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The Four Color Theorem is in a set of mathematical questions that are very simple to state but amazingly complex to answer. It goes as follows, "given any map, are any more than 4 colors required to color the map in such a way that no two areas which share a border also share a color?"(2). It was thought to be proven by Alfred Kempe for nearly a decade using a unique but unsuccessful process later referred to as Kempe chains. It wasn't until 1913, with George Birkhoff's treatment of reducibility, was true progress from the "proof" of Kempe to be made. From here, Heinrich Heesch explored reducibility with an improvement on the established A-, B-, and C-reducibilities, finding something algorithmically sound in D-reducibility and his subsequent discharging methods. Then Karl Durre introduced the first, somewhat rudimentary, computer program of D-reducibility. From here the extensive use of the supercomputers of the era helped seal the fate of the long, unfinished theorem, with Wolfgang Haken and Kenneth Appel at the helm. We seek to examine the history of this theorem from the proof of Kempe to the utilization of reducible configurations and discharging methods of Durre and Heesch and into the eventual proof of the theorem itself.

REDUCIBLE CONFIGURATIONS AND SO ON:THE FINAL YEARS OF THE FOUR COLOR THEOREM

by

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Approved by

Committee Chair

This thesis is dedicated to my mother, Denise Magee. Her pride in my studies was a constant source of motivation in the most difficult of times. I know that pride would continue if she were here today.

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I

INTRODUCTION

The origin of the Four Color Theorem is a banal one. In 1852, the mathematician Francis Guthrie noticed that a map of the counties of England was colored using only four colors. As he perused more maps he found the same results. He came up with a conjecture that four colors was the absolute maximum needed to color a map so that no countries that share a border share a color. He showed this to his younger brother, Frederick, who was studying under Augustus De Morgan. De Morgan was intrigued by this problem and wrote to William Hamilton about it. Hamilton found no interest in it, but De Morgan was enthralled. So began the 125 year journey from inception to proof of the Four Color Theorem.

1.1 Maps and Graphs

What is a map? A map itself is a special case of a graph. A graph is composed of a pair of sets, V and E. We write this as $G = (V, E)$ where V, the elements of which are the vertices of G, typically labeled v_1, v_2, etc. The other set E contains unordered pairs of vertices. A pair of vertices, say $(v_j, v_k) \in$ E is called an edge. Physically we think of this (v_j, v_k) as an arc connecting the two points v_j and v_k. Two vertices, v_j and v_k, are said to be adjacent if $(v_j, v_k) \in E$. We exclude G in which there is only one v $\in V$ or G such that E is empty. We also are only concerned with graphs that are planar, or can be drawn in R^2 such that for any two edges, (v_j, v_k) and (v_l, v_m) in E, the only intersection will occur at their endpoints. Now, putting a graph in a plane is like assigning our vertices coordinates and connecting the vertices that make up the pair in the edge sets by arcs. The resulting planar graph in R^2 is called a map.

1.2 Coloring and Dual Graphs

What then, makes a coloring? A vertex coloring is a mapping, say $f : V \to C$ where every $v_j \in V$ is assigend a color in a set of colors, say C. A proper vertex coloring is one in which for $(v_j,\, v_k) \in E,\ f(v_j) \neq f(v_k)$. A map divides the plane into a collection of disjoint sets known as regions. The borders of these regions are made up of closed circuits of adjacent vertices, $v_1,\, v_2,\, ...,\, v_{k-1},\, v_k,\, v_1$. Two regions that share a pair of adjacent vertices of this collection are said to border each other. This type of coloring and the coloring seen in maps are connected by the idea of a dual graph.

A dual graph of G is a graph $G' = (R, E)$ in which our R is composed of a vertex from each region, and our E is composed of pairs of these regions that share a border. In this scenario, a proper coloring is $g : R \to C$ where $g(r_j) \neq g(r_k)$ when r_j and r_k border each other. Thus, it is easy to see that the proper vertex coloring of a G' has an analong to a proper coloring of its regions. It should be noted, however, that as we move through our timeline of mathematicians, we will start with colorings on regions and move into colorings on vertices. The dual graph makes the process mostly trivial. (see figure 1)

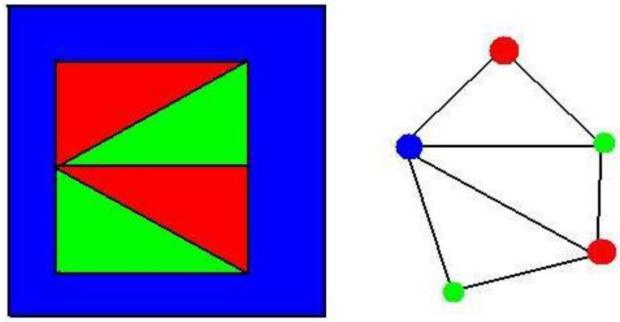

Figure 1. Proper Coloring on a Graph and its Dual

With the nature of graphs, dual graphs and colorings established, we can look at the idea of Kempe chains and their importance in the proof of the Four Color Theorem. This sets the stage for the focus of this paper, which is the history of the proof of the Four Color Theorem. It begins with Alfred Kempe, who used an interesting method to "prove" the Four Color Theorem. The trick that Kempe used, which was to create chains along pairs of colors, popped up again and again through the many steps of the final proof of the theorem. Sadly, Kempe's proof was wrong and the real race towards a comprehensive proof began.

The next mathematician to use Kempe chains was George Birkhoff. In his study of reducibility in graphs he introduced the concepts of rings and schemes and in turn opened new doors for the future proof of the theorem. He also makes a prophetic claim: that a set of reducible rings exists that every map would contain. This becomes the crux of the proof of Haken and Appel. After Birkhoff, the next mathematician to move forward with the proof was Philip Franklin.

Dr. Franklin explored the theorem from the vantage point of size. Given what was currently known about the five-chromatic map, the smallest map that requires five or more colors, he showed that this map would have at least 26 regions.

Thus, he proved that every map containing 25 regions or fewer could be properly colored. This approach proved to be too narrow in scope, though, and something more computational proved to be more appropriate.

This is where Heinrich Heesch and Alfred Durre stepped in. Heesch made considerable progress with his carefully constructed reductions of graphs known as C and D reduciblities. Heesch also constructed a process which would be known as discharging. Discharging involved assigning charges to vertices based on their degree and then moving them along edges, hoping to find a reducible configuration after positive charges have been collected. A configuration is a regular graph whose outer vertices form a circuit of size four or greater. Also, the bounded regions have triangular borders, every triangle is the border of a region, and inner vertices exist. (2, 155-156). Without this work, Haken and Appel would have found great difficulty in their approach to the proof of the theorem.

Wolfgang Haken and Kenneth Appel are the heroes of this story. Their very careful and systematic approach to the proof of the Four Color Theorem took them to many different universities and allowed them to collaborate with many different mathematicians. Their perseverance paid off, however, and in 1976 they announced their results to the world.

