Playing Games on Sets and Models

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Prologue: 0.999… = 1?

6.21 Der Satz der Mathematik drückt keinen Gedanken aus.

Ludwig Wittgenstein

Infinitesimals and Other Complex Issues

Consider the real number line. As we know, it contains a lot of numbers which lie in a certain order: of two numbers, one is always smaller than the other. It is therefore clear what we mean by saying that a number lies *between* the other two. Now, let us look at the collection of all those numbers that lie between 0 and 1, excluding 0 and 1 themselves. This collection, or *set*, is denoted by $(0, 1)$.

Which numbers do we have in this set? Is number 3 there? No, because it is not between zero and one. What about 1? No, because it was explicitly excluded. One half? Yes, it is certainly between 0 and 1. What about the number 0.954321? Yes again. But suppose I gave you the number $0.999\ldots$ in which all the infinitely many digits after the decimal point are equal to 9. Does that number $0.999\ldots$ belong to $(0, 1)$? Clearly, if $0.999\ldots$ happens to be equal to 1, then as noted above, it does not belong. On the other hand, if it is less than 1, then it does belong, because it is also a positive number and so between 0 and 1.

Here is a conversation which I made up:

Teacher: Is $0.999\ldots$ equal to 1?

Student: I am not sure, but I think *no*.

Teacher: What you say is very interesting. Why do you think so?

Student: I can imagine that one minus *an infinitely small number* is less than one but greater than $0.999\ldots$. Thus there is a number in between and so the two cannot be equal.

Teacher: There are several problems with that. You can only subtract a real number from a real number. Do you think there are *infinitely small real numbers*?

Student: I've heard about a way to define such numbers. In that theory one can define *infinitesimals* and use them for defining limits for example.

Teacher: Can you prove the existence of such numbers?

Vadim joins the conversation.

Vadim: Can you prove the existence of *any* numbers?

Teacher: Vadim, don't mix things up. I am asking whether the existence can be proved from the axioms of the real numbers. Anyway, you might want to figure out what $0.999\ldots$ *is*, if it is not 1.

Student: Umm...

Vadim: We should use the definition of $0.999\ldots$, I suppose.

Hard thinking.

Student: Aha! $0.999\ldots$ equals to the limit of the sequence $0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots$.

Teacher: And you remember the definition of a limit...

Student: Yes... the limit seems to be 1. I was wrong, wasn't I?

Vadim: You are hurrying too much. Remember that it is Teacher who taught you the definition of a limit. She might be tricking you!

Student: Are you (*looks at Vadim*) saying that you (*looks at Teacher*) made up the definition of a limit just in order to *make* $0.999\ldots = 1$?

Everyone's puzzled for $1.999\ldots$ *seconds.*

Teacher: Certainly *I* didn't make it up. Can *you* think of other definitions of a limit?

Student: Well... it is the same as the supremum of the set $\{0.9, 0.99, 0.999, 0.9999, \ldots\}$.

Vadim: Let us denote it by $1 - d$.

Student: Is d now an infinitely small number?

Vadim: Yes, for example $d = 0.000\ldots 1$. (Or it could be zero, if $0.999\ldots = 1$.)

Student: Like what? Infinitely many zeroes and *then* one?

Vadim: Yeah!

Student: Hold on. What about the number $1 - d - d$?

Vadim: You mean $1 - 0.000\ldots 2$?

Teacher: Good question, Student. It should clearly be less than $1 - d$ and hence not a supremum. Therefore there is an n such...

Student: ...that the number $0.\overbrace{99\ldots9}^{n \text{ times}}$ is greater than $1 - d - d$, but less than $1 - d$. Same holds for larger n:s, for instance if $x = 0.\underbrace{99\ldots9}_{n+1}$, then $1 - d - d < x < 1 - d$.

Teacher: Multiplying by 2 we get $2x < 2 - 2d = 2 - d - d$.

Vadim: Oops.

Student: And then subtract one! And we get $2x - 1 < 1 - d - d$. Substituting the value of x we have $1 - d - d > 2 \cdot 0.\underbrace{99\ldots9}_{n+1} - 1 = 0.\underbrace{99\ldots9}_{n}8 > 0.\underbrace{99\ldots9}_{n}$.

Teacher: That's a contradiction! Therefore $0.999\ldots = 1$.

Student: Vadim, is everything alright?

Vadim: Poor number 0.000...1. It cannot exist...

Teacher: Don't worry. It can exist, if you only discard some of the axioms of the real numbers!

As the discussion shows, it is not completely trivial to decide whether 0.999... belongs to $(0,1)$ or not, and in order to solve that, or even to define this peculiar number uniquely, one needs to use the axiom of completeness, i.e. the full machinery of the reals!

Is it possible to program a computer to decide whether or not a given number belongs to $(0,1)$ by only looking at the decimal digits of that number? Note that only a finite amount of information can be fed into a computer at a time. Suppose I have a computer and its name is Digitron. I start inputting my real number to Digitron one digit at a time: first five digits are 0, ., 9, 9 and 9. At this point Digitron cannot yet decide whether the incoming number is in $(0,1)$, because if I continue giving only nines, the number will be 1 and so Digitron should output "no", whereas if some digit in the future will be less than 9, then the output should be "yes". And so Digitron asks for more input. And I give him 9, 9, 9, 9, 9 and 9. The situation is still unchanged. Digitron cannot know. And in fact, if I continue inputting only nines, Digitron will *never* know and the program will not halt.

By the way, the same applies to the number 0.000.... Not as simple is this set $(0,1)$ as it seems to be.

However, $(0,1)$ is of the simplest kind of sets that mathematicians encounter. A bit trickier is for example the set

$$S_1 = \bigcup_{k=1}^{\infty}(2k, 2k+1).$$

It is the union of intervals from the even number $2k$ to the odd number $2k+1$ and it goes through *all* the positive even numbers! In order to know whether a given number $a_0.a_1a_2a_3\ldots$ belongs to S_1, one first needs to check whether a_0 is even and *then*, if it is even, to check whether or not $0.a_1a_2a_3\ldots$ belongs to $(0,1)$. Or consider the set

$$S_2 = \bigcup_{p \text{ is a prime}} (p, p+1)$$

which consists only of those intervals $(p, p+1)$ in which p is a prime number. Now one has to check the primeness of a number which is known to be a time consuming process.

The next paragraph is dedicated to building a complicated set T and can be omitted.

For each positive natural number $n = 1, 2, \ldots$ let P_n be the set of all those natural numbers that are *not* divisible by n. Thus for example $P_3 = \{1, 2, 4, 5, 7, 8, \ldots\}$. Above we denoted by (a, b) the set of all numbers between a and b excluding a and b. Let us now denote by $[a, b]$ the same set but *including* both a and b. Let us define

$$S_n = \bigcup_{k \in P_n} [\tfrac{k-1}{2^n}, \tfrac{k}{2^n}].$$

This set is the union of all intervals from $\frac{k}{2^n}$ to $\frac{k+1}{2^n}$ where k ranges over P_n. For example

$$S_3 = [\tfrac{1}{2^3}, \tfrac{2}{2^3}] \cup [\tfrac{2}{2^n}, \tfrac{3}{2^n}] \cup [\tfrac{4}{2^3}, \tfrac{5}{2^3}] \cup \ldots.$$

And then let us take the intersection of all these sets:

$$T = \bigcap_{n=2}^{\infty} S_n = \bigcap_{n=2}^{\infty} \bigcup_{k \in P_n} [\tfrac{k-1}{2^n}, \tfrac{k}{2^n}].$$

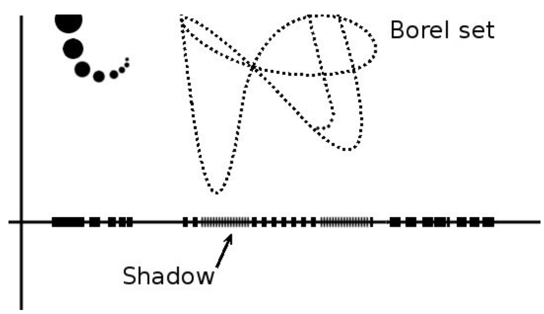
If you do not know what an intersection means, in this case it means the following: a number belongs to T, if it belongs to *every* S_n for $n \geqslant 2$.

Now, it should be fairly clear, that it is quite hard to tell for example whether the number $50224.666\ldots$ belongs to T or not. *Despite the description of T took only few lines of text.*

One set can always be described in different ways and sometimes there might be a surprisingly simple description of a set that has been given a complex description initially. For example the set $B = \bigcap_{i=1}^{\infty}(n, 10n^2)$, the intersection of all intervals from n to $10n^2$ has a simpler description, because it is empty, $B = \varnothing$.

We define *the descriptive complexity of a set* to be the simplest possible description of that set. Of course, this is vague, because we should define what it means to be *the simplest*. But there is a natural definition for that: we just count how many times we had to apply intersections and unions one after another starting from open intervals (a, b). Yes, it is that simple! The description $\bigcap_n \bigcup_m \bigcap_k (a_{nmk}, b_{nmk})$ is more complex than the description $\bigcup_m \bigcap_k (a_{mk}, b_{mk})$. This definition gives rise to the *Borel hierarchy of sets.*

Do all subsets of the reals belong to some level of the Borel hierarchy? Is it possible to express *any* collection of the real numbers by taking repeatedly unions and intersections of already defined sets? Is the descriptive hierarchy of all sets equal to the Borel hierarchy? No. There are sets way more complex than that. The descriptive hierarchy continues from Borel sets to the so called *projective sets*, the simplest of whom are the Σ^1_1-sets: the projections (shadows) of Borel sets in the plane:



Why are we so interested in the careful study of the descriptive hierarchy of sets? There are many reasons of course: the real line is one of the most central objects in whole mathematics. One of the reasons, the *logical* reason, is that many mathematical problems *can be reduced* to the question whether a certain real number belongs to a certain set. For example the question whether there are natural numbers m and n such that $\frac{n^2}{m^2} = 2$ is the same question as whether $\sqrt{2}$ belongs to the set of the rational numbers. We can code various mathematical structures to single real numbers: for instance a knot  is specified by a continuous curve and it is well known that a continuous curve is specified by its restriction to rational numbers, i.e. by a countable set, and nothing is easier than to put that set in the form of a countable binary sequence – a real number. Now the instance of the fundamental question of knot theory, whether the knots  and  are equivalent, reduces to the question, whether the code of  belongs to the set of codes of all the knots that are equivalent to .

A striking fact about the theory of reducing mathematical problems to the problems about subsets of the reals is that the aim is *not* to try to solve these problems.[1] Instead, the aim is to put mathematical problems into a hierarchy, by looking at the complexity of the corresponding sets of real numbers and their descriptive complexity. When successful, we are able to state, that a certain mathematical problem is so complex that *it cannot be solved* by means of non-complex methods. Exactly in the same way as a complex Borel set *cannot* be described by a simpler description.

Let me now return to the number $0.000\dots 1$. To emphasize that there are infinitely many zeroes, let me rewrite it: $0.000{\scriptstyle 00}{\scriptscriptstyle 00\dots}\,1$. As the discussioners above noted, this number cannot belong to the set of real numbers. However, if we drop the completeness axiom, we can add that number to our number line without any contradiction. Because sometimes mathematical objects are not countable (or even essentially countable, as knots are), the theory of descriptions has to be generalized so that dealing with uncountable mathematical structures is possible. This leads to the study of uncountably long binary sequences. These are binary sequences much longer than the ordinary real number's decimal representations and even longer than putting first infinitely many zeroes and then a one: $0.000{\scriptstyle 00}{\scriptscriptstyle 00\dots}\,1$. They look more like this:

$$0.\underbrace{01{\scriptstyle 10}{\scriptscriptstyle 10\dots}}_{\text{infinite sequence}}\,10{\scriptstyle 00}{\scriptscriptstyle 10\dots}\;11{\scriptstyle 01}{\scriptscriptstyle 10\dots 10}{\scriptscriptstyle 10\dots 11}{\scriptscriptstyle 11\dots}\;01{\scriptstyle 10}{\scriptscriptstyle 10\dots}\;10{\scriptstyle 00}{\scriptscriptstyle 10\dots 00}{\scriptscriptstyle 00\dots 00}{\scriptscriptstyle 10\dots\dots},$$

although they are much longer. Now, these sequences have uncountable length of a certain cardinality (which we choose depending on the context). And to these sequences we are able to code uncountable structures (of that fixed cardinality), and thence extend the domain of descriptive set theory. And *this* is precisely what a large part of this thesis (Chapters 4 and 5) is dealing with.

Disclaimer. This prologue does not contain any new results or facts that were not previously known nor anything surprising to scientists in this field. The field of descriptive set theory is around eighty, and its generalizations to uncountable realms around twenty years old. For more on history see section *History* of the next chapter (page 18) and Section 4.1 of Chapter 4 (page 58).

[1] Of course this matter is not that simple and sometimes problems do get solved.

2 Introduction

When we look into the deep eyes of the uncountable structures, we are perhaps starting to see there some compassion for our modest advances, our budding infinite trees, our courageous appeals to stability and our resolve to play the game to the end.

Jouko Väänänen

2.1 Overview

This thesis is about set theory and model theory, and how these two disciplines of mathematical logic are linked together. Mathematical games are used to prove many results and especially they play a role in connecting set theory with model theory. In these games players pick elements from sets or models' domains; the games are played *on* sets and models in the same, although more abstract, way as the game of chess is played *on* the chessboard. Hence the name *Playing Games on Sets and Models*.

This thesis consists of the three articles which go under Chapters 3, 4 and 5:

- *Weak Ehrenfeucht-Fraïssé Games* by Tapani Hyttinen and Vadim Kulikov, published in Trans. Amer. Math. Soc. 363 (2011), 3309-3334.
- *Generalized Descriptive Set Theory and Classification Theory* by Sy-David Friedman, Tapani Hyttinen and Vadim Kulikov, submitted (2011).
- *Borel Reductions on the Generalized Cantor Space* by Vadim Kulikov, submitted (July 2011).

Each of these articles has an introduction of its own. In this chapter I gather and explain relevant ideas, methods and central results of the whole work in a hand waving way; also the bibliographical references might not be precise in this chapter.

Despite it consists of published (and not yet published) articles, this thesis is a single unity: the page numbering runs uniformly throughout the whole book and the bibliography, index and list of symbols are common to all the chapters and are found in the end, starting on page 162. Chapter 4 is a little bit modified version of that submitted to a journal, the main difference being the presence of Theorem 4.39 which is left out from the submitted version for some reason.

Since this is my dissertation, I wish first to discuss my honest contribution to these articles. In some cases it is easy, especially if the work is not done literally together. However in most cases it is not easy, because in a mathematical joint work, when you sit down with colleagues (or stand in front of a blackboard) and discuss a mathematical problem, it is hard to tell afterwards which part was contributed by whom. Thus, what follows has to be taken with a certain precaution.

The cover page of the paperback and all the graphics in the book I made using GIMP[1]. All text has been written and typeset by me using LaTeX with the following exceptions: the proofs

[1] The GNU Image Manipulation Program

of Theorems 4.38 and 4.90 are written by Tapani Hyttinen and typeset by me, and Remark 4.45 together with the proof following it is written by Sy-David Friedman and typeset by me. I hope I didn't leave anything out.

The first paper, *Weak Ehrenfeucht-Fraïssé Games* is based on my Master's thesis and is a bit off the main theme. It is joint work with my supervisor Tapani Hyttinen, who also supervised my Master's thesis. The definition of the weak EF-game is due to Jouko Väänänen. All the major ideas of the proofs are due to Tapani, but most details are worked out by me (with grand help though), especially in the proofs of Theorems 3.36 and 3.37. Example 3.20 is entirely my invention.

The second and the largest paper, *Generalized Descriptive Set Theory and Classification Theory* constitutes my Licentiate's thesis which is joint work with my supervisor Tapani Hyttinen and Prof. Sy-David Friedman from Kurt Gödel Research Center of the University of Vienna. Most of the major ideas of the proofs are due to Sy-David Friedman and Tapani Hyttinen unless otherwise specified in the text. Most of the proofs of small lemmas and theorems, like for example most of the proofs proofs in the introductory sections 4.2 and 4.5.1, are done by me (they are not necessarily new results or even new proofs, just results that are needed later in the work). Section *The Identity Relation*, page 79 is my work. Again the results of that section are not very impressive, but later I contributed more deeply to that area in the article *Borel Reductions on the Generalized Cantor Space*, Chapter 5. The proofs of Theorems 4.35, 4.44, 4.39 and Lemma 4.89 are almost entirely my work. The rest of the article is either fair joint work or the results were proved by others and processed by me.

The third paper, *Borel Reductions on the Generalized Cantor Space* is my own work, except that Tapani Hyttinen helped to complete some details concerning the proof of Theorem 5.12. He also read the paper several times and gave me valuable comments.

2.2 A Bit of Set Theory

Ordinals are in the most fundamental role in this thesis, so let me write a few words about them. A linear order is called a *well-order* if it contains no infinite descending sequences. Ordinals are well-ordered sets and for each well-ordered set there is an ordinal that is order isomorphic to that set. Ordinals themselves are initial segments of each other and the initial segment ordering on the class of all ordinals is a well-order. We use the von Neumann ordinals which are sets which are well-ordered by the $\in$-relation. The smallest ordinal is the empty set $\varnothing$ and is denoted often by 0. If α is an ordinal, then its successor $\alpha+1$ is the set $\alpha\cup\{\alpha\}$. If A is a collection of ordinals, then $\bigcup A$ is an ordinal. Ordinals are transitive sets, so we have $\alpha\in\beta \iff \alpha<\beta\Rightarrow\alpha\subset\beta$.

If there is no bijection from any ordinal $\beta<\alpha$ to α, then α is called *a cardinal number* or just *a cardinal.* Obviously for each ordinal there is only one cardinal number with which it is in a bijective correspondence. By the well-ordering principle, *every* set A is in a bijective correspondence with *some* (unique) cardinal number and this cardinal is called the *cardinality* of A. The countable cardinal is denoted by ω or $\aleph_0$, the smallest uncountable cardinal is denoted by ω_1 or $\aleph_1$, the smallest cardinal greater than ω_1 is denoted by ω_2 or $\aleph_2$, and on and on. The Greek letter ω is used when we want to emphasize that we are thinking of the cardinal as an ordinal. The Hebrew letter $\aleph$ is used when we want to emphasize that it doesn't matter. More generally κ^+ denotes the least cardinal bigger than the cardinal κ.

By $\mathrm{cf}(\alpha)$ we denote the cofinality of the ordinal α, it is the least ordinal β for which there exists an increasing unbounded function $f\colon \beta \to \alpha$.

A subset of an ordinal $S \subset \alpha$ is *closed* if for every increasing sequence in S, the limit of that sequence is in S provided that that limit is less than α. The set S is *unbounded* if for all $\beta < \alpha$ there exists $\gamma \in S$, $\gamma > \beta$. The collection of closed unbounded (cub) sets is usually a filter[2] on α (provided that the cofinality of α is uncountable). A set $S \subset \alpha$ is *stationary* if it intersects all the closed unbounded subsets of α.

Cub sets are also of crucial importance to this work. To illustrate their applicability, let $f\colon \omega_1 \to \omega_1$ be any function. Let C be the set of those α such that $f[\alpha] \subset \alpha$. Now C is of necessity a closed unbounded set. If $\mathcal{A}$ and $\mathcal{B}$ are relational (no function symbols) structures with $\mathrm{dom}\,\mathcal{A} = \mathrm{dom}\,\mathcal{B} = \omega_1$ and f is an isomorphism between them, then C contains the set D of those α for which $f[\alpha] = \alpha$ and is therefore a set of isomorphic substructures, i.e. $\mathcal{A}\cap\alpha \cong \mathcal{B}\cap\alpha$ for all $\alpha \in D$. But it is easy to see that D is also closed unbounded. Most of our proofs for non-isomorphism are based on this fact: a counter example, that structures are isomorphic, gives us a big set (a member of the cub-filter) in which the initial segments of the models are isomorphic.

ZFC

Above we made use of the *well-ordering principle* and other set theoretic assumptions. All these follow from the axioms of ZFC.[3] Everywhere in this dissertation ZFC is assumed as the basic theory in which we work. If extra assumptions are made, they are always explicitly mentioned and if no such assumptions are mentioned, then it means that we are using ZFC. Also if we say that something is *consistent*, then we mean that it is consistent with the axioms of ZFC.

2.3 Games

Games appeared in logic in 1930's, when Leon Henkin introduced the notion of game semantics, later developed by Paul Lorenzen in the 1950's. The idea of the semantic game is to climb up the semantic tree of a logical sentence. A semantic tree branches at quantifiers and at the signs $\wedge$ and $\vee$. Here branching at a quantifier means checking all possible values of the quantified variable.

2.1 Example. Let $\psi = (\forall x_0 R(x_0)) \wedge (\exists x_1 R(x_1))$ and let the structure $\mathcal{A}$ be such that $A = \{a, b\}$ and $R^{\mathcal{A}} = \{a\}$. The semantic tree will look like this:

$$\begin{array}{ccc} R(a) \quad R(b) & & R(a) \quad R(b) \\ \backslash \quad / & & \backslash \quad / \\ \forall x_0 R(x_0) & & \exists x_1 R(x_1) \\ & \backslash \quad / & \\ & (\forall x_0 R(x_0)) \wedge (\exists x_1 R(x_1)) & \end{array}$$

The game starts at the root. If the quantifier $\forall$ or the sign $\wedge$ is in question, then player **I** chooses which branch to continue along, otherwise **II** chooses. If a negation occurs, then it

[2] A filter is a collection closed under finite intersections and taking supersets.

[3] Zermelo-Fraenkel axioms with the Axiom of Choice, for more information see [4] or [25].

is dropped and the players change roles. They end up with an element in the structure for each quantifier encountered on their way and an atomic formula into which the elements are substituted. If the atomic formula with this substitution is true, then **II** wins and otherwise **I** wins. The sentence is defined to be true if and only if **II** has a winning strategy and false if and only if **I** has a winning strategy.

An extensive treatment of the use of games in modern mathematical logic can be found in a book by Jouko Väänänen *Models and Games*, Cambridge University Press 2011.

Games play (indeed!) a major role in this thesis. We focus on infinite games of perfect information. There are four types of games that appear:

- *Ehrenfeucht-Fraïssé games.* These games represent back-and-forth systems and are designed to measure the level of similarity between two mathematical structures. They define tractable invariants of the isomorphism relation: if two structures are isomorphic, then they are EF-equivalent.
- *Semantic games.* An instance of these is described above. Semantic games generalize Tarski's definition of truth so that it can be used for a wider scope of languages.
- *Cub-games.* In cub-games players are climbing up ordinals. These games give useful characterizations of cub and related filters on uncountable cardinals and are closely connected to combinatorial principles in set theory.
- *The Borel* game.* Conventional Borel sets are built up from open sets using intersections and unions. Each Borel set can be represented as a tree which represents the sequence of intersections and unions and at the leaves of the tree there are basic open sets. By generalizing this tree-definition we get a different outcome (the Borel* sets) than by generalizing the conventional definition of closing open sets under intersections and unions.

2.3.1 Cub-games

The general form of a cub game is as follows. Let α be an ordinal and κ a cardinal greater or equal to α. Let $S \subset \kappa$. There are α moves in the game $G_\alpha(S)$ and at the move γ, first player **I** picks an ordinal $\alpha_\gamma < \kappa$ larger than any ordinal picked in the game so far and then player **II** picks an ordinal $\beta_\gamma < \kappa$ greater than α_γ. The winning criterion varies. Sometimes the winning criterion for player **II** is that the supremum of the set picked during the game is in S; sometimes the winning criterion for player **II** is that the limit points of the picked set is a subset of S. Limit points could be restricted to various cofinalities etc. The usefulness of the cub-games is that the set $\{S \subset \kappa \mid$ player **II** has a winning strategy in $G_\alpha(S)\}$ forms usually a filter on κ. This filter looks much like the cub-filter and often actually coincides with it. Therefore translating between Ehrenfeucht-Fraïssé games and cub-games gives a method of applying the theory of cub filters and stationary sets to model theory.

2.3.2 Games and Languages

Weak EF-games

Ehrenfeucht-Fraïssé games are a variant of back-and-forth systems in model theory. The standard EF-game is defined as follows:

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be structures and γ an ordinal. The *Ehrenfeucht-Fraïssé game* of length γ, $\mathrm{EF}_\gamma(\mathcal{A}, \mathcal{B})$, is played as follows. On the move α, $\alpha < \gamma$, player **I** chooses an element $a_\alpha \in A$ (or $b_\alpha \in B$). Then **II** answers by choosing an element $b_\alpha \in B$ (or $a_\alpha \in A$). **II** wins if the function f, which takes a_α to b_α for each $\alpha < \gamma$ is a partial isomorphism $\mathcal{A} \to \mathcal{B}$. Otherwise player **I** wins.

