A MAXIMAL OPERATOR, CARLESON'S EMBEDDING, AND TENT SPACES FOR VECTOR-VALUED FUNCTIONS

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Academic dissertation

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Helsinki, November 2011 Mikko Kemppainen

List of included articles

This thesis consists of an introductory part and the following three articles:

[A] M. Kemppainen. On the Rademacher maximal function. *Studia Mathematica*, Volume 203, Issue 1, 1–31, (2011).

[B] T. Hytönen, M. Kemppainen. On the relation of Carleson's embedding and the maximal theorem in the context of Banach space geometry. *Mathematica Scandinavica*, Volume 109, Issue 2, 269–284, (2011).

[C] M. Kemppainen. The vector-valued tent spaces T^1 and T^∞. *Journal of the Australian Mathematical Society* (to appear). Preprint, arXiv:1105.0261, (2011).

The author had a major part in the analysis and writing of the joint article [B].

Background

Dyadic techniques in Harmonic Analysis in their most elementary forms make use of the family $\mathcal{D}$ consisting of *dyadic intervals* $2^{-k}([0,1)+m)$ with k and m ranging through the integers $\mathbb{Z}$. The founding observation is that the family of *Haar functions* $h_I = |I|^{-1/2}(1_{I_-} - 1_{I_+})$, where I_- and I_+ are the left and right halves of a dyadic interval I, constitute an orthonormal basis for $L^2(\mathbb{R})$.

While these techniques appear at first quite specific to the Euclidean setting, it should be mentioned that versions of dyadic cubes can be constructed even in abstract metric spaces, see for instance Christ [11] and Hytönen and Kairema [26]. Moreover, operators known as *dyadic shifts* have been of central importance in the study of sharp weighted norm inequalities for singular integral operators (see Petermichl [43] and Hytönen et al. [29]). These lines, however, will not be pursued here.

The $T1$ theorem of G. David and J.-L. Journé [16] concerning the L^2-boundedness of singular integral operators is proved by T. Figiel in [20] using dyadic techniques. The proof introduces a *dyadic paraproduct operator* Π_b associated to a given function b in (dyadic) $\mathrm{BMO}(\mathbb{R})$ according to the formula

$$\Pi_b f = \sum_{I\in\mathcal{D}} \langle f\rangle_I \langle b, h_I\rangle h_I, \quad f \in L^2(\mathbb{R}), \tag{1}$$

where $\langle f\rangle_I$ denotes the average of f over a dyadic interval I and $\langle b, h_I\rangle$ stands for the pairing $\int b h_I$. To see that (1) defines a bounded operator on $L^2(\mathbb{R})$ one resorts to the inequality (*Carleson's embedding theorem*)

$$\Big(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} |\langle f\rangle_I \theta_I(x)|^2 \,\mathrm{d}x\Big)^{1/2} \le \|(\theta_I)\|_{\mathrm{Car}^2} \|Mf\|_{L^2}$$

with $\theta_I = \langle b, h_I\rangle h_I$, so that

$$\|(\theta_I)\|_{\mathrm{Car}^2} = \sup_{J\in\mathcal{D}} \Big(\frac{1}{|J|}\int_J \sum_{I\subset J} |\theta_I(x)|^2 \,\mathrm{d}x\Big)^{1/2} \eqsim \|b\|_{\mathrm{BMO}},^{1}$$

and applies the L^2-boundedness of the *dyadic maximal operator* M. Possible ways to address the boundedness of Π_b on $L^p(\mathbb{R})$ with $1<p<\infty$ include interpolation from a weak $(1,1)$ estimate or from H^1-L^1 -boundedness, and extrapolation from weighted inequalities (see Pereyra [42]).

Figiel's proof was designed to allow an extension of the $T1$ theorem to functions taking values in a Banach space X and hence we now ask if Π_b acts boundedly on the Lebesgue-Bochner space $L^p(\mathbb{R};X)$ for $p\in(1,\infty)$. The first obstruction is the lack of an orthogonality argument. If, however, X is a *UMD-space*,[2] we obtain for functions f in $L^p(\mathbb{R};X)$ that

$$\|\Pi_b f\|_{L^p(X)} \eqsim \Big(\int_{\mathbb{R}} \mathbb{E}\Big\| \sum_{I\in\mathcal{D}} \varepsilon_I \langle f\rangle_I \langle b, h_I\rangle h_I(x)\Big\|^p \,\mathrm{d}x\Big)^{1/p}, \tag{2}$$

[1] By $\alpha \lesssim \beta$ we mean that there exists a constant C such that $\alpha \le C\beta$. Quantities α and β are comparable, $\alpha \eqsim \beta$, if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

[2] UMD stands for *u*nconditional *m*artingale *d*ifferences.