CHAPTER II

KEMPE CHAINS

Kempe chains were a clever idea implemented by Alfred Kempe in an attempt to prove the Four Color Theorem. It would later be seen in his proof of the Four Color Theorem that, although Kempe's utilization of chains was very useful, Kempe had actually proven the less-strict Five Color Theorem, when his proof was found to have an error by Pearcy Heawood. For the purposes of the definition of Kempe chains, the colorings will be on vertices, where coloring the regions themselves becomes an easy analog utilizing the aforementioned dual graph. Consider a planar graph G that has a coloring and at least one degree four vertex, say x. We remove x and color the four other incident vertices with blue (b), green (g), yellow (y), and red (r). This leads us to a problem, though, since we would need a fifth color for our missing vertex x. Kempe's suggestion was to start from an arbitrarily selected blue-colored vertex and create a subgraph by following the edges from this vertex through all vertices colored b or y. (1) Note that the edges between the vertices in this subgraph and the rest of the original graph connect vertices colored b or y with vertices colored g or r. Thus, a Kempe chain is a mapping $f(v)$ from V into a finite set of colors say C wherein C contains at least two members. Thus an $a - b$ chain would be the maximally connected subgraph of G containing all vertices colored with a or b. Kempe used this chain idea for his proof of the Four Color Theorem. His proof breaks up into three main cases; two in the four degree situation and a five degree situation. (1)

2.1 Kempe's Proof

Case one simply involves the situation where our subgraph does not contain the vertex adjacent to x that we colored y. Thus we must change the coloring of all vertices colored b to y and all vertices colored y to b. This will remain a proper coloring since our beginning vertex x will now only be adjacent to vertices colored y, g and r. Thus we are left with x being able to be colored b. (1)

Case two involves a cyclical Kempe chain. In this case, the subgraph includes the vertex adjacent to x that we colored y. It seems at first that the Kempe chain might not work here, since a switching in color would still leave x adjacent to four different colors. The solution is a simple observation. If a path exists between the vertices colored b and y, then there can't be one between the vertices colored g and r. This is due to the fact that a Kempe chain that forms a cycle will separate the plane. As a result, our Kempe chains will not cross. Thus, we start our chain from r and go through all vertices colored r or g. This allows us to alternate between r and g and will leave x adjacent to b, y, and g. (1)

Now, the degree-five case. Let us create a Kempe chain starting from our b-colored vertex and move out to all the edges containing b's and y's. If the situation arises where x is not adjacent to a vertex colored y in our chain then we simply toggle our colors so that we can color x with b. (1) If we cannot create a chain containing y then we abandon that chain and instead focus on one with b and g, instead. Again, if the situation arises where x is not adjacent to a vertex colored g then we toggle the colors so that x can be colored b. Now, if this chain doesn't contain g, then we have to make two Kempe chains to make this work. We will create two chains starting from the two vertices colored r.

From the first, we create a chain from the r-colored vertex surrounded by the $b-g$ chain created earlier. This will be a $r-y$ chain. From the r-colored vertex surrounded by the b and y chain we create an r and g chain. The goal of this seemingly messy process is to free up our original vertex x, to be colored r. (1) Since the r and g chain cannot reach the vertex adjacent to x colored g, that will free up x to be colored r. Thus we have a proper coloring of our planar graph.

This was to be Kempe's proof of the Four Color Theorem for just over a decade. It was widely accepted, due partly to the general disinterest towards the theorem itself. Then, in 1889, Pearcy Heawood discovered an inconsistency in Kempes proof which breathed new life into the interest of the theorem itself. The flaw was found in the cyclical Kempe chain, utilized in case two. The two chains created, the $r - g$ chain and the $r - y$ chain can actually have nodes that are shared, so the toggling of colors could sometimes lead to an improper coloring with 4 colors, but a sufficient coloring with 5. Thus Heawood showed that Kempe had, in fact, proven a less-strict five color theorem. His work, however, was utilized later in something called unavoidable sets which would then be utilized in the the final proof of the Four Color Theorem. (1)

CHAPTER III
RINGS IN MAPS

George Birkhoff was one of the first mathematician to take Kempe chains and effectively use them in true progress towards the eventual proof of Four Color Theorem, following Heawood's upheaval of Kempe's proof. In Birkhoff's paper "The Reducibility of Maps" he goes through four well known reductions used by graph theorists towards a proof for the theorem, before introducing new reductions using Kempe chains.

3.1 Rings

He begins by discussing rings contained in a map. Rings themselves are cyclical arrangements of n regions where n > 3 and each region only shares a border with the region preceding it and the region following it. (7) Note that this allows arrangements of regions with no inner regions and those with inner regions, (see figure 2). With a ring, say R, in a map, the map will be divided into three separate areas, say A, B and R. We will refer to the partial graph of two regions in the graph by $A + R$, where A and R are the only regions included in the partial graph. Since both A and B are bordered by R, if a proper coloring of $A + R$ and $B + R$ will allow for the same arrangement of colors on R, perhaps with a permutation, then we will have a proper coloring for the map M itself. Now we move to considering the chains of paired colors. Consider a graph S, containing a ring R of size four. We define a line as a joining of two regions that are on the same Kempe chain. The physical representation of this being a line segment drawn between them. Our focus will be the collection of all the lines in pairs of colors of regions in the ring R. In other words, we will be examining colorings on R by utilizing pieces of Kempe chains. It follows then that if an $a-b$ chain connects region α of ring R to region β of ring R, and if another chain of colors $a-b$ connects region β to another region, say γ, of ring R, then there exists an $a-b$ chain connecting α to γ. (7)

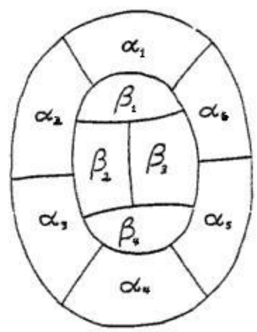

Figure 2. A Ring in a Map.

Lastly we notice that, considering our regions α and β, which are either in a single or pair of colors will either be joined by a chain containing these colors or a chain with regions γ and δ such that the regions occur in a cyclical order on the ring itself and will be joined by a chain of the complementary colors. This makes sense since we are located on a ring that is properly colored and thus we can have alternating colors on the chain or, perhaps, a cycle of all four. Also, a new coloring can easily be obtained by transposing complementary colors. This all leads to the following idea; a given collection of lines on R gives rise to a permuted collection of lines in which complementary sets of lines are unaltered and the corresponding colors are permuted on R in any way so that all those connected by these lines are transposed together. (7)

Consider our map M and region $B+R$. Let's replace $B+R$ by a set of regions, say B', such that there are k regions that retain the same n boundary lines as R. Thus we have a map $B'+A$ that we will assume is colored. We have a finite number of choices of regions for B' which gives us possible choices for the proper coloring of A. Thus we will happen upon a set of colorings via our lines for $A+R$, with the same colorings in place for R. This idea works similarly if we begin with region $A+R$ and work our way towards a set of colorings via lines for $B+R$. (7)

Consider that if any set of two colorings from lines, for the k regions, share at least one coloring, then we can say that our R is reducible! Then, for the n regions we can define a reducing number, kn for R. It is clear that, if two such colorings are shared in our sets, that such a map would be reducible in the sense that we can use a graph with fewer regions that, if colored properly, can be extended to our original, larger graph. Also, any coloring for the reduced graph could easily be extrapolated to the original graph itself.

