REAL RANK AND SQUARING MAPPING FOR UNITAL C*-ALGEBRAS

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ABSTRACT. It is proved that if X is a compact Hausdorff space of Lebesgue dimension dim(X), then the squaring mapping $\alpha_m \colon (C(X)_{sa})^m \to C(X)_+$, defined by $\alpha_m(f_1, \ldots, f_m) = \sum_{i=1}^m f_i^2$, is open if and only if $m-1 \ge \dim(X)$. Hence the Lebesgue dimension of X can be detected from openness of the squaring maps α_m . In the case m = 1 it is proved that the map $x \mapsto x^2$, from the self-adjoint elements of a unital C^* -algebra A into its positive elements, is open if and only if A is isomorphic to C(X) for some compact Hausdorff space X with dim(X) = 0.

1. INTRODUCTION

A compact Hausdorff space X is defined to have Lebesgue dimension $\leq m$ if for every closed subset F of X, each continuous map $F \to S^m$ has a continuous extension $X \to S^m$.

Various types of ranks for (unital) C^* -algebras have been inspired by corresponding prototypes in the classical dimension theory of (compact) spaces, such as the one given above. While the Lebesgue dimension of a compact space has numerous equivalent formulations, the extensions of these equivalent formulations to non-commutative C^* -algebras most often differ. Examples of such ranks for C^* -algebras are the *stable rank* defined by Rieffel in [7], the *real rank* defined by Brown and Pedersen in [1], the *analytic rank* defined by Murphy in [4], the *completely positive rank* considered by Winter in [8], and the *bounded rank* defined in [2]. Other ranks of C^* -algebras are the *tracial rank* defined by Lin in [3] and the *exponential rank* defined by Phillips in [6].

It was shown in [2] that a unital C^* -algebra A has real rank at most n if the squaring map $(x_1, \ldots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} x_i^2$, from the set of (n+1)-tuples of self-adjoint elements to the set of positive elements in A, is open; and it was asked if the reverse also holds, in which case openness of the squaring maps would determine the real rank of the C^* -algebra.

In the present note we answer this question in the affirmative in the commutative case — and in the negative in the general (non-commutative) case.

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The latter — negative — answer follows from our result that the squaring map $x \mapsto x^2$ (from the set of self-adjoint elements to the set of positive elements) is open if and only if the C^* -algebra is commutative and of real rank zero. Hence the squaring map $x \mapsto x^2$ is not open for the C^* -algebra, M_2 , of 2 by 2 matrices, but this C^* -algebra has real rank zero.

2. Results

For a C^* -algebra A we use the standard notation $A_{\rm sa}$ and A_+ to denote the set of all self-adjoint and the set of all positive elements of A, respectively. The real rank of a unital C^* -algebra A, denoted by $\operatorname{RR}(A)$, is in [1] defined as follows: For each non-negative integer n, $\operatorname{RR}(A) \leq n$ if for every (n + 1)-tuple (x_1, \ldots, x_{n+1}) in $A_{\rm sa}$ and every $\varepsilon > 0$, there exists an (n+1)-tuple (y_1, \ldots, y_{n+1}) in $A_{\rm sa}$ such that $\sum_{k=1}^{n+1} y_k^2$ is invertible and $\sum_{k=1}^{n+1} ||x_k - y_k|| < \varepsilon$. Let us say that a unital C^* -algebra A has an open m-squaring map if the map

Let us say that a unital C^* -algebra A has an open m-squaring map if the map $\alpha_m \colon (A_{sa})^m \to A_+$, defined by $\alpha_m(x_1, \ldots, x_m) = \sum_{k=1}^m x_k^2$, is open. Observe that α_m is open at (x_1, \ldots, x_m) if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $a \in A_+$ with $\|\sum_{k=1}^m x_k^2 - a\| < \delta$ there is an m-tuple (y_1, \ldots, y_m) in A_{sa} with $\sum_{k=1}^m y_k^2 = a$ and $\sum_{k=1}^m \|x_k - y_k\| < \varepsilon$.

For the readers convenience we present a shorter proof of [2, Proposition 7.1].

Proposition 2.1. Let A be a unital C^{*}-algebra. If the (n + 1)-squaring map on A is open, then $RR(A) \leq n$.

