VESIENTUTKIMUSLAITOKSEN JULKAISUJA
PUBLICATIONS OF THE WATER RESEARCH INSTITUTE 34

**VESIHALLITUS—NATIONAL BOARD OF WATERS, FINLAND**
**Helsinki 1979**

ISBN 951-46-4609-6
ISSN 0355-0982

Helsinki 1980. Valtion painatuskeskus

# A PHOSPHORUS RETENTION MODEL AND ITS APPLICATION TO LAKE PÄIJÄNNE

Kalle Matti Lappalainen[1], Jorma Niemi[2] & Kari Kinnunen[3]

LAPPALAINEN, K.M., NIEMI, J.S. & KINNUNEN, K.A.I. 1979. A phosphorus retention model and its application to Lake Päijänne. Publications of the Water Research Institute, National Board of Waters, Finland, No. 34.

A statistical, steady state phosphorus retention model developed for lakes is presented. On the basis of given values of phosphorus loading, discharge and lake volume, the model calculates a phosphorus retention coefficient and, by means of this, the annual average phosphorus concentration of the lake epilimnion. Because the retention coefficient is a function of the phosphorus loading, the model is particularly suitable for making prognoses with alternative phosphorus loadings. The calculated concentrations were found to correspond reasonably well with the observed values for five sub-basins. In addition the model was used to calculate the average phosphorus concentrations of the uppermost basin of Lake Päijänne with alternative loadings and discharges. The phosphorus loadings were allowed to vary from 2 000 to 24 000 mg $s^{-1}$ with three discharges, namely 100 $m^3 s^{-1}$, 150 $m^3 s^{-1}$, and 200 $m^3 s^{-1}$. Because of the lack of observed data the calculated results could not be verified.

Index words: Mathematical models, water quality models, phosphorus retention, pollution, Lake Päijänne.

## 1. INTRODUCTION

A number of phosphorus models developed for lakes have been described in the literature (Vollenweider 1968, 1969, 1975, Lorenzen 1973, Snodgrass and O'Melia 1975, Lappalainen 1975, and Chapra 1976). In all phosphorus retention models, an attempt is made to find a relationship between the physical and chemical factors of the lake and sedimentation of phosphorus. A widely used principle for calculating the sedimentation of phosphorus (Vollenweider 1969, 1975, Sonzogni et al. 1976) is (1)

$$\frac{dC}{dt} = \frac{Q}{V}C_o - \frac{Q}{V}C - kC \qquad (1)$$

in which

$\frac{dC}{dt}$ = change of phosphorus concentration per unit time

$Q$ = discharge

$V$ = volume of lake

---

[1] Water Protection Association of Savo, Neulamäen teollisuuskylä, 70700 Kuopio 70

[2] National Board of Waters, P.O.B. 250, 00101 Helsinki 10

[3] Oy International Business Machines Ab, P.O.B. 265, 00101 Helsinki 10

C = average phosphorus concentration of lake
$C_o$ = initial phosphorus concentration (=I/Q, total phosphorus input to the lake divided by discharge)
k = coefficient for the sedimentation rate of phosphorus

When C is calculated from Eq. (1) in a steady state (dC/dt = 0), we get

$$C = \frac{QC_o/V}{Q/V+k} \qquad (2)$$

Eq. (2) has often been presented with different symbols as follows (e.g. by Dillon and Rigler, 1974):

$$P = \frac{L_P}{z(\rho+\sigma)} \qquad (3)$$

in which
P = average phosphorus concentration in the lake
$L_P$= phosphorus loading calculated per surface area
z = average depth of the lake
$\rho$ = flushing rate (=Q/V)
$\sigma$ = coefficient for sedimentation rate

When k is calculated from Eq. (2) we get (4)

$$k = \frac{Q}{V}\left(\frac{C_o}{C} - 1\right) \qquad (4)$$

According to Eq. (4), when the volume and discharge of the lake are constant and the loading increases, there is, according to wide experience, an increase in $C_o/C$ too and thus the value of k also increases. Consequently, the coefficient for sedimentation rate is not constant. It is therefore difficult to use Eq. (1) to make reliable prognoses when there are essential changes in phosphorus loading.