Now we can show that any ring of four regions in a regular map is reducible and the reducing number is zero. We begin with a ring R of four regions, say α_1 through α_4. Consider the same areas A, and B from earlier, but where A' and B' are formed from $A+R$ and $B+R$, whilst shrinking A and B down to a vertex and joining them to α_1 and α_3. Thus, we find the colorings for the ring to be a, b, a, b or a, b, a, c. If the coloring works for both $A+B'$ and $B+A'$, then we have a coloring for R! In any other case, we have the colorings, let's assume, a, b, a, b for $A+B'$ and a, b, a, c for $B+A'$. Let's consider a second choice for A', wherein we reduce A to a vertex and connect it to α_2 and α_4. (7) This yields a coloring of a, b, a, b or a, b, c, b for $A'+B$.

The only case in need of consideration is the second one, since the first one is already in $A+B$. So, now we have a, b, a, b for $A+B'$ and a, b, a, c and

a, b, c, b for $A' + B$. Now, looking at the colorings for $A + B'$, we have either an a, d line connecting regions α_1 and α_3 for which we will obtain a coloring a, b, a, c by permutation, or an b, c line will connect our regions colored b and we can get a, b, d, b, again by permutation. Either way, our colorings will be similar to the one found in the colorings for $A+B'$. Thus we can conclude that any ring of four regions is reducible and we have a reducing number of 0. (7)

3.2 Ring of Five Regions

Now let's consider a map containing a ring of five regions instead. Since every regular map contains a five-sided figure, and considering our previous foray into reductions of four-region rings, we can assume these graphs will always contain a ring of five regions. Clearly, showing that map containing a ring of five regions is reducible to a map containing a ring of four regions will, in effect, show it is four colorable. Thus, we would wish to show the reducing number for a ring of five regions is 1. (7)

We would like to show that for our sets of colorings for $A + R$ and $B + R$, there exists at least one coloring in common, wherein $A + R$ and $B + R$ are allowed to have fewer than six regions. This will allow us to consider $A + R$ and $B + R$ to be just our ring and a single contained region. The proof of such a thing relies more on patience than cleverness. Considering the colorings we desire and our permutations along the Kempe chains we have utilized previously, a mere careful consideration of our map yields a solid proof.

The other results that Birkhoff came across include the structures of the maps which can be reduced this way, or rather, what is the nature of regular maps M containing no rings of four regions, or of five regions except about a single region? The conclusion being that a series of rings will enclose any arbitrary area of the map which you focus on. The simplest example of this is a map composed of 12 faces of a dodecahedron.

Birkhoff goes on to thoughtfully consider what this means for the progress of the Four Color Theorem, which is the area we are most keenly interested in. Considering the ring of six regions is a very different problem from the five and four region cases we've already explored. Again, it all depends on circumstance for reducibility. This is the slight flaw in Birkhoff's approach. To test all graphs, by hand, to find complete reducibilities this way would be infeasible. Though at the time, Birkhoff had great foresight and noted that "All maps can be colored in four colors, but only by means of reductions of a more extensive character applicable to sets of regions bounded by any number of rings." (7, 125) Thus, as we move into more complicated configurations, we need a more streamlined and concrete method of determining what is reducible and what isn't. Considering this work was being done around 1913, the computing power necessary for such configurations was decades away. In the meantime, Philip Franklin took the torch from Birkhoff for the next step in the proof of the Four Color Theorem.

CHAPTER IV

25 OR FEWER

Philip Franklin set about to use Birkhoff's approach to reductions to prove that any map containing 25 regions or fewer, is four-colorable. Franklin is the first in our line of mathematicians to collect all the characteristics of the concept of an irreducible map. The Birkhoff Diamond is a popular example of this. The irreducible map that we are most concerned with is the five-chromatic map. It would be the smallest (in number of regions) map such that no proper four coloring exists for it. Irreducible maps contain the following properties:

4.1 Properties of Irreducible Maps

1. Each vertex belongs to three and only three regions.
2. No group of less than five regions forms a multiply connected portion of the map. (Consequently there are no two-, three- or four-sided regions and no multiply connected regions.)
3. No group of five regions forms a multiply connected portion of the map unless the group consists of the five regions surrounding a pentagon.
4. No edge is surrounded by four pentagons.
5. No region is completely surrounded by pentagons.
6. No even-sided region is completely surrounded by hexagons. (6, 225)

Logically enough, if it is found that no such map exists, or rather, that every map contains a reducible configuration, then the Four Color Theorem would be proven. From here Franklin goes on to show that any irreducible map, if it exists, must contain more than 25 regions. Consequently, he shows that any map containing 25 regions or fewer is four-colorable.

Consider the well-known Euler formula, $a_0 - a_1 + a_2 = 2$, where a_0 is the number of vertices, a_1 is the number of edges and a_2 is the number of regions. We can make this formula more specific since we are in a irreducible map wherein any three regions only touch one vertex and where no region has less than five sides by saying $2a_1 = 3a_0 = \sum_{5 \leq v} vA_v$. A_v represents the number of regions with v sides in our map. A quick combination of these equalities can tell us that $a_1 = 3(a_2 - 2)$ and $a_0 = 2(a_2 - 2)$. (6) Now, knowing that we are in an irreducible map along with our two new facts leads us to an equality of interest to the Four Color Theorem. $A_5 = 12 + \sum_{7 \leq v} (v-6)A_v$, again where A_v is the number of regions of v sides in the map. This is an explicitly stated version of a theorem of Kempe's, which says: "Every map containing no triangles or quadrilaterals and having three regions abutting on each vertex contains at least twelve pentagons." (6, 226) Thus, we have some more information to work with regarding our irreducible map. Furthermore, Franklin states that irreducible maps must also contain one of the following: A pentagon adjacent to two other pentagons, a pentagon adjacent to a hexagon, or a pentagon adjacent to two hexagons. Considering the previous equalities we coaxed out of the irreducible maps, the proof of such a statement easily falls out.