In Chapter 3 we study a weak version of the standard Ehrenfeucht-Fraïssé game:

Definition. Let $\mathcal{A}$, $\mathcal{B}$ and γ be as in 3.2. The *weak Ehrenfeucht-Fraïssé game* of length γ, $\mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{B})$, is played as follows.

Player I chooses an element $a_\beta \in A \cup B$

Player II chooses an element $b_\beta \in A \cup B$.

Let $X = \{a_\alpha \mid \alpha < \gamma\} \cup \{b_\alpha \mid \alpha < \gamma\}$ be the set of all chosen elements. Player **II** wins if the substructures generated by $X \cap \mathcal{A}$ and $X \cap \mathcal{B}$ are isomorphic. Otherwise **I** wins.

The difference between these games in not only that EF^*_α easier or as easy to win for **II** than EF^*_α (which follows from the mere fact that the winning criterion is weaker), but also that EF_α is a closed game but EF^*_α isn't. *Closed* means basically that if **II** didn't lose at any particular move, then she didn't lose at all. Let us give an example of a EF^*_α-game which shows that it is not closed. Let $\mathcal{A} = \mathcal{B} = (\mathbb{Q}, <)$ be the rational numbers with the usual order and $\alpha = \omega$. No matter how player **II** plays, as long as she keeps the number of chosen elements in both structures the same, she doesn't lose at any move. Evidently she can still lose the whole game if she doesn't play well: the players might end up picking the whole of $\mathcal{A}$ and only the natural numbers from $\mathcal{B}$ and these are not isomorphic linear orders. A closed game of length ω is always determined; this is known as the Gale-Stewart theorem. Therefore $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ is determined for all structures $\mathcal{A}$ and $\mathcal{B}$, but is $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$ necessarily determined? At least now we cannot apply the Gale-Stewart theorem.

We show in chapter *Weak Ehrenfeucht-Fraïssé Games* that despite the differences between EF and EF^*, the game of length ω, $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$ is equivalent to the ordinary EF-game of the same length, $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$. *Equivalent* for all models $\mathcal{A}$ and $\mathcal{B}$, player **II** has a winning strategy in $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ if and only if she has one in $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$ and the same holds for player **I**. This in turn implies that weak EF-games of length ω characterize the $L_{\infty\omega}$-equivalence as this characterization result is well known for the ordinary EF-games (proved by Carol Karp). Thus we have:

Theorem ([17])**.** *Models $\mathcal{A}$ and $\mathcal{B}$ satisfy precisely the same formulas of $L_{\infty\omega}$ if and only if player* **II** *has a winning strategy in* $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$. □

The language $L_{\infty\omega}$ is obtained by closing the first-order language under arbitrary large disjunctions and conjunctions over sets of formulas with finitely many variables.

Why, games can serve as invariants of the isomorphism relations on their own, without any language being involved. This is the attitude we took when we considered longer EF-games, like $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$. Unlike the game of length ω, the longer weak EF-games can be non-determined, i.e. neither of the players has a winning strategy.

Let us return to the equivalence of EF_ω and EF^*_ω. How is it proved? The weak game is easier (or as easy) to win for player **II** and EF_ω is determined. So it is sufficient to show that if **I** has a winning strategy in $\mathrm{EF}_\omega(\mathcal{A},\mathcal{B})$, then he also has one in $\mathrm{EF}^*_\omega(\mathcal{A},\mathcal{B})$. Let τ be the strategy of **I** in $\mathrm{EF}_\omega(\mathcal{A},\mathcal{B})$ and let

$$C = \{S \subset \operatorname{dom}\mathcal{A} \cup \operatorname{dom}\mathcal{B} \mid \operatorname{card}(S) = \aleph_0 \text{ and } S \text{ is closed under } \tau\}.$$

A strategy is a function from finite sequences of $\operatorname{dom}\mathcal{A} \cup \operatorname{dom}\mathcal{B}$ to $\operatorname{dom}\mathcal{A} \cup \operatorname{dom}\mathcal{B}$, so any countable set can be closed under τ and the result is countable. Also C is closed under countable infinite unions of increasing chains.

Let us now give player **I** a winning strategy in the game $\mathrm{EF}^*_\omega(\mathcal{A},\mathcal{B})$. At each move, player **I** takes all the elements already picked in the game and closes that set under τ. He uses a bookkeeping technique and enumerates these sets by his own moves during the game. Therefore in the end, the set that has been picked, $X \subset \operatorname{dom}\mathcal{A} \cup \operatorname{dom} B$, is in C. Let us show that player **I** has won: $X \cap \mathcal{A} \not\cong X \cap \mathcal{B}$. If not, then there would be an isomorphism $f\colon X \cap \mathcal{A} \to X \cap \mathcal{B}$ and player **II** could have beaten τ in $\mathrm{EF}_\omega(\mathcal{A},\mathcal{B})$ by playing according to f, i.e. picking $f(a)$ whenever **I** picks $a \in \mathcal{A}$ and picking $f^{-1}(a)$ whenever he picks $a \in \mathcal{B}$. Player **I** cannot escape X using τ since $X \in C$, so this is a contradiction.

2.3.3 Games as Bridges Between Set Theory and Model Theory, Part I

As I explained above, the games EF_ω and EF^*_ω are equivalent and since the first one is determined, also is the second. In Chapter 3 we also ask about longer games like $\mathrm{EF}^*_{\omega_1}$, whether they can be non-determined on some structures $\mathcal{A}$ and $\mathcal{B}$.

In order to answer this question positively, we had to construct exemplifying structures $\mathcal{A}$ and $\mathcal{B}$ on which $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined. To do this we developed a method of constructing structures which made it possible to boil the question of determinacy of EF-games down to the question of determinacy of cub-games, of which much is known. By developing the idea we answered also more questions of the same nature like the following. For a given cardinal $\kappa > \omega$, are there structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ is non-determined? Is it provable in ZFC that such structures exist? Can these structures be of size κ^+? (Exercise: they cannot be of size $\leqslant \kappa$.) Are there structures $\mathcal{A}$ and $\mathcal{B}$ and cardinals $\lambda < \kappa$ such that player **II** has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ but not in $\mathrm{EF}^*_\lambda(\mathcal{A},\mathcal{B})$?

The cub-games are about climbing up the ordinals. How is that related to EF-games which are about picking elements from arbitrary mathematical structures? Assuming the Axiom of Choice, as we do, any mathematical structure of cardinality κ can be well-ordered in order type κ. Thus picking elements from that structure can be thought of as picking an ordinal below κ. If the game is long enough, or the structures are designed accordingly, the players must actually climb ***up*** the ordering during the EF-game, or else they lose.

Following these lines we defined a method of constructing the structures $\mathcal{A}(S)$ and $\mathcal{B}(S)$ for an arbitrary set $S \subset \kappa$ such that playing a weak EF-game between $\mathcal{A}(S)$ and $\mathcal{B}(S)$ is very much like playing the cub-game on the set S. The idea is that $\mathcal{A}(S)$ and $\mathcal{B}(S)$ are trees and all the branches in $\mathcal{A}(S)$ grow along closed subsets of S. $\mathcal{B}(S)$ is very similar to that, with the exception that some branches continue growing through all the levels.

If S contains a closed unbounded set, then $\mathcal{A}(S)$ and $\mathcal{B}(S)$ are in fact isomorphic, because now there is a closed unbounded set along which almost all branches can grow till the end in both structures, so the difference that $\mathcal{B}$ has a long branch disappears. Denote by $\mathcal{A}_\alpha(S)$ and $\mathcal{B}_\alpha(S)$ the trees restricted to levels $\leqslant \alpha$. Now by the same argument $\mathcal{A}_\alpha(S)$ and $\mathcal{B}_\alpha(S)$ are isomorphic if and only if $S \cap \alpha$ contains a closed set which is unbounded in α.

During the EF-game player **I** wants the game to end in a position in which non-isomorphic segments $\mathcal{A}_\alpha(S)$ and $\mathcal{B}_\alpha(S)$ have been chosen and player **II** wishes them to be isomorphic. During the game the players (or one of them) make sure that in the end initial segments are chosen and not only a part of them. If one looked only at the levels of the trees which are already covered by the game, the game would look exactly as a cub-game.

In this "cub-game" player **I** wins if they hit an ordinal α such that $\alpha \cap S$ *does not contain* a cub set and player **II** wins if they hit an ordinal α such that $\alpha \cap S$ *contains* a cub set. Now from the theory of cub games we know that this game will be non-determined (under GCH at least[4]), if $\{\alpha \mid \alpha \cap S$ contains a cub set$\}$ is bistationary, i.e. a stationary set whose complement is stationary.

The next problem is that it's non-trivial whether there exists an S which satisfies this requirement at all. But fortunately such an S can be always forced, so its existence is consistent with ZFC+GCH.

2.4 Generalized Descriptive Set Theory and Classification Theory

History

The beginning of generalized descriptive set theory dates back to the beginning of 1990's when Väänänen, Mekler, Shelah, Halko, Todorcevic and others started to look at the space 2^{ω_1} from the point of view of descriptive set theory, in other words classifying the subsets of that space according to their descriptive complexity. This required generalizations of the known concepts of Borel sets, projective sets, meager sets and other related concepts. Already at that stage the theory diverges from the classical theory on the reals, namely there are three distinct generalizations of the notion of Borel and there is no acceptable generalization of a (Lebesgue) measure. Many implications to model theory of models of size $\aleph_1$ were discovered already back then. For more on the history of this subject and precise references, see Section 4.1 starting from page 58.

2.4.1 Generalized Baire and Cantor Spaces

Standard descriptive set theory studies the space ω^ω of all functions from ω to ω equipped with the product topology. The motivation for that is explained in a hand-waving manner in the prologue, Chapter 1. The space ω^ω is called *the (universal) Baire space* and not without a reason: every Polish space, i.e. completely metrizable separable topological space, is a continuous image of ω^ω and moreover Borel isomorphic to it. For example the real line $\mathbb{R}$ is a Polish space, so to study the Borel and projective sets of reals is to study the Borel and projective subsets of ω^ω. The space 2^ω (functions from ω to $\{0,1\}$ with product topology) is a compact subspace of ω^ω and is called *the Cantor space.* Every metrizable compact space is a continuous image of the Cantor space.

These spaces are suitable also for studying isomorphism relations and other relations on countable models as explained below in section *Model Theory.* Probably this was the leading

[4]The General Continuum Hypothesis, but in fact much weaker set theoretical assumptions suffice, see Section 5.3, page 144.

line towards the generalizations from 2^ω to 2^{ω_1} (all functions from ω_1 to $\{0,1\}$) and more generally to 2^κ and from ω^ω to κ^κ (all functions from κ to κ). But once we take this step, we must answer also: How to generalize the product topology? How to generalize Borel sets? Which generalizations suits well the model theoretic purpose?

We define the topology on 2^κ to be generated by the sets

$$N_p = \{\eta \mid \eta\restriction\alpha = p\}$$

for $p \in 2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^\alpha$. This is finer than the standard product topology and as pointed out in the prologue, is similar to the topology of the reals as it (almost) comes from the lexicographical ordering of 2^κ.

The Borel sets are obtained by closing the topology under unions and intersections of size κ. This raises many questions. Are Borel sets closed under complement? Do we get even all closed sets like that? What if we explicitly close the collection under complements? Will there be more than 2^κ Borel sets then? In this work we have smashed all these questions down by assuming that $\kappa^{<\kappa} = \kappa$. This implies that closing open sets under intersections and unions of size κ gives a collection of size 2^κ closed under complements. Being of size 2^κ is important when we want to use elements of 2^κ as codes for Borel sets. Recently Hyttinen proposed another way of overcoming these questions without any restrictions on κ (which is maybe still required to be regular).

Another question is raised by the fact that this is only one out of three distinct ways to generalize Borel sets (Definition 4.16 on page 67). The other two, Δ^1_1 and Borel* sets are described below. Why we choose Borel as *the* Borel sets? First,

$$\text{Borel} \subset \Delta^1_1 \subset \text{Borel}^*,$$

([36], see Theorem 4.19 on page 68 for more) so they're at the bottom of the descriptive hierarchy among the candidates. Second, Borel sets are closed under complements (assuming $\kappa^{<\kappa} = \kappa$), but it is consistent that Borel* aren't and it is not even known whether they *can* be closed (consistently, in which case $\Delta^1_1 = \text{Borel}^*$). Third, Borel sets form precisely the collection that allows us to generalize the Lopez-Escobar theorem: they correspond to the formulas of $L_{\kappa^+\kappa}$ similarly as the standard Borel sets correspond to the formulas of $L_{\omega_1\omega}$, see Theorem 4.25 on page 71 (the original proof is due to R. Vaught). The language $L_{\kappa\lambda}$ is obtained from the first order language by allowing conjunctions and disjunctions of length less than κ and quantification over symbol-tuples of length less than λ.

Using similar intuition of relating disjunction to unions and existential quantifiers and conjunctions to intersections and universal quantifiers, one might conjecture similar things for other languages. And in fact we proved that Δ^1_1-sets correspond exactly to the formulas of $M^*_{\kappa^+\kappa}$, a generalization of $L_{\kappa^+\kappa}$; the idea of the proof is due to Sam Coskey and Philipp Schlicht and uses a separation theorem by H. Tuuri, see Theorem 4.28 on page 74. The idea of this proof is explained in the section *Games as Bridges Between Set Theory and Model Theory, Part II* below.

Besides Borel sets we also generalize other notions from descriptive set theory such as meager and co-meager sets and benefit from the generalized which say that (1) the space 2^κ is not meager (generalization of the Baire category theorem) and (2) every Borel function (see below) is continuous on a co-meager set (Theorem 4.34, page 4.34).

Borel Reductions

Suppose that κ is an infinite cardinal with $\kappa^{<\kappa} = \kappa$ and assume that 2^κ is equipped with a Borel structure as described above. Suppose that E_0 and E_1 are equivalence relations on 2^κ. We say that E_0 is *Borel reducible to* E_1, if there exists a function $f\colon 2^\kappa \to 2^\kappa$ such that $(\eta,\xi) \in E_0 \iff (f(\eta), f(\xi)) \in E_1$ for all $\eta, \xi \in 2^\kappa$ and for all open sets $U \subset 2^\kappa$, the set $f^{-1}[U]$ is Borel. Such functions are in general called *Borel functions*

The intuitive meaning of this is that E_1 serves as an invariant of E_0 modulo f which, in a sense, puts E_0 below E_1 in the descriptive hierarchy. We will explore the implications of this definition in section *Model Theory* below and will say more about it in section *The Ordering of Equivalence Relations* further below.

2.4.2 Games as Bridges Between Set Theory and Model Theory, Part II

A set $A \subset 2^\kappa$ is *Borel**, if there exists a tree t with no branches of length κ and which has at most κ successors at each node and a function

$$h\colon \{\text{Branches of } t\} \to \{\text{Basic open sets of } 2^\kappa\}$$

such that

$$\eta \in A \iff \text{Player } \mathbf{II} \text{ has a winning strategy in } BG(t,h,\eta)$$

where the game BG is played as follows. At each move the players are located at some node of t. If it is player **I**'s turn, he picks a successor of the node they're in and the players move to that picked node. If it is player **II**'s turn, she picks a successor of the node they're in and the players move to that picked node. The game starts at the root of t and so they go up until they have picked a branch b. If $\eta \in h(b)$, then player **II** wins and otherwise player **I** wins. Note that if we require t to have no infinite branches but otherwise keep the same requirements, this would become the definition of a Borel set.

A statement that a given subset of 2^κ is Δ^1_1 or Borel* belongs to set theory. A statement that a given model class is definable by a formula in a given language belongs model theory. Theorem 4.28 says: a subset A of 2^κ is Δ^1_1 if and only if the class of models coded by the elements of A is definable by a formula in $M^*_{\kappa^+\kappa}$.

A formula of $M_{\kappa^+\kappa}$ is a formula that may have infinitely long sequences of quantifiers, in chains of length less than κ. Formally, the formulas of $M_{\kappa^+\kappa}$ are labeled trees with no branches of length κ and at most κ successors at each node. This labeled tree is a direct generalization of a semantic tree of a first order sentence as described in Example 2.1 on page 14 and the definition of truth is given via the semantic game.

There is no negation in the definition of $M_{\kappa^+\kappa}$. One can define a relative of the negation: a *dual* of a formula, by switching all conjunctions to disjunctions, existential quantifiers to universal and vice versa and the atomic formulas to their first-order negations. As a matter of fact, a class definable by an $M_{\kappa^+\kappa}$-formula may not be the complement of a class of models definable by its dual, even if restricted to models of size κ. If a formula θ happens to be such that its dual defines precisely the class of all the models not in the class definable by θ, we say that these formulas are *determined*. The language $M^*_{\kappa^+\kappa}$ is the set of determined $M_{\kappa^+\kappa}$-formulas.

For one direction of Theorem 4.28, suppose that the set A consists of codes for structures definable by $\varphi \in M_{\kappa^+\kappa}$ and the complement of A is definable by $\psi \in M_{\kappa^+\kappa}$. Intuitively A

consists of those η for which *there exists* a winning strategy of player **II** in the semantic game for φ on the model coded by η. But the strategies can be coded by elements of 2^κ in a way that makes the set corresponding to the winning strategies closed, so A becomes a projection of a closed set. But the same argument goes for the complement of A, so they are both Σ_1^1 and the definition of Δ_1^1 is that it is a Σ_1^1-set whose complement is also a Σ_1^1-set.

To prove the other direction note first that if a set $A \subset 2^\kappa$ is Borel* and its complement is Borel*, then A is Δ_1^1, because Borel$^* \subset \Sigma_1^1$. The definition of a Borel* set is game theoretic as well as is the truth definition for $M_{\kappa^+\kappa}$-formulas. Moreover the class of trees used in these definitions coincide. So we would like to use that coincidence to prove that if a set is Δ_1^1, then it is definable by an $M^*_{\kappa^+\kappa}$-formula which is precisely the other direction of Theorem 4.28.

To make a long story short, using the above described game theoretic similarity of Borel* and $M_{\kappa^+\kappa}$, we prove that the set of models whose codes form a Borel* set can be defined by a formula in $\Sigma_1^1(M_{\kappa^+\kappa})$. That is, by a formula in $M_{\kappa^+\kappa}$ fronted by one extra second-order existential quantifier. We use the unary relation that is quantified to define a well-ordering of order type κ on the model's domain. This allows us to translate the Borel*-game into the $M_{\kappa^+\kappa}$-game. A separation theorem of H. Tuuri says that for any two disjoint model classes, C and D, definable by $\Sigma_1^1(M_{\kappa^+\kappa})$-formulas, there exists a formula of $M_{\kappa^+\kappa}$ which defines a model class containing C but not containing D. A bit of further work reveals that this is sufficient to complete the proof.

2.4.3 Model Theory

One logical motivation for studying the spaces 2^ω and their (standard) descriptive set theory comes from model theory of countable models. Similarly 2^κ is a way to study model classes of size κ. By thinking of all countable models as having ω as the domain, one can easily define a coding such that each $\eta \in 2^\omega$ corresponds to a countable model $\mathcal{A}_\eta$ with domain ω. One such coding is defined in section *Coding Models*, page 66. This coding is continuous in the sense that for each $\eta \in 2^\omega$ and $n < \omega$, there exists $m < \omega$ such that $\mathcal{A}_\eta \restriction m$ is isomorphic to $\mathcal{A}_\xi \restriction m$ for all ξ such that $\xi \restriction n = \eta \restriction n$. Now isomorphism classes of models and isomorphism relations of classes of models can be studied from the viewpoint of descriptive set theory being coded into subsets of 2^ω.

All this generalizes very straightforwardly to models of size κ and initial segments of length $\alpha < \kappa$ instead of $n < \omega$ etc. The isomorphism relation can be seen as a relation on the subset of 2^κ consisting of those function that code models of T. Thus for a theory T and a cardinal κ, define

$$\cong_T^\kappa = \{(\eta,\xi) \in (2^\kappa)^2 \mid \mathcal{A}_\eta \models T,\ \mathcal{A}_\xi \models T,\ \mathcal{A}_\eta \cong \mathcal{A}_\xi\}.$$

If κ is fixed by the context, then we usually drop it from the notation

This time we find that the descriptive hierarchy of the isomorphism relations seen as subsets of 2^κ, goes in synch with the model theoretic complexity of countable first-order theories. In fact, much more in synch than when dealing with countable models. Let T be a first order theory and let $M_\kappa(T)$ be the set of all models of T whose domain is κ. By defining a coding as explained in section *Coding Models* we get a one-to-one correspondence between $M_\kappa(T)$ and a subset of 2^κ. This subset is always Borel, because $\bigwedge T$ is a $L_{\kappa^+\kappa}$ sentence and as explained in the previous section defines a Borel set.

The reasons for that synch, as can be seen from our proofs, include the powerful applicability of stability theory to uncountable models and the richness of uncountable orderings.

The dividing line we draw between classifiable and unclassifiable theories is the equivalence relation modulo one or another version of the non-stationary ideal. A set is non-stationary, if its complement contains a cub-set. It is easily verified that the collection of all non-stationary subsets of a cardinal is an ideal (it is closed under finite unions and under taking subsets). Further restricting the sets to certain subsets of the cardinal one gets different versions of that ideal. Let us denote such an ambiguously defined equivalence relation by E_{NS}. We show that E_{NS} is Borel reducible to the isomorphism relations of unclassifiable theories but is not reducible to the isomorphism relation of classifiable theories.[5] The reduction of E_{NS} into the isomorphism relation of unclassifiable theories is based on various ways of building models out of linear and partial orderings. Two such methods, the well known construction of Ehrenfeucht-Mostowski models and the one presented in the proof of Theorem 4.90, are used.

Unclassifiable Theories

For reducing E_{NS} into all unclassifiable theories except those that are stable unsuperstable, EM-models are used, Theorem 4.83. We construct linear orders $\Phi(S)$ for each stationary set $S \subset \kappa$ such that the EM-models corresponding to $\Phi(S)$ and $\Phi(S')$ are isomorphic if $S \triangle S'$ is non-stationary. These orderings are κ-like, i.e. the initial segments have cardinality $< \kappa$ but the whole $\Phi(S)$ has cardinality κ. The isomorphism between $\Phi(S)$ and $\Phi(S')$ for non-stationarily-similar S and S' is obtained by extending partial isomorphisms along a cub set C in which S equals S', i.e. which satisfies $C \cap S = C \cap S'$ by using strong homogeneity of the initial segments of $\Phi(S)$ and $\Phi(S')$ at such points. The ideas here are borrowed from a paper by T. Huuskonen, T. Hyttinen and M. Rautila as of 2004.

I invite the reader to use the intuition that $\Phi(S)$ is κ-like and think of an intuitive correspondence between κ and $\Phi(S)$. Then it should make sense if I say that the ordering $\Phi(S)$ is defined such that it behaves in a slightly exceptional way at the "places" that correspond to the ordinals of κ that are in S. For each $\Phi(S)$ we build a tree which consist of increasing sequences of $\Phi(S)$, so that branches occur only where the behavior of $\Phi(S)$ is in this way "exceptional".

Then we use Shelah's Ehrenfeucht-Mostowski construction on these trees to obtain a model of the theory T for each such tree. If $S \triangle S'$ is non-stationary, then $\Phi(S) \cong \Phi(S')$ and the corresponding trees are isomorphic and so the corresponding models are of course isomorphic as well. On the other hand, assuming that there is an isomorphism f between the structures but $S \triangle S'$ is still stationary, we get a contradiction. Suppose $S \setminus S'$ is stationary and denote $S^* = S \setminus S'$. The contradiction is obtained by finding initial segments of the models such that they are isomorphic via f (i.e. closed under f) and so that they hit a "place" which is in S^*, so there is a branch in the skeleton of one of them, but not in the skeleton of the other. To witness that this is actually a contradiction we refine S^* so that the question boils down to preservation of certain types by f in a contradictory way.

In the case of a stable unsuperstable theory, the above approach didn't work, but a similar one worked: instead of EM-models over the trees mentioned above we use a prime model construction over another kind of trees, see Section 4.6.3. As in the construction described above, we define a tree $J(S)$ for each stationary $S \subset \kappa$, but this time the trees have height

[5] See Theorems 4.81 (page 119), 4.83 (page 121) and 4.90 (page 137)

ω and size κ. For such a tree J, a sequence $(J^\alpha)_{\alpha<\kappa}$ is a *filtration* of J, if it is a continuous increasing sequence of small subtrees of J whose union is J, where small means of size $< \kappa$. For each member of such a filtration, we can ask whether or not there is a branch going *through* that member, i.e. whether there is an element η of J such that $\eta \notin J^\alpha$, but $\eta \restriction n \in J^\alpha$ for all $n < \omega$. We define the set $\mathcal{S}(J)$ to be the set of those ordinals for which there is a branch going through J^α. Clearly $\mathcal{S}(J)$ depends on the filtration chosen, but we define the trees $J(S)$ such that it does not depend on the filtration modulo the ω-non-stationary sets, in fact $S \triangle \mathcal{S}(J(S))$ is always non-stationary, i.e. there are branches going through in $J(S)$ exactly at places corresponding to ordinals in S with respect to a certain filtration. Since an isomorphism of two such trees preserves a filtration on a closed unbounded set, this implies that $S(J)$ is an invariant of the isomorphism type of J.