Dillon and Rigler (1974) used a model in which the phosphorus retention coefficient R was calculated (5)

$$R = \frac{QC_o-QC}{QC_o} = \frac{I-O}{O} \qquad (5)$$

in which previously undefined symbols are
I = phosphorus input
O = annual output of phosphorus from the lake

On the basis of Eq. (5), the phosphorus concentration of a lake in steady state can thus be calculated (6)

$$C = (1-R)C_o \qquad (6)$$

Equation 6 can be used for making prognoses only if the phosphorus retention coefficient R is constant with the loading and discharge values used in the calculations.

Kirchner and Dillon (1975) found that R correlated with the hydraulic surface loading (Q/A) and suggested Eq. (7)

$$R = 0.426 \exp\left(-0.271\frac{Q}{A}\right) + 0.574 \exp\left(-0.00949\frac{Q}{A}\right) \qquad (7)$$

Larsen and Mercier (1976) constructed three models based on material found in the literature. In all three models R was a function of (Q/A) or (Q/V). These models were applied to lakes with total phosphorus concentration $\leqslant$ 25 mg $l^{-1}$. The models are (8), (9) and (10)

$$R = \frac{1}{1+\left(\frac{Q}{V}\right)^{1/2}} \qquad (8)$$

$$R = 0.482-0.112 \ln\frac{Q}{V} \qquad (9)$$

$$R = 0.86-0.143 \ln\frac{Q}{A} \qquad (10)$$

This paper presents an improved version of the phosphorus model published by Lappalainen (1975), as well as the results obtained from applying the model to the sub-basins of Lake Päijänne (Fig. 1). In addition the model was used to calculate phosphorus concentrations with alternative loadings and discharges in sub-basin No. 1 of Lake Päijänne.

## 2. THE MODEL

Lappalainen (1974, 1975) presented a model which calculates the phosphorus retention coefficient when the volume of the lake, phos-

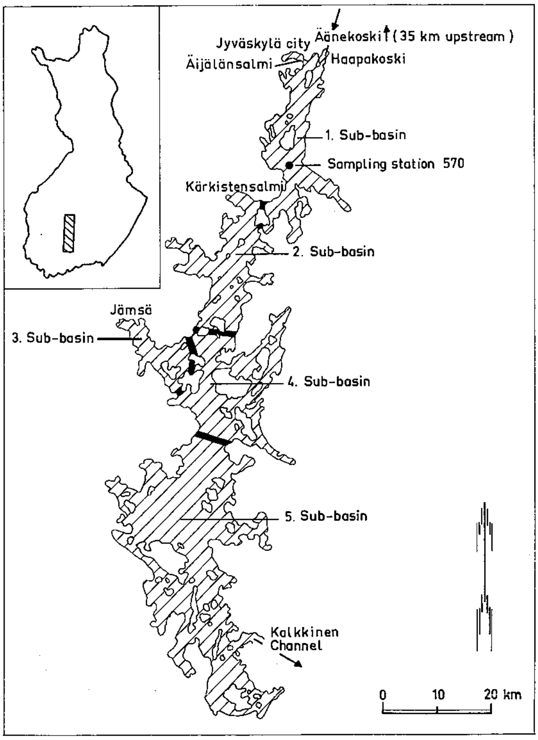

Fig. 1. Division of Lake Päijänne into sub-basins.

Table 1. Symbols used in the model.

| Symbol | Meaning | Dimension |
|---|---|---|
| I | Annual total phosphorus input to the lake | mg $s^{-1}$ |
| 0 | Annual output of phosphorus from the lake | mg $s^{-1}$ |
| S | = I–0 Annual loss of phosphorus | mg $s^{-1}$ |
| r | = S/I·100 phosphorus retention coefficient in per cent | |
| R | = S/I phosphorus retention coefficient | |
| A | surface area of the lake | $m^2$ |
| $\bar{h}$ | average depth of the lake | m |
| V | = A·$\bar{h}$ volume of the lake | $m^3$ |
| Q | average discharge | $m^3 s^{-1}$ |
| T | = V/Q theoretical detention time | month |
| C | average phosphorus concentration of the lake | mg $m^{-3}$ |
| $C_o$ | = I/Q initial phosphorus concentration | mg $m^{-3}$ |
| $C_1$ | phosphorus concentration of the water outflowing from the lake | mg $m^{-3}$ |
| k | sedimentation rate of phosphorus | $d^{-1}$ |

phorus loading entering it, and the discharge of the lake are given. By means of the retention coefficient the model calculates the annual average phosphorus concentration of the lake. The model assumes that phosphorus is a limiting factor in the lake. The symbols used in the model are presented in Table 1.