Consider a map that contains none of these combinations of regions and count the number of vertices that belong to a hexagon and pentagon. Counting shows that we have the number of vertices of hexagons which do not touch a pentagon will be more than twice the number of hexagons in the map. This is due to the fact that pentagons isolated from hexagons or other pentagons will give up five vertices apiece, two pentagons adjacent only to each other will give eight vertices up and lastly a pentagon adjacent to a hexagon will give up four vertices. So, with none of the ealier assumed combinations we would have at least $4A_5 + 2A_6$ vertices or

$v > 4A_5 + 2A_6$. But from our previous inequality, $A_5 = 12 + \sum_{7\leq v}(v-6)A_v$ and an obvious one $0 \geq \sum_{7\leq v}(7-v)A_v$ we get $\sum_{5\leq v} A_v + 12 \leq A_5$ or $\sum_{5\leq v} A_v + 12 \leq 2A_5 + A_6$. Now, since $\sum_{5} A_v = a_0/2 + 2$ we can say that $a_0/2 + 14 \leq 2A_5 + A_6$ or $v + 28 \leq 4A_5 + 2A_6$ which contradicts our earlier inequality! These inequalities are also important as they allow us to make a statement about regular graphs, particularly, $\sum_{r=1}^{v}(6-d_r) \geq 12$, where v represents the number of vertices and d_r represents the number of vertices of degree r. This statement will prove useful in the formation of discharging procedures utilized by Hakken and Appel.

These facts aren't limited to irreducible maps. As a result, if the configurations listed above could be shown to be reducible, then the Four Color Theorem would seemingly be proven. Sadly, at the time of this paper, no known reductions existed for these configurations, but other, more complicated configurations did have reductions, leaving room for hope for a proof.

As is the way in the world of mathematics, Franklin relies on the work of prior mathematicians to prove these configurations are reducible. Namely, he refers to the concept of Kempe chains to help out with these proofs. Let us first consider the configuration where a side of a hexagon surrounded by a hexagon and three pentagons. This is a reducible configuration. Clearly if this were present in an irreducible map, a few clever border erasings would yield something smaller than an irreducible map and thus, colorable map. (6) Frankin goes on to describe more and more complicated versions of this configuration and how all of them are, again, reducible using some clever erasing of borders and application of previous knowledge.

As we move through the proofs of these configurations being reducible, we can see the true scope of the difficulties that the Four Color Problem creates. There are literally an infinite number of cases to go through and even though Franklin and Birkhoff are describing ways of classifying them, we need to be much more specific and systematic to make real progress towards a solid and complete proof. Even so, the approach that Franklin takes through the myriad of pentagonal configurations, again from our concept of irreducible maps, yields the final result that"every map containing 25 or fewer regions can be colored in four colors." (7) is important. The conclusion was drawn from the idea that every configuration with 25 or less regions was reducible, so any irreducible maps must have more than 25 regions. Hence, any map with 25 or fewer regions must be four-colorable.

CHAPTER V
REDUCIBILITY

The systematic approaches that we are looking for will come from mathematician Heinrich Heesch. Heesch was observing a possible proof of the theorem from the vantage point of what are called irreducible configurations. At that time mathematicians considered looking more carefully at the five-chromatic map, that is, the smallest map that can only be properly colored with 5 or more colors. In particular, they began to look more carefully at the properties of this five-chromatic configuration and other configurations which were, in face, reducible. Since the existence of a five-chromatic map implies that it is a normal map, configurations overtook maps as a focus and become the most powerful tool in proving the theorem. We need a few definitions to fully understand the importance of this approach.

5.1 Configurations

A configuration is a regular graph whose outer vertices form a circuit of size four or greater. Also, the bounded regions have triangular borders, every triangle is the border of a region, and inner vertices exist. (2, 155-156). To clarify, a normal map is a regular map which is saturated, meaning no new vertices can be introduced, and also every face is bounded by a triangle. (2, 151) Of course, regular maps are maps where every vertex has the same degree. Primarily, these configurations appear as subgraphs of normal graphs and again become very important as our discussion progresses. (see figure 3)

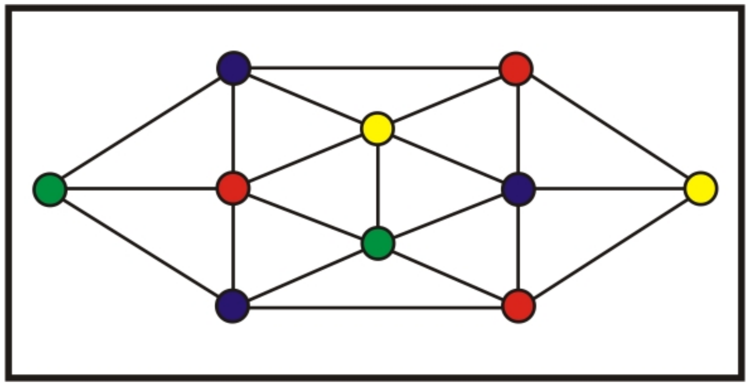

Figure 3. A Four-Colored Birkhoff Diamond.

As Heesch considered this set of reducible configurations, he remarked that it could have as many as 10,000 members (5). This set of unavoidable reducible configurations consists of all the reducible configurations that every normal configuration would contain and would be sufficient to prove our theorem. Obviously, fact-checking something like that by hand would be incredibly time consuming. Thus, reductions became very important. He was at the forefront of two methods of reduction called C, and later D and an approach that would later be coined as a "discharging procedure" by Wolfgang Haken, one of the two eventual provers of our theorem. The beauty in these approaches was that they were sufficiently algorithmic enough to lend themselves to programming, which is where the theorem eventually met its match.

5.2 Color Extendibility

Let us first look at the concept of D-reducibility. This type of reducibility has to do with what has been coined "color extendibility". Color extendibility has to do with examining a ring of the graph and its boundary coloring, a proper coloring of the ring. Essentially we will have to sift through all the boundary colorings of rings to find those which can be extended back inside the configuration with no alteration whatsoever. Those that meet this criterion will be referred to as being good from the onset which was a term coined by both Haken and Appel. An essential boundary coloring is one that uses the fewest necessary colors. The next step in establishing D-reducibility in a given configuration is finding a concise way to describe the configuration in a computer program.

5.3 Adjacency Matrix

Here we turn to a cyclic numbering of the vertices and, consequently, the oft-used adjacency matrix. An adjacency matrix is a square matrix whose rows and columns are numbered in accordance with our renumbering of our vertices in our configuration. A 1 is input into the $A_{i,jth}$ spot if and only if $(v_i,v_j) \in E$. As a result of our working with graphs which have no loops, our adjacency matrices always have a diagonal of 0s. This structure is easily input into a computer and makes an ideal avenue for the programmer. Durre, who worked closely with Heesch, developed color matrices to help with color extendibility. In these matrices, the columns correspond to the vertices of the map, where as the rows are labeled $0, 1, 2$, and 3 for our four colors. Thus, if vertex 1 has color 2, then entry $A_{2,1}$ will be a 1. Otherwise, 0's are placed in the entries. This matrix provides an easy, programmable way to look at extendibility. In fact, with the proper indexing of vertices, the first r columns of a coloring for a configuration will make up the boundary coloring. If a complete color matrix can be formed from them then we have direct color extendibility.