Proof. Let (x_1, \ldots, x_{n+1}) be an (n+1)-tuple of self-adjoint elements in A and let $\varepsilon > 0$. By openness of the (n+1)-squaring map there is $\delta > 0$ and an (n+1)-tuple (y_1, \ldots, y_{n+1}) of self-adjoint elements in A such that $\sum_{k=1}^{n+1} ||x_k - y_k|| < \varepsilon$ and $\sum_{k=1}^{n+1} y_k^2 = \sum_{k=1}^{n+1} x_k^2 + \delta \cdot 1$, and the latter element is invertible (because each x_k^2 is positive).

Next, we prove the reverse of Proposition 2.1 in the commutative case.

Proposition 2.2. If X is a compact space such that dim $X \leq n$, then C(X) has open (n + 1)-squaring map.

Proof. Let $f = f_1^2 + \cdots + f_{n+1}^2$. Put $m_i = \sup f_i$, $i = 1, \ldots, n+1$ and let $m = \max\{m_i\}$. Fix $\varepsilon > 0$ and let

$$\delta = \min\left\{\frac{(\varepsilon/3)^4}{m^2}, \left(\frac{\varepsilon}{3}\right)^2\right\} \text{ and } U = \left\{x \in X \colon f(x) > \left(\frac{\varepsilon}{3}\right)^2\right\}.$$

Let also $A = f^{-1}([0, (\varepsilon/3)^2])$ and $S = f^{-1}((\varepsilon/3)^2)$. Then A and S are closed subsets of X such that $A = X \setminus U$ and $S \subseteq A$.

Now consider the diagonal product

$$F(x) = \triangle \{f_1(x), \dots, f_{n+1}(x)\} \colon X \to \mathbb{R}^{n+1}$$

and note that

$$A = F^{-1}(B^{n+1})$$
 and $S = F^{-1}(S^n)$,

where

$$B^{n+1} = B^{n+1} \left(\mathbf{0}, \varepsilon/3 \right)$$
 and $S^n = \partial B^{n+1} \left(\mathbf{0}, \varepsilon/3 \right)$

Since dim $A \leq \dim X \leq n$, the map $F|S: S \to S^n$ admits an extension $H: A \to S^n$ (see, for instance, [5, Ch. 3, Theorem 2.2]). Represent H as the diagonal product $H = \Delta\{h_i, \ldots, h_{n+1}\}$, where each h_i maps A into \mathbb{R} . Since $H(A) \subseteq S^n$ it follows that $h_1^2 + \cdots + h_{n+1}^2 = (\varepsilon/3)^2$. Note also that since H|S = F|S we have $h_i|S = f_i|S$ for each $i = 1, \ldots, n+1$.

The last condition allows us to define for each i = 1, ..., n + 1 a continuous map h_i on X by letting

$$\widetilde{h}_i(x) = \begin{cases} f_i(x), & \text{if } x \in U \\ h_i(x), & \text{if } x \in A \end{cases}$$

Observe that the function $\tilde{h} = \tilde{h_1}^2 + \cdots + \tilde{h}_{n+1}^2$ is strictly positive on X. Notice also that $\widetilde{h}|_U = f|_U$ and $\widetilde{h}|_A = (\varepsilon/3)^2$. Next consider $g \in O(f, \delta) \cap C_+(X)$ and define a function λ on X by $\lambda(x) =$

 $(g(x)/\widetilde{h}(x))^{1/2}$. Note that $\lambda \ge 0$.

Now define g_i , i = 1, ..., n + 1, on X by the formula $g_i(x) = \tilde{h}_i(x) \cdot \lambda(x)$. Clearly

$$g_1^2(x) + \dots + g_{n+1}^2(x) = \left(\widetilde{h}_1(x) \cdot \lambda(x)\right)^2 + \dots + \left(\widetilde{h}_{n+1}(x) \cdot \lambda(x)\right)^2 = \lambda^2(x) \left(\widetilde{h}_1^2(x) + \dots + \widetilde{h}_{n+1}^2(x)\right) = \frac{g(x)}{\widetilde{h}(x)} \cdot \left(\widetilde{h}_1^2(x) + \dots + \widetilde{h}_{n+1}^2(x)\right) = g(x).$$

Next let us show that g_i is sufficiently close to f_i for each i = 1, ..., n + 1. Indeed, since $g \in O(f, \delta)$ we have for each $x \in A$ and therefore

$$g(x) < f(x) + \delta < \left(\frac{\varepsilon}{3}\right)^2 + \left(\frac{\varepsilon}{3}\right)^2$$

Since $g_i^2 \leq g$ and $f_i^2 \leq f$, the last inequality implies that

$$|g_i(x)| < \sqrt{2\left(\frac{\varepsilon}{3}\right)^2} < 2\frac{\varepsilon}{3} \text{ and } |f_i(x)| \le \frac{\varepsilon}{3}$$

for all $x \in A$. Hence

$$|f_i(x) - g_i(x)| \le |f_i(x)| + |g_i(x)| < \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon$$

as required.

