

VALDIVIA COMPACT GROUPS ARE PRODUCTS

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ABSTRACT. It is shown that every Valdivia compact group is homeomorphic to a product of metrizable compacta.

1. INTRODUCTION

A compact space is called Valdivia compact if it can be embedded into \mathbb{R}^A for some A in such a way that the image of this embedding is the closure of a subset of a Σ -product. These spaces have been studied by several authors (see [6], [7] for a nice survey of Valdivia compacta and related topics). A particular area of interest is understanding of the structure of Valdivia compact groups. There are compact connected Abelian groups which are not Valdivia compact [9]. On the other hand it was shown recently [8] that every Valdivia compact Abelian group is homeomorphic to a product of metrizable compacta. We extend this result to the noncommutative case (Theorem 3.3). This answers negatively question from [8] whether there exist Valdivia compact groups which are not products.

Our original proof was based on a recently announced result on preservation of the class of Valdivia compacta by retractions. I would like to thank the referee for pointing out that the proof of this statement contained an error and that the validity of the result itself is still open. The proof of Theorem 3.3 has been revised and as presented below does not depend on the above mentioned statement.

2. PRELIMINARIES

For the reader's convenience in this section we present some of the needed results.

2.1. Directed sets. Let $\kappa \geq \omega$. A subset B of a partially ordered directed set A is said to be κ -closed in A if for each chain $C \subseteq B$, with $|C| \leq \kappa$, we have $\sup C \in B$ whenever the element $\sup C$ exists in A ($\sup C$ denotes the lowest upper bound of elements of C). A set A is said to be κ -complete if for each chain B of elements of A , with $|B| \leq \kappa$, there exists an element $\sup B$ in

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A. A standard example of a κ -complete set is the set $\exp_\kappa A$ of all subsets of cardinality $\leq \kappa$ of any set A .

The following statement presents an important property of κ -complete sets ([2, Proposition 1.1.27]).

Proposition 2.1. *Let $\{A_i: i \in I\}$, $|I| \leq \kappa$, be a collection of κ -closed and cofinal subsets of a κ -complete set A . Then the intersection $\cap\{A_i: i \in I\}$ is also cofinal and κ -closed in A .*

2.2. Inverse Spectra. All limit projections of inverse spectra considered below are surjective and all spaces are compact. Let $\kappa \geq \omega$. An inverse spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of compact spaces is a κ -spectrum if:

- (i) $w(X_\alpha) \leq \kappa$, $\alpha \in A$;
- (ii) The indexing set A is κ -complete;
- (iii) \mathcal{S}_X is κ -continuous, i.e. for each chain $\{\alpha_i: i \in I\} \subseteq A$ with $|I| \leq \kappa$ and $\alpha = \sup\{\alpha_i: i \in I\}$, the diagonal product $\Delta\{p_{\alpha_i}^\alpha: i \in I\}: X_\alpha \rightarrow \lim\{X_{\alpha_i}, p_{\alpha_i}^{\alpha_j}, I\}$ is a homeomorphism.

One of the main results concerning κ -spectra is the following result of Ščepin (known as the Ščepin's Spectral Theorem, see [2, Theorem 1.3.4]).

Theorem 2.2. *Let $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ be two κ -spectra. Then for every map $f: \lim \mathcal{S}_X \rightarrow \lim \mathcal{S}_Y$ there exist a cofinal and κ -closed subset $B \subseteq A$ and maps $f_\alpha: X_\alpha \rightarrow Y_\alpha$, $\alpha \in B$, such that $f = \lim\{f_\alpha: \alpha \in B\}$. If f is a homeomorphism, then we may assume that each f_α , $\alpha \in B$, is also a homeomorphism.*

2.3. Σ -products. If $C \subseteq B \subseteq A$, then $\pi_B: \mathbb{R}^A \rightarrow \mathbb{R}^B$ and $\pi_C^B: \mathbb{R}^B \rightarrow \mathbb{R}^C$ denote the corresponding projections. Similarly by $i_B: \mathbb{R}^B \rightarrow \mathbb{R}^A$ and $i_C^B: \mathbb{R}^C \rightarrow \mathbb{R}^B$ we denote sections of π_B and π_C^B defined as follows:

$$i_B(\{x_t: t \in B\}) = (\{x_t: t \in B\}, \{0_t: t \in A \setminus B\})$$

and

$$i_C^B(\{x_t: t \in C\}) = (\{x_t: t \in C\}, \{0_t: t \in B \setminus C\}).$$

Below we consider the Σ -product of real lines, which is the subspace

$$\Sigma(A) = \{\{x_t: t \in A\} \in \mathbb{R}^A: |\{t \in A: x_t \neq 0\}| \leq \omega\}$$

of the product \mathbb{R}^A .