5.4 Goodness of Colorings

In this context of color extendibility, we are looking for configurations that have boundary colorings that are good from the onset and essential. A boundary coloring is a coloring of the boundary circuit of a configuration. This requires us to look at the coloring of the vertices of our configuration. Two colorings are said to be equivalent if they have the same size and differ only by a permutation. So, we then have classes of boundary colorings, and from this class we must pick a representative. This is the essential bounday coloring. The essential bounday coloring is the "smallest" coloring with respect to lexicographical order. The total number of essential boundary colorings for r regions, called, say $e(r)$, is: for r even, $\dfrac{3^{(r-1)} - 1}{8}$ and for r odd, $\dfrac{3^{(r-1)} + 5}{8}$. (2, 190) Also, if every boundary coloring is good from the onset then it is obvious we have a reducible configuration and thus a map that has a proper coloring. It is also obvious that this simplistic case will not be the case in nearly all of our configurations so, what can we do when this is not the case?

This is where we employ the concept of Kempe interchanges, which is again closely related to the Kempe chains used earlier. If any conflict occurs with two colors meeting, we simply alternate this color with another to produce a desirable outcome. This is a perfect avenue for the use of computers, as we do not want to examine the thousands of boundary colorings for a given configuration individually nor the countless permutations. So, quite a few definitions become necessary to be precise enough for a program. Suppose we have a five-chromatic map G containing a configuration C. This G is the smallest map, in number of regions, that is five-chromatic and is also known as a minimal triangulation. (2) We obtain a new graph, G' by eliminating the inner vertices and edges of C. Our new graph has the propery of having an exceptional face, that is, a region not bordered entirely by triangles. The vertices that form its edge are our bounding circuit. Examining these circuits becomes important via block decompositions.

5.5 Block Decompositions

We can use block decompositions of our vertices to help with this analysis. A block decomposition is a decomposition of our bounding circuit into blocks, say B_1, B_2, etc. (2). A block, or Kempe block, is a maximally connected subgraph of two arbitrarily chosen colors. This also brings about a definition of a color-pair choice. A color-pair choice is a partition of our colors $[0, 1, 2, 3, 4]$ into the set of a $[0, k]$, and the set of $[1, 2, 3] \backslash [k]$. Thus, by looking at our color-pair choice, we produce Kempe blocks in our boundary circuit, which we can then call Kempe sectors. (2) With the particular structures of maps we are considering, there are either 1 or an even number of Kempe sectors in a given configuration. Given the four colors we have to choose from, it is seen that there are only two types of Kempe sectors, those colored with our k and the other chosen color, and those with the two remaining colors. Thus we have only two block decompositions to consider. The amazing thing about these block decompositions is that they are only affected by the boundary coloring and the color pair choice.

Thus, the configurations themselves are unimportant in our decompositions! Let us discuss what the decompositions actually are. Supposing r is a natural number, and taking a set $M = 1, 2, 3, \ldots, r$, then M is decomposed along a set of non-empty pairwise disjoint sets $B_1, B_2, \ldots, B_r$ where the union of these sets covers M. Blocks B_k and B_l are said to abut if, for a cyclically labeled vertex t lying in B_k, $t + 1$ lies in B_l. Now we define block decomposition through a theorem (2) Theorem: Let G be a colored, connected graph without bridges or final edges all of whose faces are bordered by triangles. Suppose we have a coloring $a = (a_1, \ldots, a_r)$ and a color pair choice are given. Then a partition of the index set $(1, \ldots, r)$ into blocks $B_1, \cdots, B_s$ is a block decomposition (with respect to a and w) if and only if:

1.Each block is a union of Kempe sectors of the same type.
2.Blocks cannot mutually overlap.
3.If two blocks abut at one vertex they must abut at exactly one other vertex. (2)

This allows us to define an important term in decomposition, that is, chromodendron. If we consider a boundary coloring, a color-pair choice, k, and the block decomposition that follows, the chromodendron is the graph whose vertices are the blocks of the graph and edges are pairs of abutting blocks. These chromodendrons are trees, in the graphical sense, meaning they contain no cycles and are connected. (2)

5.6 D-Reducibility and Goodness of Colorings

Now we have enough information to move closer to a definition of D-reducible configurations. In fact, we now know enough for a rough definition. So, loosely speaking, a configuration is D-reducible when every boundary coloring can be changed, with a finite number of Kempe changes, into colorings that are good from the onset. When the set of all boundary colorings for a configuration C are good from the onset, we will call it $\phi_0(C)$. When they are not, we must use Kempe interchanges. The Kempe interchanges work in an algorithmic fashion. This algorithm helps us to find higher order of "goodness" in terms of our colorings via the notion of classes. Thus, we are taking a block decomposition and trying to take the resulting boundary colorings and make them all good from the onset or good of some stage. We have discussed being good from the onset, wherein no alteration is necessary for our boundary colorings. Intuitively then, being good of stage 1, also known as class 1 good refers to a boundary coloring whose block decomposition can be changed into a boundary coloring that is good from the onset with the choice of a color pair, i.e., letting it become a member of $\phi_0(C)$. Now suppose ϕ is a set of boundary colorings for a configuration. Let a be an element of $\phi(r)$, which we know to be our set of all boundary colorings of size r. The element a is said to be "ϕ good" where a itself isn't a member of ϕ, but instead if there exists a color-pair choice such that for the subsequent block decomposition, there is a Kempe interchange that makes a an element of ϕ. A further distinction of ϕ is $\phi_n(c)$. $\phi_n(c)$ is the set of established boundary colorings that are good of class $\leq$ n, again referring to the algorithmic process of using Kempe changes. So, we say a boundary coloring is good of stage $n+1$ if it is $\phi_n(c)$ good. (2) Finally we are in the position to define what it means for a configuration to be D-reducible:

> A configuration C is said to be D-reducible if $\overline{\phi(C)} = \phi(r)$. Note that $\overline{\phi(C)}$ is the same as $\phi_{n_0}(c)$. This means there exists an index n_0 such that no boundary colorings of goodness class $n_0 + 1$ exist. This means that every boundary coloring is good of one stage or another. Also, a configuration is said to be D-irreducible if $\overline{\phi(C)}$ is a proper subset of $\phi(r)$. (2, 205)

The D-reduction we have just discussed is the final important category of the other types of reducibilities. The other reductions are named, conveniently enough, A, B, and C. The A, B, and C are actually tributes to various mathematicians who worked on the Four Color Theorem over the years. A is for A. Errera, B symbolizing Birkhoff, and C for C. E. Winn. D-reducibility came last and was named so to keep the ordering which is never surprising in a mathematics context. As far as how they are inter-related, it turns out that A is a special case of B, B is a special case of C and D is a special case of C when thought of in terms of reducers. (2)

5.7 Reducers

A reducer is a pair (S, θ) of a graph S and a surjective mapping from the set of outer vertices of a configuration C to be reduced to the outer vertices of C. The reducer has the following properties; 1) the mapping θ must preserve the property of being a neighbor, and 2) the original distinct outer vertices of S, with respect to θ, cannot mutually overlap. By looking at the colorings as θ maps them to the vertices of our reduced map, we will establish the different classifications of reducibilities, and thus will give us a better look at D-reducibility itself.