Further, if $x \in U$, then $\tilde{h}(x) = f(x)$ and consequently

$$\widetilde{h}(x) - \delta < g(x) < \widetilde{h}(x) + \delta.$$

Hence

$$1 - \frac{\delta}{\widetilde{h}(x)} < \frac{g(x)}{\widetilde{h}(x)} < 1 + \frac{\delta}{\widetilde{h}(x)}$$

for $x \in U$. Since

$$\widetilde{h}(x) = f(x) > \left(\frac{\varepsilon}{3}\right)^2$$
 for $x \in U$ and $\delta \le \frac{(\varepsilon/3)^4}{m^2}$

we have (for $x \in U$)

$$1 - \left(\frac{\varepsilon}{3m}\right)^2 = 1 - \frac{\frac{1}{m^2} \cdot \left(\frac{\varepsilon}{3}\right)^4}{\left(\frac{\varepsilon}{3}\right)^2} < \frac{g(x)}{\widetilde{h}(x)} < 1 + \frac{\frac{1}{m^2} \cdot \left(\frac{\varepsilon}{3}\right)^4}{\left(\frac{\varepsilon}{3}\right)^2} = 1 + \left(\frac{\varepsilon}{3m}\right)^2$$

and

$$1 - \left(\frac{\varepsilon}{3m}\right)^2 < \lambda^2(x) < 1 + \left(\frac{\varepsilon}{3m}\right)^2.$$

Consequently,

$$\left|1-\lambda^2(x)\right| < \left(\frac{\varepsilon}{3m}\right)^2.$$

Since $\lambda(x) \ge 0$, this implies

$$[1-\lambda(x)]^2 \le |1-\lambda(x)| \cdot |1+\lambda(x)| = |1-\lambda^2(x)| < \left(\frac{\varepsilon}{3m}\right)^2.$$

Therefore

$$|1 - \lambda(x)| \le \frac{\varepsilon}{3m}$$
 for any $x \in U$.

Finally we have

$$|f_i(x) - g_i(x)| = |1 - \lambda(x)| \cdot |f_i(x)| < \frac{\varepsilon}{3m} \cdot m < \frac{\varepsilon}{3}$$
 for any $x \in U$.

This completes the verification of the fact that $|f_i(x) - g_i(x)| < \varepsilon$ for each $x \in X$ and any $i = 1, \ldots, n+1$.

Corollary 2.3. Let A be a unital C^* -algebra. Then the following conditions are equivalent:

- (i) The squaring map $x \mapsto x^2$ from A_{sa} to A_+ is open.
- (ii) A is commutative and RR(A) = 0.

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(iii) A is isomorphic to a C^* -algebra of the form C(X) for a compact Hausdorff space X with dim X = 0.

Proof. The equivalence of (ii) and (iii) follows from Gelfand's duality and [1, Proposition 1.1].

The implication (iii) \Rightarrow (i) follows from Proposition 2.2.

(i) \Rightarrow (ii). Assume that (i) holds. Then RR(A) = 0 by Proposition 2.1. It remains to show that A is commutative. Since A is of real rank zero it suffices to show that any two projections p, q in A commute.

Take the symmetry s = p - (1 - p). Then s is self-adjoint and $s^2 = 1$. By openness of the squaring map there are self-adjoint elements s_n in A such that $||s_n - s|| \to 0$ and $s_n^2 = 1 + n^{-1}q$. Define $\varphi \colon \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \max\{0, t\}$. For each n, the element $\varphi(s_n)$ commutes with s_n , hence with s_n^2 , and hence with q. Since $\varphi(s) = p$ we obtain

$$pq - qp = \lim_{n \to \infty} (\varphi(s_n)q - q\varphi(s_n)) = 0,$$

as desired.

