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In much of the indeterminate music composed in the 1950s and 60s, the roles of the composer and performer are blurred, the performer having been given control over musical elements previously dictated solely by the composer. Often, decisions must be made by the performer that impact a work's form and content, the composer's quiet voice heard only in the directives influencing these decisions. In many cases, these directives lack specificity, allowing for an infinite number of performance possibilities; in some cases, however, composer directives severely restrict that number, permitting it to be discretely counted.

To these latter cases we turn our attention, mathematically modeling the composer's directives to enumerate all possible realizations of certain indeterminate scores. Taking Morton Feldman's *Durations 2* and Karlheinz Stockhausen's *Klavierstück XI* as primary examples, we calculate the total number of possible realizations, generalizing each case in order to enumerate the realizations of other works with similar characteristics.

ENUMERATING INDETERMINACY

by

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CHAPTER I

INTRODUCTION

In a lecture given at Darmstadt in 1958, the composer John Cage encapsulates the function of the performer of indeterminate music with the following metaphors: "The function of the performer

> … is comparable to that of someone filling in color where outlines are given;
>
> … is that of giving form, providing, that is to say, the morphology of the continuity, the expressive content;
>
> … is that of a photographer who on obtaining a camera uses it to take a picture;
>
> … is comparable to that of a traveler who must constantly be catching trains the departures of which have not been announced but which are in the process of being announced.[1]

Clearly, in Cage's descriptions, the performer is given a level of responsibility heretofore unseen, assuming a decision-making role traditionally filled by the composer alone.[2] This surrender of control to the performer was accomplished in many ways and to many degrees. In some compositions, so much control is given the performer that the composer's voice is difficult to hear, the shouts of

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[1] John Cage, *Silence* (Middletown: Wesleyan University Press, 1973).
[2] It should come as no surprise that the movement's origins coincide with an era of total serialism, the former's surrendering of control juxtaposing the latter's exercise of it.

the performer guided by mere compositional whispers. Cage's *Concert for Piano and Orchestra* (1957), for example, leaves so many musical decisions to the hands of the performer that subsequent performances are often unrecognizable as the same work. Pierre Boulez, on the other hand, was reluctant to so drastically soften his compositional voice; in the indeterminate middle movement of his *Third Piano Sonata* (*Constellation-miroir*), the performer is given control of only the *path* through the music—a path limited to but a handful of choices at each juncture. [3]

Depending on the degree of control given to the performer, the number of realizations of an indeterminate score varies. In Cage's *Concert*, for instance, his intentional lack of specific direction and his employment of ambiguity provide the performer with a universe of possibilities, infinite in every aspect of performance. The minimal freedoms in Boulez's *Third Piano Sonata*, however, curb such far-reaching possibilities, limiting some aspects of its performance to a countable many.

This paper mathematically explores those indeterminate works whose compositional directives and high degree of compositional control render the number of possible realizations finite. Taking Morton Feldman's *Durations 2* and Karlheinz Stockhausen's *Klavierstück XI* as springboards, this paper will develop

[3] For an interesting perspective on Boulez's struggle with relinquishing compositional control to the performer (and to chance), see Jean-Jacques Nattiez, *The Boulez-Cage Correspondences* (New York: Cambridge University Press, 1993).

mathematical models of the composers' directives to enumerate all possible realizations and subsequently generalize the models for similar compositional situations.

Generalizations

Naturally, the concept of enumerating performance realizations seems like a lost cause, since many attributes of sound (duration, frequency and volume, for instance) are non-discrete and therefore do not lend themselves to counting. Although we can measure these parameters as accurately as our present instruments allow, their continuous nature thwarts any attempt to fully enumerate the number of possible sounds. Because of this infinitude, we must make some important generalizations.

In this paper, the great variety of dynamics, attack, tempo, phrasing and timbre found in performances is not considered. Here, only the *order* of notation is of interest, not the *execution* of that notation itself. Let us consider Beethoven's Ninth Symphony as an example. Though hundreds of unique recordings exist, the composer was quite strict as to the order of events. Nowhere in the score did Beethoven give performers the option of leaving a page out, reversing the order of pages or beginning in the middle of a movement. He left notation for one order of events alone, and thus for the present study, the number of *realizations* of Beethoven's masterpiece is one.

In some works of Stockhausen, Feldman, Cage and others, the order of the musical content is left up to the performer, allowing for multiple realizations of the same score. Naturally, each realization—just like the Beethoven score—can be performed an infinite number of ways. However, the exact *number* of realizations of an indeterminate score can often be determined, and it is to this end we now turn.

Some Explanatory Examples

In order to develop the mathematical tools necessary for the enumeration of *Durations 2* and *Klavierstück XI*, three manageable examples of increasing difficulty are now presented. The first is from the realm of literature, a book by the French poet and mathematician Raymond Queneau (1903-1976). Published in 1961, *Cent Mille Milliards de Poèmes* is a book of ten sonnets, each fourteen lines in length, whose pages are cut horizontally between lines.[4] This permits the reader to freely mix and match lines from the ten poems *so long as* each line in a realization occupies the same position in its respective poem (in other words, the fourth line of a realization must come from a fourth line of one of the ten sonnets).

[4] Raymond Queneau, *Cent Mille Milliards de Poèmes* (Paris: Kickshaws, 1983).

Enumerating all possible sonnet realizations found in Queneau's collection is a relatively simple procedure. There exist ten choices for line one (the first lines of the ten sonnets), ten choices for line two (the second lines of the ten sonnets), ten choices for line three, and so on, all the way to line fourteen. Because each choice is an independent event, we can simply multiply our choices together to enumerate all possible orderings:

$$(10)(10)(10)(10)(10)(10)(10)(10)(10)(10)(10)(10)(10)(10)$$

written concisely as

$$10^{14} = 100{,}000{,}000{,}000{,}000$$

possible realizations. Thus, there are one hundred million million poems embedded in Queneau's book, just as the title proclaims.

Slightly more complicated is a piece by the experimental composer Earle Brown entitled *25 Pages for 1-25 Pianos* (1953).[5] Here, the composer presents the performer(s) with 25 pages of music played in any order and either rightside-up or upside-down.[6] Enumerating all possible realizations is very similar to the Queneau book, but with one important exception: once a choice has been made for a certain page, there is one less page from which to choose the next. This situation is commonly modeled with a factorial (denoted by !) and will figure

[5] Earle Brown, *25 Pages for 1-25 Pianos* (Toronto: Universal Edition, 1975).

[6] Although the employment of multiple pianos makes possible the simultaneous performance of multiple pages, Brown does not specifically address this possibility. It is thus assumed that the 25 pages must be performed separately. Therefore, the number of performers, per the original assumptions regarding performance elements, has no effect on the number of realizations.

prominently in the remaining explorations. The number of choices made by the performer for the first few pages are as follows:

page	choices
1	25 choices of pages; 2 choices for page direction
2	24 choices of pages; 2 choices for page direction
3	23 choices of pages; 2 choices for page direction

Clearly, the number of choices for each subsequent page decreases by one until all 25 pages have been chosen. In addition, each page has a second choice attached to it—whether the page is played rightside-up or upside-down. Multiplying all of the choices together yields:

$$(25)(2)(24)(2)(23)(2)(22)(2)(21)(2)(20)(2)\ldots(3)(2)(2)(2)(1)(2) = (25!)(2^{25}) =$$

$$520\ 469842\ 636666\ 622693\ 081088\ 000000,$$

an extremely large number, known as five hundred twenty nonillion, four hundred sixty nine octillion, eight hundred forty two septillion, six hundred thirty six sextillion, six hundred sixty six quintillion, six hundred twenty two quadrillion, six hundred ninety three trillion, eighty one billion, eighty eight million.