A configuration C will be called A-reducible if it has a reducer (S, θ) such that: 1) It can only be θ-properly embedded in a minimal triangulation and 2) Each θ-compatible boundary coloring is directly color-extendible.

Now, a minimal triangulation is a normal graph that will not permit the existence of a proper four coloring. This is also known as a "minimal criminal" (2, 152) in the sense that its existence would contradict the Four Color Theorem. Consider a minimal triangulation G, a configuration C and a reducer, (S, θ). To θ-properly embed a configuration means that two outer vertices of C that have the same image under our mapping θ cannot be neighbors in our original G. Thus, an A-reducible graph is, essentially, as small a graph as is possible where in we can still extend our boundary coloring to our reduced configuration. An example of this would be the four star in three colors. (2, 210) (see figure 4)

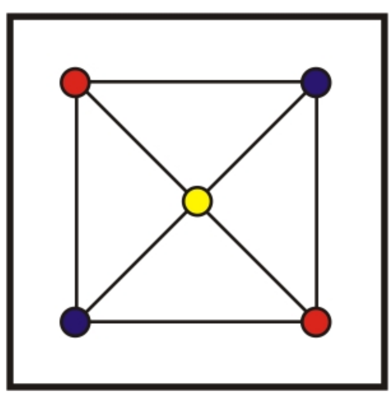

Figure 4. A 2-Colored and 4-Colored 4-Star.

Again, consider a configuration C with (S, θ) as a reducer. This reducer, just like A, exists in a way such that C can only be θ-embedded in a minimal triangulation. This configuration will be B-reducible, if: Each θ-compatible boundary coloring will be either good from the onset or good from stage 1 and will be C-reducible if: each θ-compatible boundary coloring will be good from some stage. (4, 213)

Considering D-reductions as a special case of C-reductions goes as follows. For a configuration C, consider S to be the boundary circuit. If we allow θ to be the identity mapping, then C will only be properly embedded in a minimal triangulation. By our definition of D-reduction, we can see that it is also C-reducible. This leads us to the conclusion that, in fact, C-reductions can be considered the most general of the types of reductions. This would make them seem like the best candidate for a programming approach. D-reducibility triumphed, however, as Appel put it, "All D-reducible configuration are C-reducible but, in the approach we took it was easier to prove D-reducibility and we tried to prove configurations C-reducible only if they were not D-reducible" (8) There was one more ingredient missing from a full proof of the theorem, and again Heesch was the mathematician at the helm.

CHAPTER VI
DISCHARGING

The successful approach to the four color problem involved more than reducibilites. What was necessary was a set of configurations, say S, which we call unavoidable in the sense that any minimal counterexample to the four color problem must include a member of the set. The existence of an unavoidable set S such that each member of S is reducible would imply the truth of the four color theorem. Heesch invented a clever procedure known as a "discharging method" which became crucial in the construction of an unvaoidable set. Though he later abandonded it, Haken and Appel would take and refine his idea to complete their proof.

Heesch's idea was to assign a "charge" of $6 - k$ to each vertex of degree k. It can be seen from Euler's formla that the sum of the charges in a regular planar graph will be 12. For the case of a minimal counterexample, given the proof covered in section 4.1 on Franklin's work which allows us to look only at vertices of degree 5 or greater we can expand the sum of the charges to

$$v_5 - v_7 - 2v_8 - \cdots - (s-6)v_s = 12 \tag{6.1}$$

where v_r is the number of vertices of degree r and s is the largest vertex degree. The fact that $v_r = 0$ for $r \leq 4$ is important. A discharging procedure is a set of rules that redistribute the charges among the vertices so the net charge remains the same. After the charges are redistributed, the net charge of our configuration will remain positive, so some vertices will have a positive charge. These particular vertices give way to a set of configurations that form an unavoidable set. This unavoidable set must be examined to ensure that the resulting outcomes are reducible. If they are not, the discharging procedure is refined and attempted again.

Each discharging procedure will give rise to an unavoidable set of configurations. A simple example of this can be found with a simple discharging procedure: distribute 1/5th of the charge from each degree 5 vertex to any adjacent vertex of degree 6 or more. The resulting positive configurations, which can be seen below in Figure 5, form an unavoidable set. The logic here is fairly simple. We consider what types of vertices can have a positive charge. A vertex of degree 5 cannot lose all of its charge, so it must have a neighbor of degree less than 7, or rather, of degree 5 or 6. This will give us the situation presented in Figure 5. A vertex of degree 6 has a charge of 0 and cannot change. Clearly, a vertex of degree 8 or more cannot collect enough charges to become positive. The degree 7 vertex is the interesting case. It begins with a charge of -1, so it must have at least 6 neighbors of degree 5 to become positive. The regularity of this graph implies that two of the degree 5 vertices must be adjacent, so this possibility has already been accounted for. This last case gives a good feeling for how complicated the analysis of a discharging procedure can get.

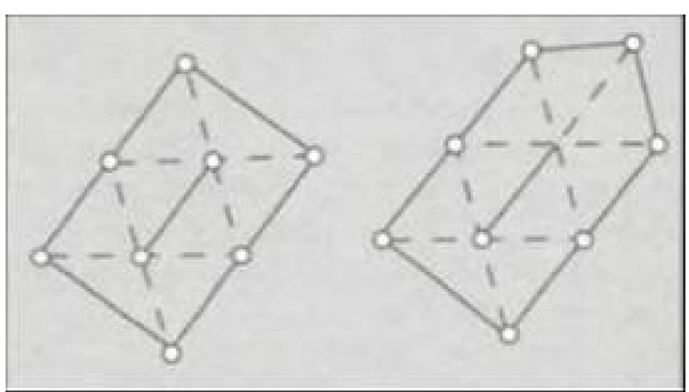

Figure 5. A 5-5 and 5-6 Chain.