3. Related comments and open problems

Existence of square roots: Suppose that A is a unital C*-algebra and that x is a self-adjoint element in A. Does there exist a continuous square root $\rho_x = \rho: \Omega \to A_{\text{sa}}$ (i.e., $\rho(a)^2 = a$ for all $a \in \Omega$) defined on an open neighborhood $\Omega \subseteq A_+$ of x^2 such that $\rho(x^2) = x$? If this is true for all self-adjoint elements a in A, then the equivalent conditions of Corollary 2.3 are satisfied.

Suppose that A = C(X) for some 0-dimensional compact Hausdorff space X (i.e., that the conditions of Corollary 2.3 are satisfied). Take a self-adjoint (i.e., real valued) $f \in C(X)$, and suppose that there is a clopen set U such that $f(x) \ge 0$ for all $x \in U$ and $f(x) \le 0$ for all $x \in X \setminus U$. Then the function $\rho_U: C(X)_+ \to C(X)_{\text{sa}}$ defined by

$$\rho_U(g) = \begin{cases} \sqrt{g(x)}, & x \in U, \\ -\sqrt{g(x)}, & x \in X \setminus U, \end{cases}$$

is a continuous square root with $\rho_U(f^2) = f$. It is not clear to the authors if there are continuous square roots at arbitrary real valued functions f in C(X).

In the case where $A = M_n$, the C^{*}-algebra of n by n matrices, if x is a selfadjoint element and if x^2 has n distinct eigenvalues, then there is a continuous square root ρ with $\rho(x^2) = x$ defined on some neighborhood of x^2 .

In the case where $A = M_2$, it follows from Corollary 2.3 (and its proof) that there is no continuous square root ρ defined on a neighborhood of I such that $\rho(I) = \text{diag}(1, -1)$. It is easily checked explicitly that if r is a (small) non-zero real number, then any square root of $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ is of the form $\begin{pmatrix} a & s \\ s & a \end{pmatrix}$, where a and s

are real numbers satisfying $a^2 + s^2 = 1$ and 2as = r, and any such square root has distance at least 1 to diag(1, -1).

We end this note by listing some open problems related to openness of the squaring maps:

Question 1. Let A be a unital C^* -algebra, let m be a positive integer, and suppose that the squaring map α_m (defined above Proposition 2.1) is open. Does it follow that α_n is open for all $n \ge m$?

The answer to Question 1 is affirmative for commutative C^* -algebras by Propositions 2.1 and 2.2. The difficulty in this question lies in the fact that if Ω is an open subset of A_+ and if $a \in A_+$, then $a + \Omega$ need not be open in A_+ . (For instance, $1 + A_+$ is not open in A_+ .)

Question 2. Are Propositions 2.1 and 2.2 valid also in the *non-unital* case? (For Proposition 2.2, this means that we will be talking about locally compact Hausdorff spaces rather than compact Hausdorff spaces.) What is the relationship between openness of α_n on a non-unital C^* -algebra A and openness of α_n on its unitization?

Question 3. Are the squaring maps α_m open for all $m \ge 2$ when A is a unital C^* -algebra of real rank zero?

Question 4. Does the class of C^* -algebras, for which the squaring map α_2 is open, have any nice properties? More generally, are there any justifications for considering the rank of a C^* -algebra defined by openness of the squaring maps; and will this rank reflect any "dimension like" properties of the C^* -algebra?

References

- L. G. Brown, G. K. Pedersen, C^{*}-algebra of real rank zero, J. Functional Anal. 99 (1991), 131-149.
- [2] A. Chigogidze, V. Valov, Bounded rank of C^{*}-algebras, preprint, 2001.
- [3] H. Lin, The tracial topological rank of C^{*}-algebras, Proc. London Math. Soc. 83 (2001), 199–234.
- [4] G. J. Murphy, The analytic rank of a C^{*}-algebra, Proc. Amer. Math. Soc. 115 (1992), 741–746.
- [5] A. R. Pears, *Dimension Theory of General Spaces*, Cambridge University Press, Cambridge, 1975.
- [6] N. C. Phillips, Simple C*-algebras with the property (FU), Math. Scand. 69 (1991), 121– 151.
- [7] M. A. Rieffel, Dimension and stable rank in the K-theory of C*-algebras, Proc. London Math. Soc. 46 (1983), 301–333.
- [8] W. Winter, Covering dimension for nuclear C*-algebras, Preprint math.OA/0107218 (2001).

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