The third example of increasing complexity is also a literary work, very similar in nature to Brown's *25 Pages*. Stéphane Mallarmé (1842-1898), the highly influential French symbolist poet, never completed his envisioned project known as *Le Livre*, a book encompassing all forms of his creative energy.[7] He did, however, write about the form of the book: he planned to leave the pages unbound, thereby allowing the reader to choose not only the order of pages but also the number of pages read. (He specifically mentions adjusting the length of "performances" for the number of people in the audience, shortening or lengthening as necessary.) This situation is a bit more complicated than *25 Pages* and requires another important mathematical tool to enumerate.

Before this enumeration can begin, one must know the number of unbound pages. Since Mallarmé never completed the work, the number of pages can be generalized with the variable x. Clearly, if only one page is read from *Le Livre*, we have exactly x possible readings. If two pages are read, however, we have x choices for page one and $(x-1)$ choices for page two, and therefore $(x)(x-1)$ possible readings. When three pages are read, there are $(x)(x-1)(x-2)$ possibilities; when all x pages are read (as in the Brown work), we have $x!$ possible readings.

Now that the number of readings for each performance length is known, they can be summed to obtain the total number of possible realizations. To do so, we employ a summation that ranges from the minimum length of realization (1) to the maximum length (x).

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[7] Sam Slote, "Imposture Book Through the Ages," http://www.antwerpjamesjoycecenter.com/ibook.html.

$$x+(x)(x-1)+(x)(x-1)(x-2)+\ldots+x!=\sum_{n=1}^{x}\frac{x!}{(x-n)!}$$

Had Mallarmé finished the book with a mere 50 unbound pages, the number of possible readings would be

$$\sum_{n=1}^{50}\frac{50!}{(50-n)!}=$$

$$82674\ 076879\ 277258\ 572496\ 581009\ 301773\ 302984\ 486449\ 338756\ 300825\ 298500\ ,$$

a number nearly as large as the number of atoms in our galaxy.

General Notation

The preceding explanatory examples required the use of some important math symbols. Because a complete understanding of these symbols is necessary to the following sections, a short summary of notation is given.

Factorial (!): The product of all positive integers up to a given number.

- Example: $5!=(5)(4)(3)(2)(1)=120$

Summation Notation (Σ): Concise notation for the sum of an indexed family of values.[8]

[8] For the mathematically untrained, the summation and product notation employ an index variable *n* underneath their respective Greek symbols. This indexing variable ranges from its starting value (*n*=1) to its ending value (5) in steps of 1. For each step, the index value is "plugged in" to the equation, resulting in a numerical solution. These solutions (there are 5 for the examples given, since the indexing variable ranges from 1 to 5) are then either summed (in the case of the summation) or multiplied (in the case of the product) to arrive at the final answer.

- Example: $\sum_{n=1}^{5} 2n = 2+4+6+8+10 = 30$

To these we add two more important symbols.

Product Notation (Π): Concise notation for the product of an indexed family of values.

- Example: $\prod_{n=1}^{5} 2n = (2)(4)(6)(8)(10) = 3840$

Combination Notation $\binom{x}{y}$**:** Shorthand notation for $\frac{x!}{y!(x-y)!}$. Used to count the number of unique groups of y objects chosen from a larger group x objects.

- Example: The number of unique 5-card poker hands in a deck of 52 cards is $\binom{52}{5} = \frac{52!}{(5!)(47!)} = 2{,}598{,}960$.[9]

Equipped with an understanding of the preceding mathematical notation, explorations into the enumeration of Morton Feldman's *Durations 2* and Karlheinz Stockhausen's *Klavierstück XI* are now possible. Chapter II enumerates the possible realizations of *Durations 2* and generalizes the results to encompass the enumeration of other indeterminate works of similar forms. These

[9] For a complete understanding of the following explorations, the identity $\binom{x}{y} = \binom{x}{x-y}$ should be mentioned. In essence, this simply states that there is the same number of 47-card groups (the complement) as 5-card groups.

explorations are followed by the enumeration of *Klavierstück XI* realizations in Chapter III, followed by some brief concluding remarks in Chapter IV.

CHAPTER II

MORTON FELDMAN'S *DURATIONS 2*

Feldman's *Durations 2* (1960) consists of 51 cello and 50 piano sonorities whose durations are chosen by the performers. The first line of the score, in addition to the composer's instructions, is shown in Figure 1. After a simultaneous entrance by both instruments, the performers are free to play their sonorities with any duration and thus have control over the order of attacks. These unique orders of cello and piano attacks are the realizations to be enumerated.

Notation for *Durations 2* Enumeration

Two additional pieces of notation will facilitate the following explorations.

(x,y): The number of possible realizations of x cello attacks and y piano attacks.

- Example: As will be discovered, the number of realizations of 2 cello attacks and 3 piano attacks is $(2,3) = 25$.

C_x* and *P_x | The x^{th} sonority of the cello and piano, respectively.

- Example: The 6th cello sonority can be referred to as C_6.

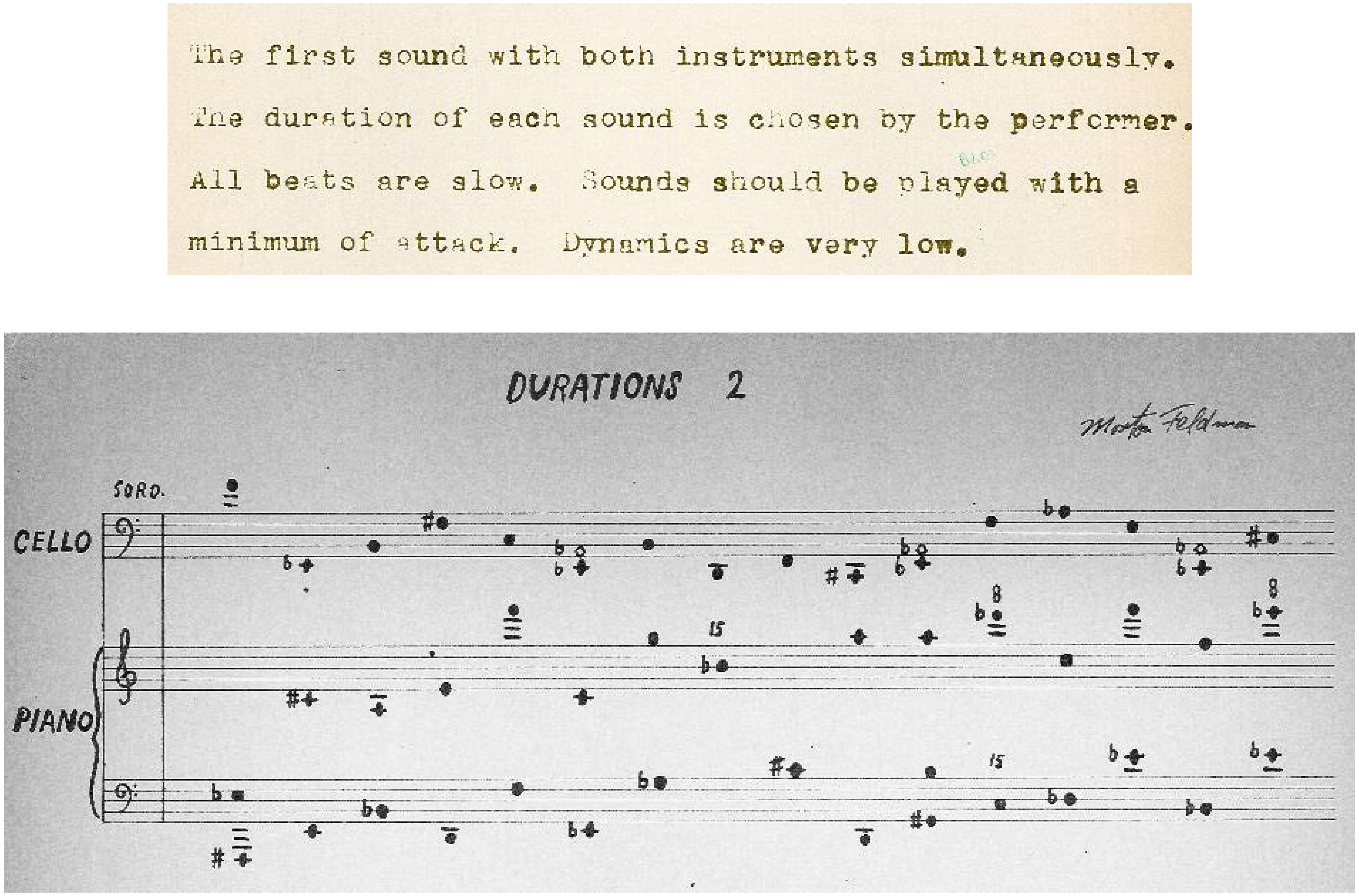
The first sound with both instruments simultaneously. The duration of each sound is chosen by the performer. All beats are slow. Sounds should be played with a minimum of attack. Dynamics are very low.