The model assumes a steady state (dC/dt=0)

$$\text{Then } S = I - 0 \qquad (11)$$

in which
I = phosphorus input
0 = phosphorus output
S = loss of phosphorus

The model calculates the annual average phosphorus concentration. The phosphorus retention coefficient in per cent (r) is defined by Eq. (12)

$$r = \frac{S}{I} \cdot 100\,\% \qquad (12)$$

Lappalainen found that the retention coefficient grew when the initial phosphorus concentration $C_o$ and the theoretical detention time $(T = \frac{V}{Q})$ increased. He introduced a new variable $C_oT$ and found that the curve of the equation of the retention coefficient in per cent, versus $C_oT$, resembled that of the equation $y = \sqrt{x}$. However, the curve did not seem to pass through the origin but obtained a value of zero with a small positive value of $C_oT$. The shape of the curve could be interpreted as indicating that phosphorus retention is possible even with small detention values, but that if the original content $C_o$ is sufficiently small, there is no retention. The upper limit of the initial phosphorus concentration, corresponding to a retention coefficient of r = 0 %, he determined graphically to be 6 mg $m^{-3}$. He introduced one further variable, $(C_o-6)T$, in which $C_o$ is given in mg $m^{-3}$ and T in months. Fig. 2 shows the interdependence of $(C_o-6)T$ and the retention coefficient r.

On the basis of a reference material from 27 Finnish lakes, Lappalainen (1975) suggested equations for the calculation of the phosphorus retention coefficient r:

$$r = 3.0\left[\frac{(C_o-6)\cdot V}{2.59\cdot 10^6\cdot Q}\right]^{1/2} \qquad (13)$$

Eq. 13 can be presented in an approximate, simplified form as follows:

$$r = 3\,[(C_o-6)T]^{1/2} \qquad (14)$$

Eqs. 13 and 14 can be used provided that the following conditions are fulfilled

$$(C_o-6)T \leqslant 500 \text{ mg m}^{-3} \text{ month} \qquad (15)$$

$$1.5 < C_o/T < 30 \text{ mg m}^{-3} \text{ month}^{-1} \qquad (16)$$

The set conditions can be explained as follows: If the ratio $C_o/T$ exceeds 30 mg m$^{-3}$ month$^{-1}$, the lake becomes very eutrophic and its bottom can no longer hold phosphorus. On the other hand, if the ratio becomes very small, a shortage of phosphorus arises in the lake, as the phosphorus is reused by plankton so quickly that it has no time to settle. When $(C_o-6)T$ fell to between 500 and 1 000 mg m$^{-3}$ month, the retention coefficient in per cent could be selected to be 70 % on the curve shown in Fig. 2, and when $(C_o-6)T$ exceeded 1 000 mg m$^{-3}$ month the coefficient was 70–80 %. This is an inaccurate method of choosing the retention coefficient, but it should be adaptable for practical work. Condition (16) is examined more closely in the discussion.

The use of Eqs. 13 and 14 is restricted by conditions (15) and (16). Lappalainen (1977) introduced the following Michaelis-Menten-type equation for the calculation of the phosphorus retention coefficient R. By means of this equation the restriction (15) concerning $(C_o-6)T$ can be avoided (17)

$$R = 0.9\,\frac{(C_o-6)T}{200+(C_o-6)T} \qquad (17)$$

The use of Eq. (17) is possible provided that condition (16) concerning $C_o/T$ is fulfilled. When $(C_o-6)T$ approaches infinity, the sedimentation rate approaches the maximum value of 0.9 estimated from the curve in Fig. 2.