where $\mathbb{E}$ denotes the expectation for the independent *Rademacher variables* ε_I attaining values 1 and -1 with an equal probability of $1/2$, so that a randomized sum replaces the square sum of the scalar case. We may now apply a vector-valued version of Carleson's embedding theorem which states that

$$\Big(\int_{\mathbb{R}} \mathbb{E}\Big\| \sum_{I\in\mathcal{D}} \varepsilon_I \langle f\rangle_I \theta_I(x)\Big\|^p \,\mathrm{d}x\Big)^{1/p} \lesssim \|(\theta_I)\|_{\mathrm{Car}^q} \|M_R f\|_{L^p},$$

where $1 < p < q < \infty$ (see Section 4 for $\|(\theta_I)\|_{\mathrm{Car}^q}$) and

$$M_R f(x) = \sup\Big\{ \Big(\mathbb{E}\Big\| \sum_{I\ni x} \varepsilon_I \langle f\rangle_I \lambda_I\Big\|^2\Big)^{1/2} : \sum_{I\ni x} |\lambda_I|^2 \le 1\Big\} \tag{3}$$

is the *Rademacher maximal function* of f. The right-hand side of equation (3) defines the *R-bound* $\mathcal{R}(\langle f\rangle_I : I \ni x)$ of the set of dyadic averages of f at x. The concept of R-boundedness, which originates from Berkson and Gillespie [2], is here applied to vectors by viewing them as operators from the scalars to X (see [A, Section 2] for definitions). Replacing square sums by randomized sums and suprema by R-bounds is a standard procedure in vector-valued Harmonic Analysis (see Bourgain [3] and McConnell [35] for discrete square functions and Weis [47] for R-boundedness). The question arises whether M_R is bounded from $L^p(\mathbb{R}; X)$ to $L^p(\mathbb{R})$ for $1 < p < \infty$. This maximal operator was first defined by T. Hytönen, A. McIntosh and P. Portal in [27], where also a vector-valued version of Carleson's embedding theorem was proven. Moreover, they discovered that the boundedness of M_R defines a non-trivial Banach space property – the *RMF-property*[3] – in the sense that not every Banach space, for instance ℓ^1, has it. It should be mentioned that in [20] Figiel announces a proof (later presented in Figiel and Wojtaszczyk [21, Section 6]) of the $L^p(X)$-boundedness of Π_b for UMD-spaces X and attributes an intermediate estimate to J. Bourgain.

Two Banach space properties: UMD and RMF

This section summarizes article [A]. In this article, the operator M_R is defined in a somewhat more general context with respect to filtrations on σ-finite measure spaces (see the next section). Its boundedness, however, does not depend on the underlying space in the sense that it suffices to study the most tractable case of unit interval, as is stated in [A, Theorem 5.1] (much in the spirit of the reduction argument in Maurey [33]). For this introductory discussion we restrict ourselves to probability spaces.

p-independence of the RMF-property

The above mentioned UMD-property of a Banach space X is often described as the requirement that every *martingale difference sequence* (δ_k) in X, i.e. a (discrete)

[3] RMF is shorthand of *R*ademacher *m*aximal *f*unction.

stochastic process with $\mathbb{E}(\delta_{k+1}|\delta_1,\dots,\delta_k)=0$,[4] satisfies

$$\mathbb{E}\Big\|\sum_k \varepsilon_k\delta_k\Big\|^p \lesssim \mathbb{E}\Big\|\sum_k \delta_k\Big\|^p \tag{UMD$_p$}$$

for any choice of (nonrandom) signs $\varepsilon_k\in\{1,-1\}$ and any $p\in(1,\infty)$. This is connected to previous considerations as

$$\delta_k(\omega)=\sum_{\substack{I\subset[0,1)\\|I|=2^{-k+1}}}\langle f,h_I\rangle h_I(\omega),\quad \omega\in[0,1),$$

is readily seen to define a martingale difference sequence in X given any $f\in L^p(0,1;X)$. Randomizing the signs ε_k allows one to deduce estimates like (2) from (UMD$_p$).

The Rademacher maximal operator M_R can be studied using *martingales*, i.e. (discrete) stochastic processes (ξ_k) for which $\mathbb{E}(\xi_{k+1}|\xi_1,\dots,\xi_k)=\xi_k$. Indeed,

$$\xi_k(\omega)=\sum_{\substack{I\subset[0,1)\\|I|=2^{-k}}}\langle f\rangle_I 1_I(\omega),\quad \omega\in[0,1),$$

defines a martingale in X for any given $f\in L^p(0,1;X)$. The L^p-norm of the maximal function will then take the form

$$\|M_Rf\|^p_{L^p(0,1)}=\mathbb{E}\mathcal{R}(\xi_k:k\geq 0)^p.$$

Note also that $\mathbb{E}\|\xi_k\|^p\leq\mathbb{E}\|\xi_{k+1}\|^p\leq\|f\|^p_{L^p(0,1;X)}$ for each k. The question of boundedness of M_R from $L^p(X)$ to L^p for some $p\in(1,\infty)$ now asks whether

$$\mathbb{E}\mathcal{R}(\xi_k:k)^p\lesssim\max_k\mathbb{E}\|\xi_k\|^p \tag{RMF$_p$}$$

holds for all martingales (ξ_k). The precise class of martingales under consideration is not relevant for this discussion and one can safely assume all martingales to be finite and simple, for instance. Assuming that (RMF$_p$) holds for some $p\in(1,\infty)$ one can derive the *weak type inequality*

$$\mathbb{P}\Big(\mathcal{R}(\xi_k:k)>\lambda\Big)\lesssim\frac{1}{\lambda}\max_k\mathbb{E}\|\xi_k\| \tag{w-RMF}$$

for all martingales (ξ_k) and all $\lambda>0$.

