional) statement and you have to write a proof. You are able to use the contrapositive method for proving because the conditional and contrapositive statements have the same “truth-value.” (Boyd et al., 2004).

Logic is not only used in mathematics, but also in everyday life. Logic is important because it is your ability to use reasoning to figure out problems. Doing mathematical proof can help increase your ability to use logic in your life. This ability increases your ability to use critical thinking. This means that proof is *not* just for students who will be pursuing a career in mathematics. It is important for all students to be able to think critically.

Difficulties Students Have With Writing Mathematical Proof

Having a strong background in proof is most advantageous to students who do seek education at the collegiate level. Making the transition to upper level mathematics courses can be extremely difficult without proper preparation. Even with some background, it can still be difficult. One of the problems with the transition to writing proof in an upper level mathematics course is the fact that geometry is often the only subject that students use proof in. Most students have not seen a general perspective of proof or even been introduced to the different methods of proving. The students who will

[4] An example of the truth table can be found in Appendix C.

be pursuing a mathematics, mathematics education, or any other related major will have to use proof in classes such as real analysis, abstract algebra, and other advanced mathematics courses such as calculus. As a mathematics education major, I was surprised with how abrupt the transition to proofs was. In fact, I could not even remember whether or not I had done proofs in high school.

Not seeing proof enough throughout early education can lead to many problems for students when they begin taking advanced mathematics courses. Moore found seven specific problems that students have associated with writing proof. The students:

- Did not know the definitions, unable to state definitions
- Had little intuitive understanding of the concepts
- Had concept images that were inadequate
- Were not able to generate own examples
- Did not know how to use definitions to obtain the overall structure of proofs
- Were not able to understand and use mathematical language/notation
- Were not able to begin the proofs

These problems stem from two main areas: concept definition and concept image. Moore found that most students were not able to accurately give definitions and that the definitions seemed abstract to the students. As Moore states it, the concept definition is "a formal verbal definition that accurately explains the concept in a noncircular way" and the concept image is "the set of all mental pictures that one associates with the concept, together with all the properties characterizing them." An example of a concept definition is the formal definition of a square: "a quadrilateral with four right angles and four congruent sides" (Boyd et al., p. 432, 2004). Concept usage is another problem area and is defined by Moore as “the ways one operates with the concept in generating or using examples or in doing proofs” (p. 252). These three things make up the concept-understanding scheme.

The biggest problem with the definitions is that they are abstract to students and students cannot relate to them if they are not presented in an informal way at least at first. The reason for this can be found in the fact that students do not realize how important it is to know a concise definition of a concept. An example of a formal definition of a triangle is “a three-sided polygon” (Boyd et al., p. 178, 2004). A student might say it is a three-sided figure, but it is important to know that it is a polygon because polygons only have straight lines.

The concept images of the students that Moore studied were built by the teacher giving illustrations of definitions and worked out examples. While this method helped the students, it did not really ensure that the students would be able to write a formal proof. The concept images were not enough, which brings us back to the need to know formal definitions. Students seem to rely too much on pictures and diagrams to understand concepts. This alone will not help when in an upper level mathematics course (Moore, 1994).

Field Study

In my own field research, I did a small sample study with a group of three Purnell Swett High School students. They were all calculus students in Mrs. Kenworthy’s class. I was once a student of Mrs. Kenworthy, so I was able to get permission from parents and the school fairly easily. The purpose of the study was to see how students in the highest level of high school mathematics would interpret an “elementary” proof problem. Before

I interviewed them about the problem, I asked them what background they all had in doing proofs. Their background was much like my own, practically none at all. Two of the students said they had proved some theorems in their geometry class, but the other could not remember writing any proofs.