When the retention coefficient R has been determined, it can be used to calculate the average phosphorus concentration of a lake (18)

$$R = \frac{S}{I} = \frac{I-O}{I} = \frac{QC_o-QC_1}{QC_o} = \frac{C_o-C_1}{C_o} \qquad (18)$$

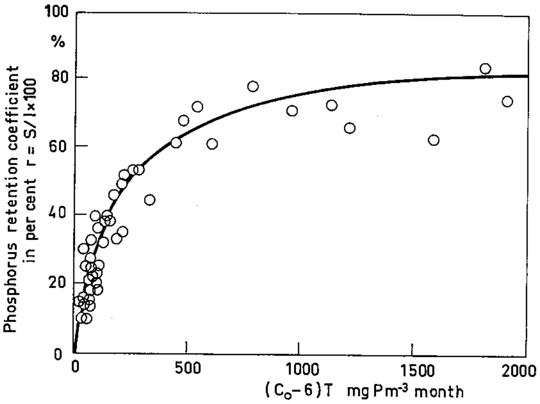

Fig. 2. Dependence of the phosphorus retention coefficient on initial phosphorus concentration and detention time.

Assuming that the phosphorus concentration of water outflowing from the lake is the same as the phosphorus concentration in the lake, C, the following equation (19) for the calculation of the average phosphorus concentration in the lake is obtained on the basis of Eq. 18

$$C = (1-R)C_o \qquad (19)$$

Eq. (19) is in principle identical with the equation presented by Dillon and Rigler (1974). With Eq. (17), however, the phosphorus retention coefficient R can also be predicted with alternative loadings.

## 3. APPLICATION OF THE MODEL TO LAKE PÄIJÄNNE

Fig. 3 shows how the model is used to calculate the average phosphorus concentration of a lake when the following data are available: discharge, phosphorus input I, and volume of lake V. This procedure was applied to the five basins of Lake Päijänne.

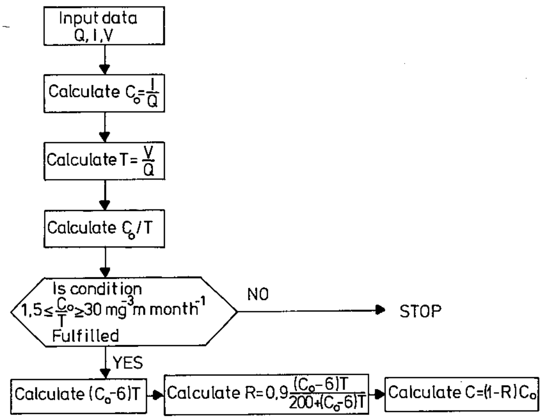

Fig. 3. Use of the model to calculate phosphorus concentration.

## 4. RESULTS AND DISCUSSION

Table 2 shows the average phosphorus concentrations calculated with the model for the various basins of Lake Päijänne during 1970–73 and 1975. The input data were: discharge, phosphorus loading, and effective volumes of the basins during the period in question. By effective volume is meant the total volume of the lake minus the volumes of narrow and shallow bays or other areas separate from the main water body. It can be seen from Table 2 that the calculated phosphorus concentrations agree very well with the observed ones.

The behaviour of the model was studied in an experiment in which it was used to calculate the phosphorus concentrations of sub-basin No. 1 with discharges of 100, 150 and 200 $m^3$ $s^{-1}$, while the phosphorus input was allowed to vary from 2 000 to 24 000 mg $s^{-1}$. The discharge values used in the experiment are realistic because the average discharge of sub-basin No. 1 was 148 $m^3$ $s^{-1}$ during 1961–1970 (National Board of Waters 1975).

Fig. 4 shows that phosphorus loadings from 2 000 to 5 000 mg $s^{-1}$, together with a large discharge, yield smaller phosphorus concentrations than the same loadings with a small discharge. This is due to the fact that with constant loading and a large discharge the initial phosphorus concentration is low.

When phosphorus loading exceeds 5 000 mg $s^{-1}$ the effect of discharge on phosphorus concentration decreases. When loading exceeds 10 000 mg $s^{-1}$, discharge has practically no effect on phosphorus concentration. The results of this experiment could not be verified because of the lack of observed data.