Figure 1. Directions for and First Line of *Durations 2*

Enumerating Realizations of *Durations 2*

In order to count the possible orderings of sounds between the cello and piano, we begin with the instruments' second sonorities (since the first sound must be performed simultaneously). There are exactly three possible relationships between these sonorities: the cello can precede the piano, the piano can precede the cello, or the sounds can be performed together. Thus, $(1,1)=3$. Expanding the possibilities to include two cello sonorities (C_1 and C_2), we next consider (2,1). Here, the piano sonority (P_1) can be performed in one of five places: before both cello sounds (option 1), together with C_1 (option 2), in between C_1 and C_2 (option 3), together with C_2 (option 4), or after both C_1 and C_2.

(2,1)		C_1		C_2	
1	P_1				
2		P_1			
3			P_1		
4				P_1	
5					P_1

Naturally, this situation can be reversed without changing the number of possible outcomes, as shown in the following table.

<table>
<tr><th>(1,2)</th><th colspan="2"></th><th>C_1</th><th colspan="2"></th></tr>
<tr><td>1</td><td>P_1</td><td>P_2</td><td></td><td colspan="2"></td></tr>
<tr><td>2</td><td colspan="2">P_1</td><td>P_2</td><td colspan="2"></td></tr>
<tr><td>3</td><td colspan="2">P_1</td><td></td><td colspan="2">P_2</td></tr>
<tr><td>4</td><td colspan="2"></td><td>P_1</td><td colspan="2">P_2</td></tr>
<tr><td>5</td><td colspan="2"></td><td></td><td>P_1</td><td>P_2</td></tr>
</table>

Thus, $(2,1) = (1,2) = 5$, and the reflexive property $(x, y) = (y, x)$ holds for the notation.

Proceeding to (3,1) and (4,1), a pattern soon emerges.

(3,1)		C_1		C_2		C_3	
1	P_1						
2		P_1					
3			P_1				
4				P_1			
5					P_1		
6						P_1	
7							P_1

(4,1)		C_1		C_2		C_3		C_4	
1	P_1								
2		P_1							
3			P_1						
4				P_1					
5					P_1				
6						P_1			
7							P_1		
8								P_1	
9									P_1

Thus, $(3,1) = (1,3) = 7$ and $(4,1) = (1,4) = 9$. Having discovered a pattern for cases of the form $(x,1)$, we move on to cases of the form $(x,2)$.

Remembering that $(2,1) = (1,2) = 5$, we can proceed to $(2,2)$. Taking the first two cello and piano sonorities as examples, the number of realizations are discovered by plotting all possibilities on the following table.

(2,2)		C_1		C_2	
1	P_1 P_2				
2	P_1	P_2			
3	P_1		P_2		
4	P_1			P_2	
5	P_1				P_2
6		P_1	P_2		
7		P_1		P_2	
8		P_1			P_2
9			P_1 P_2		
10			P_1	P_2	
11			P_1		P_2
12				P_1	P_2
13					P_1 P_2

Thus, $(2,2) = 13$. A quick exploration a bit further demonstrates that $(3,2) = (2,3) = 25$, $(4,2) = (2,4) = 41$, $(3,3) = 63$, $(4,3) = (3,4) = 129$ and $(4,4) = 321$. Clearly, all scenarios of the form $(0,x) = (x,0) = 1$, and so, we now plot all results (Table 1) as x and y range from 0 to 4 and look for patterns.

(x,y)	0	1	2	3	4
0	1	1	1	1	1
1	1	3	5	7	9
2	1	5	13	25	41
3	1	7	25	63	129
4	1	9	41	129	321

Table 1. Possible Realizations (x,y) $[0 \le x, y \le 4]$

Due to the lack of an obvious pattern, one initial reaction is to model each row or column with a suitable equation. This attempt seems manageable at first: row 0 can be modeled with the constant equation $r_0 = 1$ and row 1 with the linear equation $r_1 = 2x + 1$. Slightly more work uncovers the equations modeling rows 2, 3 and 4.

$$r_2 = 2x^2 + 2x + 1$$

$$r_3 = \frac{4}{3}x^3 + 2x^2 + \frac{8}{3}x + 1$$

$$r_4 = \frac{2}{3}x^4 + \frac{4}{3}x^3 + \frac{10}{3}x^2 + \frac{8}{3}x + 1$$

The fact that each row of our numbers requires a polynomial of different degree (each row n requiring a polynomial of degree n) is reminiscent of Pascal's Triangle. Shown in Figure 2, the triangle has applications from algebra to combinatorics and is derived using a relatively simple procedure: each number is the sum of the two numbers found diagonally above it. For example, the 10s in the fifth row are the sums of the 4 and 6 diagonally above.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Figure 2. Pascal's Triangle

The triangle shares an important property with the numbers of *Durations 2* realizations: like each row of Table 1, each diagonal column can be described with a polynomial of degree n, where n is the column number (starting at 0). For the numbers in Pascal's Triangle, the first three of those polynomials are $c_0 = 1$, $c_1 = x + 1$ and $c_2 = \frac{1}{2}x^2 + \frac{3}{2}x + 1$. Using polynomials to describe individual columns is rather cumbersome, however, and so a separate formula is used. That formula,

$$P(x, y) = \binom{x}{y} = \frac{x!}{y!(x-y)!}$$

concisely describes all numbers in Pascal's Triangle, where x is the row number and y is the diagonal column number.[10] As stated in the previous section, this notation also describes the number of unique groups of y objects chosen from a larger group of x objects. Taking row 4 ($x = 4$) as an example, there is 1 unique group of 0 objects [$\binom{4}{0} = 1$], 4 unique groups of 1 object [$\binom{4}{1} = 4$], 6 unique groups of 2 objects [$\binom{4}{2} = 6$], 4 unique groups of 3 objects [$\binom{4}{3} = 4$], and 1 unique group of 4 objects [$\binom{4}{4} = 1$].

Only after discovering the second similarity to *Durations 2* realizations, however, is this brief diversion into Pascal's Triangle fully justified. Shown as a triangle in Figure 3, the numbers of Table 1 not only require polynomials of increasing degree, they, too, can be derived using a simple procedure similar to that of Pascal's Triangle: each number is the sum of the two numbers found diagonally above it *and* the number *directly* above it. Correspondingly, each cell in Table 1 is the sum of the cell to the left, above and above-left. Only after observing the similarities of the polynomial descriptions of the numbers of Table 1 and the numbers in Pascal's Triangle does this elusive quality of the numbers become apparent.