6.1 M and N Rule

There was also a method of looking at a configuration to decide whether or not it was likely to be reducible, i.e., if it contained any obstacles. Heesch invented a procedure for which, if a configuration failed, it was not reducible at all, but passing warranted a second look. The first place we look is for any vertex, of degree d, connected to fewer than $d - 3$ vertices, the vertex may be removed to produce a smaller configuration. Then we consider pairs of degree 5 vertices which are connected to a third vertex and to one another, but to no other vertices. Then you may remove both of the degree 5 vertices to make a pair of smaller configurations. Lastly, we look for any cut-vertex of degree d available that is connected to less than $d - 2$ vertices. A cut-vertex being a vertex that, once removed, leaves the configuration disconnected. That cut-vertex is removed to form two smaller configurations. The process is repeated until no further vertices can be removed. Failure comes if, after all steps are completed, all of the remaining configurations are irreducible. This procedure is quite nice for wittling down large configurations by hand. As Hakken and Appel applied this procedure, they made note of something which became a rule in and of itself. This is called the m and n rule. For a given ring of size n, the reducibility of the ring increases very quickly as the number of vertices inside of the ring, say m, increases. If the configuration satisfies $m > \frac{3n}{2} - 6$ then it will contain a sub-configuration free of any obstacles and will almost always be reducible. The most important method, though, was the discharging procedures, honed by Hakken and Appel. (4)

6.2 R-, S-, And L-Dischargings

The discharging procedures are varied but let us look at a simple example to begin with. Suppose we have a configuration C with charge distribution, and a procedure p to move our charges in this configuration. P, as a simple discharging, would be applied as follows: for any vertex, V_5, where V_5 represents a vertex with degree 5, which is connected to a major vertex, V_k, that is a vertex whose degree $k \geq 6$, a charge of 30 will be moved from V_5 to V_k. Thus our configuration c has a new charge distribution, lets say q. Now, when we look at the individual charge on a vertex, called $V(q)$, there are only a few situations for a vertex V where it can be positive:

1) V has degree 5 and either 1 or 0 major neighbors.
2) V has degree 7 and has between 3 and 7 neighbors of degree five.
3) V has degree 8 and has between 5 and 8 neighbors of degree five.
4) V has degree 9 and has between 7 and 9 neighbors of degree five.
5) V has degree 10 and either 9 or 10 neighbors of degree five.
6) V has degree 11 and has 11 neighbors of degree five. (4)

Now the task becomes creating a set of configurations, let's call it U, such that each case from 1 to 6 is represented and each member of U is a part of the unavoidable set of configurations. The set can be constructed from subgraphs of graphs containing combinations of the criteria above. An example of such a configuration is the set containing the 6-star, the 7-star, the Birkhoff diamond, the Chojnacki configuration, and the Franklin configurations with 9, 10 and 11 inner vertices. (2, 225) The problem with the construction of such a set is that one of the potential members of U could be irreducible. Haken and Appel came across these difficulties several times and discovered that more defined procedures would help overcome these obstacles.(4)

Examining the flaws in the simple procedure p is not without difficulty. In fact, in listing the many, many, exceptions to the general distribution, one can become overwhelmed by the minutiae. By allowing these special cases to become procedures of their own, it becomes easier to manage the vast amount of dischargings and members of U. For example, let us call the situation above an R-discharging, R for regular, but allow for a new procedure called an S-discharging, which addresses a special-case configuration where we have one degree five vertex connected to a major edge along which a charge of less than 30 is transferred. We also consider another situation called an L-discharging, L for large, in which a charge greater than 30 is transferred along the edge.

So, now taking our exceptions, we define a procedure $P(S, L)$, being wary of a few situations. Being explicit about what to do when two charges are being distributed over the same edge is very important. Also, restricting the size of S and L and the ring size examined is important. Haken and Appel found that an increase in ring size from 14 to 16 increased the difficulty of deciding the reducibility of a configuration by a factor of 16. The other difficulty encountered was a 6 - 6 chain, or the edges that link V_6 to V_6. This indicated to them that an improvement was needed to the $P(S, L)$ method.

6.3 T-discharging

Thus we have what became the T-discharging procedure. This comes from allowing one further exception to R-discharging. This will address situations where V_5 is connected to a neighbor of a neighbor of a major vertex. (see figure 5) This took Haken and Appel many years of work to refine. The T is a reference to the transversal dischargings that are taking place. Thus, the charges are transferred along one or more edges that join pairs of V_6's. In fact, this kind of discharging drastically reduces the number of L-dischargings that are necessary to form an unavoidable set and is thus preferrable. (4)

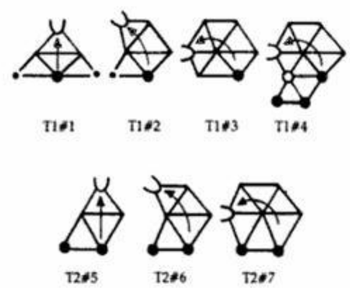

Figure 6. T-dischargings. (4)

As Haken and Appel worked out the discharging procedures by hand, now with a $P(T, S, L)$, they were able to construct their unavoidable set. This produced set S of 269 S-situations, a set L of 210 L-situations and a corresponding unavoidable set that contained 1818 reducible configurations. This number was later reduced to 1476. Haken and Appel noted that they did not believe their choice of discharging was necessarily the best way, as their final discharging procedure had more than 300 rules to it. (4) Other independent mathematicians such as Frank Allaire attempted a very different discharging procedure which contributed a much smaller unavoidable set. Also, many years later, using a conjecture of Heesch's regarding overcharged vertexes, Neil Robertson constructed an unavoidable set containing only 633 configurations. His discharging procedure only used 32 separate rules, a vast improvement over Haken and Appel. Haken and Appel's independent work cannot go overlooked, though. The final years of the theorem, although other mathematicians were working on the the theorem itself, belong to Haken and Appel. (4)

CHAPTER VII
COMPUTING POWER AND PROOF

The last stretch of the proof of the Four Color Theorem rested on the shoulders of Kenneth Appel and Wolfgang Haken and on that of the computer. Those who had already programmed the key blocks of the proof found difficulty in several places. One of the limiting factors initially was the nascent computing power that was being employed. The first programming that resembled a proof was done by Durre in 1965 (5). His D-reducibility program running a configuration containing a ring of size twelve took six hours to run on the CDC 1604A computer at Hannover. Increasing the size of the ring by one increased the running time to anywhere from 16 to 61 hours, depending on the configuration. Still, though, this was considerably better than checking the thousands of potential configurations by hand. To confuse things even more, as previously discussed, just because a configuration doesn't meet D-reducible requirements doesn't mean it isn't C-reducible. This required careful consideration by Appel as he began programming. The computer to make this approach tractable was still out of reach.

The limits of most computers of that era put a stop to most programs being used in regards to our proof. Where most computers simply weren't powerful enough, the ones that did possess the architecture to potentially make the programs tractable were at major universities or being used by the government. Unlike today, where everyone has a PC that can do an amazing amount of calculations per second, the computers of the sixties and seventies were only available on limited schedules. Interest in the solution to the Four Color Theorem had not reached a fever-pitch outside of the mathematics community, so convincing your average computer science engineer to let you use his universities supercomputer was difficult as well. Still, Haken, Appel, Heesch, and Durre continued to look for more powerful computers, in the hopes that Durre's program could be made feasible to use. (5)

At first, Haken and Appel attempted to use the ILLIAC IV, the supercomputer on the campus of University of Illinois. This computer possessed a heretofore unused parallel structure that promised to make it a very fast and powerful machine. Sadly, it was not complete enough to be used for their program. They were referred instead to a theoretical physicist who would turn out to be a major player in the final days of the theorem, Yoshio Shimamato. Shimamato had direct access to the Brookhaven 6600, a machine that was considerably more powerful than the CDC 1604A computer at Hannover. The only hiccup in the process was the translation of the original program, which was in ALGOL 60, to FORTRAN. Thankfully, this process didn't take long and many configurations were soon being tested for reducibility. (5)