In the construction of the trees $J(S)$, $J(S')$, we make sure that there are filtrations $(J^\alpha(S))_{\alpha<\kappa}$ and $(J^\alpha(S'))_{\alpha<\kappa}$ of them such that *if* there is an increasing sequence $(f_i)_{i<\alpha}$ of partial isomorphisms from $J(S)$ to $J(S')$ such that for all even $i < \alpha$ $\operatorname{dom} f_i = J^{\beta_i}(S)$ and $\operatorname{ran} f_{i+1} = J^{\beta_{i+1}}(S')$ for some $\beta_i < \beta_{i+1}$ such that $\bigcup_{i<\alpha} \operatorname{dom} f_i = J^\beta(S)$ and $\bigcup_{i<\alpha} \operatorname{ran} f_i = J^\beta(S')$ with $\beta \in \kappa \setminus (S \triangle S')$, *then* the union $\bigcup_{i<\alpha} f_i$ is a partial isomorphism. This guarantees that $\mathcal{S}(J)$ is in fact a complete isomorphism invariant. Consequently, it becomes a complete isomorphism invariant of the prime models over the trees $J(S)$ constructed in the proof of Theorem 4.90.

Classifiable Theories

On the other side of the dividing line are the classifiable theories. The equivalence relation modulo any kind of a non-stationary ideal, the vague concept of which was generally denoted above by E_{NS}, cannot be Borel reduced to an isomorphism relation of such a theory, Theorem 4.81, page 119.

Indeed, suppose there were such a reduction. The contradiction derives from the absoluteness of both, being Borel and classifiable, while being stationary is far from an absolute notion. We are talking here about absoluteness with respect to forcing. Suppose $\varphi(x)$ is a formula of set theory with one free variable. We say that φ *is an absolute property of a with respect to the forcing notion* $\mathbb{P}$, if $\varphi(a)$ holds and in all forcing extensions by $\mathbb{P}$, $\varphi(a)$ remains to hold. By a theorem of Shelah, already mentioned above, the models of a classifiable theory are distinguishable by an Ehrenfeucht-Fraïssé game of length ω where players are allowed to pick sets of size $< \kappa$ where κ is the size of the models. The existence of a winning strategy in such a game is absolute with respect to forcings that do not add small subsets, meaning subsets of size less than κ. Any move in this game is a finite sequence of sequences and partial isomorphisms of length $< \kappa$, so the player who owned a winning strategy in the ground model can use the same strategy in the generic extension and certainly it will work equally well, as no new moves (certain sequences described above) is introduced neither any finite partial isomorphism is killed.

Borel sets in turn are absolute in the following way. As noted above in section *Games as Bridges Between Set Theory and Model Theory, Part II*, Borel sets can be represented as labeled trees of size κ. These trees, as all models of size κ, can be coded into elements of 2^κ. These codes remain Borel codes in the forcing extensions, although the sets that they code in the extension might be different from those that they code in the ground model.

Now, we take a "model of ZFC" (not really) of size κ which contains the Borel code for the reduction f and $2^{<\kappa}$ and which we call M. Note that since f is a Borel function, it is continuous on a co-meager set (see section *Generalized Baire and Cantor Spaces* above). But the set of

$\mathbb{P}$-generic functions over M is also co-meager, where $\mathbb{P} = 2^{<\kappa}$. In fact it is easy to find a $\mathbb{P}$-generic G over M whose symmetric difference from the constant zero function $\bar{0}$ is stationary.[6] Without loss of generality we may assume that f is continuous also at $\bar{0}$. Now these functions, G and $\bar{0}$ must be mapped to non-isomorphic models. By the assumption, there is a winning strategy of player **I** in $\mathrm{EF}^{\kappa}_{\omega}$ on these models. That is, by the very definition of forcing, we can find a condition $p \in G$ which forces that player **I** has a winning strategy in $\mathrm{EF}^{\kappa}_{\omega}(f(G), f(\bar{0}))$ and also solves the structures $f(G)$ and $f(\bar{0})$ sufficiently far. Well, but now we just extend p in another generic way, G', so that G' is equivalent to $\bar{0}$ modulo the non-stationary ideal but preserves the winning strategy of **I** (by the absoluteness described above). For detailed proof, see 4.81.

2.4.4 Games as Bridges Between Set Theory and Model Theory, Part III

Suppose that T is a theory. The motto of Chapter 4 is that the more complex the isomorphism relation of T is model theoretically, the more complex it is set theoretically and vice versa. Let us take a look at how do we establish such a correspondence and what plays the role of bridges between set theory and model theory in this case.

The "Bridge Theorems" for this purpose are Theorems 4.68 and 4.70 on pages 112 and 115 respectively and the main ingredient of their proofs is constituted by – as the reader has surely guessed – games.

As discussed in section *Weak EF-Games* above, the EF_{ω}-equivalence is the same as the $L_{\infty\omega}$-elementary equivalence. Thus, if EF_{ω}-equivalence characterizes the isomorphism relation of a theory T, i.e. all models $\mathcal{A}$ and $\mathcal{B}$ of T are isomorphic precisely when player **II** has a winning strategy in $\mathrm{EF}_{\omega}(\mathcal{A}, \mathcal{B})$, then it means that the models of T can be fully described by the language $L_{\infty\omega}$ which makes the Model Theorist regard T as uncomplicated.

The first of these two theorems, 4.68 and 4.70, asserts that the isomorphism relation is Borel if and only if the isomorphism types are classified by an Ehrenfeucht-Fraïssé game: there exists a tree t with no infinite branches and with at most κ successors at each node such that two models of the theory $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if player **II** has a winning strategy in $\mathrm{EF}^{\kappa}_{t}(\mathcal{A}, \mathcal{B})$. This game differs from the EF-game defined above only a little. In EF^{κ}_{t} player **I** picks at his moves a subset of size $< \kappa$ of the models' domains together with an element of t above any element picked by him earlier. Player **II** in turn chooses a partial isomorphism f between the structures so that the set chosen by **I** is included in $\operatorname{dom} f \cup \operatorname{ran} f$. Additionally f has to extend all previously chosen partial isomorphisms. The game ends when player **I** cannot go up the tree anymore.

One direction (from right to left) of the proof of 4.68 uses the fact that each Borel* set is a Borel set if the defining tree has no infinite branches. Assuming that there is such a tree, we take all possible plays of the game $\mathrm{EF}^{\kappa}_{t}(\mathcal{A}, \mathcal{B})$. These plays form a tree u with no infinite branches either. We label the branches of that tree with open sets that consist of model pairs (or their codes) whose restrictions to the set picked during the game are isomorphic, i.e. player **II** has won. Using the fact that isomorphism is characterized by this game, we show that the Borel* set defined by u and the described labeling is precisely the isomorphism relation.

For the other direction, we form a model for each pair of models as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be models of T we define a pair-structure $(\mathcal{A}; \mathcal{B})$ with the property that $(\mathcal{A}; \mathcal{B}) \cong (\mathcal{A}'; \mathcal{B}') \iff$

[6] Technically the function is $\cup G$ and G is a sequence of partial functions, but we omit the difference and trust the reader.

$\mathcal{A} \cong \mathcal{A}' \wedge \mathcal{B} \cong \mathcal{B}'$. Then the isomorphism relation $\cong_T$ becomes a subset of the set of pair-structures and we assume that it is Borel. From Theorem 4.25 described above in section *Games as Bridges Between Set Theory and Model Theory, Part II*, we have that there is a sentence θ of $L_{\kappa^+\kappa}$ which defines the set $E = \{(\mathcal{A};\mathcal{B}) \mid \mathcal{A} \cong \mathcal{B}\}$, so (by the standard results 4.10, 4.11 and 4.12) there is a tree t which depends on θ such that player **II** has a winning strategy in $\mathrm{EF}^\kappa_t((\mathcal{A};\mathcal{B}),(\mathcal{A}';\mathcal{B}'))$ only if the equivalence holds: $(\mathcal{A};\mathcal{B}) \in E \iff (\mathcal{A}';\mathcal{B}') \in E$. If we assume on contrary now that there are $\mathcal{A}$ and $\mathcal{B}$ isomorphic but not distinguishable by $\mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$ (for this very same t), then we get a contradiction as follows: clearly $(\mathcal{A},\mathcal{A}) \in E$ and player **II** has a winning strategy in $\mathrm{EF}^\kappa_t((\mathcal{A};\mathcal{A}),(\mathcal{A};\mathcal{B}))$, but $(\mathcal{A};\mathcal{B}) \notin E$.

The second "Bridge Theorem", 4.70, states a result that is, in a way, a converse to 4.68: it states that if for every tree t of a certain kind there are non-isomorphic models of T that cannot be distinguished by EF^κ_t, then the isomorphism relation of T cannot be Δ^1_1. Since Borel $\subset \Delta^1_1$, the result is stronger than the corresponding direction of Theorem 4.68, although the assumptions are stronger as well. The proof is based on a Lemma given by A. Mekler and J. Väänänen in their joint work on 2^{ω_1} in 1993 (Lemma 4.69 on page 115) which characterizes the Δ^1_1-sets via existence of certain trees.

The proof is quite detailed and its explanation in this introduction is unnecessary. However the main idea (apart from Lemma 4.69) is based on the following. Let t be a tree of (almost) all partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. Now, assuming that the tree is carefully defined, if player **II** has a winning strategy in $\mathrm{EF}^\kappa_u(\mathcal{A},\mathcal{B})$ for some tree u, then it is possible to construct an order preserving function from u to t using the winning strategy of player **II**: going through all possible games in which player **I** goes up u, look at how player **II** goes up the isomorphism tree and define the function accordingly.

2.5 The Ordering of the Equivalence Relations

In section *Generalized Baire and Cantor Spaces* above it was defined what it means for an equivalence relation E_0 to be Borel reducible to an equivalence relation E_1. Given any class $\mathcal{E}$ of equivalence relations on 2^κ, a legitimate question goes: "What kind of an ordering $\langle \mathcal{E}, \leqslant_B\rangle$ is?"

The model theoretic part of the work is interested in the case where $\mathcal{E}$ is the set of all isomorphism relations on model classes of various theories. Our contribution to that question is somewhat roughly and incompletely explained in the sections above.

But these contributions do not tell us much about the structure of this ordering. Is there any hope of finding long chains, not only of isomorphism relations, but even of any Σ^1_1-relations whatsoever? Can we generalize well known theorems from classical descriptive set theory such as the Glimm-Effros dichotomy and the Silver dichotomy? In case $\kappa = \omega$ it is known that the ordering of Borel equivalence relations $\langle \mathcal{E}^B_\omega, \leqslant_B\rangle$ is very complicated: it contains a copy of the ordering of Borel subsets of the reals ordered by inclusion (Adams-Kechris 2000). An older result by Louveau and Velickovic from 1994 tells us that it contains a copy of the power set of ω ordered by inclusion modulo bounded sets.

The proofs of these theorems are not generalizable to the case $\kappa > \omega$ (at least we didn't see them to be), because they rely a lot on the induction principle on natural numbers, ergodic theory, measure theory or even computability theory. The inductive proofs either fail at limit ordinals or are based for example on the usage of regressive functions which are not supposed to be constant on a large set, so anyone in the know realizes that the generalizations are hopeless.

2.5.1 On the Silver Dichotomy

The Silver Dichotomy for a class of equivalence relations $\mathcal{E}$ containing the identity relation, states that if an equivalence relation $E \in \mathcal{E}$ has more than κ equivalence classes, then the identity relation id is reducible to it. Our account on that issue is summarized below. Recall that $\cong_T^\kappa$ is the isomorphism relation of the models of T of size κ seen as a relation on 2^κ via coding.

- Suppose $\mathcal{E} = \{\cong_T^\kappa \mid T$ is countable complete FO-theory$\}$. If κ is inaccessible, then the Silver Dichotomy for $\mathcal{E}$ holds, Theorem 4.37. The proof uses stability theory. If the theory is not classifiable, we use a similar argument as that which allowed us to reduce the equivalence modulo a version of a non-stationary ideal to $\cong_T^\kappa$ for successor κ. If it is classifiable, then, once the number of models is greater than κ, the depth of the theory is of necessity greater than 1. This allows us construct primary models $\mathcal{A}_S$ for each $S \subset \kappa$ such that $\mathcal{A}_S$ cannot be isomorphic to $\mathcal{A}_{S'}$, if $S \triangle S'$ is stationary (roughly similar argumentation as in the above section *Unclassifiable Theories* by looking at "filtrations").
- There are theories on the edge: theories whose isomorphism relation is bireducible with the identity, see Theorems 4.38 and 4.39.
- Suppose $\mathcal{E}$ is the set of Borel equivalence relations. Then it is consistent that the Silver Dichotomy fails for $\mathcal{E}$. The counter example is constructed from a Kurepa tree, which is a closed subset of 2^κ, still being of cardinality between κ and 2^κ, or a version of that.

2.5.2 Above Borel

As pointed out above, we didn't find it useful to try to generalize the proofs of Velickovic-Louveau or Adams-Kechris theorems in order to show that $\langle \mathcal{E}, \leqslant_B \rangle$ is complex for some $\mathcal{E}$.

However, adopting other (set theoretical) methods we first proved that if $\mathcal{E}$ is the set of Borel* equivalence relations, then starting from GCH one can force that this ordering contains a copy of the power set of κ ordered by inclusion, Theorem 4.55 page 99. Recall the theorem which says that the equivalence modulo the non-stationary ideal is not reducible to the isomorphism relation of a classifiable theory (Theorem 4.81) whose proof was explained above under the caption *Classifiable Theories*. The proof here is similar. Only now we take a stationary set S and declare $\eta \subset \kappa$ and $\xi \subset \kappa$ equivalent, if $(\eta \triangle \xi) \cap S$ is non-stationary. Denote this equivalence relation by N_S (that is not how it is denoted in the text).

Now, if S and S' are sufficiently different stationary sets (satisfy some non-reflecting requirements), then we can use similar idea as in the proof of Theorem 4.81 to show that $N_S \not\leqslant_B N_{S'}$. On the other hand, if $S \subset S'$, our relation N_S is easily seen to be reducible to $N_{S'}$.

I said *similar idea as in the proof of Theorem 4.81*. But in that proof the idea was based on the fact that the other equivalence relation had in some sense more forcing absoluteness than the other, so that we could falsify the reduction by a forcing argument. But now both relations are equally non-absolute. The trick is that we choose our forcing always depending on S and S' and put all our effort to make the forcing change N_S but preserve $N_{S'}$. This certainly makes the proof much more complicated and factually it is almost five pages longer.

The same idea is used then to show that it is consistent that the equivalence relations modulo λ-stationary ideals are all incomparable to each other, where λ runs through all regular cardinals below κ, Theorem 4.59, page 104.

On the other hand, the existence of a certain diamond sequence implies a converse, namely that the equivalence relation modulo μ_1-non-stationary ideal is reducible to the equivalence relation modulo μ_2-non-stationary ideal when $\mu_1 < \mu_2 < \kappa$ are regular. Thus the consistency of a weakly compact cardinal (which guarantees the needed diamond) implies that it is consistent that the equivalence relation modulo the ω-non-stationary ideal is continuously reducible to the equivalence relation modulo ω_1-non-stationary ideal on 2^{ω_2} and some related results, see Theorem 4.58.

All equivalence relations so far are not Borel, because they contain some version of the equivalence modulo the non-stationary ideal which cannot be Borel by Theorem 4.53, page 99..

2.5.3 Borel Equivalence Relations

Finally, in Chapter 5 the answer to the question concerning the complexity of $\langle \mathcal{E}, \leqslant_B \rangle$ is improved. It is shown that the power set of κ ordered by inclusion modulo the ω-non-stationary ideal can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$, the order of Borel equivalence relations ordered by Borel reductions. This result holds in ZFC, assuming as always $\kappa^{<\kappa} = \kappa > \omega$. Further results are proved with some extra assumptions. If $\square_\lambda$ holds and κ is the successor of λ, then $\mathcal{P}(\kappa)$ ordered by inclusion modulo the non-stationary ideal can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$. If κ is not a successor of an ω-cofinal cardinal or else $\kappa = \omega_1$ and $\diamondsuit_{\omega_1}$ holds, then $\mathcal{P}(\kappa)$ ordered by inclusion modulo bounded sets can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$.

Prior to the appearance of these ideas, it was observed by T. Hyttinen and S. D. Friedman, that the Glimm-Effros dichotomy fails for $\kappa > \omega$ in the sense that there exists a Borel equivalence relation not reducible to the identity but to which the equivalence relation modulo bounded sets cannot be embedded either. This is strengthened in Chapter 5, because all relations in the ranges of the embeddings described above are strictly between the identity and E_0, the equivalence relation modulo bounded sets.

For $\eta, \xi \in 2^\kappa$, let $\eta \triangle \xi$ be the function in 2^κ such that for all $\alpha < \kappa$, $(\eta \triangle \xi)(\alpha) = 0 \iff \eta(\alpha) = \xi(\alpha)$. For each set $S \subset \kappa$ define the equivalence relation E_S as follows: $\eta, \xi \in 2^\kappa$ are E_S-equivalent, if and only if for all ordinals $\alpha \in S \cup \{\kappa\}$ there exists $\beta < \alpha$ such that $(\eta \triangle \xi)(\gamma)$ has the same value for all $\gamma \in (\beta, \alpha)$, and if $\alpha = \kappa$, the value is 0 (Definition 5.19).

The rough idea is that we want to show that

1. if $S' \setminus S$ is stationary, then $E_S \not\leqslant_B E_{S'}$ and
2. if $S' \setminus S$ is non-stationary, then $E_S \leqslant_B E_{S'}$.

If we proved this, then the function $S \mapsto E_S$ would be an embedding from $\mathcal{P}(\kappa)$ into the Borel equivalence relations and would preserve the reverse ordering modulo the non-stationary ideal. Moreover, by taking $S' = \varnothing$, we get from (2) that $E_S \leqslant_B E_0$, since $E_0 = E_\varnothing$. On the other hand by (1), $E_0 \not\leqslant_B E_S$ for all stationary S and the identity relation reduces to each E_S via the same reduction as the identity is normally reduced to E_0.

Well, item (1) can indeed be proved with "stationary" replaced by "ω-stationary", that is Theorem 5.27.1a for $\lambda = \omega$. The idea is as follows. Suppose that there is a continuous reduction f from E_S to $E_{S'}$. The proof for a Borel reduction uses precisely the same argument using the existence of a co-meager set in which f is continuous, but care should be taken in order to hit that co-meager set. (This is done in detail in the actual proof of Theorem 5.27.) If $p \in 2^\alpha$, $\alpha < \kappa$, let $p^\frown\bar{1}$ denote the function $\eta \in 2^\kappa$ such that $\eta \restriction \alpha = p$ and $\eta(\beta) = 1$ for all $\beta \geqslant \alpha$ and similarly $p^\frown\bar{0}$. Player **I** and **II** play the cub-game of length ω, see section *Cub-games* on page 15. Player **I** wins if they hit an element of $S' \setminus S$. Let us define a strategy for player **II**. At each move n she defines elements $p^0_n \in 2^{\gamma_n}$, $p^1_n \in 2^{\gamma_n}$, $q^0_n \in 2^{\gamma'_n}$ and $q^1_n \in 2^{\gamma'_n}$ as follows.

At even moves n she puts $\eta^0 = p^0_{n-1}{}^\frown\bar{0}$ and $\eta^1 = p^1_{n-1}{}^\frown\bar{1}$. Now η^0 and η^1 are not E_S-equivalent, so she can find, by continuity of f, a $\gamma_n > \gamma'_{n-1}$ and q^0_n and q^1_n with $\operatorname{dom} q^0_n = \operatorname{dom} q^1_n = \gamma'_n$ such that for some $\gamma'_{n-1} < \beta < \gamma'_n$, $q^0_n(\beta) \neq q^1_n(\beta)$ and $f[N_{p^0_n}] \subset N_{q^0_n}$ and $f[N_{p^1_n}] \subset N_{q^1_n}$, where N_p is the basic open set determined by p. After she has completed that, she replies in the cub game by γ'_n.

At odd moves n she puts $\eta^0 = p^0_{n-1}{}^\frown\bar{0}$ and $\eta^1 = p^1_{n-1}{}^\frown\bar{0}$. Now η^0 and η^1 are E_S-equivalent, so she can find, by continuity of f, a $\gamma_n > \gamma'_{n-1}$ and q^0_n and q^1_n with $\operatorname{dom} q^0_n = \operatorname{dom} q^1_n = \gamma'_n$ such that for some $\gamma'_{n-1} < \beta < \gamma'_n$, $q^0_n(\beta) = q^1_n(\beta)$ and $f[N_{p^0_n}] \subset N_{q^0_n}$ and $f[N_{p^1_n}] \subset N_{q^1_n}$. After she has completed that, she replies in the cub-game by γ'_n.

Denote this strategy by σ.

Now player **I** takes an ordinal α^* from $S' \setminus S$ that is closed under σ. This is possible, because the set of ordinals that are closed under σ is cub and $S' \setminus S$ is stationary. In that way player **I** can win the game by playing towards that chosen ordinal. During the game player **II** has constructed elements $p^0 = \bigcup_{n<\omega} p^0_n$, $p^1 = \bigcup_{n<\omega} p^1_n$, $q^0 = \bigcup_{n<\omega} q^0_n$ and $q^1 = \bigcup_{n<\omega} q^1_n$ such that $\operatorname{dom} p^0 = \operatorname{dom} p^1 = \operatorname{dom} q^0 = \operatorname{dom} q^1 = \alpha^*$, p^0 and p^1 take cofinally same and different values as well as q^0 and q^1 take cofinally same and different values. Additionally $f[N_{p^0}] \subset N_{q^0}$ and $f[N_{p^1}] \subset N_{q^1}$, but this is a contradiction, because p^0 and p^1 can be extended to E_S-equivalent elements, since $\alpha^* \notin S$, but q^0 and q^1 cannot be extended to $E_{S'}$-equivalent elements, since $\alpha^* \in S'$.

However item (2) cannot be proved in its present form. The relations need to be modified first. That is why we define a product of two equivalence relations on page 149.

Using this method $\mathcal{P}(\kappa)$ modulo the λ-non-stationary ideal can be embedded into Borel relations, provided GC_λ-characterization holds (the cub-game characterization of λ-stationary sets). So when we embed $\mathcal{P}(\kappa)$ modulo the general non-stationary ideal, more work is needed.

In order to reduce the problem to fixed cofinalities, we split stationary set S into parts of fixed cofinalities. The idea is to take the sum (a disjoint union, Definition 5.26) of the corresponding equivalence relations. Let us call the equivalence relations that form the sum *building blocks* and if the building block corresponds to, say cofinality λ, call it *building block of cofinality* λ.

Before we take the sum, we have to make sure that the building blocks of coordinates of different cofinalities cannot be reduced to each other. This is done by adding (taking a union with) ω-stationary test sets, so that they are disjoint for different cofinalities. This raises the problem of what should be done with the building blocks of cofinality ω and this problem is solved by taking products of relations in an appropriate way, see the equation on page 156.

Since we are assuming in that proof that κ is not inaccessible, if $S' \setminus S$ is stationary, then there is a cofinality λ in which $S' \setminus S$ is stationary. The λ-cofinal building block cannot be reduced to other than the λ-cofinal building block, because of the test sets and neither it can be reduced to the λ-cofinal building block by the λ-stationarity of $S' \setminus S$. Therefore the building block of cofinality λ cannot be reduced to any coordinate. However it is conceivable that it can be reduced to the sum of products in some other nasty way, but we show that at least on some non-meager set the building block has to be reduced fully to some other building block and this is enough to carry out the contradiction described above.

2.6 Summary

Historically, millennia ago, the real line was but an abstract yardstick to measure nature. Nowadays it also codes classes of countable groups and orderings, it gives differential structures to manifolds, hosts probability distributions and forcing notions, serves as a building block to a vast majority of applied mathematical models, gives us an intuition of the infinity and large cardinals and exploits the transcendental limits of our understanding.

No matter where we grasped our motivation to study the "uncountable version" of the reals, as John von Neumann puts, there might be surprises:

> A large part of mathematics which becomes useful [is] developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so.