In [A] the relation between (RMF$_p$) and (w-RMF) is studied. To see that the validity of (RMF$_p$) does not depend on p, it is shown that (w-RMF) is also sufficient for (RMF$_p$). The argument in [A] follows a similar one for UMD by D. L. Burkholder (see [4, Section 1]) and proceeds via a distributional inequality

$$\mathbb{P}\Big(\mathcal{R}(\xi_k:k)>3\lambda,\quad\max_k\|\xi_k\|\leq\gamma\lambda\Big)\leq\alpha(\gamma)\mathbb{P}\Big(\mathcal{R}(\xi_k:k)>\lambda\Big),$$

where $\alpha(\gamma)\to 0$ as $\gamma\searrow 0$. From this we arrive at

[4] $\mathbb{E}(\xi|\eta_1,\dots,\eta_k)$ denotes the conditional expectation of a random variable ξ with respect to the σ-algebra generated by the random variables $\eta_1,\dots,\eta_k$.

Theorem. *The following are equivalent for a Banach space X:*

1. *for all $p \in (1, \infty)$, (RMF_p) holds for all martingales (ξ_k) in X,*
2. *for some $p \in (1, \infty)$, (RMF_p) holds for all martingales (ξ_k) in X,*
3. *(w-RMF) holds for all martingales (ξ_k) in X and all $\lambda > 0$.*

The p-independence of the RMF-property was proven in the dyadic case already in [27, Proposition 7.1] by an interpolation argument.

Concave functions

Concave functions provide another way to study both UMD and RMF properties. The former was considered by Burkholder in [4] and [5], where the UMD-property of a Banach space X was characterized by the existence of a suitable biconcave function $u : X \times X \to \mathbb{R}$ and by a related notion of *ζ-convexity*. The method was tailored to provide sharp constants and as such allows one to determine the unconditional constant for the Haar basis on $L^p(0, 1)$ with $1 < p < \infty$. See also Burkholder [6, 7].

In [A] these techniques are applied to the case of RMF. Namely, for a fixed $p \in (1, \infty)$, the validity of

$$\mathbb{E}\mathcal{R}(\xi_k : k)^p \leq C\mathbb{E}\|\xi_\infty\|^p$$

for (finite) martingales (ξ_k), whose final member we denote by ξ_∞, is equivalent to

$$\mathbb{E}f(\{\xi_k\}, \xi_\infty) \leq 0,$$

where the function $f(S, x) = \mathcal{R}(S)^p - C\|x\|^p$ is defined on finite subsets S of X and vectors $x \in X$. Here also the constant C remains fixed. The RMF-property of X is then characterized by the existence of a majorant u of f:

Theorem. *The following are equivalent for a Banach space X:*

1. *$\mathbb{E}f(\{\xi_k\}, \xi_\infty) \leq 0$ holds for all martingales (ξ_k) in X,*
2. *there exists a real-valued function u such that*
   - *$u(S, x) \geq f(S, x)$,*
   - *$u(\emptyset, x) \leq 0$,*
   - *$u(S \cup \{x\}, x) = u(S, x)$,*
   - *$u(S, \cdot)$ is concave,*

   for all finite subsets S of X and all vectors $x \in X$.

The concave function argument has the technical advantage that the transition from certain dyadic martingales to more general martingales can be handled by the elementary fact that locally bounded midpoint concave functions are actually concave.

Similar methods have recently been used by F. Nazarov, S. Treil and A. Volberg under the name of *Bellman functions* to prove results, old and new: The dyadic version of Carleson's embedding theorem and the dyadic maximal function are the first two introductory examples in [38], the two-weight problem for Haar multipliers is considered in [39] and the dyadic shifts of [29] are studied in [46].

Relation with Banach space geometry

Both UMD- and RMF-properties have implications for the geometry of the underlying Banach space X. A Banach space is said to have *type* $p \in [1, 2]$ if

$$\Big(\mathbb{E}\Big\|\sum_k \varepsilon_k x_k\Big\|^2\Big)^{1/2} \lesssim \Big(\sum_k \|x_k\|^p\Big)^{1/p}$$

for any choice of vectors $x_k \in X$. The larger the p, the stronger the requirement for type p. It has been shown by B. Maurey and G. Pisier (see [34] or Albiac and Kalton [1, Chapter 11]) that a Banach space X has no greater type than the trivial type $p = 1$ if and only if ℓ^1 is *finitely representable* in X, which in turn is a requirement for uniform containment of finite dimensional subspaces of ℓ^1 in X. Once it has been shown that ℓ^1 has neither UMD nor RMF it is not difficult to see that both these properties imply that the underlying Banach space has type greater than 1. This observation motivates the more general framework for RMF in which the Banach space X is assumed to lie inside a space $\mathcal{L}(E, F)$ of operators so that a different, more intrinsic notion of R-boundedness is available. Indeed, for infinite dimensional Banach spaces E and F the space $\mathcal{L}(E, F)$ has only trivial type (see Diestel, Jarchow and Tonge [17, Proposition 19.17] for a proof of an even stronger result that the subspace of compact operators has only infinite *cotype*) and could not have RMF in the original framework.