[10] Setting y equal to a constant in this equation will produce a polynomial for describing the numbers of the y^{th} diagonal column.

1

1 1

1 3 1

1 5 5 1

1 7 13 7 1

1 9 25 25 9 1

Figure 3. Numbers of Table 1 in Triangle Form

Due to the abundant study of such patterns in the field of mathematics, it is not surprising that the Table 1 numbers have been previously encountered. First discovered in 1889 by the French mathematician Henri Delannoy, they appeared in his explorations of lattice path enumeration.[11] When counting the number of king's paths from a corner square of a chessboard to any other square (*x*,*y*) (prohibiting the king from moving backwards), Delannoy preempted our musical situation of free duration: whereas the king may move right, up, or diagonally up to the right from the bottom left-hand corner square, the cello may play before, after, or simultaneously with the piano. Viewing the *x* axis as the piano and the *y* axis as the cello, the congruence of such paths and free duration becomes clear, as shown in Figure 4. Here, a lattice path is employed to depict the following realizations of (4,4): C_1 and P_1 simultaneously, then C_2, then C_3 and P_2 simultaneously, then P_3, then P_4, and finally C_4.

[11] For an overview of Henri Delannoy's mathematical pursuits, see Banderier et al., "Why Delannoy Numbers?" *Journal of Statistical Planning and Inference* 135 (2004): 40-54.

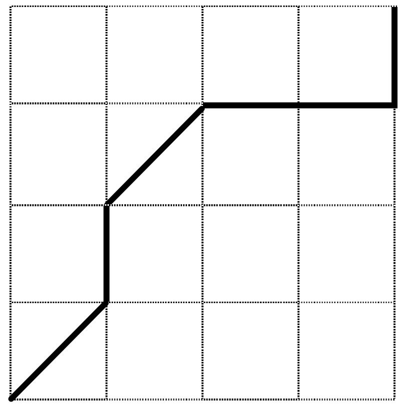

Figure 4. One of 321 Possible Realizations of (4,4)

Since their discovery, these numbers, called the Delannoy numbers, have been shown to have applications outside the realm of lattice path enumeration in fields as disparate as construction (counting floor tiles) to molecular biology (counting alignments between DNA sequences). To their many known applications, the enumeration of free duration between two instruments can now be added.

Of the multiple equations used to succinctly describe all Delannoy numbers, $D(x,y)=\sum_{n=0}^{x}\binom{x}{n}\binom{y}{n}2^{n}$ will be utilized due to its similarity to the equation that describes the numbers of Pascal's Triangle. Because the index of the summation notation ranges from 0 to x, the stipulation $x \le y$ must be made to ensure the bottom number of each combination notation is less than or equal to the top number. This requirement is of no burden, however, due to our notation's reflexive property. Therefore, Feldman's *Durations 2*, consisting of 51 cello attacks and 50 piano attacks, is modeled by (50,49) (one less for each instrument because of their simultaneous first attack), which, without loss of generality, can be reversed to (49,50), giving us the grand total of

$$D(49,50) = \sum_{n=0}^{49} \binom{49}{n}\binom{50}{n} 2^n = 6\,328374\,505422\,427820\,349527\,307448\,224645$$

realizations. This number is deceptively large: were every person living on earth today capable of performing two billion realizations a second with another person, it would take twice as long as the universe is old to perform them all.

Similar Works and Possible Extensions

Durations 2 is hardly the only work employing free duration whose realizations can be similarly enumerated. In the final movement of Feldman's *Last Pieces* (1959)[12], for example, the directions "durations are free for each hand" give sole control of the realization process to the single pianist. In *Durations 3* (1961)[13], a trio for violin, tuba and piano, Feldman introduces a third voice and therefore a third dimension to the lattice path analogy. Viewing the violin, tuba and piano as the *x*, *y* and *z* dimensions of a cube, the number of realizations of the 28 violin attacks, 40 tuba attacks and 42 piano attacks can be modeled by the

[12] Morton Feldman, *Solo Piano Works 1950-64* (New York: C. F. Peters, 1998).
[13] Morton Feldman, *Durations 3* (New York: C. F. Peters, 1961).

number of lattice paths from $(0,0,0)$ to $(28,40,42)$ in which seven "moves" are allowed:

$(1,0,0)$	the violin attacks
$(0,1,0)$	the tuba attacks
$(0,0,1)$	the piano attacks
$(1,1,0)$	the violin and tuba attack simultaneously
$(1,0,1)$	the violin and piano attack simultaneously
$(0,1,1)$	the tuba and piano attack simultaneously
$(1,1,1)$	the violin, tuba and piano attack simultaneously

As an example of just how many realizations this third dimension adds, consider a smaller work of twenty attacks for each of the three instruments. Modeled by the formula enumerating 3-dimensional Delannoy lattice paths, we have

$$\sum_{i=0}^{20}\sum_{j=0}^{20-i}\sum_{k=0}^{20}\binom{20}{i}\binom{20-i}{j}\binom{20}{k}\binom{20+i}{i+j}\binom{20}{i+k}2^{i+j+k}$$

or 1785 928841 771860 542254 045576 118961 realizations. Compared to the number of realizations of twenty attacks for only *two* instruments (like the original problem),

$$\sum_{n=0}^{20}\binom{20}{n}\binom{20}{n}2^{n} = 260\ 543813\ 797441,$$

one notices that the addition of an instrument (dimension) *more than squares* the number of realizations. Taking this one step further, Feldman's *Piece for Four Pianos* (1957)[14] is a score of 74 attacks given to four pianists who perform the identical score with personally chosen durations. Although an explicit formula modeling 4-dimensional Delannoy lattice paths is not available, a recursive algorithm can be employed to enumerate these multi-dimensional situations. A super-computer at Indiana University was required to calculate the number of realizations of *Piece for Four Pianos* (the algorithm can be found in Appendix A):

2813 984819
522048 638528 753141 769695 784932 097032 987905 749502 728120 279768 877480
339807 616963 263084 366030 231317 492576 803137 210956 605599 129442 997810
332485 506721 469756 898072 827415 581434 905013 243568 406405 506904 151051

The name of this 208-digit number, the largest by far found in this paper, begins two octosexagintillion, eight hundred thirteen septensexagintillion…

Morton Feldman is by no means the only composer employing free duration. Late in his compositional life, John Cage began writing free-durational works referred to by Pritchett as the "number pieces."[15] The titles of the pieces simply refer to the number of performers: *One* (1987)[16] is for a single pianist; *Five* (1988)[17] is for a string quintet. With few exceptions, these works employ "time

[14] Morton Feldman, *Piece for Four Pianos* (New York: C. F. Peters, 1961).
[15] James Pritchett, *The Music of John Cage* (New York: Cambridge University Press, 1996).
[16] John Cage, *One* (New York: C. F. Peters, 1988).
[17] John Cage, *Five* (New York: C. F. Peters, 1988).

brackets" at the beginning and end of each staff, denoting the timespans during which the performer must actualize the sonorities. For example, the first staff of *Two* for flute and piano includes the time bracket [0'00"–0'45"] at the beginning of the staff and [0'30"–1'15"] at the end, requiring the performer to begin playing the sonorities within the first 45-second span and finish playing them sometime within the second 45-second span. When the ending time bracket on one staff does *not* overlap the beginning time bracket on the subsequent staff, the situation can be modeled with n-dimensional Delannoy lattice paths, n referring to the number of separate voices (this is the case for *Two*^{2} for two pianos). However, more often than not Cage allows for a *blending* of staves by setting one staff's ending time *later* than the following staff's beginning time. This greatly complicates any enumeration of possible realizations and is fertile grounds for further study.