I used one of the first proof problems I was given in my Introduction to Advanced Mathematics course. The problem was “Your Age By Eating Out.” It is seen as a math “trick,” but a proof can be written to show why it works. The problem is shown below:

- Follow the instructions bellow and show your work on this page.
- Pick a number from 1 to 10 and the number will represent how many times you want to go out to eat
- Multiply the number by 2
- Add 5 to the previous product
- Multiply this sum by 50
- If you haven’t had your birthday yet this year add 1759, if you have add 1760
- Subtract your birth year from the previous sum
- The last two digits are your age!
- Your job now is to prove why this works (Hint: think of this algebraically and use a variable(s))

I gave the three students a full week to complete the problem and to add motivation (because this was a voluntary activity) Mrs. Kenworthy offered the students who participated extra credit.

I was not present when the students attempted to write a proof of the problem, but I did go over the problem with the students after a week’s time. Out of the three students, only one was able to get anywhere near what the actual proof of the problem looks like. In the problem, I gave the hint of using a variable to write the proof, but this actually caused conflict instead of helping as I had hoped it would. They focused on the variable and went back to the habit of having to solve for *x* when this was not the purpose of using the variable. The student[5] who got the closest to doing the problem stopped right before the end because she did not know what to do with *x*. This showed me that students are drilled with the idea that they are always trying to find some kind of specific number. This type of teaching does not allow the students to think creatively, nor does it allow them to think critically.

Another aspect of the student's work that stood out to me was the fact that none of the students wrote out the reasons they were doing something. When writing a proof it is ideal to show why you are able to do something to get you to the next step of the proof. Once I went over the actual proof of the problem, the students did not seem confused and the problem seemed simple to them.

The proof (for if you have had your birthday, substitute 1759 if you have not):

- Your age can be represented by 2010-y, your year of birth.
- So the expression I wrote above (50(2x+5)+1760-y) is equal to 100x+(2010-y).
- Let's look at this equation:
  - 50(2x+5)+1760-y = 100x+(2010-y)
- 100x+250+1760-y = 100x+2010-y
- By canceling like terms, we find that:
  - 250+1760 = 2010

[5] A scanned copy of what this student did can be found in Appendix D.

- 2009 = 2009
- This is an identity function

They all agreed that having a class or more of a background in writing proofs would have helped them with this problem.

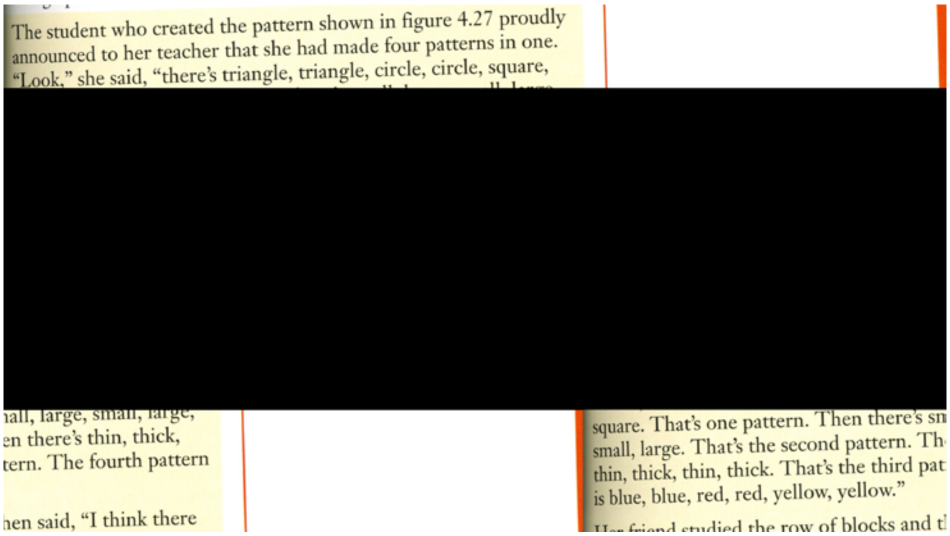
National Standards for Proof

This led me to research why there was not a greater background in proof. Are there standards for proof throughout school? The National Council of Teachers of Mathematics (NCTM) outlines the standards of proof from pre-kindergarten through the twelfth grade (NCTM, 2000). Proof builds on the reasoning skills that students already have and it allows them to apply it to mathematics. It is also important for students to understand that proof is fundamental. In the beginning, you can simply prompt students to understand mathematical truths after asking them what makes something true. This will lead to a greater ability to think mathematically and then students will be able to progress and use mathematical reasoning to solve problems.