Every Banach space X with type 2 has RMF, since in this case R-boundedness coincides with uniform boundedness and the standard dyadic maximal operator is always bounded from $L^p(\mathbb{R}; X)$ to $L^p(\mathbb{R})$ whenever $1 < p \le \infty$. The RMF-property of X is also inherited to $L^p(\mathbb{R}; X)$ for all $p \in (1, \infty)$ (see [A, Proposition 4.3]). For more examples of RMF-spaces, see [27, Section 7] and [A] for the more general framework.

An example of a non-reflexive Banach space with type 2 by R. C. James in [31] shows that RMF does not imply UMD.[5] The converse is not known:

Problem. *Does UMD imply RMF?*

Going back to the previous discussion about the vector-valued dyadic paraproduct we see that an affirmative answer to the problem above would remove the need for an additional assumption on the RMF-property in our argument. Also the solution of Kato's square root problem in $L^p(\mathbb{R}^n; X)$ still relies on both UMD- and RMF-properties of the Banach space X (see Hytönen, McIntosh and Portal [27]).

Carleson's embedding theorem and discrete tent spaces

Carleson's inequality originates from the work of L. Carleson on analytic functions (see [9, Theorem 1] and [8, Theorem 2]). It has found its way to the real-variable theory in Fefferman and Stein [19, Theorem 2] with a formulation similar to the one we present in the next section. The inequality (or embedding) discussed in this section is a modification of the earlier dyadic version to a more general discrete setting. After gathering the results from article [B] we present a toy model of tent spaces which are the topic of the next section.

[5] It is known that all UMD spaces are reflexive.

Carleson's embedding theorem and article [B]

The vector-valued Carleson's embedding theorem [27, Theorem 8.2] was the original reason for defining the new maximal operator. In [B] this embedding is formulated in a more general setting and related to Banach space geometry through the concept of type.

Suppose that $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space equipped with a *filtration* $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ of σ-finite sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_k \subset \mathcal{F}_{k+1}$. We denote by E_k the conditional expectation operator with respect to $\mathcal{F}_k$. The Rademacher maximal function of an $f : \Omega \to X$ is defined in this context by

$$M_R f(x) = \mathcal{R}(E_k f(x) : k \in \mathbb{Z}), \quad x \in \Omega.$$

The prime example of such a setting is of course the dyadic filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$, where $\mathcal{F}_k$ is the σ-algebra generated by the collection $\mathcal{D}_k$ of dyadic cubes $2^{-k}([0,1)^n + m)$, $m \in \mathbb{Z}^n$. In this case the conditional expectations are given by

$$E_k f(x) = \sum_{Q\in\mathcal{D}_k} \langle f\rangle_Q 1_Q(x), \quad x \in \mathbb{R}^n.$$

Let $1 < p, q < \infty$ and consider, for a family $\theta = (\theta_k)_{k\in\mathbb{Z}}$ of real-valued functions on Ω and an $f \in L^p(\Omega; X)$, the inequality

$$\Big(\int_\Omega \mathbb{E}\Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k E_k f(x)\theta_k(x)\Big\|^p \,\mathrm{d}\mu(x)\Big)^{1/p} \lesssim \|\theta\|_{\mathrm{Car}^q} \|f\|_{L^p(X)}, \tag{$\mathrm{CAR}_{q,p}$}$$

where

$$\|\theta\|_{\mathrm{Car}^q} = \sup_{m\in\mathbb{Z}} \sup_{A\in\mathcal{F}_m} \Big(\frac{1}{\mu(A)} \int_A \Big(\sum_{k\geq m} |\theta_k(x)|^2\Big)^{q/2} \mathrm{d}\mu(x)\Big)^{1/q}.$$

Observe that $\|\theta\|_{\mathrm{Car}^p} \leq \|\theta\|_{\mathrm{Car}^q}$ whenever $p \leq q$, so that the collections of θ for which these quantities are finite satisfy $\mathrm{Car}^q(\Omega) \subset \mathrm{Car}^p(\Omega)$ for $p \leq q$. Note also, that $(\mathrm{CAR}_{q,p})$ cannot hold for all $f \in L^p(\Omega; X)$ unless $\theta \in \mathrm{Car}^p(\Omega)$, as can be seen by choosing $f = 1_A \otimes \xi$ with $A \in \mathcal{F}_m$ and $\|\xi\| = 1$.

The main result of [B] is the following:

Theorem. *Let $1 < p < \infty$ and suppose that X is a Banach space. The inequality $(\mathrm{CAR}_{q,p})$ holds for all $f \in L^p(\Omega; X)$ and $\theta \in \mathrm{Car}^q(\Omega)$*

- *with $q > p$ if and only if X has RMF,*
- *with $q = p$ if and only if X has RMF and type p.*

As no Banach space can have type greater than 2, the validity of $(\mathrm{CAR}_{q,p})$ for equal indices is restricted even in the scalar case to $q = p \leq 2$. The article [B] is written in the setting of operator-valued f and vector-valued θ_k.