For each $B \subseteq A$, $\Sigma(B)$ is identified with the subspace $\Sigma(A) \cap i_B(\mathbb{R}^B)$ of $\Sigma(A)$. Note that $\Sigma(A) = \bigcup\{i_B(\mathbb{R}^B): B \in \exp_\omega A\}$.

Retractions $r_B = i_B \pi_B: \Sigma(A) \rightarrow \Sigma(B)$, $B \subseteq A$, play an important role below. We refer the reader to [5] for relevant definitions and results. For a given closed subset $F \subseteq \Sigma(A)$ we are interested in finding subsets $B \subseteq A$ such that $r_B(F) \subseteq$

F or, equivalently, $F \cap \Sigma(B) = r_B(F)$. Note that if $Y = \text{cl}_{\mathbb{R}^A} F$ is compact, then $r_B(F) \subseteq F$ if and only if $r_B(Y) \subseteq Y$ or, equivalently, $\text{cl}(Y \cap \Sigma(B)) = r_B(Y)$. In this case B is called an F -good (or Y -good) subset of A . It turns out (see [4, Lemma 1], [1, Lemma 1.2]) that there are many Y -good subsets. Precise statement we use below is recorded in the following lemma.

Lemma 2.3. *Let $\omega \leq \kappa < \tau$, $|A| = \tau$ and $Y = \text{cl}_{\mathbb{R}^A}(Y \cap \Sigma(A))$ be compact. Then*

- (a) *the set \mathcal{A}_κ^V of Y -good subsets of A of cardinality $\leq \kappa$ is cofinal and κ -closed in $\text{exp}_\kappa A$;*
- (b) *union of an increasing collection of Y -good subsets of A is again a Y -good subset of A ;*
- (c) *if B is a Y -good subset of A , then $r_B|_Y: Y \rightarrow Y_B$ is a retraction.*

3. PROOF OF THE MAIN RESULT

In this section we prove our main result - Theorem 3.3. First we need the following statement.

Lemma 3.1. *Let $\omega \leq \kappa < \tau$, $|A| = \tau$ and a compact group X be embedded into \mathbb{R}^A . Then the set $\text{exp}_\kappa A$ contains a cofinal and κ -closed subset \mathcal{A}_κ^G such that $\pi_B(X)$ is a topological group and the projection $\pi_B|_X: X \rightarrow \pi_B(X)$ is a homomorphism for each $B \in \mathcal{A}_\kappa^G$.*

Proof. Let $\lambda: X \times X \rightarrow X$ and $\mu: X \rightarrow X$ denote group operations in X , i.e. $\lambda(x, y) = x \cdot y$ and $\mu(x) = x^{-1}$, $x, y \in X$. Let also $X_B = \pi_B(X)$, $p_B = \pi_B|_X$ and $p_C^B = \pi_C^B|_{X_B}$ for any $C, B \subseteq A$ with $C \subseteq B$.

Consider the standard κ -spectrum $\mathcal{S}_X = \{X_B, p_C^B, \text{exp}_\kappa A\}$ and note that $X = \lim \mathcal{S}_X$. By Theorem 2.2, applied to the map $\mu: X \rightarrow X$ and the κ -spectrum \mathcal{S}_X , there exist a cofinal and κ -closed subset \mathcal{A}_κ^μ of $\text{exp}_\kappa A$ and maps $\mu_B: X_B \rightarrow X_B$, $B \in \mathcal{A}_\kappa^\mu$, such that $p_B \mu = \mu_B p_B$ for each $B \in \mathcal{A}_\kappa^\mu$ and $\mu = \lim\{\mu_B: B \in \mathcal{A}_\kappa^\mu\}$.