CHAPTER III

KARLHEINZ STOCKHAUSEN'S *KLAVIERSTÜCK XI*

Karlheinz Stockhausen's *Klavierstück XI* consists of 19 musical fragments spread around a poster-sized page. The performance directions, found on the reverse side of the score, can be paraphrased as follows:

- Randomly glance at any musical fragment and play it, choosing the tempo, dynamic and attack at will.
- At the end of the fragment are three symbols providing the tempo, dynamic and attack of the next fragment. Choose the next fragment at random and perform it according to these indications.
- Continue this process until a fragment is reached for the third time, at which moment one realization has ended.

These rules permit many possible realizations employing each of the 19 fragments once, twice or not at all.

<u>The Stockhausen Problem</u>

Naturally, one might wonder how many possible realizations of this indeterminate work exist. Were each fragment permitted only one occurrence, we'd have a congruent situation with Mallarmé's *Le Livre*, the performer permitted to play any number of the 19 fragments in any order, giving us

$$\sum_{n=1}^{19} \frac{19!}{(19-n)!} = 330665\ 665962\ 403999$$

possible realizations. Clearly, permitting a *second* occurrence of the musical fragments will dramatically increase this number. As per one of the initial premises, the three designations following each fragment (tempo, dynamic and attack) can be ignored, since they inform aspects of performance unrelated to the ordering of the musical fragments.

There are multiple facets to enumerating the possible realizations of *Klavierstück XI* that must be addressed before proceeding with any exploration. First, one must decide the vantage point from which the problem is approached. Since the last fragment is not played, the listener of a performance of the work will not be aware of that fragment's identity; the *performer*, however, will know, since he/she chooses it. In the following explorations, the vantage point of the listener is assumed unless otherwise stated. Second, there is a small amount of ambiguity in the directives given by Stockhausen which leads to two possible interpretations:

- <u>Interpretation one</u>—When Stockhausen writes "pick another fragment," he is prohibiting the performer from *immediately* choosing the preceding fragment again.

- Interpretation two—"Another fragment" means any fragment, including the one just performed.

Naturally, interpretation one seems more likely. When asked to pick a number one through ten, then to pick *another* number one through ten, most people choose *two* distinct numbers. However, that the choices of fragments are to be random lends credence to the second interpretation, since a truly random choice would equalize the probabilities of *all* fragments, including the one just performed.

The number of realizations of *Klavierstück XI* under the first interpretation of Stockhausen's directives is a difficult combinatorics problem and was solved by Lily Yen[18] and later generalized by Yen and Ronald Read.[19] Their solution provides a total of

1024 937361 666644 598071 114328 769317 982974

possible realizations assuming that fragments cannot be immediately repeated. Clearly, allowing fragments to be immediately repeated under interpretation two relaxes the situation, leading to a larger number of expected realizations.

[18] Lily Yen, "A Combinatorial Proof for Stockhausen's Problem," *SIAM Journal on Discrete Mathematics* 10, no. 3 (1997).
[19] Ronald Read and Lily Yen, "A Note on the Stockhausen Problem," *Journal of Combinatorial Theory* 76, no. 1 (1996): 1-10.

Notation for *Klavierstück XI* Enumeration

The following notation will be employed during the explorations of *Klavierstück XI*.

A, B, C, etc.: Uppercase letters denote repeated fragments in the underlying structure of a realization. In order to prevent double-counting, the first instance of A must precede the first instance of B, and so on.

- Example: A realization involving three repeated fragments can be notated ABACCB, but not BCBAAC. Although these two examples are apparently distinct, they reflect the same underlying structure and therefore only the first is used.

x: Lowercase *x*s denotes unrepeated fragments in the underlying structure of a realization.

- Example: A realization involving three repeated fragments and two unrepeated fragments can be notated xABACxCB.

Together, the uppercase letters and the lowercase *x*s denote the *configuration* of a realization. Configurations consisting solely of uppercase letters are called *base configurations*; these configurations contain only repeated fragments. The configurations themselves do not denote the fragments chosen for a specific realization, but rather show the overall *shape* or *form* of a realization. The specific fragments filling in the letters of the configuration is a separate matter.

Enumerating Realizations with Immediate Repetitions Permissible

In order to enumerate all possible realizations under the second interpretation of Stockhausen's directions (allowing for such realizations as xxAAxxx and AABBCCDD, uncounted in Yen and Read's enumeration), the realizations are broken up into 19 cases:

- Case 1. This case includes all realizations in which exactly one (1) fragment is repeated. In order for a performance to be complete, one fragment must be chosen for the third time, requiring at least one fragment to be repeated in each realization. Case 1 counts those realizations that meet this requirement at its very minimum. Realizations from Case 1 will range in length from 2 fragments (AA—the shortest overall form of realization) to 20 fragments (the realizations which include a single instance of each of the other 18 fragments [xxxxxAxxxxxxxxxxAxxx, for example]). Other examples of Case 1 realizations are xxAAx, AxxxxxxxxxxxAx and xAxxxxxA.
- Case 2. This case includes all realizations in which exactly two (2) fragments are repeated. Such realizations can range from 4 in length (ABBA, for example) to 21 (in which the other 17 fragments are all present [xxxxAxxxBBxxxxxxAxxx, for example]). Other examples of Case 2 realizations are xxABxxxBxxA, AAxxxBxxBx and ABABxxxxx.

- Case 3. This case includes all realizations in which exactly three (3) fragments are repeated. These realizations range from 6 in length (ABACCB, for example) to 22 in length (xxxAxxBxACBxxxxxCxxxxx, for example).

Cases 4-18 continue in the pattern, including those realizations in which exactly four, five, six, … , eighteen fragments are repeated, bringing us to Case 19.

- Case 19. This case includes all realizations in which each of the nineteen (19) fragments is repeated. Every realization in Case 19 will be of length 38, since each fragment will appear exactly twice. Once each fragment has made its second appearance, any choice for the next fragment would affect its third instance and thus result in the end of the performance.

By design, Cases 1-19 are mutually exclusive: all possible realizations fit nicely into *exactly* one of them. Therefore, by counting all possible realizations in each case and adding them all together, we are assured that 1) no realization will be double-counted, and 2) all possible realizations are accounted for. To begin, let us consider Case 1.

The shortest of all Case 1 realizations is 2 fragments in length, having the configuration AA. Here, any of the fragments is chosen, immediately repeated, and chosen again for the third time, thereby terminating the performance. A Case 1 realization of length 3 takes one of the three following configurations: AAx, AxA or xAA. A Case 1 realization of length 4 takes any one of six configurations: AAxx, AxAx, AxxA, xAAx, xAxA or xxAA. Such calculation quickly becomes cumbersome, since Case 1 realizations range to 20 in length. Therefore, we mathematically describe the number of configurations for each length n of Case 1 realizations ($2 \le n \le 20$) using a combination, since the repeated fragment A must hold 2 positions in any realization of length n.

length n	sample configuration	number of configurations
2	AA	1
3	AAx	3
4	xAxA	6
5	xAAxx	10
$2 \le n \le 20$		$\binom{n}{2}$