In this introduction, Carleson's embedding theorem has so far been motivated by its application to the classical paraproduct operator, where L^p-estimates are obtained directly without interpolation or extrapolation. It is worth noting that this, however, is not the only reason to study these inequalities. Indeed, the dyadic version of the vector-valued Carleson's embedding theorem (and hence RMF) played an important rôle in the analysis of the principal part of the operator sequence appearing in the "quadratic $T1$ theorem" [27, Theorem 6.1], which in turn was a step towards the solution of the Kato's square root problem in $L^p(\mathbb{R}^n; X)$. The RMF assumption was also present in an earlier version of T. Hytönen's paper concerning the vector-valued Tb theorem in the non-homogeneous setting (see [24, Theorem 3.5]). To the best of my knowledge, RMF is still present in an ongoing work regarding a local version of the Tb theorem.

Discrete tent spaces

The setting of a σ-finite measure space with a filtration allows us to define discrete versions of tent spaces. This discussion aims to give an idea of how the UMD- and RMF-properties appear in standard operations on these spaces.

For $1 \le p < \infty$, the space $dT^p_\infty(X)$ consists of functions $F : \Omega \times \mathbb{Z} \to X$ such that $F(\cdot, k)$ is $\mathcal{F}_k$-measurable and

$$\|F\|_{dT^p_\infty(X)} = \Big(\int_\Omega \mathcal{R}(F(x,k) : k \in \mathbb{Z})^p \,\mathrm{d}\mu(x) \Big)^{1/p} < \infty.$$

For RMF-spaces X it follows that every function f in $L^p(\Omega; X)$ with $1 < p < \infty$ lifts to $dT^p_\infty(X)$ by the formula

$$F(x,k) = E_k f(x), \quad (x,k) \in \Omega \times \mathbb{Z}.$$

For $p = 1$ we see that $H^1(\Omega; X)$-functions (when defined using atoms or maximal functions as in the Euclidean case, see Garsia [22]) extend to $dT^1_\infty(X)$-functions.

Moreover, a function $F : \Omega \times \mathbb{Z} \to X$ belongs to $dT^p(X)$ with $1 \le p < \infty$ if $F(\cdot, k)$ is $\mathcal{F}_k$-measurable and

$$\|F\|_{dT^p(X)} = \Big(\int_\Omega \mathbb{E} \Big\| \sum_{k \in \mathbb{Z}} \varepsilon_k F(x,k) \Big\|^p \,\mathrm{d}\mu(x) \Big)^{1/p} < \infty.$$

If X is UMD, then every f in $L^p(\Omega; X)$ with $1 < p < \infty$ lifts to $dT^p(X)$ by the formula

$$F(x,k) = E_k f(x) - E_{k-1} f(x), \quad (x,k) \in \Omega \times \mathbb{Z}.$$

As before, for $p = 1$ we see that $H^1(\Omega; X)$-functions extend to $dT^1(X)$-functions.

Let us denote by $\mathrm{Rad}(X)$ the Banach space of sequences $(\xi_k)_{k \in \mathbb{Z}}$ in X for which the series $\sum_{k \in \mathbb{Z}} \varepsilon_k \xi_k$ converges almost surely so that the norm

$$\|(\xi_k)_{k \in \mathbb{Z}}\|_{\mathrm{Rad}(X)} = \Big(\mathbb{E} \Big\| \sum_{k \in \mathbb{Z}} \varepsilon_k \xi_k \Big\|^2 \Big)^{1/2}$$

is finite. For more details on these *almost unconditionally summable sequences*, see [17, Chapter 12]. For $1 < p < \infty$, the space $dT^p(X)$ can be studied as a complemented subspace of $L^p(\Omega; \mathrm{Rad}(X))$. Indeed, the vector-valued *Stein's inequality*[6]

[6]The scalar case can be found in Stein [45, Theorem 8]; the vector-valued version is stated without proof already in Bourgain [3, Lemma 8].

(see Clément et al. [12, Proposition 3.8] for a proof) states that any increasing sequence of conditional expectation operators is R-bounded on $L^p(\Omega; X)$ when X is UMD so that for all $F = (F_k)_{k\in\mathbb{Z}} \in L^p(\Omega; \mathrm{Rad}(X))$ we have

$$\int_\Omega \mathbb{E}\Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k E_k F_k(x)\Big\|^p \,\mathrm{d}\mu(x) \lesssim \int_\Omega \mathbb{E}\Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k F_k(x)\Big\|^p \,\mathrm{d}\mu(x).$$

From this we infer that $NF(x,k) = E_k F_k(x)$ defines a bounded projection N on $L^p(\Omega; \mathrm{Rad}(X))$ whose range is $dT^p(X)$.

Carleson's embedding theorem also generalizes to

$$\Big(\int_\Omega \mathbb{E}\Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k F(x,k)\theta(x,k)\Big\|^p \,\mathrm{d}\mu(x)\Big)^{1/p} \lesssim \|\theta\|_{\mathrm{Car}^q} \|F\|_{dT^p_\infty(X)},$$

which says that the function $(x,k) \mapsto F(x,k)\theta(x,k)$ is in $dT^p(X)$ whenever F is in $dT^p_\infty(X)$ and θ is in $\mathrm{Car}^q(\Omega)$ with $q > p$.