On one hand this work continues a long standing tradition of searching for invariants of model classes or proving that certain invariants cannot exist. The first paper, Chapter 3, is wholly dedicated to such an invariant; it tells how strong that invariant is, how weak it is, how it differs from the other known invariants and what are its boundaries. The second paper, Chapter 4, draws a connection between the model theoretical invariant searching and the descriptive set theory of generalized Baire spaces whose development started twenty years ago. Finally, Chapters 4 and 5 drive further the set theory of the generalized Baire and Cantor spaces. Since most of the proofs of the standard descriptive set theory do not generalize to this context, we had to look at the questions with a fresh attitude. In particular we have found some new proofs for some classical theorems and those proofs do generalize; on the other extreme we have falsified many generalization attempts, such as the Silver dichotomy, see section *Failures of Silver's Dichotomy*, page 88.

On the other hand this thesis has a potential to give a basis for a *new* research tradition. The picture of the generalized descriptive theory has been made clearer; some questions that were obvious to ask are now answered and new questions that haven't been asked before are found. It is, if only a little, clearer now, which directions of this research area are promising and which on contrary less so.

The next major step on the side of model theory would be to understand better the ordering $\leqslant_B$ in the set of the isomorphism relations of countable complete first-order theories on models of some fixed cardinality. This could greatly improve and refine our understanding of model theory and more generally, why some problems are easier than others. Our contribution here is that this ordering is at least in harmony with the well established principles of stability theory and is worth looking at. The dividing line between classifiable and unclassifiable theories, the non-stationary ideal (see section *Model Theory*, page 21), can be seen as a set theoretic strengthening of Shelah's Main Gap Theorem [39].

The set theoretic questions are countless. What else can we learn about the ordering of the equivalence relations? What dichotomies are there? Despite that the obvious generalization of the Glimm-Effros dichotomy fails, maybe there is another equivalence relation so that if E_0 is replaced by it, then a dichotomy holds? What about the complexity hierarchy? What happens if $2^\omega > \kappa$? Are there other important implications than the model theoretic ones?

A Personal Remark

Now, as this work is complete and I look back, I see that this process was of great impact on me.

Although far from all results being mine, I learned a lot from comprehending, processing and putting them onto paper. Never before have I practiced anything as intensely nor imagined that so much is possible to learn and understand.

The skill that I practiced is the skill of abstract thinking. It was a difficult psychological process which gave awesome results. It is like developing a sixth sense; with this sense I can now reliably look at abstract mathematical objects, probe them, modify them, discard them or develop them.

3

Weak Ehrenfeucht-Fraïssé Games

The argument "I may be dreaming" is senseless for this reason: if I am dreaming, this remark is being dreamed as well – and indeed it is also being dreamed that these words have any meaning.

Ludwig Wittgenstein

3.1 Introduction

Abstract

In this paper we define a game which is played between two players **I** and **II** on two mathematical structures $\mathcal{A}$ and $\mathcal{B}$. The players choose points from both structures in α moves and in the end of the game the player **II** wins if the chosen structures are isomorphic. Thus the difference of this to the ordinary Ehrenfeucht-Fraïssé game is that the isomorphism can be arbitrary whereas in ordinary EF-game it should be determined by the moves of the players. We investigate determinacy of the weak EF-game for different α (the length of the game) and its relation to the ordinary EF-game.

3.1.1 History and Motivation

The following question arises very often in mathematics: Does a given description of a mathematical structure describe the structure up to isomorphism? Or equivalently: Is the structure satisfying given conditions unique? And if it is unique, can we further weaken the description or the conditions? Or if it is not unique, then how good the description still is? Model theory and mathematical logic in general has a long history in studying these questions, in particular classifying those ways of description which never lead to a unique solution, studying how much information those descriptions provide, studying various equivalence relations between structures which are weaker than (but as close as possible to) isomorphism, constructing strongly equivalent non-isomorphic models and giving methods to establish such weak equivalences between structures, which under some conditions may lead to a unique description.

On the other hand mathematicians often seek for methods to distinguish between structures (invariants), which would be mathematically simple but which would still classify the structures of a certain class well enough. In many cases, for example, isomorphism is too hard an invariant, though it is the best possible for distinguishing structures. If one can show that a strong invariant does not distinguish between structures of a certain class of structures, then one knows that any invariant that would distinguish should be even more powerful.

One of the most celebrated solved problems in this area which was also one of the starting points for further investigation was the Whitehead's problem, which asks whether all Whitehead groups[1] are free abelian. Saharon Shelah proved in 1974 that the answer is independent of ZFC. Similar question that has been studied is whether an almost free (abelian) group is free

[1]A group G is Whitehead, if it is abelian and: For all abelian B and surjective homomorphism $f\colon B \to G$ with $\ker(f) \cong \mathbb{Z}$ there exists a homomorphism $g\colon G \to B$ with $f \circ g = \mathrm{id}_G$

(abelian). An *almost free (abelian) group* is such a group that all its countable subgroups (or more generally all subgroups of size $< \kappa$ for κ an uncountable cardinal) are free (abelian). Many other properties of free and almost free groups are studied in this context; they appear also in the present chapter (Section 3.5.2, page 41).

In the 1950's A. Ehrenfeucht and R. Fraïssé introduced back-and-forth systems and what we know today as Ehrenfeucht-Fraïssé games. They showed that player $\mathbf{II}$ has a winning strategy in this game of length $n < \omega$ on structures $\mathcal{A}$ and $\mathcal{B}$ in a finite vocabulary if and only if the structures satisfy exactly the same first-order formulas of quantifier rank n. Carol Karp proved in 1965 that having a winning strategy (of player $\mathbf{II}$) in EF-game of length ω is equivalent to $L_{\infty\omega}$-equivalence. These characterizations have already proved to be very useful. Instead of having the fact that the structures satisfy the same $L_{\infty\omega}$-formulas which is very subtle and difficult to handle, we have back-and-forth systems or winning strategies, for which things are (almost) always easier to prove and which are intuitive concepts.

In 1977, Kueker introduced countable approximations, which are closely related (as appears in the present article) to EF-games. Kueker studies how much information about a model can we obtain by looking at its countable submodels. It turns out that two structures have a closed unbounded set of isomorphic countable substructures if and only if they are $L_{\infty\omega}$-equivalent which by the above discussion is equivalent to a winning strategy of player $\mathbf{II}$ in the EF-game of length ω.

Kueker's result can be reformulated in terms of games. If one does this reformulation, one notices that the new game played is a natural modification of the EF-game, which at first sight is easier for player $\mathbf{II}$ i.e. provides a weaker equivalence. But as the results show it is not the case (see Theorem 3.17, page 40). This article can be seen as a development of the idea of this new game, generalizing the concept of countable approximations to "uncountable approximations", giving new viewpoints on characterizations of equivalences, introducing new similarity relations between structures and finally constructing models with interesting properties with respect to the given similarities. For example we give a method to construct structures on which the weak game of length κ can be non-determined for certain κ and this method also provides structures with non-reflecting winning strategies (see Section 3.6, page 55).

The authors wish to express their gratitude to Jouko Väänänen who suggested them the topic of the paper.

3.1.2 The Weak Game and a Sketch of the Results.

We introduce a similarity[2] relation on the class of first order L-structures for some (usually relational) vocabulary L. We define a two player game, the *weak Ehrenfeucht-Fraïssé game*, which defines this relation in the same manner as the ordinary Ehrenfeucht-Fraïssé game defines the EF-similarity relations[3]. In the weak Ehrenfeucht-Fraïssé game of length α on structures $\mathcal{A}$ and $\mathcal{B}$ players $\mathbf{I}$ and $\mathbf{II}$ choose points from both structures and in the end player $\mathbf{II}$ wins if and only if the chosen substructures of size $\leqslant |\alpha|$ are isomorphic; notably the isomorphism can be arbitrary to contrast the ordinary EF-game. We denote the weak EF-game of length α on structures $\mathcal{A}$ and $\mathcal{B}$ by $\mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$.

[2] We use the word similarity relation instead of equivalence relation, because not all of them are equivalence relations as shown later in this article.

[3] The relations being "*player* $\mathbf{I}$ *does not have a winning strategy in the EF game between* $\mathcal{A}$ *and* $\mathcal{B}$" and "*player* $\mathbf{II}$ *has a winning strategy in the EF game between* $\mathcal{A}$ *and* $\mathcal{B}$".

In the case of game length ω, the question of whether EF^*_ω is determined and whether it has any difference to the ordinary Ehrenfeucht-Fraïssé game was solved – in a slightly different context and formulation – in [30]. It turns out that a player wins EF_ω if and only if he or she wins EF^*_ω and since EF_ω is determined, also EF^*_ω is determined.

Using this game we are able to generalize Kueker's equivalence relation to longer games. In fact we define two weak games. The other one is denoted EF°. EF° is weaker than EF and EF^* is weaker than EF°. We are more concentrated on studying EF^*, because it has clear model theoretic and set theoretic interpretations (see Theorem 3.12, page 38 and Section 3.5.4 page 47, where a connection to the cub-game is drawn), it is easier to study and most importantly, since the game EF° falls in between of the two other games, many results for EF^* imply results for EF°.

When we say *the* weak EF-game, we mean EF^*. To sum up, we give the following results. If the player X wins the game G if and only if he wins G', we say that these games are equivalent, and if not, we say that they are different. Here X is of course **I** or **II**.

- (Theorem 3.15 on page 38) If $\kappa^{<\lambda} = \kappa$, then $\mathbf{I}\uparrow \mathrm{EF}_\lambda(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{I}\uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$.

- (Theorem 3.17 on page 40) The games EF_ω and EF^*_ω are equivalent.

- (Examples 3.18 and 3.19 pages 40 and 40) If $\omega < \alpha < \omega_1$, then EF^*_α is properly weaker than EF_α.

- (Theorem 3.22 on page 41) It was shown in [35] that it is consistent with ZFC that GCH and EF_{ω_1} is determined on structures of size $\leqslant \aleph_2$. This implies (using 3.15 page 38) that it is consistent that all the games EF_{ω_1}, $\mathrm{EF}^\circ_{\omega_1}$ and $\mathrm{EF}^*_{\omega_1}$ are equivalent on structures of size $\leqslant \aleph_2$ and are all determined.

- (Theorems 3.28 and 3.29 on pages 42 and 43) Assuming $\Box_{\omega_1}$ in [35] groups $\mathcal{F}$ and $\mathcal{G}$ of cardinality $\aleph_2$ were constructed such that $\mathrm{EF}_{\omega_1}(\mathcal{F},\mathcal{G})$ is not determined. On these structures $\mathrm{EF}^*_{\omega_1}$ is determined and **II** wins. It is easy to generalize to $\Box_\kappa$ and EF_κ, EF^*_κ.

- (Theorems 3.30, 3.31, 3.34, 3.33) Using these structures $\mathcal{F}$ and $\mathcal{G}$ we can construct structures $\mathcal{F}'$, $\mathcal{G}'$, $M(\mathcal{F})$ and $M(\mathcal{G})$ (under GCH all are of cardinality $\aleph_2$) such that $\mathrm{EF}_{\omega_1}(\mathcal{F}',\mathcal{G}')$ is non-determined, but player **II** wins $\mathrm{EF}^\circ_{\omega_1}(\mathcal{F}',\mathcal{G}')$; the game $\mathrm{EF}^\circ_{\omega_1}(M(\mathcal{F}),M(\mathcal{G}))$ is non-determined, but **II** wins $\mathrm{EF}^*_{\omega_1}(M(\mathcal{F}),M(\mathcal{G}))$.

- (Theorem 3.39) It is consistent with ZFC that there are structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is not determined.

- (Theorem 3.40) In ZFC, there are structures $\mathcal{A}$ and $\mathcal{B}$ (of course bigger than $\aleph_2$) such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined.

- (Example 3.20 and theorems 3.41, 3.42) In ZFC there are such structures that player **II** has a winning strategy in $\mathrm{EF}^*_\beta(\mathcal{A},\mathcal{B})$ but not in $\mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$, where $\alpha < \beta$ are ordinal numbers. It is consistent with ZFC that the above holds with α and β being both cardinals.

3.2 Definitions

In this paper structures are ordinary structures of a first order vocabulary L unless stated otherwise and are denoted by letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and their domains respectively by A, B, C. Also $\mathrm{dom}(\mathcal{A})$ is the domain of $\mathcal{A}$. If $f\colon X \to Y$ is a function, we denote $X = \mathrm{dom}(f)$ the domain of f, $f[A]$ or fA the image of a set $A \subset X$ as well as $f^{-1}B = f^{-1}[B]$ the inverse image of a set $B \subset Y$. Range is denoted $\mathrm{ran}(f) = f[X]$.

3.1 Definition. A *game* $G_\gamma(S)$ consists of a set S, game length γ (an ordinal) and a winning set $W \subset (S \times S)^\gamma$. It is played between two players, **I** (he) and **II** (she). On the move $\beta < \gamma$ player **I** chooses $a_\beta \in S$ and then **II** chooses $b_\beta \in S$. Player **II** wins if and only if $(a_i, b_i)_{i<\gamma} \in W$. Otherwise player **I** wins.

3.2 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be structures and γ an ordinal. The *Ehrenfeucht-Fraïssé game* of length γ, $\mathrm{EF}_\gamma(\mathcal{A}, \mathcal{B})$, is played as follows. On the move α, $\alpha < \gamma$, player **I** chooses an element $a_\alpha \in A$ (or $b_\alpha \in B$). Then **II** answers by choosing an element $b_\alpha \in B$ (or $a_\alpha \in A$). **II** wins if the function f, which takes a_α to b_α for each $\alpha < \gamma$ is a partial isomorphism $\mathcal{A} \to \mathcal{B}$. Otherwise player **I** wins.

3.3 Definition. Let $\mathcal{A}$, $\mathcal{B}$ and γ be as in 3.2. The *weak Ehrenfeucht-Fraïssé game* of length γ, $\mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{B})$, is played as follows.

Player I chooses an element $a_\beta \in A \cup B$

Player II chooses an element $b_\beta \in A \cup B$.

Let $X = \{a_\alpha \mid \alpha < \gamma\} \cup \{b_\alpha \mid \alpha < \gamma\}$ be the set of all chosen elements. Player **II** wins if the substructures generated by $X \cap \mathcal{A}$ and $X \cap \mathcal{B}$ are isomorphic. Otherwise **I** wins.

3.4 Definition. The game, which is exactly as in Definition 3.3, but where **II** has to play from the different structure than **I** did on the same move, will be denoted $\mathrm{EF}^\circ_\gamma(\mathcal{A}, \mathcal{B})$.

By *the* weak Ehrenfeucht-Fraïssé game we will refer to the game EF^* defined in 3.3 and by the weak EF-*games* we will refer to both EF^* and EF°.

3.5 Definition. A *strategy* of player **I** in some game $G_\gamma(S)$ is a function $\tau\colon S^{<\gamma} \to S$. A strategy τ of player **I** is *winning* if player **I** always wins the game $G_\gamma(S)$ by playing the element $\tau((b_\alpha)_{\alpha<\beta})$ on the β:th move, where b_α are the elements that player **II** has chosen before the β:th move, for each $\beta < \gamma$.

Note that in the case of Ehrenfeucht-Fraïssé games on structures $\mathcal{A}$ and $\mathcal{B}$, a strategy is a function $\tau\colon (A \cup B)^{<\gamma} \to (A \cup B)$. The concepts of a strategy and a winning strategy are defined analogously for player **II**. A game is said to be *determined* if one of the players has a winning strategy, otherwise not determined or non-determined.

3.6 Definition. Assume that τ is a strategy of player **I** and σ is a strategy of player **II**. Consider the game where **I** uses τ and **II** uses σ. If **II** wins, we say that σ *beats* τ and vice versa.

3.7 Lemma. *A game G is non-determined if and only if for every strategy τ of* **I** *there exists a strategy of* **II** *that beats τ and for every strategy σ of* **II** *there exists a strategy of* **I** *that beats σ.*

Proof. Straight from the definitions. □

Let us introduce some notations that will be used throughout the paper:

$X\uparrow G$	Player X has a winning strategy in the game G.
$\mathcal{A} \cong \mathcal{B}$	$\mathcal{A}$ and $\mathcal{B}$ are isomorphic.
$\mathcal{A} \sim_\gamma \mathcal{B}$	means the same as $\mathbf{II}\uparrow \mathrm{EF}_\gamma(\mathcal{A},\mathcal{B})$.
$\mathcal{A} \sim_\gamma^\circ \mathcal{B}$	means the same as $\mathbf{II}\uparrow \mathrm{EF}_\gamma^\circ(\mathcal{A},\mathcal{B})$.
$\mathcal{A} \sim_\gamma^* \mathcal{B}$	means the same as $\mathbf{II}\uparrow \mathrm{EF}_\gamma^*(\mathcal{A},\mathcal{B})$.

All of the relations, $\sim_\gamma$, $\sim_\gamma^\circ$ and $\sim_\gamma^*$ are equivalence relations on the class of L-structures. It is clear that

$$\mathbf{II}\uparrow \mathrm{EF}_\gamma(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{II}\uparrow \mathrm{EF}_\gamma^\circ(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{II}\uparrow \mathrm{EF}_\gamma^*(\mathcal{A},\mathcal{B})$$

and

$$\mathbf{I}\uparrow \mathrm{EF}_\gamma(\mathcal{A},\mathcal{B}) \Leftarrow \mathbf{I}\uparrow \mathrm{EF}_\gamma^\circ(\mathcal{A},\mathcal{B}) \Leftarrow \mathbf{I}\uparrow \mathrm{EF}_\gamma^*(\mathcal{A},\mathcal{B}).$$

The converses are those which are hard to prove or disprove.

An easy example shows that $\mathrm{EF}_k(\mathcal{A},\mathcal{B})$ and $\mathrm{EF}_k^*(\mathcal{A},\mathcal{B})$ are non-equivalent games for finite $k > 1$.

3.8 Example. Let $A = \mathbb{N}$ and $B = \mathbb{Z}$ equipped with the usual ordering on both. Then **I** wins $\mathrm{EF}_k(\mathcal{A},\mathcal{B})$ by playing first $0 \in \mathbb{N}$ and then $n-1 \in \mathbb{Z}$, where n is the first move by **II**, so $\mathbf{I}\uparrow \mathrm{EF}_k(\mathcal{A},\mathcal{B})$. On the other hand all finite linear orderings are isomorphic if and only if their cardinality is the same. Thus $\mathbf{II}\uparrow \mathrm{EF}_k^\circ(\mathcal{A},\mathcal{B})$ and, $\mathbf{II}\uparrow \mathrm{EF}_k^*(\mathcal{A},\mathcal{B})$. In fact $\mathbf{II}\uparrow \mathrm{EF}_k^*(\mathcal{A},\mathcal{B})$ holds for all $k < \omega$ and linear orders $\mathcal{A}$ and $\mathcal{B}$.

Let us turn now our attention to infinite games. Let κ be a cardinal. Consider the game $\mathrm{EF}_\kappa^*(\mathcal{A},\mathcal{B})$. Let $S = \{X \subset A \cup B \mid |X| \leqslant \kappa,\ X \cap \mathcal{A} \cong X \cap \mathcal{B}\}$. Under the assumption $\kappa^{<\kappa} = \kappa$ player **II** has a winning strategy in $\mathrm{EF}_\kappa^*(\mathcal{A},\mathcal{B})$ if and only if S contains a κ-cub set, and player **I** has a winning strategy if and only if the complement of S, e.g. $[A \cup B]^{<\kappa^+} \setminus S$ contains a κ-cub set. The used concepts will be defined first.

3.9 Definition. Let $(X,<)$ be a partial order. We say that a subset $C \subset X$ is a λ-cub if the following conditions are satisfied:

Closedness Assume that $(c_i)_{i<\lambda}$ is an $<$-increasing chain of elements of C and there exists an element $c \in X$ such that $\forall(i<\lambda)(c_i < c)$ and for all $c' \in X$ if $c' < c$, then $c' < c_i$ for some $i < \lambda$. Then $c \in C$. The element c is called the *supremum* of the chain $(c_i)_{i<\lambda}$.

Unboundedness For each $c \in X$ there exists $c' \in C$ such that $c < c'$.

Notation: $[X]^{<\kappa^+} = \{Y \subset X \mid |Y| < \kappa^+\}$. This is not to be confused with already used $(X)^{<\gamma} = \{f\colon \alpha \to X \mid \alpha < \gamma\}$. The set $[X]^{<\kappa^+} = \{Y \subset X \mid |Y| < \kappa^+\}$ equipped with the proper subset relation $Y < Y' \iff Y \subsetneq Y'$ is a partially ordered set and it is understood what is meant by a λ-cub subset of $[X]^{<\kappa^+}$. A set $C \subset [X]^{<\kappa^+}$ is cub if it is λ-cub for all $\lambda < \kappa^+$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures and let

$$S = \{X \subset A \cup B \mid |X| \leqslant \kappa,\ X \cap \mathcal{A} \cong X \cap \mathcal{B}\} \subset [A \cup B]^{<\kappa^+} \qquad (*).$$

Continuing this approach let us define:

3.10 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be some structures of the same vocabulary and $\lambda, \mu \leqslant \kappa$ non-zero cardinals, the length of the game κ is infinite. Let us define the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}(\mathcal{A}, \mathcal{B})$, which is played between **I** and **II** as follows. On the move $\alpha < \kappa$,

Player I chooses $X_\alpha \subset A \cup B$ such that $|X_\alpha| \leqslant \lambda$ and then

Player II chooses $Y_\alpha \subset A \cup B$ such that $|X_\alpha| \leqslant \mu$

In the end **II** wins if the substructures generated by $A \cap \bigcup_{\alpha<\kappa} X_\alpha \cup Y_\alpha$ and $B \cap \bigcup_{\alpha<\kappa} X_\alpha \cup Y_\alpha$ are isomorphic. Otherwise **I** wins.

In Definition 3.3, EF^*_α was defined for ordinals α. We shall see now that when $\alpha = \kappa$ is an infinite cardinal, the defined games coincide.

3.11 Theorem. *Let λ, μ and κ be non-zero cardinals such that $\lambda, \mu \leqslant \kappa$ and κ infinite. Player* **I** *(**II**) wins the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}(\mathcal{A}, \mathcal{B})$ if and only if he (she) wins the game $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$.*

Proof. Fix a bijective map $f\colon \kappa \times \kappa \to \kappa \setminus \{0\}$ such that for each α we have $f(\alpha, \beta) > \alpha$.

Assume first that **II** has a winning strategy in the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$. Then the strategy of **II** in $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$ is as follows. She imagines that she is playing $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$ against **I**. On each move she chooses $X_\alpha \subset A \cup B$ according to her strategy in the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$, and when he chooses an element $x_\alpha \in A \cup B$, she considers it as the set $\{x_\alpha\}$ being played by **I** in her imaginary game. Also, she enumerates all these sets $X_\alpha = \{x_{\alpha,\beta} \mid \beta < \kappa\}$ (enumeration need not be one-to-one) and on the γ:th move she plays $x_{f^{-1}(\gamma)}$ in the actual game. Thus she eventually picks the same set as she would in $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$.

On the other hand, if **II** wins $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$ the strategy for her in $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$ is a reasoning somewhat converse to the previous: she imagines that they are playing EF^*_κ. Every time he chooses a set $X_\alpha \in A \cup B$, she enumerates it: $X_\alpha = \{x_{\alpha,\beta} \mid \beta < \kappa\}$ and imagines that he played $x_{f^{-1}(\alpha)}$ in the game EF^*_κ and in the actual game she plays $\{x_\gamma\}$, where x_γ is according to the winning strategy in EF^*_κ. Eventually the same sets are enumerated as they were playing the imaginary game of **II**. So the resulting substructures are isomorphic as she used a winning strategy.

The proofs for player **I** are completely analogous. □

Remark. This shows that actually all games $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}(\mathcal{A}, \mathcal{B})$, $\lambda, \mu \leqslant \kappa$ are equivalent to the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\kappa,\kappa}(\mathcal{A}, \mathcal{B})$.

It is also not difficult to see that in $\overset{*}{\mathrm{EF}}{}^{\kappa,\kappa}_{\kappa}(\mathcal{A},\mathcal{B})$ we could require player **II** to choose on each move such an $X \subset A \cup B$ that $X \cap \mathcal{A} \cong X \cap \mathcal{B}$ and it would not change the game (i.e. **II** wins exactly on the same structures as before as well as **I**).