Next consider the κ -spectrum $\mathcal{S}_X \times \mathcal{S}_X = \{X_B \times X_B, p_C^B \times p_C^B, \text{exp}_\kappa A\}$. Obviously, $X \times X = \lim \mathcal{S}_X \times \mathcal{S}_X$. Applying Theorem 2.2 to the map $\lambda: X \times X \rightarrow X$ and to the spectra $\mathcal{S}_X \times \mathcal{S}_X$ and \mathcal{S}_X , we conclude that there exist a cofinal and κ -closed subset $\mathcal{A}_\kappa^\lambda$ of $\text{exp}_\kappa A$ and maps $\lambda_B: X_B \times X_B \rightarrow X_B$, $B \in \mathcal{A}_\kappa^\lambda$ such that $p_B \lambda = \lambda_B(p_B \times p_B)$ for each $B \in \mathcal{A}_\kappa^\lambda$ and $\lambda = \lim\{\lambda_B: B \in \mathcal{A}_\kappa^\lambda\}$.

By Proposition 2.1, the intersection $\mathcal{A}_\kappa^G = \mathcal{A}_\kappa^\mu \cap \mathcal{A}_\kappa^\lambda$ is still cofinal and κ -closed in $\text{exp}_\kappa A$. Note that for each $B \in \mathcal{A}_\kappa^G$ we have two maps $\lambda_B: X_B \times X_B \rightarrow X_B$ and $\mu_B: X_B \rightarrow X_B$. These maps define a group structure on X_B as follows: $x_B \cdot y_B = \lambda_B(x_B, y_B)$ and $x_B^{-1} = \mu_B(x_B)$, $x_B, y_B \in X_B$. The unit element in X_B is defined as follows: $e_B = p_B(e)$, where e is the unit in X . It is easy to see that the projection $p_B: X \rightarrow X_B$ becomes a group homomorphism. Indeed, for $x, y \in X$ we have

$$p_B(x \cdot y) = p_B \lambda(x, y) = \lambda_B(p_B \times p_B)(x, y) = \lambda_B(p_B(x), p_B(y)) = p_B(x) \cdot p_B(y).$$

□

Lemma 3.2. *Let X be a topological group which is a retract of a Valdivia compact. Then there exists a well-ordered continuous spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\}$ such that*

- (1) $\tau = w(X)$;
- (2) $w(X_\alpha) \leq |\omega| \cdot |\alpha|$, $\alpha < \tau$;
- (3) X_α is a topological group and a retract of a Valdivia compact, $\alpha < \tau$;
- (4) The limit projection $p_\alpha: X \rightarrow X_\alpha$ is a topological homomorphism and a retraction, $\alpha < \tau$.

Proof. Let Y be a Valdivia compact of suitably embedded into \mathbb{R}^A with $|A| = \tau > \omega$. Suppose also that $s: Y \rightarrow X$ is a retraction and X is a topological group. Without loss of generality we may assume that $w(X) = \tau$.

Let $\omega \leq \kappa < \tau$. By Lemma 2.3(a), the collection $\mathcal{S}_Y^\kappa = \{Y_B, q_C^B, C, B \in \mathcal{A}_\kappa^V\}$, where $q_C^B = r_C^B|_{Y_B}$, whenever $C, B \in \mathcal{A}_\kappa^V$ and $C \subseteq B$, forms a κ -spectrum. Also consider the κ -spectrum $\mathcal{S}_X^\kappa = \{X_B, p_C^B, C, B \in \mathcal{A}_\kappa^V\}$, where $X_B = r_B(X)$ and $p_C^B = r_C^B|_{X_B}$, whenever $C, B \in \mathcal{A}_\kappa^V$ and $C \subseteq B$. Clearly, $Y = \lim \mathcal{S}_Y^\kappa$ and $X = \lim \mathcal{S}_X^\kappa$. Note that projections $q_C^B: Y_B \rightarrow Y_C$ and $q_B: Y \rightarrow Y_B$ of the spectrum \mathcal{S}_Y^κ are retractions (Lemma 2.3(c)). Theorem 2.2, applied to the κ -spectra \mathcal{S}_Y^κ and \mathcal{S}_X^κ and to the retraction $s: Y \rightarrow X$, guarantees existence of a κ -closed and cofinal subset \mathcal{A}_s^κ of \mathcal{A}_κ^V such that for each $B \in \mathcal{A}_s^\kappa$ there is a map $s_B: Y_B \rightarrow X_B$ satisfying the equality $p_B s = s_B q_B$.