An *atomic decomposition* for a dyadic version of scalar dT^1 can be found in Meyer [37, Chapter 5, Section 3].

Tent spaces and article [C]

Tent spaces were introduced by R. R. Coifman, Y. Meyer and E. M. Stein in [15] for the purpose of serving as a unified framework for non-tangential maximal functions and conical square functions arising in Harmonic Analysis.

Paraproduct operator

We will find tent spaces useful when studying a *continuous time paraproduct operator* Π_b, which as its dyadic analogue is associated to a function $b \in \mathrm{BMO}(\mathbb{R}^n)$ and defined (formally) by

$$\Pi_b f = \int_0^\infty \Psi_t * ((\Phi_t * f)(\Psi_t * b)) \,\frac{\mathrm{d}t}{t}, \quad f \in L^2(\mathbb{R}^n),$$

where Ψ and Φ are, say, smooth radial real-valued functions supported in the unit ball with $\int \Psi = 0$ and $\int \Phi = 1$. Here, as is usual, $\Phi_t(x) = t^{-n}\Phi(x/t)$ and likewise for Ψ. The above expression is handled by pairing $\Pi_b f$ with a $g \in L^2(\mathbb{R}^n)$ in which case

$$\langle \Pi_b f, g\rangle = \int_{\mathbb{R}^n} \int_0^\infty \Phi_t * f(x) \Psi_t * b(x) \Psi_t * g(x) \,\frac{\mathrm{d}t}{t} \,\mathrm{d}x. \tag{4}$$

The L^2-boundedness of Π_b follows then from a Carleson's embedding theorem, which in this setting states, for suitable Φ, that a measure ν on the upper half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ satisfies

$$\int_{\mathbb{R}^{n+1}_+} |\Phi_t * f(x)|^2 \,\mathrm{d}\nu(x,t) \lesssim \int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x$$

for all $f \in L^2(\mathbb{R}^n)$ if and only if it satisfies for every ball $B \subset \mathbb{R}^n$ the *Carleson condition*

$$\nu(\widehat{B}) \lesssim |B|,$$

where $\widehat{B} = \{(y,t) \in \mathbb{R}^{n+1}_+ : B(y,t) \subset B\}$ is the *tent* over B. One can show that the measure

$$\mathrm{d}\nu = |b * \Psi_t(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$$

satisfies the Carleson condition when b is in $\mathrm{BMO}(\mathbb{R}^n)$ (see Duoandikoetxea [18, Theorem 9.6]).

This version of the paraproduct operator was introduced by Coifman and Meyer in the book [14] and was later used in the original proof of the $T1$ theorem by David and Journé in [16].

Scalar-valued tent spaces

To obtain L^p-boundedness for Π_b we appeal to the tent space formalism. For $1 \le p < \infty$, the space T^p_∞ consists of functions F on $\mathbb{R}^{n+1}_+$ for which the *non-tangential maximal function* is in L^p meaning that

$$\|F\|_{T^p_\infty} = \Big(\int_{\mathbb{R}^n} \sup_{(y,t)\in\Gamma(x)} |F(y,t)|^p \,\mathrm{d}x \Big)^{1/p} < \infty,$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ denotes the *cone* at $x \in \mathbb{R}^n$. The space T^p, on the other hand, is defined by requiring that the *conical square function* is L^p-integrable, that is,

$$\|F\|_{T^p} = \Big(\int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |F(y,t)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \Big)^{p/2} \mathrm{d}x \Big)^{1/p} < \infty.$$

Finally, a function H on $\mathbb{R}^{n+1}_+$ belongs to T^∞ if

$$\|H\|_{T^\infty} = \sup_B \Big(\frac{1}{|B|} \int_{\widehat{B}} |H(y,t)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t} \Big)^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

The idea is that for f in $L^p(\mathbb{R}^n)$ with $1 < p < \infty$, the extensions $F(y,t) = \Phi_t * f(y)$ and $F(y,t) = \Psi_t * f(y)$ reside in T^p_∞ and T^p, respectively. This holds also in the case $p = 1$ if f is in $H^1(\mathbb{R}^n)$. Moreover, functions b in $\mathrm{BMO}(\mathbb{R}^n)$ extend to T^∞ via $H(y,t) = \Psi_t * b(y)$. When $1 < p < \infty$ and $1/p + 1/p' = 1$, we can use (4) to write

$$\begin{aligned}
|\langle \Pi_b f, g\rangle| &\le \Big(\int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |\Phi_t * f(y) \Psi_t * b(y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \Big)^{p/2} \mathrm{d}x \Big)^{1/p} \\
&\quad \times \Big(\int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |\Psi_t * g(y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \Big)^{p'/2} \mathrm{d}x \Big)^{1/p'} \\
&\le \|f\|_{L^p} \|b\|_{\mathrm{BMO}} \|g\|_{L^{p'}},
\end{aligned}$$

where the first inequality can be interpreted as a consequence of the *tent space duality* $(T^p)^* \simeq T^{p'}$ given by the pairing

$$\langle F, G\rangle = c \int_{\mathbb{R}^{n+1}_+} F(y,t) G(y,t) \frac{\mathrm{d}y\,\mathrm{d}t}{t},^{7}$$

[7] Here c denotes a dimensional constant.

and the second inequality rests on the estimate $\|FH\|_{T^p} \lesssim \|F\|_{T^p_\infty}\|H\|_{T^\infty}$ (see Cohn and Verbitsky [13, Lemma 2.1]) arising from Carleson's embedding theorem. Both the tent space duality and the previous estimate remain true for $p = 1$, which allows us to use the strategy above in order to prove the boundedness of Π_b on $H^1(\mathbb{R}^n)$.