Using this new definition it is easy to see that (recall the definition of S from $(*)$):

3.12 Theorem. *If S (resp. $[A \cup B]^{<\kappa^+} \setminus S$) contains a κ-cub set, then* **II** *(resp.* **I***) has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$. If $\kappa^{<\kappa} = \kappa$, then the converse is also true: if* **II** *(resp.* **I***) wins the game $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$, then S (resp. $[A \cup B]^{<\kappa^+} \setminus S$) contains a κ-cub set.* □

3.13 Corollary. *If* **I** *(resp.* **II***) does not have a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$, then S (resp. $[A \cup B]^{<\kappa^+} \setminus S$) is κ-stationary (intersects all κ-cub sets).* □

3.3 Similarity of EF_κ and EF^*_κ

Since the weak game is easier for the second player, the implications which are shown on the Figure 3.1 are immediately verified.

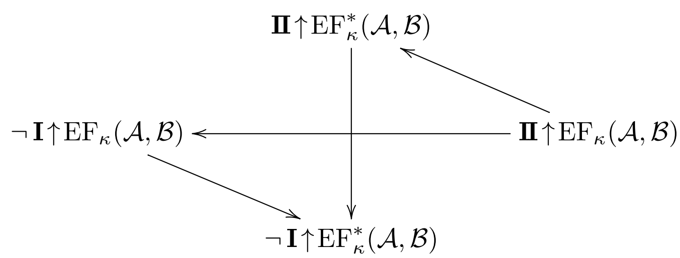


Figure 3.1: Implications that follow directly from the definitions of the games.

One more implication can be proved under $\kappa^{<\kappa} = \kappa$:

3.14 Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures and κ a cardinal such that $\kappa^{<\kappa} = \kappa$. Then* $\mathbf{I}{\uparrow}\mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{I}{\uparrow}\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$.

For later needs we shall prove a slightly more general result:

3.15 Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures, κ a cardinal and α an ordinal such that $\kappa^{<\alpha} = \left|\bigcup_{\beta<\lambda} \kappa^\beta\right| = \kappa$. Then* $\mathbf{I}{\uparrow}\mathrm{EF}_\lambda(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{I}{\uparrow}\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$.

Proof. Assume that $\tau\colon (A \cup B)^{<\alpha} \to (A \cup B)$ is the winning strategy of player **I** in $\mathrm{EF}_\alpha(\mathcal{A},\mathcal{B})$. We now claim that the set

$$W = \{X \in [A \cup B]^{<\kappa^+} \mid X \text{ is closed under } \tau \text{ and } \tau(\varnothing) \in X\} \subset [A \cup B]^{\kappa^+}$$

is κ-cub. To see this, note that:

1. If $X \in [A \cup B]^{\kappa^+}$, then by $\kappa^{<\alpha} = \kappa$ there exist $X' \subset A \cup B$, such that $|X'| = \kappa$, X' is closed under τ and $X \cup \{\tau(\varnothing)\} \subset X'$. So $X < X' \in W$.

2. Assume $(X_\beta)_{\beta<\kappa}$ is increasing and each X_β is closed under τ. To see that $\bigcup_{\beta<\kappa} X_\beta$ is also closed under τ, let $k \in \left(\bigcup_{\beta<\kappa} X_\beta\right)^{<\alpha}$. Then $k \in (X_\beta)^\gamma$ for some $\beta < \kappa$ and $\gamma < \alpha \leqslant \kappa$, but X_β is closed under τ.

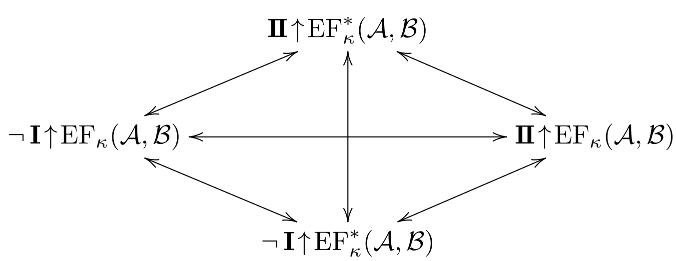
Now it remains to show that if $X \cup Y \in W$ ($X \subset A$, $Y \subset B$) then X and Y cannot be isomorphic. By definition of W the set $X \cup Y$ is closed under τ, the winning strategy of **I** in $\mathrm{EF}_\alpha(\mathcal{A},\mathcal{B})$. If there were an isomorphism $f\colon X \cong Y$, then **II** could win the game $\mathrm{EF}_\alpha(\mathcal{A},\mathcal{B})$ when **I** uses τ: she plays according to the isomorphism f. Note that the first move of **I** $\tau(\varnothing)$ is in $X\cup Y$ again by definition of W, and since W is closed under this strategy, also all subsequent moves are there. A contradiction. So W is a κ-cub set outside the set S of Theorem 3.12.

Now by theorem 3.12 **I** has a winning strategy in the game $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ and so also in the game $\mathrm{EF}^\circ_\kappa(\mathcal{A},\mathcal{B})$. □

3.16 Corollary. *If κ is such that $\kappa^{<\kappa} = \kappa$ and $\mathrm{EF}_\kappa(\mathcal{A},\mathcal{B})$ is determined, then $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ as well as $\mathrm{EF}^\circ_\kappa(\mathcal{A},\mathcal{B})$ are determined and*

$$\mathcal{A} \sim \mathcal{B} \iff \mathcal{A} \sim^\circ \mathcal{B} \iff \mathcal{A} \sim^* \mathcal{B}.$$

Proof. When EF-game is determined, we can add the implication $\neg\,\mathbf{I}\uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \to \mathbf{I}\uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B})$ to the diagram of Figure 3.1 and by theorem 3.15 we can add the implication $\neg\,\mathbf{I}\uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) \to \neg\,\mathbf{I}\uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B})$. After completing all implications which follow by combining th existing ones we obtain:



□

3.4 Countable Games

3.4.1 The Shortest Infinite Game EF^*_ω

Let $S = \{X \subset A\cup B \mid X\cap\mathcal{A} \cong X\cap\mathcal{B} \text{ and } |X| \leqslant \omega\} \subset [A\cup B]^{<\omega_1}$ for some structures $\mathcal{A}$ and $\mathcal{B}$. Recall that $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ means that for all $\varphi \in L_{\infty\omega}$, $\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$. It was proved in [30] (Theorem 3.5) that

(a) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff S$ contains a cub-set

(b) $\mathcal{A} \not\equiv_{\infty\omega} \mathcal{B} \iff [A\cup B]^{<\omega_1} \setminus S$ contains a cub-set.

This can be reformulated by Theorem 3.12 as follows:

(a) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \mathbf{II}\uparrow \mathrm{EF}^*_\omega(\mathcal{A},\mathcal{B})$

(b) $\mathcal{A} \not\equiv_{\infty\omega} \mathcal{B} \iff \mathbf{I}\uparrow \mathrm{EF}^*_\omega(\mathcal{A},\mathcal{B})$

3.17 Corollary. *The games* $\mathrm{EF}^\circ_\omega(\mathcal{A},\mathcal{B})$ *and* $\mathrm{EF}^*_\omega(\mathcal{A},\mathcal{B})$ *are determined for every* $\mathcal{A}$ *and* $\mathcal{B}$ *and*

$$\mathcal{A} \sim_\omega \mathcal{B} \iff \mathcal{A} \sim^\circ_\omega \mathcal{B} \iff \mathcal{A} \sim^*_\omega \mathcal{B}.$$

Proof. Because $\omega^{<\omega} = \omega$, we can apply 3.16. □

3.4.2 Counterexamples for Game Length α, $\omega < \alpha < \omega_1$

As mentioned, the result of Theorem 3.17 does not work for finite ordinals and it does not generally extend for example to ordinals $\omega < \alpha < \omega_1$ either.

3.18 Example. Let $A = B = \omega_1$, R a unary relation such that $R^\mathcal{A} = \omega$, $R^\mathcal{B} = \omega_1 \setminus \omega$. Now clearly $\mathcal{A} \sim_\omega \mathcal{B}$. Also if $\mathbf{I}$ fills the set $\omega \subset A$ during the first ω moves, the second player loses the ordinary EF-game on the next move i.e. $\mathbf{I}\uparrow \mathrm{EF}_{\omega+1}(\mathcal{A},\mathcal{B})$. But $\mathbf{II}$ survives in the weak game. She survives as long as the length of the game is countable, because the only thing she has to do is to choose the same amount of points with properties R and $\neg R$ as $\mathbf{I}$ does.

3.19 Example. Consider the structures constructed in [37]: For $B \subset \omega_1$ let

$$\Phi(B) = \bigcup_{\alpha<\omega_1} \{\alpha\} \times \tau_\alpha,$$

where $\tau_\alpha = 1 + \mathbb{Q}$ if $\alpha \in B$ and $\tau_\alpha = \mathbb{Q}$ if $\alpha \notin B$. The order on Φ is lexicographical, that is $(\alpha, q) < (\beta, p)$ if $\alpha < \beta$ or $\alpha = \beta$ and $q < p$. We set now $\mathcal{A} = \Phi(\varnothing)$ and $\mathcal{B} = \Phi(\omega_1 \setminus \omega)$. The game $\mathrm{EF}_{\omega+2}(\mathcal{A},\mathcal{B})$ is a win for $\mathbf{I}$, which implies the same for $\mathrm{EF}_{\omega+n}(\mathcal{A},\mathcal{B})$, where $n \geqslant 2$.

On the other hand it is easy to see that $\mathbf{II}\uparrow \mathrm{EF}^*_{\omega+n}(\mathcal{A},\mathcal{B})$.

Another example is given to manifest that player $\mathbf{II}$ can loose a shorter game but win a longer one on the same structures.

3.20 Example. Let $\mathcal{A} = \langle \mathbb{R}, < \rangle$ be the real numbers with the usual ordering and $\mathcal{B}$ with domain $B = \mathbb{R} \times \omega_1$ and lexicographical ordering $((x,\alpha) < (y,\beta) \iff \alpha < \beta \vee (\alpha = \beta \wedge x < y))$. These are dense linear orderings and are EF_ω-equivalent as a simple back-and-forth argument shows, thus $\mathbf{II}\uparrow \mathrm{EF}^*_\omega(\mathcal{A},\mathcal{B})$. However $\mathbf{I}\uparrow \mathrm{EF}^*_{\omega+1}(\mathcal{A},\mathcal{B})$: he can play such that an unbounded set of $\mathcal{A}$ is chosen during the first ω moves. But since any countable subset of $\mathcal{B}$ is bounded, $\mathbf{I}$ can play an upper bound on the last move $\omega + 1$. But when the length of the game is increased again to $\omega + \omega$, $\mathbf{II}$ wins again by picking countable elementarily equivalent substructures. In fact $\mathbf{I}\uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$ for successors $\omega < \alpha < \omega_1$ and $\mathbf{II}\uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$ for limits $\omega \leqslant \alpha < \omega_1$.

3.5 Longer Games

In this section we will show that it is consistent with ZFC that

- EF_{ω_1} and $\mathrm{EF}^*_{\omega_1}$ are equivalent on structures of cardinality $\leqslant \aleph_2$ and are both determined. (This requires the consistency of a weakly compact cardinal)

- there are structures $\mathcal{A}$ and $\mathcal{B}$ such that $|A| = |B| = \aleph_2$ and $\mathcal{A} \not\sim_{\omega_1} \mathcal{B}$ but $\mathcal{A} \sim^*_{\omega_1} \mathcal{B}$.
- there are structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and $\mathcal{B}'$ such that $|A| = |B| = |\mathcal{A}'| = |\mathcal{B}'| = \aleph_2$ and $\mathcal{A} \not\sim_{\omega_1} \mathcal{B}$ but $\mathcal{A} \sim^\circ_{\omega_1} \mathcal{B}$ and $\mathcal{A}' \not\sim^\circ_{\omega_1} \mathcal{B}'$ but $\mathcal{A}' \sim^*_{\omega_1} \mathcal{B}'$.
- there are structures $\mathcal{A}$ and $\mathcal{B}$ such that $|A| = |B| = \aleph_2$ and $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$ is not determined.
- there are structures $\mathcal{A}$ and $\mathcal{B}$ and cardinals $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots$, such that $|A| = |B| = \aleph_{\omega\cdot\omega+1}$, for all $n < \omega$, α_n is regular and β_n is singular and $\mathcal{A} \not\sim^*_{\alpha_n} \mathcal{B}$ but $\mathcal{A} \sim^*_{\beta_n} \mathcal{B}$ for all $n < \omega$.

And finally in ZFC we prove that there are structures $\mathcal{A}$ and $\mathcal{B}$ (of course bigger than $\aleph_2$) such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined.

3.5.1 All Games Can Be Determined on Structures of Size $\aleph_2$

In [24] the following was proved (Corollary 13):

3.21 Theorem. *It is consistent relative to the consistency of a weakly compact cardinal, that CH and the game* $\mathrm{EF}_{\omega_1}(\mathcal{A}, \mathcal{B})$ *is determined for all* $\mathcal{A}$ *and* $\mathcal{B}$ *of cardinality* $\leqslant \aleph_2$. □

3.22 Corollary. *It is consistent relative to the consistency of a weakly compact cardinal that CH and the games* EF_{ω_1} *and* $\mathrm{EF}^*_{\omega_1}$ *are equivalent and both games are determined on structures of cardinality* $\aleph_2$.

Proof. By Theorem 3.21 and CH we can use Corollary 3.16 to obtain the result. □

3.5.2 $\mathcal{A} \sim^*_\kappa \mathcal{B} \not\Rightarrow \mathcal{A} \sim_\kappa \mathcal{B}$ on Structures of Size κ^+

Let us fix an uncountable regular cardinal κ. We shall construct groups $\mathcal{F}$ and $\mathcal{G}$ such that $\mathrm{EF}_\kappa(\mathcal{F}, \mathcal{G})$ is non-determined. In fact $\mathcal{F}$ is the free abelian group of cardinality κ^+ and $\mathcal{G}$ will be an almost free abelian group of the same cardinality constructed using the combinatorial principle $\Box_\kappa$. This construction was done in [35] in the case $\kappa = \omega_1$ and is almost identical. The proof that $\mathrm{EF}_\kappa(\mathcal{F}, \mathcal{G})$ is non-determined is exactly the same as is the proof for $\kappa = \omega_1$ in [35]. Formally in this section, these groups will be models of a relational vocabulary.

3.23 Definition. The statement $\Box_\kappa$ says that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa^+, \cup\alpha = \alpha\rangle$ of sets such that

1. C_α is a closed and unbounded subset of α.
2. If $\mathrm{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.
3. If γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$.

For the proof of the next theorem the reader is referred to [25] or to the primary source of this result by Jensen [26].

3.24 Theorem. *If* $V = L$ *then* $\Box_\kappa$ *holds.* □

This square principle, $\Box_\kappa$, implies the existence of a non-reflecting stationary set E on κ^+, which we will use to construct our groups. Recall the notation $S^{\kappa^+}_\omega = \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}$.

3.25 Lemma. *Assume $\square_\kappa$. Then there exists an ω-stationary set $E \subset S^{\kappa^+}_\omega$ such that for every ordinal $\gamma < \kappa^+$ of cofinality κ, the set $E \cap \gamma$ is non-stationary on γ.*

Proof. This is standard and can be found for example in [25]. □

Now we are ready to construct the groups we talked about at the beginning of this section. We shall use some well known facts about free abelian groups, direct products etc. As we already noted, in this section groups will be models of a relational vocabulary. Substructures are not necessarily groups.

As both, $\square_\kappa$ and GCH hold if $V = L$, the use of GCH makes no contradiction. The first group $\mathcal{F}$ will be the free abelian group generated by κ^+:

$$\mathcal{F} = \bigoplus_{i<\kappa^+} \mathbb{Z}.$$

Another group will be a so-called *almost free* abelian group. The idea is that an almost free group G is the union $G = \cup_{i<\kappa^+} G_i$ of its subgroups G_i such that

- Each G_i is free.
- $G_i \subset G_j$ whenever $i < j$
- G is not free.

3.26 Definition. A subgroup S of an abelian group G (write it additively) is *pure* if for all $x \in S$ $(\exists y \in G(ny = x)) \to (\exists y \in S(ny = x))$. That is, if $x \in S$ is divisible in G, it has to be divisible in S.

Let $\mathbb{Z}^{\kappa^+}$ stand for the direct product $\Pi_{\alpha<\kappa^+}\mathbb{Z}$ of κ^+ copies of integers. By x_γ we shall denote the element of $\mathbb{Z}^{\kappa^+}$ which is zero on coordinates $\neq \gamma$ and 1 on the coordinate γ.

For each $\delta \in E$ (of Lemma 3.25) let us fix an increasing cofinal function $\eta_\delta \colon \omega \to \delta$ such that $\eta_\delta[\omega] \cap E = \varnothing$ (for instance take successor ordinals only). Define

$$z_\delta = \sum_{n=0}^{\infty} 2^n x_{\eta_\delta(n)} \in \mathbb{Z}^{\kappa^+}.$$

For each $\alpha \leqslant \kappa^+$ let $\mathcal{G}_\alpha$ be the smallest pure subgroup of $\mathbb{Z}^{\kappa^+}$ which contains the set $\{x_\gamma \mid \gamma < \alpha\} \cup \{z_\delta \mid \delta \in E \cap \alpha\}$. We set $\mathcal{G} = \mathcal{G}_{\kappa^+}$. Let also $\mathcal{F}_\alpha$ be the free abelian group generated by $\{x_\gamma \mid \gamma < \alpha\}$ and set $\mathcal{F} = \mathcal{F}_{\kappa^+}$. We shall denote by $\langle y_\alpha \mid \alpha < \beta\rangle$ the group generated by the set $\{y_\alpha \mid \alpha < \beta\}$.

The proof of the following lemma and the following theorem are exactly as in [35], ω_1 changed to κ.

3.27 Lemma. *For each $\alpha < \kappa^+$ the group $\mathcal{G}_\alpha$ is free and if $\beta \in \alpha \setminus E$, then any free basis of $\mathcal{G}_\beta$ can be extended to a free basis of $\mathcal{G}_\alpha$.* □

3.28 Theorem. *If $\square_\kappa$ and GCH, in particular if $V = L$, then* $\mathrm{EF}_\kappa(\mathcal{F}, \mathcal{G})$ *is not determined.*

Remark. GCH can be avoided, see [35].

Proof. (Sketch.) Player **I** does not win: The set $S = \{\alpha \mid E\cap\alpha \text{ is non-stationary.}\}$ is stationary. Given a strategy τ of **I**, the set $\{\alpha \mid \mathcal{F}_\alpha \cup \mathcal{G}_\alpha \text{ is closed undet } \tau\}$ intersects S being cub and there is an isomorphism $\mathcal{F}_\alpha \cong \mathcal{G}_\alpha$. So **II** just follows the isomorphism.

Player **II** doe not win: Assume that σ is a winning strategy of player **II**. Player **I** takes an $\alpha \in E$ such that $\mathcal{F}_\alpha \cup \mathcal{G}_\alpha$ is closed under first ω moves of **II**. In those first ω moves player **I** picks $\{x_{\eta_\alpha(n)} \mid n < \omega\}$ and a direct summand of $\mathcal{F}_\alpha$. Let J be the set played so far in $\mathcal{G}_\alpha$. In the next ω moves **I** picks the smallest pure subgroup of $\mathcal{G}$ containing $J \cup \{z_\delta\}$. Denote it by A. Now A/J is not a free group, but the corresponding structure K/I in $\mathcal{F}$ (I are the first ω moves in $\mathcal{F}$ and K are the first $\omega+\omega$ moves) is free. In the ordinary EF-game the isomorphism has to respect the order of moves, hence a contradiction. □

3.29 Theorem. *Player* **II** *wins* $\mathrm{EF}^*_\kappa(\mathcal{F},\mathcal{G})$.

Proof. Recall Theorem 3.11, page 37. In the game $\overset{*}{\mathrm{EF}}{}^{1,\kappa}_\kappa$ player **II** can choose on each move the set $\mathcal{F}_\beta \cup \mathcal{G}_\beta$, where β is such that all elements played before this move are in $\mathcal{F}_\beta \cup \mathcal{G}_\beta$. Eventually substructures $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are picked at the end of the game. By Lemma 3.27 they are isomorphic. □

3.5.3 $\mathcal{A} \sim^*_\kappa \mathcal{B} \not\Rightarrow \mathcal{A} \sim^\circ_\kappa \mathcal{B}$ and $\mathcal{A} \sim^\circ_\kappa \mathcal{B} \not\Rightarrow \mathcal{A} \sim_\kappa \mathcal{B}$ if $|\mathcal{A}| = |\mathcal{B}| = \kappa^+$

Here we shall show that all these games can be different on structures of size κ^+. GCH is assumed in all parts and κ is a regular uncountable cardinal.

To prove that EF°_κ is different from EF_κ, we use a vocabulary with function symbols.

$\mathcal{A} \sim^\circ_\kappa \mathcal{B}$ Does Not Imply $\mathcal{A} \sim_\kappa \mathcal{B}$

In this section we will use groups as models of a functional vocabulary. Thus instead of relation $+_R$ we have function symbols $+$ and $-$ whose interpretations satisfy $+(x,y) = z \iff (x,y,z) \in +_R$ etc.

3.30 Theorem. *Let $\mathcal{F}'$ and $\mathcal{G}'$ be the groups constructed in the previous section presented with function symbols $+$, $-$. Then $\mathrm{EF}_\kappa(\mathcal{F}',\mathcal{G}')$ is non-determined.*

Proof. The same reason as why $\mathrm{EF}_\kappa(\mathcal{F},\mathcal{G})$ is non-determined. □

3.31 Theorem. *Let $\mathcal{F}'$ and $\mathcal{G}'$ be the groups constructed in the previous paragraph presented with function symbols $+$, $-$. Then player* **II** *wins* $\mathrm{EF}^\circ_\kappa(\mathcal{F}',\mathcal{G}')$.

Proof. Note that now any substructure is a subgroup. Let us provide a winning strategy for **II** by induction. Assume that on the move α the position of the game is such that the players chose $X \subset \mathcal{F}'$ and $Y \subset \mathcal{G}'$ and the subgroups $\langle X\rangle$ and $\langle Y\rangle$ are isomorphic. Assume that **I** picks next $x \in \mathcal{F}'$. Dimension of a free abelian group is the cardinality of the basis. Note that it is unique, and in the case of abelian groups the dimension of a subgroup is always less or equal to the dimension of the supergroup. If

$$\dim\langle X \cup \{x\}\rangle > \dim\langle X\rangle,$$

then obviously

$$\dim\langle X \cup \{x\}\rangle = \dim\langle X\rangle + 1$$

wherefore let **II** pick an element $y \in \mathcal{G}'$ such that

$$\dim\langle Y \cup \{y\}\rangle = \dim\langle X \cup \{x\}\rangle$$

(it is possible since X and Y are still subsets of $\mathrm{dom}(\mathcal{F}')$ and $\mathrm{dom}(\mathcal{G}')$ of size κ, while $|\mathrm{dom}(\mathcal{F}')| = |\mathrm{dom}(\mathcal{G}')| = \kappa^+$). On the other hand, if x is such that $\dim\langle X \cup \{x\}\rangle = \dim\langle X\rangle$, then we have three cases:

C1: $\dim\langle X\rangle < \omega$. **II** has to pick an element, which is already in $\langle Y\rangle$.

C2: $\dim\langle X\rangle \geqslant \omega$ and $x \in \langle X\rangle$. **II** has to pick an element, which is already in $\langle Y\rangle$.

C3: $\dim\langle X\rangle \geqslant \omega$ and $x \notin \langle X\rangle$. **II** has to pick an element, which is in $\mathcal{G}' \setminus \langle Y\rangle$.

If **I** picks an element from $\mathcal{G}'$ instead of $\mathcal{F}'$, the reasoning for player **II** would be exactly the same with the structures switched.

This strategy guarantees that at each move the groups generated by the played sequences remain isomorphic and simultaneously it guarantees that if **I** picks at the end of the game κ points from one of the structures, then the same amount is picked from the other one and moreover the chosen groups are isomorphic, because their sets of generators are of the same cardinality. □

Thus $\mathcal{F}' \sim_\kappa^\circ \mathcal{G}'$, however by Theorem 3.30, we have $\mathcal{F}' \not\sim_\kappa \mathcal{G}'$. Thus the intended result is proved.

$\mathcal{A} \sim_\kappa^* \mathcal{B}$ Does Not Imply $\mathcal{A} \sim_\kappa^\circ \mathcal{B}$

Let us consider two structures, $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$ is non-determined, but $\mathbf{II} \uparrow \mathrm{EF}_\kappa^*(\mathcal{A}, \mathcal{B})$. Using these structures, we shall construct new structures $M(\mathcal{A})$ and $M(\mathcal{B})$ such that $\mathrm{EF}_\kappa^\circ(M(\mathcal{A}), M(\mathcal{B}))$ is non-determined but $\mathbf{II}\uparrow\mathrm{EF}_\kappa^*(M(\mathcal{A}), M(\mathcal{B}))$. Such structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality κ^+ were constructed in the previous section, thus we can assume that $\mathcal{A} = \mathcal{F}$ and $\mathcal{B} = \mathcal{G}$ (the free and almost free abelian groups of cardinality κ^+). Under GCH, we will have $|M(\mathcal{A})| = |M(\mathcal{B})| = \kappa^+$.