Proof for Pre-K-2

During this time, students have little mathematical knowledge, but are still able to use reasoning that they have gained through experience. They are able to use pattern-recognition and classification skills in order to find mathematical truths.

Students' abilities to show proof should be apparent by their ability to recognize patterns. This will start with noticing only one attribute and over time the students will be able to identify multiple properties. By the end of the second grade students should be able to recognize patterns in numbers, not just shapes. NCTM gives an example of pattern recognition from these early years of education:



The student who created the pattern shown in figure 4.27 proudly
announced to her teacher that she had made four patterns in one.
“Look,” she said, “there’s triangle, triangle, circle, circle, square,
[redacted]
nall, large, small, large,
en there’s thin, thick,
tern. The fourth pattern

hen said, “I think there

square. That’s one pattern. Then there’s sm
small, large. That’s the second pattern. Th
thin, thick, thin, thick. That’s the third pat
is blue, blue, red, red, yellow, yellow.”
Her friend studied the row of blocks and t

Figure 3. An idea of what proof should look like from NCTM (2000, p. 163).

This era of a student’s education should be guided by teachers who show them the importance of giving examples and counterexamples. This allows students to see why or why not the generalizations they make are accurate.

Proof for 3-5

Students begin to think with more mathematical reasoning rather than experience (as in having seen the problem before). The focus during this time should be “reasoning about mathematical relationships” (p. 188). Students will work more with classes of objects instead of the basic individual mathematical objects they were working with before.

During this time, proof should be shown by the students’ abilities to “define relationship, analyze why it is true, and determine to what group of mathematical objects…it can be applied” (p. 189). An example can be found in the algebraic work students do at these grade levels. A student was asked to do 74x6 and broke it into two parts: 74x2 and 74x4. This is an example of the distributive property (you could compute the problem by doing 74[2 + 4]). Students should be encouraged use different mathematical properties and to be able to recognize when they are using them. This will help them make the mathematical connections.

By encouragement, NCTM means that teachers should encourage students to talk about their ideas and allow them to be examined. The teacher also must create the feeling of expectance. The students should expect to have to know that they have to be able to apply mathematical conjectures (conjectures are statements that are believed to be true, but have not been proven) about relationships within their mathematical lessons. The way that one can create this expectation is questioning the students' reasoning. Another thing that teachers can do at this level is to make the students responsible for their own reasoning. This will make the lessons more meaningful to the students and help create true understanding.

Proof for 6-8

According to NCTM, by the time students reach the middle grades, they should know that "mathematics involves examining patterns and noting regularities, making conjectures about possible generalizations, and evaluating the conjectures" (p. 262). The sixth to eighth students will need to strengthen their reasoning skills and be able to use this to form mathematical arguments.

During this time proof should be shown by students discussing their reasoning frequently. This will allow them to use inductive and deductive reasoning at the appropriate times. An example of inductive reasoning is recognizing a pattern to predict a future happening. NCTM uses triangular numbers to show what patterns the students should be able to recognize:

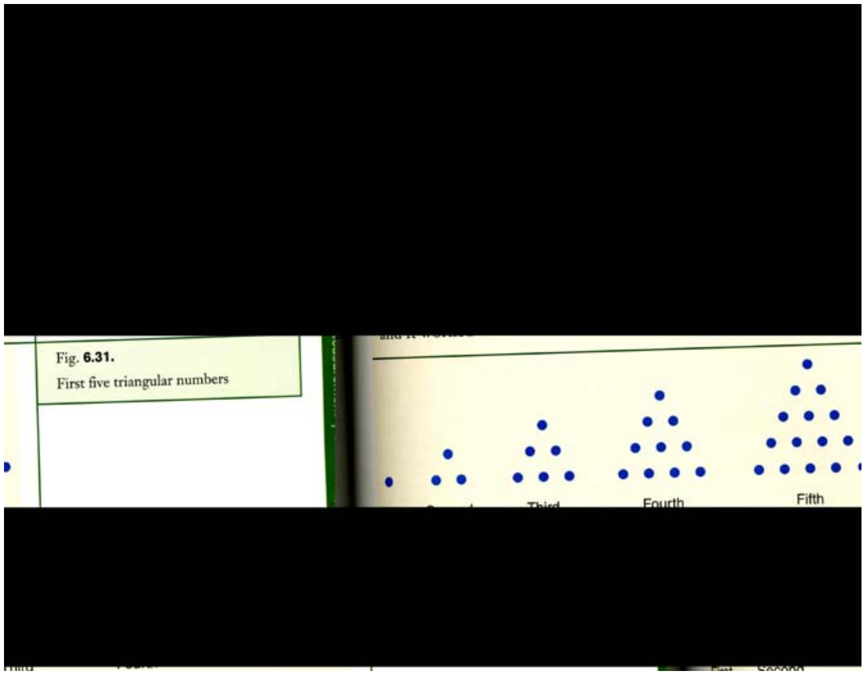


Figure 4. An idea of what proof should look like from NCTM (2000, p. 263).

During this time teachers can encourage the development of advanced mathematical reasoning by keeping the students interested in the material. One could make the problem relevant to the students' lives and ask them to find mathematical relationships within the problem. Induction is used a lot during this time so it is important for the teacher to make the students understand that it cannot always be relied on. This will allow the students to see that not everything can be generalized, meaning they will have to do more critical thinking.

Proof for 9-12

By the time students enter high school, they should have a higher appreciation for the reasoning involved with mathematics. Ninth through twelfth grade students should be able to give a more "sophisticated" method of proof.

Proof at these grades should have explanations/justifications for conclusions. This includes the use of mathematical terms and an understanding of mathematical definitions. Students will need to understand definitions in order to use them as proof of the conclusions they reach. The importance of understanding definitions was also given by Moore (1994). An example of how a teacher can emphasize proof in the high school grades is requiring them to explain why something is true. For example, when a student does an algebraic problem:

$35(x + 5) = 180$	
$35x + 175 = 180$	Distributive Property
$35x = 5$	Subtraction Property
$x = {}^{1}/_{7}$	Division Property

Students will understand why they are able to work through problems instead of just following a "standard format."

An important factor in teachers teaching students about proof is their own understanding of the matter. Teachers have to emphasize the importance of knowing mathematical truths, which are found in doing proof. There are many other factors that come into play during this era of schooling: teachers must have a classroom filled with discussions and questioning; students should be expected to be able to explore and find answers. Even if an exploration leads to a dead end, new mathematical understanding can be discovered.

Conclusion of National Standards

The NCTM gives a detailed outline of proof for each category of students from Pre-K-12 grade. With all of these standards, it leaves one wondering, how could there still be a problem for students in the area of proof? I have come to the conclusion that it is because reasoning and proof are only emphasized during high school in geometry classes. This is simply not enough time for students to gain mathematical knowledge.

International Comparison

The NCTM has clearly outlined what needs to be done in regards to proof. Not only do they show what needs to be done in this area, but in all of the other areas of mathematics. Students' inability to do proof is evidence that these standards are not being followed. Another form of evidence is that American students fail in comparison to other countries. Clarke (2003) gives a statement from the *Washington Post*: "international tests results…show the nation's eighth-graders are just above average when compared with peers around the world" (p. 146).