It should be noted here that the paraproduct operator can also be studied as a Calderón–Zygmund operator, providing another way to deduce for instance its boundedness on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ (see Christ [10, Chapter III, Section 3]).

Vector-valued tent spaces and article [C]

The article [C] studies tent spaces of functions taking values in a (real) Banach space X. The main adjustment to the scalar-valued case is the replacement of the square integrals

$$\int_{\Gamma(x)} |F(y,t)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}$$

by stochastic integrals much in analogue with the discrete case where square sums are replaced by randomized sums. This is done by associating a *Gaussian random measure* W to the measure $\mathrm{d}y\,\mathrm{d}t/t^{n+1}$ on the upper half-space and extending the *stochastic integral* (arising from this random measure) to the algebraic tensor product $L^2(\mathbb{R}^{n+1}_+) \otimes X$. The completion of this space with respect to the norm

$$\|F\|_{\gamma(X)} = \Big(\mathbb{E}\Big\| \int_{\mathbb{R}^{n+1}_+} F(y,t)\,\mathrm{d}W(y,t)\Big\|^2\Big)^{1/2}$$

is denoted by $\gamma(X)$. One of the technical problems with stochastic integrals is that, unlike $\mathrm{Rad}(X)$, the natural class of stochastically integrable functions is not generally complete in the norm above, except in the case of X being isomorphic to a Hilbert space. The vector-valued case of square functions have been studied for instance in Kalton and Weis [32] and Hytönen [25]. A detailed account of the theory of stochastic integration can be found in van Neerven and Weis [41]. Closely related to this is the theory of *γ-radonifying operators*, which was surveyed by J. M. A. M. van Neerven in [40].

For $1 \leq p < \infty$, the tent space $T^p(X)$ of functions $F : \mathbb{R}^{n+1}_+ \to X$ is now equipped with the norm

$$\|F\|_{T^p(X)} = \Big(\int_{\mathbb{R}^n} \mathbb{E}\Big\| \int_{\Gamma(x)} F(y,t)\,\mathrm{d}W(y,t)\Big\|^p \mathrm{d}x\Big)^{1/p}.$$

The space $T^\infty(X)$, on the other hand, is taken to consist of functions $H : \mathbb{R}^{n+1}_+ \to X$ for which

$$\|H\|_{T^\infty(X)} = \sup_B \Big(\frac{1}{|B|}\int_B \mathbb{E}\Big\| \int_{\Gamma(x;r_B)} H(y,t)\,\mathrm{d}W(y,t)\Big\|^2 \mathrm{d}x\Big)^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ each of whose radius r_B defines the *truncated cone* $\Gamma(x; r_B) = \{(y,t) \in \Gamma(x) : t < r_B\}$ at $x \in \mathbb{R}^n$. This quantity was shown by T. Hytönen and L. Weis (see [30]) to be comparable with scalar version of T^∞ norm for $X = \mathbb{R}$. We have glossed over the technical difficulties of stochastic integrability and completeness in the definitions above.

E. Harboure, J. L. Torrea and B. E. Viviani studied in [23] the scalar-valued tent spaces T^p by embedding them into $L^p(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))$ for $1 < p < \infty$ and extended this to the endpoint cases $p = 1$ and $p = \infty$ by embedding T^1 and T^∞, respectively, into $H^1(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))$ and $\mathrm{BMO}(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))$. This approach was carried out in the vector-valued case for $1 < p < \infty$ by T. Hytönen, J. M. A. M. van Neerven and P. Portal (see [28, Section 4]) who embedded $T^p(X)$ into $L^p(\mathbb{R}^n; \gamma(X))$ assuming that X is UMD. The endpoint cases $T^1(X)$ and $T^\infty(X)$ are considered in [C].