3.32 Definition. Let $\mathcal{A}$ be an L-structure. Let

$$L^+ = L \cup \{<\} \cup \{P_\alpha \mid \alpha < \kappa,\ P_\alpha \text{ is a unary relation symbol}\},$$

where the new symbols are not in L. See remark in the end of this section for how to get rid of an infinite vocabulary. We define $M(\mathcal{A})$ to be the L^+-structure with the domain

$$\mathrm{dom}(M(\mathcal{A})) = \{f\colon \alpha + 1 \to A \mid \alpha < \kappa\}$$

and if $f_i \in \mathrm{dom}(M(\mathcal{A}))$, $i < n$ and R is an n-place relation symbol of the vocabulary, we define

$$(f_0, \ldots, f_{n-1}) \in R^{M(\mathcal{A})} \iff (f_0(\alpha_0), \ldots, f_{n-1}(\alpha_{n-1})) \in R^{\mathcal{A}},$$

where α_i is the maximum of the domain of f_i. The partial order $f < g$ is defined for $f, g \in M(\mathcal{A})$ such that $f <^{M(\mathcal{A})} g$ if $f \subset g$, that is $g \restriction \mathrm{dom}(f) = f$. The relations P_α are interpreted as $P_\alpha^{M(\mathcal{A})} = \{f \mid \mathrm{dom}\, f = \alpha + 1\}$.

Note that if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then $M(\mathcal{A})$ and $M(\mathcal{B})$ are isomorphic. Also if $(f_i)_{i<\alpha}$ is an increasing chain, then the reduction of the substructure $\{f_i \mid i < \alpha\} \subset M(\mathcal{A})$ to L is isomorphic to the substructure $\{f_i\big(\max(\text{dom}(f_i))\big) \mid i < \alpha\} \subset \mathcal{A}$. But if we have a chain $\{f_i \mid i < \alpha\}$ in $M(\mathcal{A})$ and another chain $\{g_i \mid i < \alpha\}$ in $M(\mathcal{B})$, then if there is an isomorphism $\{f_i \mid i < \alpha\} \to \{g_i \mid i < \alpha\}$, then it has to be order preserving.

We claim now that player **II** does not win $\text{EF}^{\circ}_{\kappa}(M(\mathcal{A}), M(\mathcal{B}))$.

3.33 Theorem. *Player* **II** *does not have a winning strategy in* $\text{EF}^{\circ}_{\kappa}(M(\mathcal{A}), M(\mathcal{B}))$.

Proof. Assume that σ is a winning strategy of **II**. Player **I** will play so that the played elements form a $<$-chain. This will force σ to do the same: if on some move **II** plays such that the chosen elements of say $M(\mathcal{A})$ fail to form a chain, the chosen elements of $M(\mathcal{B})$ still form a chain and **I** will play all subsequent moves in $M(\mathcal{B})$ continuing that chain. Apparently, in the end, the structures will not be isomorphic with respect to $<$. Also, if player **I** plays an element f on the move α, then $\text{dom}(f) = \alpha + 1$. This forces **II** to do the same because of the unary relations P_α, $\alpha < \kappa$.

Now player **I**, as playing $\text{EF}^{\circ}_{\kappa}(M(\mathcal{A}), M(\mathcal{B}))$, imagines that they are playing the game $\text{EF}_{\kappa}(\mathcal{A}, \mathcal{B})$: whenever **II** picks $f \in M(\mathcal{A})$ or $M(\mathcal{B})$, he imagines that she played $f(\max \text{dom}(f))$ from $\mathcal{A}$ or $\mathcal{B}$. Let τ be a strategy of **I** that wins the game $\text{EF}_{\kappa}(\mathcal{A}, \mathcal{B})$ (strategy of **II** is fixed by σ). He will pick elements according to this strategy except that he interprets them as functions in the structures $M(\mathcal{A})$ and $M(\mathcal{B})$ in the way described above.

Because τ wins in $\text{EF}_{\kappa}(\mathcal{A}, \mathcal{B})$, the chosen structures are not isomorphic by the isomorphism which respects the order of moves. But the order of moves is the same as that induced by the ordering in $M(\mathcal{A})$ and $M(\mathcal{B})$. □

However it is necessary for **I** to be able to choose from which structure to play:

3.34 Theorem. *Player* **II** *has a winning strategy in* $\text{EF}^{*}_{\kappa}(M(\mathcal{A}), M(\mathcal{B}))$.

Proof. Again, the only thing we use about $\mathcal{A}$ and $\mathcal{B}$ is that $\text{EF}_{\kappa}(\mathcal{A}, \mathcal{B})$ is non-determined but $\mathbf{II}\!\uparrow\!\text{EF}^{*}_{\kappa}(\mathcal{A}, \mathcal{B})$.

If $X \subset \mathcal{A} \cup \mathcal{B}$, let

$$N(X) = \{f \in M(\mathcal{A}) \cup M(\mathcal{B}) \mid \text{ran}\, f \subset X\}$$

and if $Y \subset M(\mathcal{A}) \cup M(\mathcal{B})$, then

$$N^{-1}(Y) = \{x \in \mathcal{A} \cup \mathcal{B} \mid x \in \text{ran}\, f \text{ for some } f \in Y\}.$$

Realize that for all $X, X' \subset \mathcal{A} \cup \mathcal{B}$, $Y, Y' \subset M(\mathcal{A}) \cup M(\mathcal{B})$ we have

- $|X| \leqslant \kappa \iff N(X) \leqslant \kappa$
- $N(N^{-1}(Y)) \supset Y$
- $N^{-1}(N(X)) = X$
- $N(X \cap \mathcal{A}) = N(X) \cap M(A)$ and $N(X \cap \mathcal{B}) = N(X) \cap M(B)$
- $X \cong X' \iff N(X) \cong N(X')$.

By 3.12 it is enough to show that there is an κ-cub set

$$C \subset S = \{X \subset M(\mathcal{A}) \cup M(\mathcal{B}) \mid X \cap M(\mathcal{A}) \cong X \cap M(\mathcal{B}),\, |X| \leqslant \kappa\}.$$

We know that $S' = \{X \subset \mathcal{A} \cup \mathcal{B} \mid X \cap \mathcal{A} \cong X \cap \mathcal{B},\, |X| \leqslant \kappa\}$ contains a cub set. Let it be denoted by C'. We claim that the set

$$C = \{Y \subset M(\mathcal{A}) \cup M(\mathcal{B}) \mid Y = N(X),\, X \in C'\}$$

is cub and contained in S. Because $X \cong Y \Rightarrow N(X) \cong N(Y)$, it is clear that $C \subset S$. Let us show that it is cub.

Let $Y \in C$. Then there is $X \in C'$ such that $X \supset N^{-1}(Y)$. Then $N(X) \supset N(N^{-1}(Y)) \supset Y$. And on the other hand, because $X \cap \mathcal{A} \cong X \cap \mathcal{B}$, we get

$$N(X) \cap M(\mathcal{A}) = N(X \cap \mathcal{A}) \cong N(X \cap \mathcal{B}) = N(X) \cap M(\mathcal{B}).$$

Thus C is unbounded.

Assume that $(Y_i)_{i<\kappa} = (N(X_i))_{i<\kappa}$ is an increasing chain in C. Then X_i is in fact an increasing chain in C'. Thus we know $\bigcup_{i<\kappa} X_i \in C'$. But then $N\left(\bigcup_{i<\kappa} X_i\right) \in C$ and it easy to see that

$$N\left(\bigcup_{i<\kappa} X_i\right) = \bigcup_{i<\kappa} N(X_i).$$

It is easy to see because the functions have always a domain of cardinality less than κ, so if $f \in N\left(\bigcup_{i<\kappa} X_i\right)$, then surely $f \in N\left(\bigcup_{i<\alpha} X_i\right)$ for some $\alpha < \kappa$ and since the chain is increasing this implies $f \in X_\alpha$. □

Remark. We used an uncountable vocabulary L^+ as the vocabulary of $M(A)$ and $M(B)$ because we wanted to fix the levels of the $<$-tree. However we can do that by only a finite extension of the vocabulary assuming that κ is a successor cardinal. By Theorem 0.4 of Chapter VIII of [39] if T is not a superstable theory, then there are models $\mathcal{A}_i$ of T, $i < 2^\kappa$ such that $|\mathcal{A}_i| = \kappa$ for all i and for all distinct indices i, j the model $\mathcal{A}_i$ cannot be elementarily embedded in $\mathcal{A}_j$.

Because the theory of dense linear orderings without end points is unstable and has quantifier elimination, there are 2^κ (we need only κ) linear orderings of cardinality κ such that they are pairwise non-embeddable to each other. Let $\{Q_i \mid i < \kappa\}$ be a collection of such linear orderings. Let L, $\mathcal{A}$ and $\mathcal{B}$ be as in the beginning of this section and define $L^+ = L \cup \{<, <^*, R\}$, where the new symbols are binary relations. Let $M(\mathcal{A})$ and $M(\mathcal{B})$ be the structures defined in this section except that without the relations P_α. Let us now define $M'(\mathcal{A})$ ($M'(B)$ is similar). The domain is the disjoint union

$$\mathrm{dom}(M'(\mathcal{A})) = \mathrm{dom}(M(\mathcal{A})) \cup \bigcup \{Q_i \mid i < \kappa\}.$$

The symbol $<^*$ is interpreted as the ordering of the linear orderings Q_i and R is interpreted as follows:

$$(f, q) \in R \iff f \in \mathrm{dom}(M(\mathcal{A})) \wedge \mathrm{dom}(f) = i + 1 \wedge q \in Q_i,$$

i.e. we fix the $(i+1)$:st level by the linear ordering Q_i. Now if at any move player **II** plays at a different level than **I**, then he will play the corresponding linear ordering and **II** will not be able to embed it to any other than the same one, thus losing the game.

3.5.4 $\mathrm{EF}^*_{\omega_1}$ Can Be Non-determined on Structures of Size $\aleph_2$

Recall that, by 3.13, page 38, in order to construct $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined, we have to find models $\mathcal{A}$ and $\mathcal{B}$ such that the set $\{X \subset A \cup B \mid X \cap \mathcal{A} \cong X \cap \mathcal{B}\}$ is at least ω_1-bistationary i.e. a stationary set whose complement is also stationary (if CH, then it is enough).

3.35 Definition. Let $\omega \leqslant \lambda \leqslant \alpha < \mu$ be such that λ and μ are regular cardinals and α an ordinal. Then let $S \subset \mu$. The cub-game $G^\alpha_\lambda(S)$ is the following game played by players **I** and **II**. On the move $\gamma < \alpha$ first player **I** picks $x_\gamma \in \mu$, such that x_γ is greater than any element played so far in the game and then player **II** chooses $y_\gamma \in S$ such that $y_\gamma > x_\gamma$. Finally sequences $(x_\gamma)_{\gamma<\alpha}$ and $(y_\gamma)_{\gamma<\alpha}$ are formed. Player **II** wins if

(1) she has played according to the rules and

(2) $\mathrm{cl}_\lambda\{y_\gamma \mid \gamma < \alpha\} \subset S$,

Where $\mathrm{cl}_\lambda B$ is the smallest λ-closed set which contains B.

More on these games, see [22] and [16].

Let us consider the following construction. Let μ be an uncountable cardinal and $S \subset S^\mu_\omega$. In the following $\mu \times \omega$ is equipped with reversed lexicographical order and pr_1 and pr_2 are projections respectively onto μ and ω. Then let

$$
\begin{aligned}
A(\mu,S) \quad = \{f\colon \alpha+1 \to \quad & \mu\times\omega \mid \alpha<\mu,\\
& f \text{ is strictly increasing, according to the reversed alphaberical order}\\
& \text{for each } n<\omega \text{ the set } \mathrm{pr}_1[\mathrm{ran}(f)\cap(\mu\times\{n\})]]\\
& \quad \text{is } \omega\text{-closed in } \mu \text{ and is contained in } S\}
\end{aligned}
$$

and

$$
\begin{aligned}
B(\mu,S) \quad = \{f\colon \alpha+1 \to \quad & \mu\times\omega \mid \alpha<\mu,\\
& f \text{ is strictly increasing,}\\
& \text{for each } n<\omega \text{ the set } \mathrm{pr}_1[\mathrm{ran}(f)\cap(\mu\times\{n\})]]\\
& \quad \text{is } \omega\text{-closed as a subset of } \mu \text{ and } \textbf{if } \ n>0, \textbf{ then} \text{ is contained in } S\}.
\end{aligned}
$$

The structures $\mathcal{A}(\mu,S)$ and $\mathcal{B}(\mu,S)$ are L-structures with universes $A(\mu,S)$ and $B(\mu,S)$, $L = \{\leqslant\}$ and $f \leqslant g \iff f \subset g$. Their cardinality is $2^{<\mu}$. In $\mathcal{B}(\mu,S)$ there is a branch which goes through the tree, it consists of the functions $f\colon \alpha+1 \to \mu\times\omega$ such that $f(\beta) = (\beta,0)$. Let us denote such function by $\mathrm{id}_{\alpha+1}$, it is an element of $\mathcal{B}(\mu,S)$.

Because we need to mark the levels, we will temporarily add μ unary relation symbols to the vocabulary $\{P_\alpha \mid \alpha<\mu\}$ and interpret them to fix the levels:

$$P_\alpha^{\mathcal{A}(\mu,S)} = \{f \in A(\mu,S) \mid \mathrm{dom}(f) = \alpha+1\}$$

and

$$P_\alpha^{\mathcal{B}(\mu,S)} = \{f \in B(\mu,S) \mid \mathrm{dom}(f) = \alpha+1\}.$$

In the end we will show how this can be avoided and done with a finite vocabulary. The idea is of the same nature as that of Theorems 3.33, 3.34 and the remark which followed.

The idea here is that the structures $\mathcal{A}(\mu, S)$ and $\mathcal{B}(\mu, S)$ are trees and the subtrees $\mathcal{A}_\alpha = \{f \in \mathcal{A} \mid \operatorname{ran}(f_1) \subsetneq \alpha\}$ and $\mathcal{B}_\alpha = \{f \in \mathcal{B} \mid \operatorname{ran}(f_1) \subsetneq \alpha\}$ are isomorphic if and only if $\alpha \cap S$ contains a cub. If S is complicated enough we get structures on which $\mathrm{EF}^*_{\omega_1}$ is not determined.

3.36 Theorem. *Let $\mu > \omega_1$ and $S \subset S^\mu_\omega$. If player* **I** *does not have a winning strategy in $G^{\omega_1}_\omega(S)$ and S contains arbitrarily long ω-cub sets, then he does not have one in* $\mathrm{EF}^*_{\omega_1}\big(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S)\big)$.

Remark. The existence of an arbitrarily long cub sets means that for every $\alpha < \mu$, $\operatorname{cf}(\alpha) \geqslant \omega_1$ there exists a subset of S which is ω-closed and of order type α. Using the cub game and the fact[4], that player **I** does not have a winning strategy in the games $G^\alpha_\omega(S)$ for $\alpha < \mu$, $\alpha \geqslant \omega_1$, we can find ordinals $\alpha \in \mu$ such that there is an ω-cub set of order type α in $S \cap \alpha$.

Proof. If $f\colon \gamma \to \mu \times \omega$, denote by $f_1 = \mathrm{pr}_1 \circ f$ and $f_2 = \mathrm{pr}_2 \circ f$. Also for simplicity denote $\mathcal{A} = \mathcal{A}(\mu, S)$ and $\mathcal{B} = \mathcal{B}(\mu, S)$,

$$\mathcal{A}_\alpha = \{f \in \mathcal{A} \mid \operatorname{ran}(f_1) \subsetneq \alpha\}$$

and similarly

$$\mathcal{B}_\alpha = \{f \in \mathcal{B} \mid \operatorname{ran}(f_1) \subsetneq \alpha\}.$$

First we prove two claims. A map $g\colon \alpha \to \alpha$ is ω-continuous if for every increasing sequence $(x_k)_{k<\omega}$ in α $g(\cup_{k<\omega} x_k) = \cup_{k<\omega} g(x_k)$. Thus the image of such a function is ω-closed. Define $\mathfrak{C}$ to be the set of such functions h:

$$\mathfrak{C} = \{h\colon \alpha \to S \cap \alpha \mid \alpha \in S, h \text{ is } \omega\text{-continuous increasing and unbounded}\}$$

and

$$\mathfrak{C}_\alpha = \{h \in \mathfrak{C} \mid \operatorname{dom}(h) < \alpha\}$$

Claim 1: For each $h \in \mathfrak{C}$ with $\operatorname{dom}(h) = \alpha$. there exists an isomorphism $F_h\colon \mathcal{A}_\alpha \cong \mathcal{B}_\alpha$ in such a way that if $h \subset h'$, then $F_h \subset F_{h'}$.

Proof of Claim 1. Let $h\colon \alpha \to S \cap \alpha$ be as in the assumption. Then in particular h is an order isomorphism $\alpha \to h[\alpha]$ and the former is an ω-closed unbounded subset of α. Hence we can write h^{-1} for the inverse $h[\alpha] \to \alpha$. For defining the isomorphism $F_h\colon \mathcal{A}_\alpha \to \mathcal{B}_\alpha$, let $f \in \mathcal{A}_\alpha$ be arbitrary, say $f\colon \delta \to S \times \omega$, $\delta < \alpha$. Put

$$\beta_f = \min\{\beta \mid f(\beta) \notin h[\alpha] \times \{0\}\} \cup \{\delta\}.$$

Now for all $\gamma < \beta_f$ let $F_h(f)(\gamma) = (h^{-1}(f_1(\gamma)), 0)$ and for all $\gamma \geqslant \beta_f$ define

$$F_h(f)(\gamma) = \begin{cases} (f_1(\gamma), f_2(\gamma) + 1), & \text{if } f_1(\beta_f) \notin h[\alpha], \\ (f_1(\gamma), f_2(\gamma)) = f(\gamma), & \text{if } f_1(\beta_f) \in h[\alpha]. \end{cases}$$

Clearly $F_h(f) \in \mathcal{B}_\alpha$ and in fact $F_h(f)\colon \delta \to \alpha \times \omega$ (same domain as that of f). We will show that F_h is an isomorphism.

[4] if **I** has a winning strategy in a game of length α, he has one also in the game of length $\operatorname{cf}(\alpha)$, see [22] and for more detailed approach part 2 of the proof of theorem 4.3 of [16].

(1) F_h is one-to-one and onto. It suffices to define a working inverse map. Here we go: Let $g \in \mathcal{B}_\alpha$ be arbitrary, $g\colon \delta \to \mu \times \omega$. Let $\beta_0 = \min\{\beta \mid g_2(\beta) \neq 0\} \cup \{\delta\}$ and let $F^{-1}(g) = f\colon \delta \to S \times \omega$ be such that

$$f(\gamma) = \begin{cases} h(g(\gamma)), & \text{if } \gamma < \beta_0, \\ g(\gamma), & \text{if } \gamma \geqslant \beta_0 \text{ and } g_1(\beta_0) \in h[\alpha], \\ (g_1(\gamma), g_2(\gamma) - 1), & \text{if } \gamma \geqslant \beta_0 \text{ and } g_1(\beta_0) \notin h[\alpha], \end{cases}$$

It is not difficult to check that $f \in \mathcal{A}_\alpha$ and $F_h(f) = g$.

(2) F_h preserves ordering and relations P_α. For the P_α it is already mentioned, that $\mathrm{dom}(f) = \mathrm{dom}(F_h(f))$. Assume $f \leqslant g$. If $\beta_g \geqslant \mathrm{dom}(f)$, then for all $\gamma < \mathrm{dom}(f)$ we have $F_h(f)(\gamma) = h^{-1}(f(\gamma)) = h^{-1}(g(\gamma)) = F_h(g)(\gamma)$, thus $F_h(f) \leqslant F_h(g)$. So assume then $\beta_g < \mathrm{dom}(f)$, in which case $\beta_f = \beta_g$ and $f_1(\beta_f) \in h[\alpha] \iff g_1(\beta_g) \in h[\alpha]$. Hence clearly $F_h(f)(\gamma) = F_h(g)(\gamma)$ whenever $\beta_f \leqslant \gamma < \mathrm{dom}(f)$. The case $\gamma < \beta_f$ as above.

By (1) and (2) F_h is an isomorphism.

Assume that $h \subset h'$. Then by definition $F_{h'\restriction \mathrm{dom}\, h} = F_h$, so the claim follows. □

Claim 2: Let $h \in \mathfrak{C}$ and $\gamma \geqslant \mathrm{dom}(h)$. Then there exists $h' \in \mathfrak{C}$, which extends h and $\gamma \leqslant \mathrm{dom}(h')$.

Proof of Claim 2. Denote $\alpha = \mathrm{dom}\, h$ and let β be such that

- $\beta > \gamma$
- $\mathrm{cf}(\beta) = \omega_1$,
- There is an ω-cub-set $W \subset S \cap \beta$ of order type β,
- $h \in \mathfrak{C}_\beta$.

This is possible by the assumption of the theorem. Assume $\eta\colon \beta \to W$ is an ω-continuous order isomorphism. Let $\alpha_0 = \min(W \setminus \gamma)$ and

$$\alpha_{n+1} = \eta(\alpha_n) \text{ and } \alpha_\omega = \cup_{n<\omega}\alpha_n.$$

Then $\eta \restriction (\alpha, \alpha_\omega)$ is a function from (α, α_ω) to $W \cap (\alpha, \alpha_\omega)$. Thus we can define

$$h' = h \cup \{\alpha, \alpha\} \cup \eta \restriction (\alpha, \alpha_\omega).$$

Then $h'\colon \alpha_\omega \to S \cap \alpha_\omega$ (note, that because $h \in \mathfrak{C}$, $\alpha = \mathrm{dom}\, h \in S$) and $h' \in \mathfrak{C}_\beta$. □

Let us define a function $K(\gamma)\colon h \mapsto h'$, where $h' = h$ if $\gamma < \mathrm{dom}\, h$ and if $\gamma \geqslant \mathrm{dom}\, h$, then h' is obtained from h using Claim 2 and choice.

Let now τ be any strategy of player **I** in $\overset{*}{\mathrm{EF}}{}^{\omega_1,\omega_1}_{\omega_1}(\mathcal{A}, \mathcal{B})$. For simplicity let us assume without loss of generality that $\tau(\langle X_i\rangle_{i<\beta}) \subset \tau(\langle X_i\rangle_{i<\alpha})$, whenever $\beta < \alpha$.

Recall that $[A \cup B]^{<\mu} = \{F \subset A \cup B \mid |F| < \mu\}$. Define a function $G\colon [A \cup B]^{<\mu} \to \mu$ such that $G(F) = \sup\{\mathrm{ran}(f_1) \mid f \in F\}$

Notation: if $f\colon X \to X$ is a function and $J \subset X$, let $f_{\mathrm{cl}}[J]$ denote the closure of J under f:

$$f_{\mathrm{cl}}[J] = \text{the smallest subset of } X\text{, which contains } J \text{ and is closed under } f.$$

Let τ^* be a strategy of **I** in $G_\omega^{\omega_1}(S)$ which will be defined using τ.