Having mathematically modeled the number of configurations for realizations of Case 1, the choices for specific fragments are considered. There are 19 possible performances of length 2, since there are 19 fragments and only 1 configuration (AA) from which to choose. For length 3 realizations, there are 19 choices for the repeated fragment, 18 choices for the unrepeated fragment, and 3 choices for the configuration (AAx, AxA or xAA), giving us a total of $(19)(18)(3) = 1026$ realizations of length 3. Length 4 realizations follow in this vein, with 19 choices for the repeated fragment, 18 and 17 choices for the two unrepeated fragments, and 6 choices for the configuration, providing a total of $(19)(18)(17)(6) = 34884$

realizations of length 4. As can be seen in Table 2, the number of realizations at each length can be found by multiplying the number of fragment choices by the number of configurations.

length n	sample configuration	fragment choices	number of configurations	number of realizations
2	AA	19	1	(19)(1)
3	AAx	(19)(18)	3	(19)(18)(3)
4	xAxA	(19)(18)(17)	6	(19)(18)(17)(6)
5	xAAxx	(19)(18)(17)(16)	10	(19)(18)(17)(16)(10)
$2 \le n \le 20$		$\frac{19!}{(20-n)!}$	$\binom{n}{2}$	$\frac{19!}{(20-n)!}\binom{n}{2}$

Table 2. Summary of Case 1 Realizations

Having derived a general equation that counts all possible Case 1 realizations of length n ($2 \le n \le 20$), summation notation is now employed, providing the total number of Case 1 realizations.

$$\sum_{n=2}^{20} \frac{19!}{(20-n)!}\binom{n}{2} = 56\,709161\,712552\,286009$$

Let us now consider Case 2 realizations, where exactly 2 fragments are repeated. As stated before, the lengths of Case 2 realizations range from 4 to 21; however, the number of configurations is less straightforward than in Case 1. Starting with the shortest length realization ($n = 4$), there are three base configurations: AABB, ABAB or ABBA. A Case 2 realization of length 5 includes one unrepeated musical fragment and can take any one of the five positions in any of the three aforementioned base configurations, granting a total of 15 possibilities.

AABBx	ABABx	ABBAx
AABxB	ABAxB	ABBxA
AAxBB	ABxAB	ABxBA
AxABB	AxBAB	AxBBA
xAABB	xABAB	xABBA

For realizations of length 6, the two unrepeated fragments can occur in 15 different positions in each of the three base configurations, increasing the possible configurations to $(3)(15) = 45$.

AABBxx	ABABxx	ABBAxx
AABxBx	ABAxBx	ABBxAx
AAxBBx	ABxABx	ABxBAx
AxABBx	AxBABx	AxBBAx
xAABBx	xABABx	xABBAx
AABxxB	ABAxxB	ABBxxA
AAxBxB	ABxAxB	ABxBxA
AxABxB	AxBAxB	AxBBxA
xAABxB	xABAxB	xABBxA
AAxxBB	ABxxAB	ABxxBA
AxAxBB	AxBxAB	AxBxBA
xAAxBB	xABxAB	xABxBA
AxxABB	AxxBAB	AxxBBA
xAxABB	xAxBAB	xAxBBA
xxAABB	xxABAB	xxABBA

To mathematically describe the configurations for Case 2, we notice that the two repeated fragments occupy four places in a string of length n ($4 \le n \le 21$) and can occur in one of three orders (the base configurations AABB, ABAB and ABBA). Therefore, a combination can be similarly employed to describe the number of possible configurations for each length of Case 2 realizations.

length n	sample configuration	number of configurations
4	AABB	3
5	ABxAB	(3)(5)
6	xABBxA	(3)(15)
$4 \le n \le 21$		$(3)\binom{n}{4}$

Just as in Case 1, we now multiply the number of fragment choices by the possible configurations at each length, as shown in Table 3.

length n	sample configuration	fragment choices	number of configurations	number of realizations
4	AABB	(19)(18)	3	(19)(18)(3)
5	ABxAB	(19)(18)(17)	(3)(5)	(19)(18)(17)(3)(5)
6	xABBxA	(19)(18)(17)(16)	(3)(15)	(19)(18)(17)(16)(3)(15)
$4 \le n \le 21$		$\frac{19!}{(21-n)!}$	$(3)\binom{n}{4}$	$(3)\frac{19!}{(21-n)!}\binom{n}{4}$

Table 3. Summary of Case 2 Realizations

To arrive at the total number of Case 2 realizations, summation notation is again employed to add the number of realizations of each length n ($4 \le n \le 21$).

$$\sum_{n=4}^{21}(3)\frac{19!}{(21-n)!}\binom{n}{4} = 4888\,106540\,297482\,431276$$

Exploring Case 3 realizations will elucidate the emerging patterns, allowing us to predict the behaviors of Cases 4-19. Since Case 3 realizations require three repeated fragments, the shortest realizations will be of length 6. The base configurations for Case 3 can be found by placing a new fragment at the beginning of the base configurations for Case 2 and placing its repetition in each of the 5 remaining positions. Using each of the three base configurations from Case 2 (relabeled as BBCC, BCBC and BCCB below), we arrive at 15 base configurations for Case 3.

AABBCC	AABCBC	AABCCB
ABABCC	ABACBC	ABACCB
ABBACC	ABCABC	ABCACB
ABBCAC	ABCBAC	ABCCAB
ABBCCA	ABCBCA	ABCCBA

Now that the base configurations for Case 3 have been determined, the configurations can be lengthened with unrepeated fragments. An unrepeated fragment in a realization of length 7 can be placed in any one of seven positions of the 15 base configurations (using the base form AABBCC as an example: AABBCCx, AABBCxC, AABBxCC, AABxBCC, AAxBBCC, AxABBCC and xAABBCC), giving us a total of $(15)(7) = 105$ different configurations of length 7. Case 3 realizations of length 8 require two unrepeated fragments; for each of the 15 base configurations, these fragments can appear in 28 arrangements (choosing 2 positions from a string of 8), thereby providing a total of $(15)(28) = 420$

different configurations of length 8. Table 4 can now be constructed to help attain a generalized equation for enumerating Case 3 realizations.

length n	sample configuration	fragment choices	number of configurations	number of realizations
6	ABACCB	(19)(18)(17)	15	(19)(18)(17)(!5)
7	ABACxCB	(19)(18)(17)(16)	(15)(7)	(19)(18)(17)(16)(15)(7)
8	AxBACxCB	(19)(18)(17)(16)(15)	(15)(28)	(19)(18)(17)(16)(15)(15)(28)
$6 \le n \le 22$		$\frac{19!}{(22-n)!}$	$(15)\binom{n}{6}$	$(15)\frac{19!}{(22-n)!}\binom{n}{6}$

Table 4. Summary of Case 3 Realizations

Just as in the previous Cases 1 and 2, all Case 3 realizations of length n ($6 \le n \le 22$) can be enumerated using summation notation.

$$\sum_{n=6}^{22}(15)\frac{19!}{(22-n)!}\binom{n}{6} = 280394\ 075656\ 508817\ 201930$$

Having created a generalized formula for counting the realizations of Cases 1, 2 and 3, patterns are now observed in order to generalize all Cases 1 through 19. Viewing the generalized formulas for Cases 1-3 together, their common attributes become apparent.

case c	generalized equation
1	$\sum_{n=2}^{20} \frac{19!}{(20-n)!}\binom{n}{2}$
2	$\sum_{n=4}^{21} (3)\frac{19!}{(21-n)!}\binom{n}{4}$
3	$\sum_{n=6}^{22} (15)\frac{19!}{(22-n)!}\binom{n}{6}$

Subsequently, we can write an equation for a generalized Case c.

$$\sum_{n=min\,length}^{max\,length} (base\,configurations)\frac{19!}{(max\,length-n)!}\binom{n}{min\,length}$$