Data can be found in the Trends in International Mathematics and Science Study (TIMSS) that backs what Clarke found. The average scores for fourth grades in America are significantly below eight other countries, which includes: Hong Kong SAR, Singapore, Chinese Taipei, and Japan. Eighth graders also fell below these listed countries. Studying TIMSS, I noticed that our ranking is dropping, while other countries are passing us. Between 1995 and 2007 there was only a significant difference between us and four other countries for fourth grades and the most recent comparison has shown that number increase to eight. While eight might not seem like a lot of other countries that rank higher, it is still negative, considering it means the number has doubled

(Gonzales et al., 2008). The TIMSS studies are done to gain “a deeper understanding of the effects of policies and practices within and across educational systems” (Kloosterman & Lester, 2004, p. 386). Kloosterman & Lester have also found that it is a common belief that a country’s success in the world market positively correlates with mathematics. This relationship also suggests that it is important for American students to do well in mathematics because their economy might depend on it.

Our inability to do mathematical proof stems from the emphasis put on the subject. Countries like the United States and Japan put emphasis on recalling mathematical information, but countries like Hong Kong and Israel put emphasis on justification and proof. Perhaps to learn proof better, we should follow some of the other countries around the world’s standards regarding proof (Clarke, 2003).

Changing Teacher Conception of Proof

The NCTM stated that “teachers themselves need to understand mathematics well” (2000, p. 345). The problem with proofs is not only the students’ difficulties, but also with the difficulties that teachers have with teaching proof and their own conceptions of proof. In a study done by Knuth, it was found that most teachers viewed proof as a means of establishing why a concept is true. All of the teachers also agreed that proofs could be viewed as generalizations. In other words proof is used for a general concept and there is not a specific *x* or *n* that you are writing a proof for.

Knuth found that these teachers had the knowledge of what roles proofs have in mathematics. He proposed that if they focused on those roles then they would be able to teach their students in a way that would allow the students to have a better understanding of proofs than they normally do. Another key to helping the students would be for the teachers to teach *understanding* rather than just showing a proof. In his conclusion he encourages mathematicians and mathematics educators to better prepare future teachers. He also encourages the use of the modified Moore method (MMM) to teach proof, which will explained in a later section titled "Modified-Moore Method" (Knuth, 2001).

Transitional Course

While emphasizing proof according to NCTM, and changing teacher conceptions are great solutions to the problems students have with proof, there is still another solution. The other solution that seems to work effectively is having a transitional course to advanced mathematics. At UNC-Pembroke, there is a course titled Introduction to Advanced Mathematics. It was in this course that I learned the different methods of proving. I was also able to relearn the definitions and properties used to do proof. It has helped with Real Analysis, Calculus 1-3, and Probability and Statistics 1. It has also helped me in geometry as far as knowing the necessary properties.

In Moore's study, he too, found that with a transitional course students are not overloaded by the higher cognitive level it takes to be successful in and advanced course. Students who enter an advanced mathematics class without a transitional course and only a small background in proof have a lot of information to cope with. They do not really have time to relearn definitions with all of the other required work. Also, methods of proving are expected knowledge upon entering an upper level mathematical course. A transitional course allows the students to have a refresher course. This method ensures that students are better prepared for the more difficult proofs they will be expected to do in the upper level mathematics courses.

Modified-Moore Method

A current style of teaching that has been shown to improve students' understanding of proof is the modified-Moore method (MMM). The MMM proposes that students will learn a lot more about mathematical concepts and proofs if they are given the chance to construct the proofs themselves. When using the Moore method, the teacher presents definitions and a few examples before allowing the students to prove or disprove examples about the given concepts. The teacher and students then analyze the student's work. If there is too much trouble with a proof, the teacher asks the students to prove something simpler and minimal assistance is given (Weber, 2003).

Jennifer C. Smith did a study, which showed the difference in how students in an MMM-based classroom and a lecture-style based classroom did with proofs. One difference can be found in what students' focused on when learning proof. Students in the MMM focused more on the understanding of the proofs, whereas the students in the traditional setting focused more on the correct method for doing the proofs.