Still, though Shimamato had access to the Brookhaven 6600, he did not have the authority to monopolize its computing power. In this idle time he became more and more interested in the Four Color Theorem. In fact, he later remarked that during a particular boring faculty meeting, he began drawing out a particular configuration that was almost the end of the Four Color Theorem. The figure was the aptly named "Shimamato Horsehoe". (figure 6)

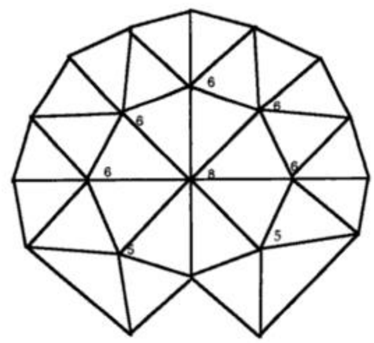

Figure 7. The Shimamato Horseshoe. (5)

In essence, Shimamato had constructed a figure that, if proven D-reducible, would prove the Four Color Theorem once and for all. Given some time, the horseshoe was run through Durre's program. The announcement was a sad one, the horseshoe was not, in fact, D-reducible. Many reruns of the program solidified this fact. It seemed, for a brief moment, that the progress of this proof had stalled entirely.

Haken did not consider himself a regular mathematician. He was more of a physicist by trade and even mused on one occasion that he "could not pass any one of those exams that are required today" for mathematics professors (5). It was at the urging of Kenneth Appel for a lecture on the methods of one of Haken's graduate students, Thomas Osgood, that the two began working together. Osgood was, in fact, working on the Four Color Theorem himself, under the supervision of Haken, in the area of reducible configurations. In this lecture, Haken admitted to the gathered group that he may return to the Four Color problem, but for now "I'm quitting"(5, 130). Appel, after the lecture, found himself very interested in the problem, given his background in programming and as a logician.

At his urging, Haken agreed to work on the problem more, with Appel's aid as a programmer. They both recognized that the brute force methods that others had applied just wouldn't work. Something slightly more elegant and clever needed to be used. The trick of searching for reduction obstacles became an intricate part in making this proof a reality. In essence, these were characteristics of irreducible graphs that had been noticed by Haken, who possessed an amazing knack for predicting whether or not a given configuration would be reducible. One of these reduction obstacles he noticed was that no reducible configuration contained "at least two vertices, a vertex adjacent to four vertices of the ring, and no smaller configuration that was reducible." (4) Haken was able to identify three more of these obstacles which were easily describable. This breathed new life into the programming aspect of things, and helped reduce computer time, giving a more narrow focus in the search for reducible configurations. With this and the discharging methods, Haken and Appel would have everything they needed, from a computer standpoint.

Still, though, there were things to be done in this proof that require work by hand. While the computer and programs showed that the selected configurations were reducible, showing these configurations formed an unavoidable set became something done by Haken, Appel, and their families. Also, hand-checking was done on the various discharging methods, of which the final method was completely describable by hand. The method of this section of the proof became a modification of the discharging procedure every time the associated configurations could not be proved reducible. As the proof drew to a close, Haken and Appel both noticed something strange. "There are literally thousands of proofs of the Four Color Theorem in the sense that many possible discharging procedures and their associated unavoidable sets would yield proofs." (8) It turns out that as they were modifying their discharging methods, they were downsizing to a smallest size of acceptable proof.

Another method of proof Haken and Appel employed was fact-checking against other resources. As they were verifying the outputs of their program, they looked at the work for Heesch and Durre, Frank Allaire and E. R. Swart. Since this part of the proof had to be correct, they took no chances. Thankfully, with correspondence they found all parties in agreement about the reducibilities that they had checked with the computer.

Even with all of this careful work, some found problems with the proof when it was announced in 1976. The problems that most mathematicians had came from the computer itself. Hand checking pages upon pages of work was routine to mathematicians of the age, but the use of computers as workhorses in proofs was new and, to some, scary. Appel addressed this in a paper published in 1977 entitled "Computers and The Four Color Theorem."

In his estimation, there are two kinds of proof that fall under the category of benign. One of them is something called pseudo-benign. A pseudo-benign proof is one that can be hand-verified by a single mathematician with a lifetime of work. The other is called easily replicable, which refers to a proof that requires a small number of easily programmable algorithms and can be verified by an interested party without an overwhelming amount of effort. The Four Color Theorem is a proof that can fall into either category. Appel notes that, although mistakes will be made by humans in transcribing and perhaps in thought as a proof grows in size, most of the time the errors are easily corrected. Even so, looking over a pseudo-benign proof of the Four Color Theorem would require a patience and work ethic that very few could muster for the amount of time necessary for thorough checks. Thus, there is an inevitability to these kinds of proofs. Computers, when programmed correctly, have these attributes in spades. Acceptance came in small doses at first, but the acceptance of the proof by William Tutte was considered a major step towards a true consensus. His almost comic poem entitled "Some Recent

Progress in Combinatorics" cooled some fires in the mathematical community:

Wolfgang Haken,
Smote the Kraken,
One! Two! Three! Four!
Quoth he: The monster is no more.(3)

Since this proof was announced at a Summer meeting of the American Mathematical Society in 1976, many other mathematicians have improved upon the methods of Haken and Appel but produced the same elegant outcome. Some sought to reduce the size of the unavoidable set, which Haken and Appel predicted would be possible. The proof given by Neil Robertson in 1993 used only 633 configurations, but very similar methodology. (9) All the proofs that followed Haken and Appel only solidified what was announced to that stunned audience; to properly color a planar graph, one needs, at most, four colors!

BIBLIOGRAPHY

[1] Saaty, Thomas L. and Paul C. Kainen. The Four Color Problem. Great Britain: McGraw-Hill, 1977.

[2] Fritsch, Gerda and Rudolf. The Four-Color Theorem. New York: Springer-Verlag, 1998.

[3] Chartrand, Gary and Ping. Introduction to Graph Theory. New York: McGraw Hill, 2005.

[4] Appel, Kenneth and Wolfgang Haken. Every Planar Map is Four-Colorable. American Mathematical Society, 1992.

[5] MacKenzie, Donald. Computing, Risk, and Trust. MIT Press, 2001.

[6] Franklin, P. "The Four Color Problem", American Journal of Mathematics, 44.3 (1922), 225 236.

[7] Birkhoff, G. "The Reducibility of Maps", American Journal of Mathematics, 35.2 (1913), 115 - 128.

[8] Appel, Kenneth. "The Use of the Computer in the Proof of the Four Color Theorem", Proceedings of the American Philosophical Society, 128.1 (1984), 35 39.

[9] Robertson, N. et al. "The Four Color Theorem", Journal of Combinatorial Theory, 70, (1997), 2 - 44.