First step:

$$\tau^*(\varnothing) = G(\tau(\varnothing))$$

Next define $\tau^*(\langle y_i\rangle_{i<\alpha})$ for $\alpha = \beta + 1 < \omega_1$, where y_i are answers of **II**:
If $\beta = 0$, then let h_0 be an arbitrary element of $\mathfrak{C}$, such that $y_0 < \operatorname{dom}(h_0)$. Because $y_0 > \tau^*(\varnothing)$ this implies $\tau(\varnothing) \subset \mathcal{A}_{\operatorname{dom}(h_0)} \cup \mathcal{B}_{\operatorname{dom}(h_0)}$. Then (independently of whether $\beta = 0$ or not) define

$$\begin{aligned} X_\beta &= (F_{h_\beta} \cup F_{h_\beta}^{-1})_{\mathrm{cl}}\left[\bigcup_{\delta\leqslant\beta} \tau(\langle X_i\rangle_{i<\delta}) \cup \{\mathrm{id}_{y_\beta}\}\right] \\ \tau^*((y_i)_{i<\alpha}) &= G(\tau(\langle X_i\rangle_{i\leqslant\beta})) \\ h_\alpha &= K(y_\alpha)(h_\beta) \end{aligned}$$

Finally define $\tau^*(\langle y_i\rangle_{i<\alpha})$ for α a limit $< \omega_1$:

$$\begin{aligned} X_\alpha &= \bigcup_{i<\alpha} X_i \cup \{\mathrm{id}_{y_\alpha}\} \\ \tau^*((y_i)_{i<\alpha}) &= G(\tau(\langle X_i\rangle_{i<\alpha}) \\ h_\alpha &= \bigcup_{i<\alpha} h_i \text{ if } \bigcup_{i<\alpha} \operatorname{dom} h_i \in S \text{ i.e. such exists and otherwise arbitrary.} \end{aligned}$$

Let now σ^* be a strategy of **II**, which beats τ^* and finally the strategy σ of **II** in $\overset{*}{\mathrm{EF}}{}_{\omega_1}^{\omega_1,\omega_1}$ is obtained from σ^* by induction as follows:

$$\sigma((X_i)_{i<\alpha}) = X_\alpha \text{ as defined above.}$$

Because σ^* beats τ^*, it is obvious that h_α exists for all limit α, since $\bigcup_{i<\alpha} \operatorname{dom} h_i \in S$. Thus for all $i < \omega_1$ we have $X_i \cap \mathcal{A} \cong X_i \cap \mathcal{B}$ and moreover the isomorphisms extend each other i.e.

$$i < j \Rightarrow X_i \subset X_j \text{ and } F_i \subset F_j,$$

where F_i is the isomorphism between $X_i \cap \mathcal{A} \cong X_i \cap \mathcal{B}$ and F_j is the isomorphism between $X_j \cap \mathcal{A} \cong X_j \cap \mathcal{B}$. Thus σ beats τ and τ is not winning. □

3.37 Theorem. *Let μ be a cardinal, $S \subset S_\omega^\mu$ and $\hat{S} = \{\alpha \in S_{\omega_1}^\mu \mid \alpha \cap S$ contains a cub$\}$. If player* **II** *does not have a winning strategy in*

$$G_{\omega_1}^{\omega_1}(\hat{S}),$$

then she does not have one in $\mathrm{EF}_{\omega_1}^*\big(\mathcal{A}(\mu,S),\mathcal{B}(\mu,S)\big)$.

Proof. Let σ be any strategy of **II** in $\overset{*}{\mathrm{EF}}{}_{\omega_1}^{\omega_1,\omega_1}\big(\mathcal{A}(\mu,S),\mathcal{B}(\mu,S)\big)$. Without loss of generality, assume that whenever a sequence $(E_i)_{i<\gamma}$ is played, it holds that $i < j \to E_i \subset E_j$.

Let C be the cub set $\{\alpha < \mu \mid \forall\beta < \alpha(\beta + \beta < \alpha)\}$. Let $G\colon [A \cup B]^{<\mu} \to \mu$ be as in the proof of the previous theorem and $\hat{G}$ a similar function with a little modification:

$$\hat{G}(F) = \min\{\alpha \in \hat{S} \mid \alpha \geqslant G(F) \wedge \alpha \geqslant \min(C \setminus G(F))\}.$$

In the first part it only matters that $\hat{G}(F) \in \hat{S}$ and $\hat{G}(F) \geqslant G(F)$.

Let σ^* be the strategy of player **II** in $G^{\omega_1}_{\omega_1}(\hat{S})$ which is obtained from σ and $\hat{G}$ as follows:

$$\sigma^*((\alpha_i)_{i<\gamma}) = \hat{G}\big(\sigma((\underbrace{\{\mathrm{id}_{\alpha_i+1}\}}_{\subset\mathcal{B}})_{i<\gamma})\big),$$

i.e. **II** imagines that **I** played the set $\{\mathrm{id}_{\alpha_i+1}\}$ instead of α_i in $G^{\omega_1}_{\omega_1}(\hat{S})$. Let τ^* be the strategy of **I** in $G^{\omega_1}_{\omega_1}(\hat{S})$, which beats σ^*. And then let the strategy τ be such that if $E_i \subset A \cup B$ for each $i < \gamma$ are the moves of **II** in $\overset{*}{\mathrm{EF}}{}^{\omega_1,\omega_1}_{\omega_1}$, then

$$\tau((E_i)_{i<\gamma}) = \{\mathrm{id}_{\beta+1}\} \subset \mathcal{B}, \text{ where } \beta = \tau^*((\hat{G}(E_i))_{i<\gamma})\}.$$

Assume the players picked $X \subset \mathcal{A} \cup \mathcal{B}$. Because τ^* beats σ^*, $X \cap \mathcal{B} \subset \mathcal{B}_{G(X)}$ contains an unbounded branch of length ω_1: $\{\mathrm{id}_{\beta_i+1} \mid i < \omega_1\}$, but there is no unbounded branch of such length in the structure $X \cap \mathcal{A} \subset \mathcal{A}_{G(X)}$ (because there is no ω-cub set in $G(X)$).

It remains to show that the unbounded branch $I = \{\mathrm{id}_{\beta_i+1} \mid i < \omega_1\}$ would be mapped to an unbounded branch by an isomorphism. For a contradiction assume F to be an isomorphism. It preserves levels and the level of id_{β_i+1} is β_i, i.e. $\mathrm{id}_{\beta_i+1} \in P^{\mathcal{B}}_{\beta_i}$. So if $F(\mathrm{id}_{\beta_i+1}) = f_i$, then $\mathrm{dom}(f_i) = \beta_i + 1$. Thus $\beta = \sup\{\mathrm{dom}(f) \mid f \in F[I]\} = \bigcup_{i<\omega_1} \mathrm{dom}(\mathrm{id}_{\beta_i+1}) = \bigcup_{i<\omega_1} \beta_i$ and its cofinality is ω_1. From the definition of $\hat{G}$ it follows that β is in C, hence

$$(\forall\gamma < \beta)(\gamma + \gamma < \beta)$$

and hence if $\bigcup_i \mathrm{pr}_1(\mathrm{ran}(f_i)) < \beta$, then we had an increasing function $\beta \to \alpha$ with $\alpha < \beta$, which is a contradiction. □

By the two theorems above it is enough to find a set $S \subset S^\mu_\omega$ for which

ND1 Player **I** does not have a winning strategy in $G^{\omega_1}_\omega(S)$

ND2 S contains arbitrarily long ω-cub sets.

ND3 Player **II** does not have a winning strategy in $G^{\omega_1}_{\omega_1}(\hat{S})$.

where $\hat{S} = \{\alpha \in S^\mu_{\omega_1} \mid \alpha \cap S \text{ contains a cub}\}$. Then $\mathrm{EF}^*_{\omega_1}(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S))$ is non-determined.

Stationary sets whose complement satisfies ND1 are called strongly bistationary, see [22]. A generic set $S \subset S^{\omega_2}_\omega$ obtained by standard Cohen forcing provides an example of a set, which has intended properties ND1 and ND3. ND2 can then be obtained with the use of the following lemma.

3.38 Lemma. *Let $S \subset \mu$ satisfy the properties ND1 and ND3. Then there exists $S^* \subset \mu$ which satisfies ND1, ND2 and ND3.*

Proof. Let $f\colon \mu \to \mu$ be the continuous map defined as follows:

$$f(0) = 0, \quad f(\alpha+1) = f(\alpha) + \alpha, \quad f(\gamma) = \bigcup_{\alpha<\gamma} f(\alpha), \text{ when } \gamma \text{ is a limit.}$$

This function is clearly continuous. Let

$$S^* = \mu \setminus f[\mu \setminus S],$$

Let us show that S^* has the intended properties ND1-ND3. Note that $f[S] \subset S^*$.

ND1 By the assumption, player **I** does not have a winning strategy in $G^{\omega_1}_{\omega}(S)$. Because $f[S] \subset S^*$, it is enough to show, that **I** does not have a winning strategy in $G^{\omega_1}_{\omega}(f[S])$. Define $f^{-1}\colon \mu \to \mu$ as follows:

$$f^{-1}(x) = \min\{y \in \mu \mid f(y) \geqslant x\}.$$

Let τ be any strategy of **I** in $G^{\omega_1}_{\omega}(f[S])$. Then $\tau^* = f^{-1} \circ \tau \circ f$ is a strategy of **I** in $G^{\omega_1}_{\omega}(S)$. Now by the assumption there is a strategy σ^* of player **II** which beats τ^*. Now $f \circ \sigma^* \circ f^{-1}$ beats τ.

ND2 This is clear from the definitions of S^* and f.

ND3 For any set $A \subset S^{\mu}_{\omega}$ denote $A^* = \mu \setminus f[\mu \setminus A]$ and $\hat{A} = \{\alpha \in S^{\mu}_{\omega_1} \mid \alpha \cap A \text{ contains a cub}\}$. Then because f is one-to-one and continuous, we have that

$$(\hat{S})^* = \widehat{(S^*)}.$$

Then a similar deduction as for ND1 from the fact that ND3 holds for S follows.

□

Notation. If $(A, <)$ is a well order, or A is a subset of an ordinal with the induced ordering, then $\mathrm{OTP}(A)$ means the ordinal order isomorphic to $(A, <)$, the *order type*.

3.39 Theorem. *It is consistent that there are structures of cardinality $\aleph_2$ such that the game $\mathrm{EF}^*_{\omega_1}$ is non-determined.*

Proof. Forcing with $\{p\colon \alpha \to \omega_2 \mid \alpha < \omega_2\}$ starting with ground model in which GCH holds, gives a generic set S such that $\{\alpha \in S^{\omega_2}_{\omega} \mid \alpha \cap S \text{ contains cub}\}$ is ω_1-bistationary. Now using GCH it is easy to show the intended properties ND1 and ND3. That for it is enough to note that the sets S and $\{\alpha \mid S \cap \alpha \text{ contains cub}\}$ are bistationary. Then using GCH players can take closures of each others strategies and beat them this way. For ND2 one can simply use Lemma 3.38 but in this case it is not necessary.

The conditions ND1 − ND3 i.e. the assumptions of Theorems 3.36 and 3.37 on pages 48 and 50 are now satisfied. □

3.40 Theorem. *Let $\mu = \max\{(2^{\omega})^+, \omega_4\}$. From ZFC it follows that there are models $\mathcal{A}$ and $\mathcal{B}$ of cardinality $2^{<\mu}$ such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined.*

Proof. It was shown in [3], lemma 7.7, that if $\mu > \omega_3$ (as ours) then there are: a stationary $X \subset S^{\mu}_{\omega_2}$ and sets $D_\alpha \subset \alpha$, for each $\alpha \in X$ such that:

1. D_α is cub in α,
2. $\mathrm{OTP}(D_\alpha) = \omega_2$,
3. if $\alpha, \beta \in X$ and $\gamma < \min\{\alpha, \beta\}$ is a limit of both D_α and D_β, then $D_\alpha \cap \gamma = D_\beta \cap \gamma$.
4. if $\gamma \in D_\alpha$, then γ is a limit point of D_α if and only if γ is a limit ordinal.

Define $X' = X \cup \{\gamma \mid \exists \alpha > \gamma(\gamma \in \lim D_\alpha = \text{ the limit points of } D_\alpha)\}$ and for each β in X' let

$$g(\beta) = \min\{\gamma \in X \mid \gamma \geqslant \beta \wedge \beta \text{ is a limit point of } D_\gamma\} \in X.$$

Clearly if $\beta \in X$, then $g(\beta) = \beta$. Then let

$$C_\beta = \beta \cap \lim D_{g(\beta)}.$$

We now have the coherence property: if $\beta \in C_\alpha$, then $C_\beta = \beta \cap C_\alpha$. Moreover each C_α is closed and if $\mathrm{cf}(\alpha) \geqslant \omega_1$, then it is unbounded in α and $\mathrm{OTP}(C_\alpha) \leqslant \omega_2$. For each $\alpha < \omega_2$ define

- $S_\alpha = \{\beta \in X' \mid \mathrm{OTP}(C_\beta) = \alpha\}$,
- $S_{\geqslant \alpha} = \bigcup_{\alpha \leqslant \beta < \omega_2} S_\beta$.

First we observe that for all $\alpha < \omega_2$, $S_{\geqslant\alpha}$ is ω-stationary and ω_1-stationary. To see this let C be an ω_1-cub set (ω-case is similar). Because X is stationary, there exists a point $\xi \in X \cap \lim C$. Thus now $C \cap \xi$ is cub in ξ. Hence also $C \cap C_\xi$ is cub and its order type is obviously ω_2 ($\xi \in X \subset S^\mu_{\omega_2}$ and $\mathrm{OTP}(C_\xi)$ is at most ω_2). This implies the existence of $\beta \in C_\xi \cap C$ such that C_β is of order type $\geqslant \alpha$ and thus an element of $S_{\geqslant\alpha}$.

Because $S_{\geqslant\alpha}$ is stationary and is a union of ω_2 disjoint sets, one of them must be stationary itself. Thus for every $\alpha < \omega_2$ there exists $\gamma > \alpha$ such that S_γ is ω-stationary.

Now we refer to theorem 3.7 of [22] which states applied to our case:
Let $A \subset S^\mu_\omega$ and assume $A = \bigcup_{i<\omega_2} A_i$, where each A_i is stationary and $A_i \cap A_j = \varnothing$ if $i \neq j$. Then there is an ordinal $j < \omega_2$ such that **I** does not have a winning strategy in $G^{\omega_1}_\omega(S^\mu_\omega \setminus \bigcup_{j \leqslant i < \omega_2} A_i)$.

In our case A_i are those sets $\bigcup_{\gamma_i < \xi \leqslant \gamma_{i+1}} S^\mu_\omega \cap S_\xi$ where $(\gamma_i)_{i<\omega_2}$ is a sequence such that each S_{γ_i} is ω-stationary. There is ω_2 of them as concluded and all disjoint. Let now γ be such that **I** does not have a winning strategy in $G^{\omega_1}_\omega(S^\mu_\omega \setminus S_{\geqslant\gamma})$ and

$$S = S^\mu_\omega \setminus S_{\geqslant\gamma}.$$

The set S clearly satisfies the intended property ND1.

For ND3 we have to show that player **II** does not have a winning strategy in

$$G^{\omega_1}_{\omega_1}(\hat{S}),$$

where $\hat{S} = S \cup \{\alpha \in S^\mu_{\omega_1} \mid \alpha \cap S \text{ contains a cub}\}$. Let us show first that $\{\alpha \in S^\mu_{\omega_1} \mid \alpha \cap S \text{ does not contain cub}\}$ is ω_1-stationary. We know that in the complement of S there is $S_{\geqslant\gamma}$. Let us show that if C is an ω_1-cub, then there is a point $\alpha \in C$ such that $S_{\geqslant\gamma} \cap \alpha$ contains a cub, which is more than enough. Let $\beta \in X \cap \lim C$ and let α be the $(\gamma + \omega_1)$:st element of C_β

and α' the γ:th element. Then all points of $C_\beta \cap [\alpha', \alpha)$ are in $S_{\geqslant\gamma}$, because for these points, say $\delta \in C_\beta \cap [\alpha', \alpha)$, we have $C_\delta = C_\beta \cap \delta$ and it has order type $\geqslant \gamma$. This implies that the set $\{\alpha \in S^\mu_{\omega_1} \mid \alpha \cap S$ does not contain cub$\}$ is stationary.

Assume now that σ is a strategy for **II** in $G^{\omega_1}_{\omega_1}(\hat{S})$. The set

$$R = \{\xi \in \mu \mid \xi \text{ is closed under } \sigma\}$$

is ω_1-cub ($\lambda < \mu \to \lambda^{<\omega_1} < \mu^{<\omega_1} = \mu$). Consequently there is $\alpha \in R \cap \{\beta \in S^\mu_{\omega_1} \mid \beta \cap S$ does not contain cub$\}$. Player **I** can now ensure that they play towards α, so σ cannot be winning. Thus ND1 and ND3 are satisfied and so by Lemma 3.38 page 51 and Theorems 3.36 and 3.37 the game $\mathrm{EF}^*_{\omega_1}(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S))$ is non-determined. □

Remark. In the beginning of this section we promised to show how the vocabulary can be made finite. In order to do this, we have to construct μ structures $(\mathcal{C}_i)_{i<\mu}$ such that for $i \neq j$ $\mathbf{I}\!\uparrow\!\mathrm{EF}^*_{\omega_1}(\mathcal{C}_i, C_j)$ and add these structures to the levels using one binary relation. This replaces the use of a unary relation P_α for each level. During the game player **I** will make sure that if levels α and β are played, then a 'subgame' between $\mathcal{C}_\alpha$ in $\mathcal{A}$ and $\mathcal{C}_\beta$ in $\mathcal{B}$ is played to show that they are different levels. In the end an isomorphism between the picked substructures can only take $\mathcal{C}_\alpha$ in $\mathcal{A}$ to $\mathcal{C}_\alpha$ in $\mathcal{B}$, because it otherwise contradicts the fact that **I** won all those 'subgames'.

It remains to find structures $\mathcal{C}_i$, $i < \mu$ for those μ for which we proved our theorems, i.e. $\mu = \omega_2$ and $\mu = \max\{(2^\omega)^{++}, \omega_4\}$.

In the case $\mu = \omega_2$ just take all dense linear orders of cardinality ω_1. There are 2^ω of them and all different. Because of the small size, also $\mathbf{I} \uparrow \mathrm{EF}^*_{\omega_1}(\mathcal{C}_i, \mathcal{C}_j)$ if $\mathcal{C}_i$ and $\mathcal{C}_j$ are two non-isomorphic representatives.

Assume now that $\mu = \max\{(2^\omega)^{++}, \omega_4\}$. It is enough to show that there are $(2^{\omega_1})^{++} \geqslant \mu$ models for which the intended property holds.

Let the vocabulary consist of four binary relation symbols and one unary relation P:

$$L = \{R, <, <^*, <^\#, P\}.$$

Let $\mathcal{Q}$ be the disjoint set of well orderings $\{\alpha \mid 2^{\omega_1} \leqslant \mathrm{OTP}(\alpha) < (2^{\omega_1})^+\}$ and let $\mathcal{W}$ be the disjoint set of well orderings $\{\alpha \mid (2^{\omega_1})^+ \leqslant \mathrm{OTP}(\alpha) < (2^{\omega_1})^{++}\}$. Disjoint means that $\alpha \cap \beta = \varnothing$ for all distinct elements $\alpha, \beta \in \mathcal{Q}$ or $\mathcal{W}$. We have:

- $\forall \alpha \in \mathcal{Q}(|\alpha| = 2^{\omega_1})$
- $|\mathcal{Q}| = (2^{\omega_1})^+$
- $\forall \alpha \in \mathcal{W}(|\alpha| = (2^{\omega_1})^+)$
- $|\mathcal{W}| = (2^{\omega_1})^{++}$.

For each $\alpha \in \mathcal{Q}$ let $F_\alpha \colon \mathcal{P}(\omega_1) \to \alpha$ be a fixed bijection and for each $i \in \mathcal{W}$ let $G_i \colon i \to \mathcal{Q}$ be another fixed bijection. For each $i \in \mathcal{W}$ define $\mathcal{C}_i$ as follows:

- $\mathrm{dom}(\mathcal{C}_i) = \omega_1 \cup \mathcal{Q}$ (disjoint union).
- $x {<^\#}^{\mathcal{C}_i} y \iff x, y \in \omega_1 \wedge x < y$ (in ω_1)

- $x <^{\mathcal{C}_i} y \iff \exists \alpha \in \mathcal{Q}(x, y \in \alpha) \wedge x < y$ (in α)
- $x <^{*\mathcal{C}_i} y \iff \exists \alpha, \beta \in \mathcal{Q}\big(G_i^{-1}(\alpha) < G_i^{-1}(\beta) \wedge x \in \alpha \wedge y \in \beta\big)$
- $(\alpha, x) \in R^{\mathcal{C}_i} \iff (\exists X \in \mathcal{P}(\omega_1))(\exists \beta \in \mathcal{Q})(\alpha \in X \wedge x \in \beta \wedge F_\beta(X) = x)$
- $P^{\mathcal{C}_i} = \omega_1$

Now we claim that $\mathbf{I} \uparrow \overset{*}{\mathrm{EF}}{}^{\omega_1, \omega_1}_{\omega_1}(\mathcal{C}_i, \mathcal{C}_j)$ (the game, where the players can choose sets of size ω_1, see Theorem 3.11 page 37) whenever $i \neq j$. On the first move player **I** chooses $P^{\mathcal{C}_i} \cup P^{\mathcal{C}_j}$. After that player **I** picks α and β in $\mathcal{Q}$ such that $G_i^{-1}(\alpha) < G_i^{-1}(\beta)$ and $G_j^{-1}(\alpha) > G_j^{-1}(\beta)$, i.e. $x \in \alpha \wedge y \in \beta \Rightarrow x <^* y$ in $\mathcal{C}_i$ and $y <^* x$ in $\mathcal{C}_j$. Such exist, because i and j are non-isomorphic orders. Now player **I** must make sure that if there is an isomorphism between the played substructures in the end, then it takes β in $\mathcal{C}_i$ to β in $\mathcal{C}_j$ and α in $\mathcal{C}_i$ to α in $\mathcal{C}_j$. This will result in a contradiction and there cannot be any isomorphism. Because every order ζ in $\mathcal{Q}$ is different from β (provided of course $\zeta \neq \beta$) the task is easy for player **I**. Every time an element is played from an ordering ζ, player **I** picks two elements $x, y \in \zeta$ and $x', y' \in \beta$ such that $x < y$, $y' < x'$, $F_\zeta^{-1}(x) = F_\beta^{-1}(x')$ and $F_\zeta^{-1}(y) = F_\beta^{-1}(y')$. Because of the relation R it follows that β cannot be mapped to ζ by an isomorphism. Similarly he manages with α.

3.6 Structures with Non-reflecting Winning Strategies

In this section GCH is assumed. Let $\mu = \aleph^+_{\omega\cdot\omega}$. Put $\mathcal{A} = \mathcal{A}(\mu, S)$ and $\mathcal{B} = \mathcal{B}(\mu, S)$, where $S \subset S^\mu_\omega$ is the generic set obtained by Cohen forcing as mentioned in the proof of Theorem 3.39. It has the following property: the set

$$E_\lambda = \{\alpha \in S^\mu_\lambda \mid \alpha \cap S \text{ contains a cub}\} \qquad (***)$$

is λ-bistationary for each regular $\lambda < \mu$.

Let $\alpha_n = \omega_{\omega\cdot n+1}$ (regular) and $\beta_n = \omega_{\omega\cdot(n+1)}$ (singular).

3.41 Theorem. *If $\lambda < \mu$ is regular (for example α_n), then player **II** cannot have a winning strategy in the game $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$.*

Proof. One can show as in theorem 3.37 that it is enough that player **II** does not have a winning strategy in $G^{\alpha_n}_{\alpha_n}(E_{\alpha_n})$ (see $(***)$ above). Let σ be any strategy of **II** in this game. Then the set

$$\{\alpha \in S^\mu_{\alpha_n} \mid \alpha \text{ is closed under } \sigma\}$$

is α_n-cub (by GCH) and thus the complement of E_{α_n} of $(***)$ intersects it being stationary. Player **I** can now easily play towards an element in this intersection. □

3.42 Theorem. *Assume GCH. If $\mathrm{cf}(\lambda) = \omega$, $\lambda < \mu$ (for example $\lambda = \beta_n$), then player **II** has a winning strategy in the game $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$.*

Proof. Let $\eta\colon \omega \to \lambda$ be a cofinal increasing map. As in the proof of Theorem 3.36, page 48, there are isomorphisms $F_\beta\colon \mathcal{A}_\beta \to \mathcal{B}_\beta$ for each β in E_{ω_1}. In the game $\overset{*}{\mathrm{EF}}{}^{1,\lambda}_\lambda$ player **II** will play as follows: assume that X_n is the set of already picked elements. By the methods of the proof of theorem 3.36 she can choose an isomorphism F_{β_n} such that β_n is greater than $\sup\{\operatorname{dom} f \mid f \in X_n\}$ and $F_{\beta_0} \subset F_{\beta_1} \subset \cdots$. Then she chooses the set $(F_{\beta_n} \cup F_{\beta_n}^{-1})[X_n]$. At the end of the game $\cup_{k<\omega} F_{\beta_k}$ should be a partial isomorphism. □

Thus the sequence

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots ,$$

where $\alpha_n = \omega_{\omega \cdot n+1}$ and $\beta_n = \omega_{\omega \cdot (n+1)}$ is such that $\mathcal{A} \not\sim^*_{\alpha_n} \mathcal{B}$ but $\mathcal{A} \sim^*_{\beta_n} \mathcal{B}$.

4

Generalized Descriptive Set Theory and Classification Theory

Your strength as a rationalist is
your ability to be more confused
by fiction than by reality... [He]
was confused. Therefore,
something he believed was fiction.