The students in the MMM class used strategies for proving that they had used earlier. These strategies were based on similar concepts, which contrasted with the students in the traditional class. The traditional students used previous problems that had the same methods rather than the same concepts. Another apparent difference was in the use of concrete examples. The MMM students would use concrete examples often in order to stimulate their thinking and the traditional students rarely used concrete examples.

The differences between the students in the different settings can be summarized by saying that the students in the MMM class wanted to make sense of what they were doing. This allowed the MMM students to have an easier time with writing proofs because they were able to understand the concepts the proofs were based on (Smith, 2006).

Conclusion

There is *proof* of a problem in regards to students' conceptions of writing proofs. Students often do not have much of a background in proving other than the few theorems they "prove" in their high school geometry classes. I have presented a few options to help with this problem. Having proof taught more throughout primary school is one solution. The way I have offered is simply following standards set forth by the NCTM. They have outlined many ways throughout the school years that teachers can help students with reasoning and proof. It also shows what the students' proofs should look like in each era of education. Doing the outlined things would build students' backgrounds in proof and would make the transition to upper level mathematics courses in college smoother.

If this is not done, then there is the option of having an "Introduction to Proofs" or "Introduction to Advanced Mathematics" course. This would lighten the load of students who will have to use proofs in a class such as Real Analysis. There is not enough time to teach the basics in an upper level course so having a transitional course can be vital for students.

Even within the upper level courses there should be a change in the way proofs are taught. First, there is a need for better *understanding* of the definitions. Students should not just know definitions of key concepts, but they should *understand* them and be able to state them formally and informally. The modified-Moore method reinforces the idea of students truly *understanding* the concepts they are being taught. It allows the students to approach proofs based off of concept *understanding* rather than the methodical approach or the memorization approach. Through all of my research I have found that *understanding* is the key to learning proofs. If students can *understand* key concepts and ideas, then they will be able to construct proofs independently and accurately.

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Appendix A

Summary of Methods of Writing Proof as Given by Solow (2004)

Proof Technique	**When to Use It**	**What to Assume**	**What to Conclude**	**How to do it**
Forward-Backward	As a first attempt, or when B does not have a recognizable form.	A	B	Work forward from A and apply the backward process to B.
Contrapositive	When B has the word “no” or “not” in it.	NOT B	NOT A	Work forward from NOT B and backward from NOT A.
Contradiction	When B has the word “no” or “not” in it.	A and NOT B	Some contradiction	Work forward from A and NOT B to reach a contradiction.
Construction	When B has the words “there is,” “there are,” and so on.	A	That there is the desired object	Guess, construct, and so on, the object. Then show that it has the certain property and that the something happens.
Choose	When B has the words “for all,” “for each,” and so on.	A, and choose an object with the certain property	That the something happens	Work forward from A and the fact that the object has the certain property. Also work backward from the something that happens.
Specialization	When A has the words “for all,” “for each,” and so on.	A	B	Work forward by specializing A to one particular object having the certain property.
Forward Uniqueness	When A has the key word “unique” in it.	There is such an object, X	X and Y are the same, that is, X = Y	Look for another object Y with the same properties as X.

Direct Uniqueness	When B has the key word "unique" in it.	There are two such objects, and A	The two objects are equal	Work forward using A and the properties of the objects. Also work backward to show the objects are equal.
Indirect Uniqueness	When B has the key word "unique" in it.	There are two different objects, and A	Some contradiction	Work forward from A using the properties of the two objects and the fact that they are different.
Induction	When a statement P(n) is true for each integer $n \geq n_0$	P(n) is true for n	P(n+1) is true; also prove $P(n_0)$ is true	First prove $P(n_0)$. Then use the assumption that P(n) is true to prove P(n + 1)
Proof by Cases	When A has the from "C or D."	Case 1: C Case 2: D	B B	First prove that C implies B; then prove that D implies B.
Proof by Elimination	When B has the form "C or D."	A and NOT C Or A and NOT D	D Or C	Work forward from A and NOT C, and backward from D Or Work forward from A and NOT D, and backward from C.
Max/Min 1	When B has the form "max $S \leq x$" or "min $S \geq x$"	Choose an $s \in S$, and A	$s \leq x$ or $s \geq x$	Work forward from A and the fact that $s \in S$. Also work backward.
Max/Min 2	When B has the form "max $S \geq x$" or "min $S \leq x$"	A	There is an $s \in S$ for which $s \geq x$ or $s \leq x$	Use A and the construction method to produce the desired $s \in S$

Appendix B

Example proofs

Example of Proof By Contrapositive

Contrapositive works from NOT B to NOT A. You prove that the negative of B is the negative of A, which means that the A implies B.