Next observe that $s_B: Y_B \rightarrow X_B$, $B \in \mathcal{A}_s^\kappa$, is a retraction. Indeed, let $x_B \in X_B$ and $x \in X$ is such that $p_B(x) = x_B$. Then we have

$$s_B(x_B) = s_B(p_B(x)) = s_B(q_B(x)) = p_B(s(x)) = p_B(x) = x_B.$$

Next note that since in the diagram ($B \in \mathcal{A}_s^\kappa$)

$$\begin{array}{ccc} X & \xleftarrow{s} & Y \\ p_B \downarrow & & \downarrow q_B \\ X_B & \xleftarrow{s_B} & Y_B \end{array}$$

maps s , s_B and q_B are retractions, it follows that p_B is also a retraction.

By Lemma 3.1, we may assume without loss of generality that there exists a κ -closed and cofinal subset \mathcal{A}^κ of $\mathcal{A}_s^\kappa \cap \mathcal{A}_\kappa^G$ such that for each $B \in \mathcal{A}^\kappa$ compactum X_B is a topological group and the projection $p_B: X \rightarrow X_B$ is a topological homomorphism.

Thus, for any κ with $\omega \leq \kappa < \tau$ we have a κ -closed and cofinal subset \mathcal{A}^κ of $\exp_\kappa A$ such that the following conditions are satisfied for each $B \in \mathcal{A}^\kappa$:

- (i) Y_B is a Valdivia compact
- (ii) $q_B: Y \rightarrow Y_B$ is a retraction

- (iii) There exists a retraction $s_B: Y_B \rightarrow X_B$ such that $p_B s = s_B q_B$.
- (iv) $p_B: X \rightarrow X_B$ is a retraction
- (v) X_B is a topological group
- (vi) $p_B: X \rightarrow X_B$ is a topological homomorphism.

Let us now define subsets A_α , $\alpha < \tau$, of A . Fix a well-ordering $\{B_\alpha: \alpha < \tau\}$ of the set \mathcal{A}^κ , let $A_0 = B_0$, $s_0 = s_{A_0}$ and assume that the sets A_β and retractions $s_\beta: Y_{A_\beta} \rightarrow X_{A_\beta}$ have been constructed for each $\beta < \alpha$, where $\alpha < \tau$, in such a way that the following conditions are satisfied:

- (a) $_\beta$ $|A_\beta| \leq |\omega| \cdot |\beta|$
- (b) $_\beta$ $A_\gamma \subseteq A_\beta$ whenever $\gamma \leq \beta < \alpha$
- (c) $_\beta$ $A_\beta = \cup\{A_\gamma: \gamma < \beta\}$ whenever β is a limit ordinal.
- (d) $_\beta$ $A_{\beta+1} \supseteq A_\beta \cup B_{\beta+1}$, whenever $\beta + 1 < \alpha$.
- (e) $_\beta$ There exists a retraction $s_\beta: Y_{A_\beta} \rightarrow X_{A_\beta}$ such that $p_{A_\beta} s = s_\beta q_{A_\beta}$
- (f) $_\beta$ X_{A_β} is a topological group
- (g) $_\beta$ $p_{A_\beta}: X \rightarrow X_{A_\beta}$ is a retraction and a topological homomorphism

If α is a limit ordinal, then let $A_\alpha = \cup\{A_\beta: \beta < \alpha\}$. Note that for each γ, β with $\gamma < \beta < \alpha$ the following diagram

$$\begin{array}{ccc} X_{A_\beta} & \xleftarrow{s_\beta} & Y_{A_\beta} \\ p_{A_\gamma}^{A_\beta} \downarrow & & \downarrow q_{A_\gamma}^{A_\beta} \\ X_{A_\gamma} & \xleftarrow{s_\gamma} & Y_{A_\gamma} \end{array}$$

is commutative. Note also that $X_{A_\alpha} = \lim\{X_{A_\beta}, p_{A_\gamma}^{A_\beta}, \alpha\}$ and $Y_{A_\alpha} = \lim\{Y_{A_\beta}, q_{A_\gamma}^{A_\beta}, \alpha\}$. Consequently, collection $\{s_\beta: \beta < \alpha\}$ defines a map $s_\alpha: Y_{A_\alpha} \rightarrow X_{A_\alpha}$ such that $p_{A_\beta} s_\alpha = s_\beta q_{A_\beta}$ for each $\beta < \alpha$. Since s_β is a retraction, for any $x \in X_{A_\alpha}$ we have

$$p_{A_\beta}(s_\alpha(x)) = s_\beta(q_{A_\beta}(x)) = s_\beta(p_{A_\beta}(x)) = p_{A_\beta}(x),$$

which shows that $s_\alpha(x) = x$. This verifies condition (e) $_\alpha$. By (b) $_\beta$ and Lemma 2.3(b), A_α is a Y -good subset of A . Consequently, by Lemma 2.3(c), $q_{A_\alpha}: Y \rightarrow Y_{A_\alpha}$ is a retraction. But then it is easy to conclude that $p_{A_\alpha}: X \rightarrow X_{A_\alpha}$ is also a retraction.