In Fig. 5, the models by Kirchner and Dillon (1975), Lappalainen (1975) and Larsen and Mercier (1976) are compared. It is assumed that the models are applied to a lake with a mean depth of 10 m. The Figure shows that all the models give very similar curves. When the models are applied to lakes with mean depths of less than 10 m the curves of equations (7) and (10) with R = f(Q/A) are elevated, and when the mean depth becomes 1 m they run above the curve of the model by Lappalainen ($C_o$ = 150). If the mean depth exceeds 10 m, the curves of equations (7) and (10) are depressed, and with a mean depth of 100 m they approach the curve of the model by Lappalainen ($C_o$ = 10). The curves generated by equations (7) and (10) with different values of mean depths greatly resemble the curves of the model by Lappalainen calculated with different values of initial phosphorus concentration.

In the model, a condition (Eq. 16) has been set concerning $C_o/T$ (=I/V), which must be fulfilled before the model can be used. It is probable that the upper phosphorus limit, 30 mg $m^{-3}$ $month^{-1}$ is too high. The lower limit of the condition, 1.5 mg $m^{-3}$ $month^{-1}$ may also be too high, because in basin 570 (Table 2) $C_o/T$ was 1.5 mg $m^{-3}$ $month^{-1}$, in spite of which the phosphorus concentrations calculated were rather close to the observed values.

Frisk (1978) investigated various phosphorus retention models developed for lakes. He found that in addition to $C_o/T$ the applicability of the investigated models depends on the phosphorus concentration of lake water. According to his investigations the more suitable practical condition for the use of Eqs. (13) and (14) could be that the phosphorus concentration of lake water must be less than 40–50 mg $m^{-3}$.

The use of the model as an aid in decision-making is restricted by the fact that in common with other statistical phosphorus models, the

Table 2. Phosphorus concentrations of the sub-basins of Lake Päijänne in 1970–1973 and 1975 calculated with the model and the corresponding observed values.

| Sub-basin | Year | Phosphorus loading mg $s^{-1}$ | Discharge[3] $m^3$ $s^{-1}$ | Effective volume x $10^6 m^3$ | Phosphorus concentrations mg $m^{-3}$ | |
|---|---|---|---|---|---|---|
| | | | | | calculated | observed[2] |
| 1 | 1970 | 5 080 | 137.2 | 2 150 | 20.9 | 22.0 |
| | 1 | 5 810 | 180.2 | | 21.3 | 23.0 |
| | 2 | 5 260 | 156.7 | | 20.8 | 23.0 |
| | 3 | 4 810 | 147.7 | | 20.0 | 25.0 |
| | 5 | 5 350 | 207.3 | | 19.2 | 19.0 |
| 2 | 1970 | 3 180 | 144.0 | 2 800 | 14.6 | 16.0 |
| | 1 | 4 080 | 186.5 | | 15.7 | 17.0 |
| | 2 | 4 050 | 162.8 | | 16.3 | 15.0 |
| | 3 | 3 580 | 153.3 | | 15.4 | 18.0 |
| | 5 | 4 270 | 220.0 | | 15.1 | 17.3 |
| 3 | 1970 | 2 270 | 58.1 | 600 | 25.1 | 21.0 |
| | 1 | 2 100 | 61.1 | | 23.6 | 25.0 |
| | 2 | 2 460 | 60.5 | | 26.1 | 24.0 |
| | 3 | 2 020 | 54.9 | | 23.8 | 26.0 |
| | 5 | 2 420 | 77.5 | | 23.5 | 27.8 |
| 4 | 1970 | 3 580 | 164.0 | 2 000 | 16.5 | 16.0 |
| | 1 | 4 470 | 208.7 | | 17.1 | 15.0 |
| | 2 | 3 370 | 184.3 | | 14.9 | 14.0 |
| | 3 | 3 570 | 169.2 | | 16.2 | 16.0 |
| | 5 | 4 910 | 248.4 | | 16.6 | 16.5 |
| 5 | 1970 | 3 650 | 199.0 | 7 200 | 10.7[1] | 12.0 |
| | 1 | 4 150 | 240.0 | | 11.1[1] | 13.0 |
| | 2 | 3 790 | 214.0 | | 10.8[1] | 9.0 |
| | 3 | 3 760 | 195.4 | | 10.8[1] | 14.0 |
| | 5 | 8 180 | 306.8 | | 15.1 | 13.9 |
| 3+4 | 1970 | 4 110 | 164.0 | 2 600 | 16.8 | |
| | 1 | 4 910 | 208.7 | | 17.2 | |
| | 2 | 4 510 | 184.3 | | 17.1 | |
| | 3 | 4 050 | 169.2 | | 16.5 | |
| | 5 | 5 830 | 248.4 | | 18.0 | |