Prove: that If 3n-8 is odd, then n is odd.

Assume: If n is even, then 3n-8 is even.

1. n is even	1. Given
2. n = 2k	2. Definition of an even number
3. 3(2k) – 8	3. Substitution
4. 6k – 8	4. Distribution
5. 2(3k – 4)	5. Divisible by 2
6. Let 3k – 4 = m	6. Equality Axiom
7. 2m	7. Substitution. Since 3n-8 is even when n is even, then it is odd when n is odd by Contrapositive.

Example of Proof by Mathematical Induction

Prove: $8^n - 3^n$ is divisibly by 5.

Step 1 (prove statement is true for n = 1):

$8^{(1)} - 3^{(1)}$	Substitution
= 8 – 3	Simplify
= 5	Subtract

5 is divisible by 5 so true for n = 1

Step 2 (Substitute for n = k):

Proof of a Problem

$8^k – 3^k = 5j$

where j is any real number

Step 3 (Prove that it is divisible for any k: (k+1):

$8^{(k + 1)} - 3^{(k + 1)}$

$8^{(k + 1)} = 8(8^k)$ and $3^{(k+1)} = 3(3^k)$

$= 8(8^k) – 3(3^k)$

$= 8(8^k) – (3 * 8^k) + (3 * 8^k) – 3(3^k)$

$= 8^k(8 – 3) + 3(8^k – 3^k)$

$= 8^k(5) + 3(5j)$

$(8^k – 3^k) = 5j$ from Step 2

$= 5(8^k + 3j)$

It is a multiple of 5 therefore $8^n – 3^n$ is divisible by 5 when n is any number.

Example of Proof By Cases

Proof by cases shows that C implies B and that D implies B.

Prove: There is no integer *n* for which $n^2 + 4$ is exactly divisible by 7.

Case 1: *n* is even		**Case 2: *n* is odd**	
n = 2a	Definition of an even integer	n = 2b + 1	Definition of an odd integer
$(2a)^2 + 4$	Substitution	$(2b + 1)^2 + 4$	Substitution
$4a^2 + 4$	Power Rule	$(4b^2 + 4b + 1) + 4$	Power Rule
$4(a^2 + 1)$	Divisible by 4	$4b^2 + 4b + 4 + 1$	Commutative Property
So when *n* is even, it is not divisible by 7		$4(b^2 + b + 1) + 1$	Divisible by 4

	So when *n* is odd, it is not divisible by 7 either.

Example of Proof By Contradiction

Contradiction works from A to NOT B. If the proof is correct you will reach a contradiction in the end.

Prove: If 3n – 8 is odd, then n is odd

Assume: If 3n-8 is odd, then n is even

1. 3n – 8 is odd n is even	1. Given
2. n = 2b	2. Definition of and even number
3. 3(2b) – 8	3. Substitution
4. 6b – 8	4. Distribution
5. 2(3b – 4)	5. Divisible by 2
6. Let 3b – 4 = m	6. Equality Axiom
7. 2m	7. Substitution. This means that 3n – 8 is even, which is a contradiction of the given statement that it is odd. This proves that n has to be odd.