In our generalized equation, there are three variables: minimum length, maximum length and number of base configurations. Before the equation can be of use, each variable must be related to a common variable—the case number c. Relating two of these variables to c is relatively simple: the minimum length is twice the number of repeated fragments ($2c$) and the maximum length is $c+19$. Relating the number of base configurations to c, however, is not quite as straightforward. In finding the number of base configurations for Case 3 realizations, we noticed that the new repeated fragment could occupy five positions in each of the three base configurations from Case 2, resulting in a total of $(3)(5)=15$ base configurations for Case 3. The base configurations for Case 4 can be attained similarly, since the new repeated fragment could hold seven positions in each of the 15 base configurations for Case 3. This pattern can be generalized with an equation employing product notation.

case c	sample base configuration	number of base configurations
1	AA	1
2	ABBA	(1)(3)
3	ABCBAC	(1)(3)(5)
4	ABACDDBC	(1)(3)(5)(7)
$1 \le c \le 19$		$\prod_{y=1}^{c}(2y-1)$

Having related the three variables to the common variable c, a generalized equation for any Case c ($1 \le c \le 19$) is now constructed

$$\sum_{n=2c}^{c+19}\left[\left(\prod_{y=1}^{c}(2y-1)\right)\frac{19!}{(c+19-n)!}\binom{n}{2c}\right]$$

and summation notation is again employed, enumerating all realizations of Cases 1-19 by setting c to range from 1 to 19.

$$\sum_{c=1}^{19}\left\{\sum_{n=2c}^{c+19}\left[\left(\prod_{y=1}^{c}(2y-1)\right)\frac{19!}{(c+19-n)!}\binom{n}{2c}\right]\right\}$$

Evaluating this expression gives all possible realizations when immediate repetitions of a musical fragment are permissible, the second interpretation of Stockhausen's directives. The total, 2748 629142 904964 305737 593042 179938 584355,

is about 2.5 times larger than when immediate repetitions are prohibited (interpretation one). Its size is quite difficult to grasp; if we possessed that many cups of water, we could fill up the oceans roughly as many times as there are cups of water in the oceans.

Viewing the problem from a performer's point of view, the number of realizations will naturally increase, since the performer has multiple choices as to which fragment he/she returns to for the third time, thereby ending the performance. Because the performer has exactly c choices for this culminating fragment, the generalized equation is multiplied by c and Cases 1-19 are summed.

$$\sum_{c=1}^{19}\left\{\sum_{n=2c}^{c+19}\left[\left(\prod_{y=1}^{c}(2y-1)\right)\frac{19!}{(c+19-n)!}\binom{n}{2c}(c)\right]\right\}$$

This yields a number nearly 18 times as large as the number of realizations from a listener's perspective.

Varying the Number of Starting Fragments

In order to fully generalize our equation, a different number of starting fragments and required repetitions must be allowed, permitting a composer wishing to begin with 10 fragments or requiring 6 repetitions to calculate the total number of possible realizations. To begin, varying the number of starting fragments f is considered. The variable f will be the new upper bound to the outermost summation, since the equation sums case by case until the final case (the number of starting fragments) is reached; therefore, $\sum_{c=1}^{19}$ is replaced with $\sum_{c=1}^{f}$. The inside summation, $\sum_{n=2c}^{c+19}$, sums all lengths for each case, starting from the minimum length (2*c*) to the maximum length ($c+19$). As was discovered before, the maximum length for each case is simply the case plus the number of fragments, so substituting $c+f$ for $c+19$ completes this generalization.

$$\sum_{c=1}^{f}\left\{\sum_{n=2c}^{c+f}\left[\left(\prod_{y=1}^{c}(2y-1)\right)\frac{f!}{(c+f-n)!}\binom{n}{2c}\right]\right\}$$

Clearly, letting $f=19$ models the original problem. Letting $f=10$ models a piece with ten original starting fragments, only allowing for

6630 796791 915670

realizations, a number dwarfed by the outcomes when $f=19$.

<u>Varying the Number of Required Repetitions</u>

Enumerating the possible realizations of similar works with a different number of required repetitions is undoubtedly the most difficult task yet encountered. Although the approach taken in the original explorations is manageable and clear, a similar approach to the situation of four required repetitions, for example, quickly becomes burdensome. Not only does the equation for base configurations need rethinking, the entire case system on which the solution to the original problem rests requires heavy modification. The previous case system separating realizations by their number of repeated fragments is no longer enough: because realizations will now have at least one fragment repeated *three* times, subcases are necessary to further separate these realizations by their number of *twice*-repeated fragments. For example, Case 3 realizations—those with three fragments repeated three times—require separate subsets: Case 3.1 includes those realizations with one fragment repeated *twice*, Case 3.2 includes those realizations with two fragments repeated *twice*, and so on.

Although cumbersome, this method does indeed work. Once the equation for base configurations is modified to $\prod_{y=1}^{c}\binom{3y-1}{2}$ (a derivation left to the reader), one can enumerate the realizations of Case 1.0 (one thrice-repeated fragment, no other repeated fragment), Case 1.1 (one thrice-repeated fragment, one twice-repeated fragment), Case 1.2 (one thrice-repeated fragment, two twice repeated fragments), all the way to Case 1.18 (one thrice-repeated fragment, eighteen twice-repeated fragments), then sum them all. Cases 2 through 19 can be similarly enumerated, each subsequent case requiring one less subcase.

Such bottom-up explorations (those employing equations that recursively build on other equations) do not rival the power and elegance of a top-down approach—one in which *every* possible number of required repetitions is considered *at once*. This elegance, though, does have a drawback: grasping such an approach requires mathematics training far beyond that of the vast majority of musicians. The approach taken in this paper, *specifically* designed for an audience of musicians, has served its purpose for *Klavierstück XI*; we leave the generalized equations for *all* possible numbers of required repetitions to higher-level combinatorics. (The referenced generalized equation can be found in Appendix B.)

CHAPTER IV

CONCLUDING REMARKS

After much effort deriving equations that mathematically model the number of possible realizations of an indeterminate score, it is fair to question the overall purpose of such a study. Most musicians would agree that a performance of a work is more than the specific pitches played or the specific bowings and fingerings employed. Although a substantial amount of a musician's training is spent mastering the execution of these skills, these efforts are augmented by studies in historical contexts, notational procedures and theoretical grammars—in short, the elements that *inform* an interpretation, thereby separating the performances of robots from those of sentient human musicians. It seems obvious that a poet, having written a poem for recitation before an audience, would desire his/her performer to be able to not only *speak* the individual words of the poem, but also to have reached some understanding of their *meaning*. Likewise, musical performances tend to fall flat when notes are simply *played*, the performer giving the audience no sense of personal voice, direction, context or signification.

Before performing any work (a Beethoven sonata, for example), the list of concepts with which a well-informed musician must first become acquainted is staggering, including (at the minimum) form, harmony, tradition, mechanics of the instrument, purpose for writing and mental state of the composer. Only with a suitable understanding of the preceding elements can a performer produce a well-informed performance, personally choosing which elements he/she wishes to highlight.

The mathematical explorations in this paper provide an additional element by which a performer of indeterminate music can be informed. Just as the performer of a Beethoven sonata chooses to highlight some elements and not others, the performer of indeterminate music chooses to realize the score in one way and not another. Without an understanding of the indeterminate score's possibilities, the performance is like that of a Beethoven sonata without an understanding of sonata form.

Much musical instruction is devoted to the study of the musical universe in which a work resides, including its musical language and grammar. For indeterminate music, that language and grammar is not found solely in the pitches but also in the performance directions given by the composer. Its musical universe, in addition to the musical language found on the score, is created by these performance directions and thereby warrants the same treatment given to the musical universes of composers from the past.