[1] Values have been calculated although $C_0/T < 1.5$ mg $m^{-3}$ $month^{-1}$.

[2] Sources: Values of 1970–1973 for all sub-basins (Mäkinen et al. 1976), values of 1975 for sub-basins 1 and 2 (Granberg et al. 1976), for sub-basin 3 (Granberg and Nyrönen 1976), and for sub-basin 4 and 5 (National Board of Waters 1977).

[3] Source: Frisk (1978)

model predicts annual average phosphorus concentration with no time factor involved, and consequently does not indicate how long it will take to reach a new steady state in a lake after a change in phosphorus input. The advantage of the model is the small number of input data needed and a simple mathematical structure which allow it to be used in making prognoses with varying load alternatives quickly and at low cost.

## ACKNOWLEDGEMENTS

This work was performed within the KVT-project of the National Board of Waters (National

Board of Waters 1978). The authors would like to thank project leader Dr. Pertti Heinonen and computer scientist Juhani Eloranta for their help as well as Mr. Michael Bailey for revising the English.

Helsinki, September 1979

Kalle Matti Lappalainen, Jorma Niemi,
Kari Kinnunen

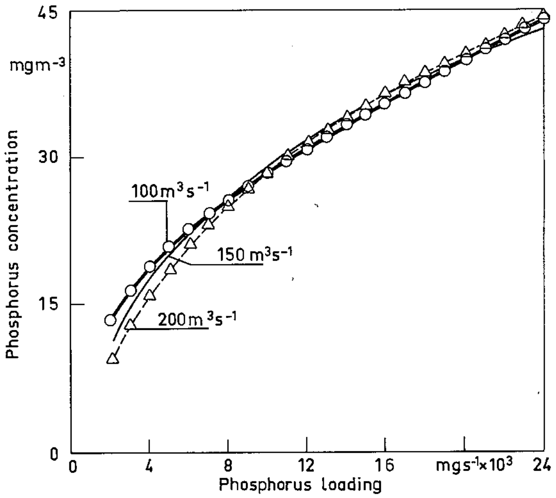

Fig. 4. Phosphorus concentrations calculated by the model with different loading alternatives and discharges.