Appendix C

Example of Truth Tables as Given by Boyd et al (2004)

Example of a Contrapositive Truth Table:

p	**q**	***~p**	**~q**	***p → q**	**~q → ~p**
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

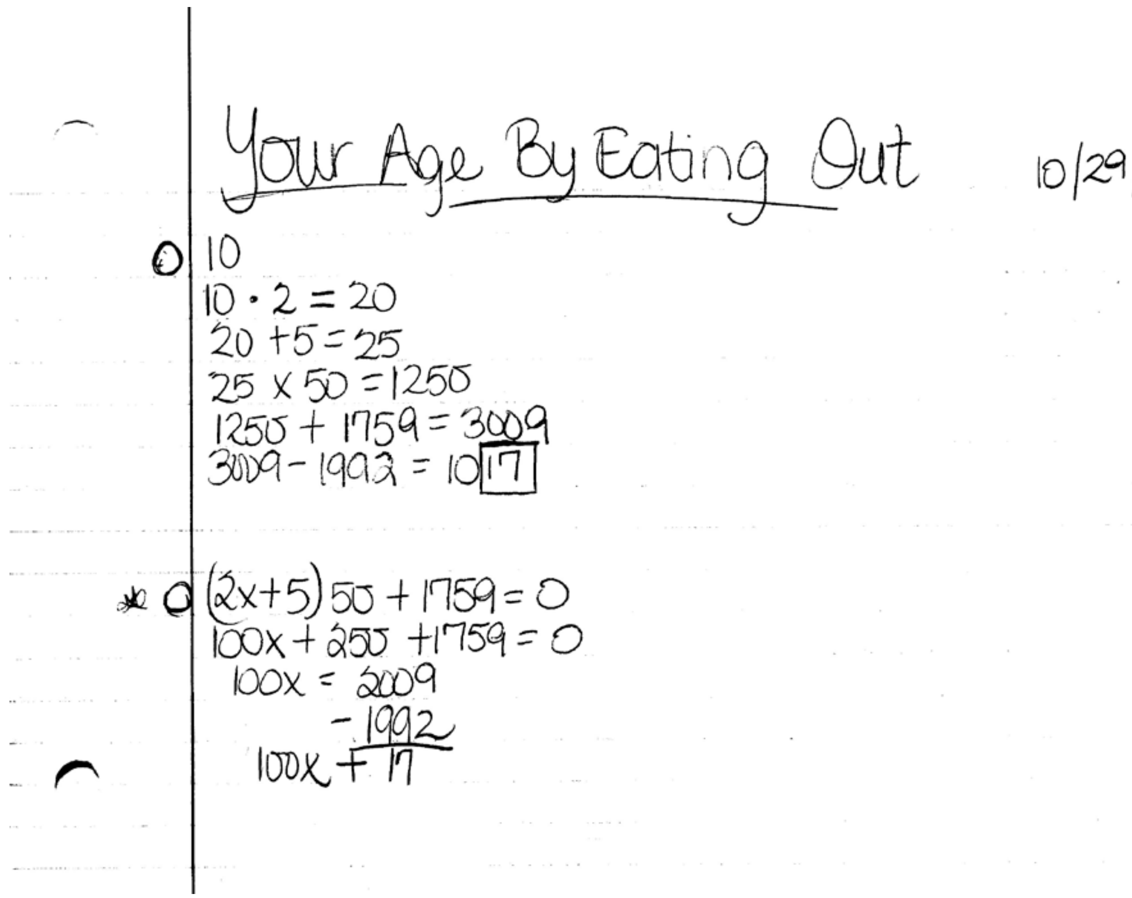
*The symbol ~ means "not" so it is the negation of the statement.

**The symbol → means implies so one statement implies the other.

Note: From this you can see that the conditional statement has the same truth value as the contrapositive therefore if you find the solution using the contrapositive method of proof, then its conditional will also hold true.

Appendix D

Example of Field Study Student's Work


Your Age By Eating Out 10/29

10
10 · 2 = 20
20 + 5 = 25
25 x 50 = 1250
1250 + 1759 = 3009
3009 − 1992 = 10 17

(2x+5) 50 + 1759 = 0
100x + 250 + 1759 = 0
100x = 2009
− 1992
100x + 17