By conditions (f) $_\beta$ and (g) $_\beta$, it follows that for $\beta < \alpha$, $X_{A_\alpha} = \lim\{X_{A_\beta}, p_{A_\gamma}^{A_\beta}, \alpha\}$ is also a topological group with naturally defined group operations and the projection $p_{A_\alpha}: X \rightarrow X_{A_\alpha}$ is a topological homomorphism.

Next consider the case $\alpha = \beta + 1$. Since, by condition (a) $_\beta$, $|A_\beta| \leq |\omega| \cdot |\beta|$ and since \mathcal{A}^κ (with $\kappa = |\omega| \cdot |\beta|$) is cofinal in $\exp_\kappa A$, there exists $A_\alpha \in \mathcal{A}^\kappa$ such that $A_\alpha \supseteq A_\beta \cup B_\alpha$. Then there exists a retraction $s_\alpha: Y_{A_\alpha} \rightarrow X_{A_\alpha}$ such that

$p_{A_\alpha} s = s_\alpha q_{A_\alpha}$, $p_{A_\alpha}: X \rightarrow X_{A_\alpha}$ is a retraction, X_{A_α} is a topological group and p_{A_α} is a topological homomorphism.

This completes the construction. Let $X_\alpha = X_{A_\alpha}$ and $p_\alpha = p_{A_\alpha}$ for each $\alpha < \tau$. It only remains to note that $X = \lim\{X_\alpha, p_\alpha^{\alpha+1}, \tau\}$ and that all the required conditions (1)–(4) are satisfied. \square

Theorem 3.3. *Let X be a compact group. Then the following conditions are equivalent:*

- (1) X is a Valdivia compact;
- (2) X is a retract of a Valdivia compact;
- (3) X is homeomorphic to a product of metrizable compacta;

Proof. Implications (1) \implies (2) and (3) \implies (1) are trivial. Let us prove implication (2) \implies (3). We proceed by induction. For metrizable groups which are retracts of Valdivia compacta there is nothing to prove. Let X denote a topological group which is a retract of a Valdivia compact. Suppose that the statement is true in cases when $w(X) < \tau$, $\tau > \omega$, and consider X of weight $w(X) = \tau$. Let $\mathcal{S}_X = \{X_\alpha, p_\alpha^{\alpha+1}, \omega_0 \leq \alpha < \tau\}$ be a spectrum supplied by Lemma 3.2. Note that by property (4) in Lemma 3.2 each limit projection $p_\alpha: X \rightarrow X_\alpha$ is a retraction and a group homomorphism. This implies that each short projection $p_\alpha^{\alpha+1}: X_{\alpha+1} \rightarrow X_\alpha$ is a group homomorphism (and obviously a retraction). Consequently, $X_{\alpha+1}$ is homeomorphic to the product $X_\alpha \times \ker(p_\alpha^{\alpha+1})$ (see [3, Lemma 0.2]) (corresponding homeomorphism is provided by the formula $h(x) = (p_\alpha^{\alpha+1}(x), x \cdot (p_\alpha^{\alpha+1}(x))^{-1})$, $x \in X_{\alpha+1}$). Thus X is homeomorphic to the product $X_{\omega_0} \times \prod\{\ker(p_\alpha^{\alpha+1}): \omega_0 \leq \alpha < \tau\}$. Finally note that $\ker(p_\alpha^{\alpha+1})$ is a retract of $X_{\alpha+1}$ and $X_{\alpha+1}$, by Lemma 3.2(3), is a retract of a Valdivia compact. Therefore $\ker(p_\alpha^{\alpha+1})$ is a retract of a Valdivia compact. By Lemma 3.2(2), $w(\ker(p_\alpha^{\alpha+1})) \leq w(X_{\alpha+1}) \leq |\omega| \cdot |\alpha + 1| < \tau$. Then the inductive assumption guarantees that each $\ker(p_\alpha^{\alpha+1})$ is homeomorphic to a product of metrizable compact groups. Proof is completed. \square

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