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APPENDIX A. ALGORIGHTM FOR FOUR-DIMENSIONAL DELANNOY NUMBERS

The following program, written by Bob Sulanke for the mathematics program Maple, provides a list of the first 75 central 4-dimensional Delannoy numbers. Although designed to compute 4-dimensional Delannoy numbers, it can be used to compute the 2- and 3-dimensional numbers as well.

```
dela := proc(x,y,z,w)
local ret
option remember

if x < 0 or y < 0 or z < 0 or w < 0 then ret := 0
else
if x = 0 and y = 0 and z = 0 and w = 0 then ret := 1
else
ret := dela(x-1,y,z,w)+dela(x,y-1,z,w)+dela(x,y,z-1,w)+
dela(x,y,z,w-1)+dela(x-1,y-1,z,w)+dela(x-1,y,z-1,w)+
dela(x-1,y,z,w-1)+dela(x,y-1,z-1,w)+dela(x,y-1,z,w-1)+
dela(x,y,z-1,w-1)+dela(x,y-1,z-1,w-1)+dela(x-1,y,z-1,w-1)+
dela(x-1,y-1,z,w-1)+dela(x-1,y-1,z-1,w)+dela(x-1,y-1,z-1,w-1)
end if
end if;

ret
end proc;

for i from 0 to 75 do print(dela(i,i,i,i)) end do;
```

Everyday computers, however, cannot handle the amount of recursion required to calculate the 74th central 4-dimensional Delannoy number; therefore, a supercomputer was used. The following program, written and run by Thom Sulanke at the University of Indiana, provided the 208-digit number of *Piece for Four Pianos* realizations.

```
main()
{

//    n := 80

//    OLD := array(1..n, 1..n,1..n);
 mpz_t OLD[n][n][n];
//    NEW := array(1..n, 1..n,1..n);
 mpz_t NEW[n][n][n];

 int i,j,k,L;

//    for i from 1 to n do
//    for j from 1 to n do
//    for k from 1 to n do
//    OLD[i,j,k] := 0;  NEW[i,j,k] := 0;
//    end do end do end do;

 for (i = 0; i < n; i++)
   for (j = 0; j < n; j++)
     for (k = 0; k < n; k++)
       {
         mpz_init(OLD[i][j][k]);
         mpz_init(NEW[i][j][k]);
       }

//    OLD[1,1,1] := 1;
 mpz_add_ui(OLD[0][0][0], OLD[0][0][0], 1);

//    for L from 2 to n do
//    for i from 2 to n do
//    for j from 2 to n do
//    for k from 2 to n do

 for (L = 1; L < n; L++) {
   for (i = 1; i < n; i++)
     for (j = 1; j < n; j++)
       for (k = 1; k < n; k++) {
//    NEW[i,j,k] :=
NEW[i-1,j,k]+NEW[i,j-1,k]+NEW[i,j,k-1]+NEW[i-1,j-1,k]+NEW[i-
1,j,k-1]+
//              NEW[i,j-1,k-1]+NEW[i-1,j-1,k-1]+
//              OLD[i,j,k]+OLD[i-1,j,k]+OLD[i,j-1,k]+OLD[i,j,k-
1]+
//              OLD[i,j-1,k-1]+OLD[i-1,j,k-1]+OLD[i-1,j-1,k]+
                OLD[i-1,j-1,k-1]
//    end do end do end do;
         mpz_add(NEW[i][j][k],NEW[i-1][j][k],NEW[i][j-1][k]);
         mpz_add(NEW[i][j][k],NEW[i][j][k],NEW[i][j][k-1]);
         mpz_add(NEW[i][j][k],NEW[i][j][k],NEW[i-1][j-1][k]);
         mpz_add(NEW[i][j][k],NEW[i][j][k],NEW[i-1][j][k-1]);
         mpz_add(NEW[i][j][k],NEW[i][j][k],NEW[i][j-1][k-1]);
         mpz_add(NEW[i][j][k],NEW[i][j][k],NEW[i-1][j-1][k-1]);
         mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i][j][k]);
```

```
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i-1][j][k]);
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i][j-1][k]);
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i][j][k-1]);
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i][j-1][k-1]);
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i-1][j][k-1]);
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i-1][j-1][k]);
          mpz_add(NEW[i][j][k],NEW[i][j][k],OLD[i-1][j-1][k-1]);
        }
//    for i from 1 to n do
//    for j from 1 to n do
//    for k from 1 to n do
//    OLD[i,j,k] := NEW[i,j,k]
//    end do end do end do;
   for (i = 0; i < n; i++)
     for (j = 0; j < n; j++)
       for (k = 0; k < n; k++)
         mpz_set(OLD[i][j][k],NEW[i][j][k]);

//    print(L-1,NEW[L,L,L]):
   fprintf(stdout,"%d ", L);
   mpz_out_str(stdout,10,NEW[L][L][L]);
   fprintf(stdout,"\n");
//    end do;
 }
}
```

APPENDIX B. GENERALIZED EQUATION FOR ANY NUMBER OF REQUIRED REPETITIONS

The following formula was provided by Carlos Nicolas after proofreading an earlier version of the paper.

> Assuming that there are 19 fragments to choose from, the number of realizations with $r \geq 2$ repetitions having c_i fragments repeated exactly i times $(1 \leq i \leq r-1)$ is equal to
>
> $$\frac{(c_1 + 2c_2 + \ldots + (r-1)c_{r-1})!}{(1!)^{c_1}(2!)^{c_2}\ldots((r-1)!)^{c_{r-1}} c_1! c_2! \ldots c_{r-1}!} \frac{19!}{(19 - c_1 - \ldots - c_{r-1})!}.$$
>
> Therefore, the total number of realizations with $r \geq 2$ repetitions is
>
> $$\sum \frac{(c_1 + 2c_2 + \ldots + (r-1)c_{r-1})!}{(1!)^{c_1}(2!)^{c_2}\ldots((r-1)!)^{c_{r-1}} c_1! c_2! \ldots c_{r-1}!} \frac{19!}{(19 - c_1 - \ldots - c_{r-1})!}$$
>
> where the sum ranges over all $c_1, \ldots, c_{r-1}$ satisfying:
> (1) $c_1 + c_2 + \ldots + c_{r-1} \leq 19$.
> (2) $c_i \geq 0$ for $1 \leq i \leq r-1$.
> (3) $c_{r-1} \geq 1$.

Also provided by Dr. Nicolas was a derivation showing how to transform the above formula into the equation derived in this paper.

$$\sum_{\substack{c_1 + c_2 \leq 19 \\ c_1 \geq 0, c_2 \geq 1}} \frac{(c_1 + 2c_2)!}{(1!)^{c_1}(2!)^{c_2} c_1! c_2!} \frac{19!}{(19 - c_1 - c_2)!}$$

$$= \sum_{\substack{n \leq 19 + c_2 \\ c_2 \geq 1, n \geq 2c_2}} \frac{n!}{(1!)^{n-2c_2}(2!)^{c_2}(n - 2c_2)! c_2!} \frac{19!}{(19 - n + c_2)!}$$

$$= \sum_{c_2=1}^{19} \sum_{n=2c_2}^{19+c_2} \frac{n!}{2^{c_2}(n-2c_2)!c_2!} \frac{19!}{(19-n+c_2)!}$$

$$= \sum_{c=1}^{19} \sum_{n=2c}^{19+c} \left[\frac{(2c)!}{2^c c!}\right] \frac{19!}{(19-n+c)!} \frac{n!}{(2c)!(n-2c)!}$$

$$= \sum_{c=1}^{19} \sum_{n=2c}^{19+c} \left[\prod_{y=1}^{c} (2y-1)\right] \frac{19!}{(c+19-n)!} \binom{n}{2c}$$