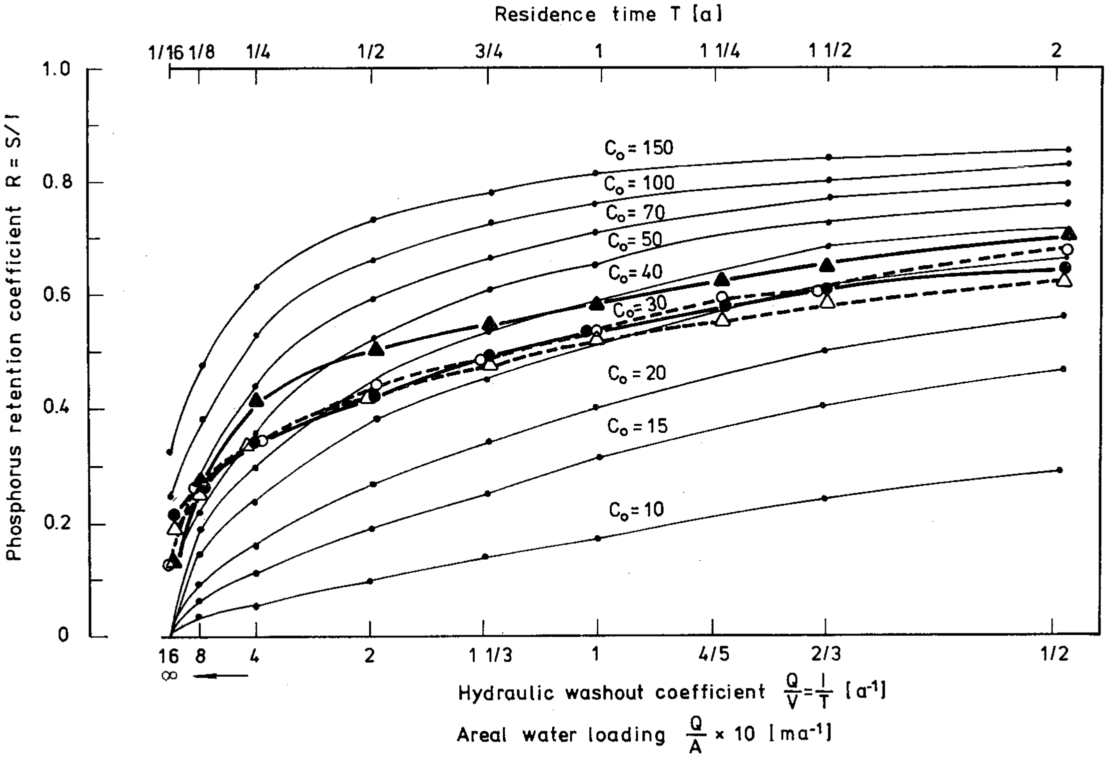

Fig. 5. Comparison of the models of Kirchner and Dillon (1975), Lappalainen, Larsen and Mercier (1976). The figure shows the curve of the model Eq. (7) by Kirchner and Dillon with various values of A/Q and a group of curves that are obtained with the model of Lappalainen Eq. (17) when T is given values at 0.2–2 years (3–24 months) and $C_0$ values from 10 to 150 mg $m^{-3}$. The models by Larsen and Mercier Eqs. (8), (9) and (10) with different values of Q/A and Q/V are presented. The ratio of the scales Q/A and Q/V is numerically equal to average depth of the lake, in this example 10 m.

▲—▲ Model of Kirchner and Dillon, Eq. (7)
•—• Model of Lappalainen, Eq. (17) with various values of $C_0$ and T
●—● Model of Larsen and Mercier, Eq. (8)
△—△ Model of Larsen and Mercier, Eq. (9)
○—○ Model of Larsen and Mercier, Eq. (10)

## LOPPUTIIVISTELMÄ

Työssä esitetään tilastollinen, järville soveltuva fosforimalli. Järveen tulevan kokonaisfosforikuormituksen, järven tilavuuden ja järven luusuasta lähtevän keskimääräisen virtaaman avulla malli laskee fosforin pidättymiskertoimen ja sen avulla järven päällysveden keskimääräisen vuotuisen fosforipitoisuuden. Mallia voidaan soveltaa laskettaessa erilaisten kuormitusvaihtoehtojen vaikutuksia veden laatuun.

Mallia sovellettiin Päijänteen viiteen osa-alueeseen. Altaitten keskimääräiset fosforipitoisuudet laskettiin vuosien 1970–1973 ja 1975 tiedoista. Mallilla lasketut arvot vastasivat hyvin havaittuja arvoja.

Lisäksi mallia kokeiltiin laskemalla Päijänteen ylimmän osa-altaan fosforipitoisuuksia eri kuormitus- ja virtaama-arvoilla. Fosforikuormitusten annettiin vaihdella välillä 2 000–24 000 mg s$^{-1}$ kolmella virtaamalla, jotka olivat 100 m$^3$ s$^{-1}$, 150 m$^3$ s$^{-1}$ ja 200 m$^3$ s$^{-1}$. Mallin laskemia tuloksia ei voitu kuitenkaan verifioida, koska tällaisesta tilanteesta ei ole olemassa havaittuja arvoja.

## REFERENCES

Chapra, S.C. 1976. Total phosphorus model for Great Lakes. Journal of the Environmental Engineering Division ASCE, Vol. 103 No. EE2:147–161.