The main result of [C] decomposes a $T^1(X)$ function into *atoms* $A : \mathbb{R}^{n+1}_+ \to X$ each of which possesses a ball $B \subset \mathbb{R}^n$ so that $\operatorname{supp} A \subset \widehat{B}$ and

$$\int_B \mathbb{E}\Big\| \int_{\Gamma(x)} A(y,t)\,\mathrm{d}W(y,t)\Big\|^2 \,\mathrm{d}x \le \frac{1}{|B|}.$$

Theorem. *For every function F in $T^1(X)$ there exist countably many atoms A_k and real numbers λ_k such that*

$$F = \sum_k \lambda_k A_k \quad \textit{and} \quad \sum_k |\lambda_k| \lesssim \|F\|_{T^1(X)}.$$

Such a decomposition was provided in the scalar case already in [15], but with a proof that does not seem to be applicable in the case of X-valued functions. The atomic decomposition is a crucial tool when embedding $T^1(X)$ into $H^1(\mathbb{R}^n; \gamma(X))$. In analogue to [23], $T^\infty(X)$ is embedded in $\mathrm{BMO}(\mathbb{R}^n; \gamma(X))$. As for the duality results in the vector-valued case, it was shown in [28] that $T^p(X)^* \simeq T^{p'}(X^*)$ when $1 < p < \infty$ and $1/p + 1/p' = 1$, whereas in [C] the author has settled for a partial duality result stating that $T^\infty(X^*)$ is isomorphic to a norming subspace of $T^1(X)^*$. Both in the embeddings and in the duality results, it has been assumed that X is UMD.

Vector-valued tent spaces were called upon in [28] to provide a framework for Hardy spaces associated with *bisectorial operators* and to examine their H^∞*-functional calculus* – a technique introduced in McIntosh [36]. This was studied mostly in the case $1 < p < \infty$ and it is expected that the results for $T^1(X)$ and $T^\infty(X)$ in [C] find applications in these topics.

Vector-valued paraproduct

In order to address the boundedness of the paraproduct operator for vector-valued functions, we introduce one more tent space, namely $T^p_\infty(X)$ consisting of functions $F : \mathbb{R}^{n+1}_+ \to X$ for which

$$\|F\|_{T^p_\infty(X)} = \Big(\int_{\mathbb{R}^n} \mathcal{R}(F(y,t) : (y,t) \in \Gamma(x))^p \,\mathrm{d}x \Big)^{1/p} < \infty.$$

Now, for $f \in L^p(\mathbb{R}^n; X)$ and $g \in L^{p'}(\mathbb{R}^n; X^*)$ with $1 < p < \infty$ we obtain

$$
\begin{aligned}
|\langle \Pi_b f, g\rangle| &\le \Big(\int_{\mathbb{R}^n} \mathbb{E} \Big\| \int_{\Gamma(x)} \Phi_t * f(y) \Psi_t * b(y) \, \mathrm{d}W(y,t) \Big\|^p \, \mathrm{d}x \Big)^{1/p} \\
&\quad \times \Big(\int_{\mathbb{R}^n} \mathbb{E} \Big\| \int_{\Gamma(x)} \Psi_t * g(y) \, \mathrm{d}W(y,t) \Big\|^{p'} \, \mathrm{d}x \Big)^{1/p'} \\
&\le \|N_R f\|_{L^p} \|b\|_{\mathrm{BMO}} \|g\|_{L^{p'}(X^*)},
\end{aligned}
$$

where the first inequality comes from the tent space duality, the second rests on an estimate by T. Hytönen and L. Weis (see [30, Corollary 6.3]) stating that $\|FH\|_{T^p(X)} \lesssim \|F\|_{T^p_\infty(X)} \|H\|_{T^\infty}$ and

$$
N_R f(x) = \mathcal{R}(\Phi_t * f(y) : (y,t) \in \Gamma(x))
$$

is the *non-tangential Rademacher maximal function* of f. The question arises whether $N_R f$ is controlled by $M_R f$ so that X being RMF would guarantee that $L^p(\mathbb{R}^n; X)$ functions extend to $T^p_\infty(X)$ as in the scalar-valued case. Choosing $\Phi = P$, the *Poisson kernel*, a representation theorem of G.-C. Rota (see [44] or Stein [45, Chapter IV, Section 4]) allows one to express the Poisson semigroup $(P_t * \cdot)_{t>0}$ in terms of conditional expectations which in turn implies that for every $t > 0$,

$$
\mathcal{R}(P_{2kt} * f(x) : k \in \mathbb{Z}_+) \lesssim M_R f(x).^8
$$

It remains unknown if in this vector-valued case the vertical maximal operator controls the non-tangential:

Problem. *Suppose that a Banach space X has RMF and let $1 < p < \infty$. Does $F(y,t) = \Phi_t * f(y)$ define a function in $T^p_\infty(X)$ when $f \in L^p(\mathbb{R}^n; X)$ and $\int \Phi = 1$? In particular, does*

$$
\int_{\mathbb{R}^n} \mathcal{R}(P_t * f(y) : (y,t) \in \Gamma(x))^p \, \mathrm{d}x \lesssim \int_{\mathbb{R}^n} \mathcal{R}(P_t * f(x) : t > 0)^p \, \mathrm{d}x
$$

hold for all f in $L^p(\mathbb{R}^n; X)$?

Whether the paraproduct operator is bounded on $H^1(\mathbb{R}^n; X)$ is also an interesting question. In this case it is not known if $\|FH\|_{T^1(X)} \lesssim \|F\|_{T^1_\infty(X)} \|H\|_{T^\infty}$ when X is UMD nor if $\|N_R f\|_{L^1} \lesssim \|f\|_{H^1(X)}$ when X is RMF.

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[8] Here M_R is not the dyadic Rademacher maximal operator, but a modified one.

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