Dillon, P.J. & Rigler, R.H.1974. A test of a simple nutrient budget model predicting the phosphorus concentration in lake water. J. Fis. Res. Board Can. 31:1771–1778.

Frisk, T. 1978. Järvien fosforimallit. (Phosphorus retention models of lakes, in Finnish). Report of the National Board of Waters, Finland 146. 113 p.

Granberg, K. & Nyrönen, J. 1976. Keski-Päijänteen velvoitetarkkailu vuonna 1975. Summary: The limnological studies in Central Päijänne in 1975. Reports of the Hydrobiological Research Institute of Jyväskylä, No. 75. 72 p.

Granberg, K., Selin, P. & Nyrönen, J. 1976. Pohjois-Päijänteen velvoitetarkkailu v. 1975. Summary: The limnological studies of Northern Päijänne 1975. Reports of the Hydrobiological Research Institute of Jyväskylä, No. 74. 80 p.

Kirchner, W.B. & Dillon P.J. 1975. An empirical method of estimating the retention of phosphorus in lakes. Water Resour. Res. 11:182–183.

Lappalainen, K.M. 1974. Kehitysarviot eri vaihtoehdoilla, Kallaveden reitti ja Haukivesi (in Finnish). Report of the National Board of Waters, Finland 59. 119 p.

Lappalainen, K.M. 1975. Phosphorus loading capacity of lakes and a mathematical model for water quality prognoses. In Eutrofiering, Tioende Nordiska Symposiet om vattenforskning, Vaerløse 20–22 Maj 1974. Norforsk Miljövårdssekretariatet Publication 1975 (1):425–441.

Lappalainen, K.M. 1977. Matemaattisia apukeinoja vesistötutkimuksen tulosten käsittelyyn. (The use of mathematical methods in processing water quality data, in Finnish). Vesi- ja kalatalousmiehet ry:n täydennyskoulutuspäivät 25.26.10.1976. Vesistömallit ja niiden sovaltaminen käytäntöön, 107–121. Helsinki.

Larsen, D.P. & Mercier, H.T. 1976. Phosphorus retention capacity of lakes. J. Fish. Res. Board Can. 33:1742–1750.

Lorenzen, M.W. 1973. Predicting the effects of nutrient diversion on lake recovery. In: E.J. Middlebrooks, D.H. Falkenborg and T.E. Maloney (Eds.), Modeling the Eutrophication Process. An Arbor Science Science Publishers Inc. Michigan, p. 205–210.

Mäkinen, P., Kortelainen, S. & Saarinen, K. 1976. Päijänteen veden laadun fysikaalis-kemialliset tutkimukset v. 1970–1974 osa II. Kausikeskiarvot ja vuosikeskiarvot (in Finnish). Reports of the Hydrobiological Research Institute of Jyväskylä, No. 55.

National Board of Waters, Finland. 1975. Hydrologinen vuosikirja 1971. Hydrological Yearbook 1971. Publications of the Water Research Institute, No. 10. 115 p.

National Board of Waters, Finland. 1977. Unpublished water quality data of Lake Päijänne.

National Board of Waters, Finland, 1978. Final Report for the International Bank for Reconstruction and Development of the Research Project carried out in 1975–1978 by the National Board of Waters. Publications of the National Board of Waters No. 26. 153 p.

Snodgrass, W.J. & O'Melia, C.R. 1975. Predictive model for phosphorus in lakes. Envir. Sci. Technol. 9:937–944.

Sonzogni, W.C., Uttormark, P.C. & Lee, G.F. 1976. A phosphorus residence time model: Theory and application. Water Research 10:429–435.

Vollenweider, R.A. 1968. Scientific fundamentals of the eutrophication of lakes and flowing waters, with particular reference to nitrogen and phosphorus as factors in eutrophication. Tehnical Report OECD, Paris DAS/CSI/68 27. 159 p.

Vollenweider, R.A. 1969. Möglichkeiten und Grenzen elementarer Modelle der Stoffbilanz von Seen. Arch Hydrobiol. 66(1):1–36.

Vollenweider, R.A. 1975. Input-output models with special reference to the phosphorus loading concept in limnology. Schweiz. Hydrol. 37:53–84.