Eliezer Yudkowski

4.1 History and Motivation

There is a long tradition in studying connections between Borel structure of Polish spaces (descriptive set theory) and model theory. The connection arises from the fact that any class of countable structures can be coded into a subset of the space 2^ω provided all structures in the class have domain ω. A survey on this topic is given in [13]. Suppose X and Y are subsets of 2^ω and let E_1 and E_2 be equivalence relations on X and Y respectively. If $f\colon X \to Y$ is a map such that $E_1(x,y) \iff E_2(f(x),f(y))$, we say that f is a *reduction of* E_1 *to* E_2. If there exists a Borel or continuous reduction, we say that E_1 is Borel or continuously *reducible* to E_2, denoted $E_1 \leqslant_B E_2$ or $E_1 \leqslant_c E_2$. The mathematical meaning of this is that f *classifies* E_1*-equivalence in terms of* E_2*-equivalence.*

The benefit of various reducibility and irreducibility theorems is roughly the following. A reducibility result, say $E_1 \leqslant_B E_2$, tells us that E_1 is at most as complicated as E_2; once you understand E_2, you understand E_1 (modulo the reduction). An irreducibility result, $E_1 \nleqslant_B E_2$ tells that there is no hope in trying to classify E_1 in terms of E_2, at least in a "Borel way". From the model theoretic point of view, the isomorphism relation, and the elementary equivalence relation (in some language) on some class of structures are the equivalence relations of main interest. But model theory in general does not restrict itself to countable structures. Most of stability theory and Shelah's classification theory characterizes first-order theories in terms of their uncountable models. This leads to the generalization adopted in this paper. We consider the space 2^κ for an uncountable cardinal κ with the idea that models of size κ are coded into elements of that space.

This approach, to connect such uncountable descriptive set theory with model theory, began in the early 1990's. One of the pioneering papers was by Mekler and Väänänen [36]. A survey on the research done in 1990's can be found in [50] and a discussion of the motivational background for this work in [49]. A more recent account is given the book [51], Chapter 9.6.

Let us explain how our approach differs from the earlier ones and why it is useful. For a first-order complete countable theory in a countable vocabulary T and a cardinal $\kappa \geqslant \omega$, define

$$S_T^\kappa = \{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models T\} \text{ and } \cong_T^\kappa = \{(\eta,\xi) \in (S_T^\kappa)^2 \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$$

where $\eta \mapsto \mathcal{A}_\eta$ is some fixed coding of (all) structures of size κ. We can now define the partial order on the set of all theories as above by

$$T \leqslant^\kappa T' \iff \cong_T^\kappa \leqslant_B \cong_{T'}^\kappa .$$

As pointed out above, $T \leqslant^\kappa T'$ says that $\cong_T^\kappa$ is at most as difficult to classify as $\cong_{T'}^\kappa$. But does this tell us whether T is a simpler theory than T'? Rough answer: *If* $\kappa = \omega$*, then no but if* $\kappa > \omega$*, then yes.*

To illustrate this, let $T = \mathrm{Th}(\mathbb{Q}, \leqslant)$ be the theory of the order of the rational numbers (DLO) and let T' be the theory of a vector space over the field of rational numbers. Without loss of generality we may assume that they are models of the same vocabulary. It is easy to argue that the model class defined by T' is strictly simpler than that of T. (For instance there are many questions about T, unlike T', that cannot be answered in ZFC; say existence of a saturated model.) On the other hand $\cong^\omega_T \leqslant_B \cong^\omega_{T'}$ and $\cong^\omega_{T'} \not\leqslant_B \cong^\omega_T$ because there is only one countable model of T and there are infinitely many countable models of T'. But for $\kappa > \omega$ we have $\cong^\kappa_T \not\leqslant_B \cong^\kappa_{T'}$ and $\cong^\kappa_{T'} \leqslant_B \cong^\kappa_T$, since there are 2^κ equivalence classes of $\cong^\kappa_T$ and only one equivalence class of $\cong^\kappa_T$.

Another example, introduced in Martin Koerwien's Ph.D. thesis and his article [29] shows that there exists an ω-stable theory without DOP and without OTOP with depth 2 for which $\cong^\omega_T$ is not Borel, while we show here that for $\kappa^{<\kappa} = \kappa > 2^\omega$, $\cong^\kappa_T$ is Borel for all classifiable shallow theories (*shallow* is the opposite of *deep*). The converse holds for all κ with $\kappa^{<\kappa} = \kappa > \omega$: if $\cong^\kappa_T$ is Borel, then T is classifiable and shallow, see Theorems 4.66, 4.71 and 4.72 starting from page 112.

Our results suggest that the order $\leqslant^\kappa$ for $\kappa > \omega$ corresponds naturally to the classification of theories in stability theory: the more complex a theory is from the viewpoint of stability theory, the higher it seems to sit in the ordering $\leqslant^\kappa$ and vice versa. Since dealing with uncountable cardinals often implies the need for various cardinality or set theoretic assumptions beyond ZFC, the results are not always as simple as in the case $\kappa = \omega$, but they tell us a lot. For example, our results easily imply the following (modulo some mild cardinality assumptions on κ):

- If T is deep and T' is shallow, then $\cong_T \not\leqslant_B \cong_{T'}$.
- If T is unstable and T' is classifiable, then $\cong_T \not\leqslant_B \cong_{T'}$.

4.2 Introduction

4.2.1 Notations and Conventions

Set Theory

We use standard set theoretical notation:

- $A \subset B$ means that A is a subset of B or is equal to B.
- $A \subsetneq B$ means proper subset.
- Union, intersection and set theoretical difference are denoted respectively by $A \cup B$, $A \cap B$ and $A \setminus B$. For larger unions and intersections $\bigcup_{i\in I} A_i$ etc..
- The symmetric difference: $A \triangle B = (A \setminus B) \cup (B \setminus A)$
- $\mathcal{P}(A)$ is the power set of A and $[A]^{<\kappa}$ is the set of subsets of A of size $< \kappa$

Usually the Greek letters κ, λ and μ will stand for cardinals and α, β and γ for ordinals, but this is not strict. Also η, ξ, ν are usually elements of κ^κ or 2^κ and p, q, r are elements of

$\kappa^{<\kappa}$ or $2^{<\kappa}$. $\mathrm{cf}(\alpha)$ is the cofinality of α (the least ordinal β for which there exists an increasing unbounded function $f\colon \beta \to \alpha$).

By S^κ_λ we mean $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$. A λ-*cub set* is a subset of a limit ordinal (usually of cofinality $> \lambda$) which is unbounded and contains suprema of all bounded increasing sequences of length λ. A set is *cub* if it is λ-cub for all λ. A set is *stationary* if it intersects all cub sets and λ-*stationary* if it intersects all λ-cub sets. Note that $C \subset \kappa$ is λ-cub if and only if $C \cap S^\kappa_\lambda$ is λ-cub and $S \subset \kappa$ is λ-stationary if and only if $S \cap S^\kappa_\lambda$ is (just) stationary.

If $(\mathbb{P}, \leqslant)$ is a forcing notion, we write $p \leqslant q$ if p and q are in $\mathbb{P}$ and q forces more than p. Usually $\mathbb{P}$ is a set of functions equipped with inclusion and $p \leqslant q \iff p \subset q$. In that case $\varnothing$ is the weakest condition and we write $\mathbb{P} \Vdash \varphi$ to mean $\varnothing \Vdash_{\mathbb{P}} \varphi$. By *Cohen forcing* or *standard Cohen forcing* we mean the partial order $2^{<\kappa}$ of partial functions from κ to $\{0,1\}$ ordered by inclusion, where κ depends on the context.

Functions

We denote by $f(x)$ the value of x under the mapping f and by $f[A]$ or just fA the image of the set A under f. Similarly $f^{-1}[A]$ or just $f^{-1}A$ indicates the inverse image of A. Domain and range are denoted respectively by $\mathrm{dom}\, f$ and $\mathrm{ran}\, f$.

If it is clear from the context that f has an inverse, then f^{-1} denotes that inverse. For a map $f\colon X \to Y$ *injective* means the same as *one-to-one* and *surjective* the same as *onto*.

Suppose $f\colon X \to Y^\alpha$ is a function with range consisting of sequences of elements of Y of length α. The projection pr_β is a function $Y^\alpha \to Y$ defined by $\mathrm{pr}_\beta((y_i)_{i<\alpha}) = y_\beta$. For the coordinate functions of f we use the notation $f_\beta = \mathrm{pr}_\beta \circ f$ for all $\beta < \alpha$.

By support of a function f we mean the subset of $\mathrm{dom}\, f$ in which f takes non-zero values, whatever "zero" means depending on the context (hopefully never unclear). The support of f is denoted by $\mathrm{sprt}\, f$.

Model Theory

In section *Coding Models* on page 66 we fix a countable vocabulary and assume that all theories are theories in this vocabulary. Moreover we assume that they are first-order, complete and countable. By $\mathrm{tp}(\bar a/A)$ we denote the complete type of $\bar a = (a_1, \dots, a_{\mathrm{length}\,\bar a})$ over A where $\mathrm{length}\,\bar a$ is the length of the sequence $\bar a$.

We think of models as tuples $\mathcal{A} = \langle \mathrm{dom}\,\mathcal{A}, P_n^{\mathcal{A}} \rangle_{n<\omega}$ where the P_n are relation symbols in the vocabulary and the $P_n^{\mathcal{A}}$ are their interpretations. If a relation R has arity n (a property of the vocabulary), then for its interpretation it holds that $R^{\mathcal{A}} \subset (\mathrm{dom}\,\mathcal{A})^n$. In section *Coding Models* we adopt more conventions concerning this.

In Sections *The Silver Dichotomy for Isomorphism Relations* (page 81) and *Complexity of Isomorphism Relations* (page 111) we will use the following stability theoretical notions: stable, superstable, DOP, OTOP, shallow and $\kappa(T)$. Classifiable means superstable with no DOP nor OTOP, the least cardinal in which T is stable is denoted by $\lambda(T)$.

Reductions

Let $E_1 \subset X^2$ and $E_2 \subset Y^2$ be equivalence relations on X and Y respectively. A function $f\colon X \to Y$ is *a reduction* of E_1 to E_2 if for all $x, y \in X$ we have that $xE_1y \iff f(x)E_2f(y)$.

Suppose in addition that X and Y are topological spaces. Then we say that E_1 is *continuously reducible to* E_2, if there exists a continuous reduction from E_1 to E_2 and we say that E_1 is *Borel reducible to* E_2 if there is a Borel reduction. For the definition of Borel adopted in this paper, see Definition 4.16. We denote the fact that E_1 is continuously reducible to E_2 by $E_1 \leqslant_c E_2$ and respectively Borel reducibility by $E_1 \leqslant_B E_2$.

We say that relations E_2 and E_1 are (Borel) *bireducible* to each other if $E_2 \leqslant_B E_1$ and $E_1 \leqslant_B E_2$.

4.2.2 Ground Work

Trees and Topologies

Throughout the paper κ is assumed to be an uncountable regular cardinal which satisfies

$$\kappa^{<\kappa} = \kappa \qquad (*)$$

(For justification of this, see below.) We look at the space κ^κ (the generalized Baire space), i.e. the functions from κ to κ and the space formed by the initial segments $\kappa^{<\kappa}$. It is useful to think of $\kappa^{<\kappa}$ as a tree ordered by inclusion and of κ^κ as a topological space of the branches of $\kappa^{<\kappa}$; the topology is defined below. Occasionally we work in 2^κ (the generalized Cantor space) and $2^{<\kappa}$ instead of κ^κ and $\kappa^{<\kappa}$.

4.1 Definition. A *tree* t is a partial order with a root in which the sets $\{x \in t \mid x < y\}$ are well ordered for each $y \in t$. A *branch* in a tree is a maximal linear suborder.

A tree is called *a* $\kappa\lambda$*-tree*, if there are no branches of length λ or higher and no element has $\geqslant \kappa$ immediate successors. If t and t' are trees, we write $t \leqslant t'$ to mean that there exists an order preserving map $f : t \to t'$, $a <_t b \Rightarrow f(a) <_{t'} f(b)$.

Convention. *Unless otherwise said, by a tree* $t \subset (\kappa^{<\kappa})^n$ *we mean a tree with domain being a downward closed subset of*

$$(\kappa^{<\kappa})^n \cap \{(p_0, \dots, p_{n-1}) \mid \operatorname{dom} p_0 = \dots = \operatorname{dom} p_{n-1}\}$$

ordered as follows: $(p_0, \dots, p_{n-1}) < (q_0, \dots, q_{n-1})$ *if* $p_i \subset q_i$ *for all* $i \in \{0, \dots, n-1\}$*. It is always a* $\kappa^+, \kappa+1$*-tree.*

4.2 Example. Let $\alpha < \kappa^+$ be an ordinal and let t_α be the tree of descending sequences in α ordered by end extension. The root is the empty sequence. It is a $\kappa^+\omega$-tree. Such t_α can be embedded into $\kappa^{<\omega}$, but note that not all subtrees of $\kappa^{<\omega}$ are $\kappa^+\omega$-trees (there are also $\kappa^+, \omega+1$-trees).

In fact the trees $\kappa^{<\beta}$, $\beta \leqslant \kappa$ and t_α are universal in the following sense:

4.3 Fact ($\kappa^{<\kappa} = \kappa$). *Assume that* t *is a* $\kappa^+, \beta+1$*-tree,* $\beta \leqslant \kappa$ *and* t' *is* $\kappa^+\omega$*-tree. Then*

1. *there is an embedding* $f : t \to \kappa^{<\beta}$*,*
2. *and a strictly order preserving map* $f : t' \to t_\alpha$ *for some* $\alpha < \kappa^+$ *(in fact there is also such an embedding* f*).* □

Define the topology on κ^κ as follows. For each $p \in \kappa^{<\kappa}$ define the basic open set

$$N_p = \{\eta \in \kappa^\kappa \mid \eta \restriction \mathrm{dom}(p) = p\}.$$

Open sets are precisely the empty set and the sets of the form $\bigcup X$, where X is a collection of basic open sets. Similarly for 2^κ.

There are many justifications for the assumption $(*)$ which will be most apparent after seeing the proofs of our theorems. The crucial points can be summarized as follows: if $(*)$ does not hold, then

- the space κ^κ does not have a dense subset of size κ,
- there are open subsets of κ^κ that are not κ-unions of basic open sets which makes controlling Borel sets difficult (see Definition 4.16 on page 67).
- Vaught's generalization of the Lopez-Escobar theorem (Theorem 4.25, page 71) fails, see Remark 4.26 on page 73.
- The model theoretic machinery we are using often needs this cardinality assumption (see e.g. Theorem 4.31, page 75, and proof of Theorem 4.74, page 117).

Initially the motivation to assume $(*)$ was simplicity. Many statements concerning the space $\kappa^{<\kappa}$ are independent of ZFC and using $(*)$ we wanted to make the scope of such statements neater. In the statements of (important) theorems we mention the assumption explicitly.

Because the intersection of less than κ basic open sets is either empty or a basic open set, we get the following.

Fact ($\kappa^{<\kappa} = \kappa$)**.** *The following hold for a topological space $P \in \{2^\kappa, \kappa^\kappa\}$:*

1. *The intersection of less than κ basic open sets is either empty or a basic open set,*
2. *The intersection of less than κ open sets is open,*
3. *Basic open sets are closed,*
4. $|\{A \subset P \mid A \text{ is basic open}\}| = \kappa$,
5. $|\{A \subset P \mid A \text{ is open}\}| = 2^\kappa$.

In the space $\kappa^\kappa \times \kappa^\kappa = (\kappa^\kappa)^2$ we define the ordinary product topology.

4.4 Definition. A set $Z \subset \kappa^\kappa$ is Σ^1_1 if it is a projection of a closed set $C \subset (\kappa^\kappa)^2$. A set is Π^1_1 if it is the complement of a Σ^1_1-set. A set is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

As in standard descriptive set theory ($\kappa = \omega$), we have the following:

4.5 Theorem. *For $n < \omega$ the spaces $(\kappa^\kappa)^n$ and κ^κ are homeomorphic.* □

Remark. This standard theorem can be found for example in Jech's book [25]. Applying this theorem we can extend the concepts of Definition 4.4 to subsets of $(\kappa^\kappa)^n$. For instance a subset A of $(\kappa^\kappa)^n$ is Σ^1_1 if for a homeomorphism $h\colon (\kappa^\kappa)^n \to \kappa^\kappa$, $h[A]$ is Σ^1_1 according to Definition 4.4.

Ehrenfeucht-Fraïssé Games

We will need Ehrenfeucht-Fraïssé games in various connections. It serves also as a way of coding isomorphisms.

4.6 Definition (Ehrenfeucht-Fraïssé games)**.** Let t be a tree, κ a cardinal and $\mathcal{A}$ and $\mathcal{B}$ structures with domains A and B respectively. Note that t might be an ordinal. The game $\mathrm{EF}_t^\kappa(\mathcal{A},\mathcal{B})$ is played by players **I** and **II** as follows. Player **I** chooses subsets of $A \cup B$ and climbs up the tree t and player **II** chooses partial functions $A \to B$ as follows. Suppose a sequence

$$(X_i, p_i, f_i)_{i<\gamma}$$

has been played (if $\gamma = 0$, then the sequence is empty). Player **I** picks a set $X_\gamma \subset A \cup B$ of cardinality strictly less than κ such that $X_\delta \subset X_\gamma$ for all ordinals $\delta < \gamma$. Then player **I** picks a $p_\gamma \in t$ which is $<_t$-above all p_δ where $\delta < \gamma$. Then player **II** chooses a partial function $f_\gamma \colon A \to B$ such that $X_\gamma \cap A \subset \operatorname{dom} f_\gamma$, $X_\gamma \cap B \subset \operatorname{ran} f_\gamma$, $|\operatorname{dom} f_\gamma| < \kappa$ and $f_\delta \subset f_\gamma$ for all ordinals $\delta < \gamma$. The game ends when player **I** cannot go up the tree anymore, i.e. $(p_i)_{i<\gamma}$ is a branch. Player **II** wins if

$$f = \bigcup_{i<\gamma} f_i$$

is a partial isomorphism. Otherwise player **I** wins.

A *strategy* of player **II** in $\mathrm{EF}_t^\kappa(\mathcal{A},\mathcal{B})$ is a function

$$\sigma\colon ([A\cup B]^{<\kappa} \times t)^{<\mathrm{ht}(t)} \to \bigcup_{I\in[A]^{<\kappa}} B^I,$$

where $[R]^{<\kappa}$ is the set of subsets of R of size $<\kappa$ and $\mathrm{ht}(t)$ is the *height* of the tree, i.e.

$$\mathrm{ht}(t) = \sup\{\alpha \mid \alpha \text{ is an ordinal and there is an order preserving embedding } \alpha \to t\}.$$

A strategy of **I** is similarly a function

$$\tau\colon \Big(\bigcup_{I\in[A]^{<\kappa}} B^I\Big)^{<\mathrm{ht}(t)} \to [A\cup B]^{<\kappa} \times t.$$

We say that a strategy τ of player **I** *beats* strategy σ of player **II** if the play $\tau * \sigma$ is a win for **I**. The play $\tau * \sigma$ is just the play where **I** uses τ and **II** uses σ. Similarly σ beats τ if $\tau * \sigma$ is a win for **II**. We say that a strategy is a *winning strategy* if it beats all opponents strategies.

The notation $X \uparrow \mathrm{EF}_t^\kappa(\mathcal{A},\mathcal{B})$ means that player X has a winning strategy in $\mathrm{EF}_t^\kappa(\mathcal{A},\mathcal{B})$

Remark. By our convention $\operatorname{dom}\mathcal{A} = \operatorname{dom}\mathcal{B} = \kappa$, so while player **I** picks a subset of $\operatorname{dom}\mathcal{A} \cup \operatorname{dom}\mathcal{B}$ he actually just picks a subset of κ, but as a small analysis shows, this does not alter the game.

Consider the game $\mathrm{EF}_t^\kappa(\mathcal{A},\mathcal{B})$, where $|\mathcal{A}| = |\mathcal{B}| = \kappa$, $|t| \leqslant \kappa$ and $\mathrm{ht}(t) \leqslant \kappa$. The set of strategies can be identified with κ^κ, for example as follows. The moves of player **I** are members of $[A\cup B]^{<\kappa} \times t$ and the moves of player **II** are members of $\bigcup_{I\in[A]^{<\kappa}} B^I$. By our convention $\operatorname{dom}\mathcal{A} = \operatorname{dom}\mathcal{B} = A = B = \kappa$, so these become $V = [\kappa]^{<\kappa} \times t$ and $U = \bigcup_{I\in[\kappa]^{<\kappa}} \kappa^I$. By our cardinality assumption $\kappa^{<\kappa} = \kappa$, these sets are of cardinality κ.

Let

$$\begin{aligned} f\colon U &\to \kappa \\ g\colon U^{<\kappa} &\to \kappa \\ h\colon V &\to \kappa \\ k\colon V^{<\kappa} &\to \kappa \end{aligned}$$

be bijections. Let us assume that $\tau\colon U^{<\kappa} \to V$ is a strategy of player **I** (there cannot be more than κ moves in the game because we assumed $\mathrm{ht}(t) \leqslant \kappa$). Let $\nu_\tau\colon \kappa \to \kappa$ be defined by

$$\nu_\tau = h \circ \tau \circ g^{-1}$$

and if $\sigma\colon V^{<\kappa} \to U$ is a strategy of player **II**, let ν_σ be defined by

$$\nu_\sigma = f \circ \sigma \circ k^{-1}.$$

We say that ν_τ *codes* τ.

4.7 Theorem ($\kappa^{<\kappa} = \kappa$). *Let $\lambda \leqslant \kappa$ be a cardinal. The set*

$$C = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \textit{ codes a w.s. of } \mathbf{II} \textit{ in } \mathrm{EF}^\kappa_\lambda(\mathcal{A}_\eta, \mathcal{A}_\xi)\} \subset (\kappa^\kappa)^3$$

is closed. If $\lambda < \kappa$, then also the corresponding set for player **I**

$$D = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \textit{ codes a w.s. of } \mathbf{I} \textit{ in } \mathrm{EF}^\kappa_\lambda(\mathcal{A}_\eta, \mathcal{A}_\xi)\} \subset (\kappa^\kappa)^3$$

is closed.

Remark. Compare to Theorem 4.14.

Proof. Assuming $(\nu_0, \eta_0, \xi_0) \notin C$, we will show that there is an open neighborhood U of (ν_0, η_0, ξ_0) such that $U \subset (\kappa^\kappa)^3 \setminus C$. Denote the strategy that ν_0 codes by σ_0. By the assumption there is a strategy τ of **I** which beats σ_0. Consider the game in which **I** uses τ and **II** uses σ_0.

Denote the γ^{th} move in this game by (X_γ, h_γ) where $X_\gamma \subset A_{\eta_0} \cup A_{\xi_0}$ and $h_\gamma\colon A_{\eta_0} \to A_{\xi_0}$ are the moves of the players. Since player **I** wins this game, there is $\alpha < \lambda$ for which h_α is not a partial isomorphism between $\mathcal{A}_{\eta_0}$ and $\mathcal{A}_{\xi_0}$. Let

$$\varepsilon = \sup(X_\alpha \cup \mathrm{dom}\, h_\alpha \cup \mathrm{ran}\, h_\alpha)$$

(Recall $\mathrm{dom}\,\mathcal{A}_\eta = A_\eta = \kappa$ for any η by convention.) Let π be the coding function defined in Definition 4.13 on page 66. Let

$$\beta_1 = \pi[\varepsilon^{<\omega}] + 1.$$

The idea is that $\eta_0 \restriction \beta_1$ and $\xi_0 \restriction \beta_1$ decide the models $\mathcal{A}_{\eta_0}$ and $\mathcal{A}_{\xi_0}$ as far as the game has been played. Clearly $\beta_1 < \kappa$.

Up to this point, player **II** has applied her strategy σ_0 precisely to the sequences of the moves made by her opponent, namely to $S = \{(X_\gamma)_{\gamma<\beta} \mid \beta < \alpha\} \subset \mathrm{dom}\, \sigma_0$. We can translate this set to represent a subset of the domain of ν_0: $S' = k[S]$, where k is as defined before the statement of the present theorem. Let $\beta_2 = (\sup S') + 1$ and let

$$\beta = \max\{\beta_1, \beta_2\}.$$

Thus $\eta_0 \restriction \beta$, $\xi_0 \restriction \beta$ and $\nu_0 \restriction \beta$ decide the moves $(h_\gamma)_{\gamma<\alpha}$ and the winner.

Now

$$\begin{aligned} U &= \{(\nu, \eta, \xi) \mid \nu \restriction \beta = \nu_0 \restriction \beta \wedge \eta \restriction \beta = \eta_0 \restriction \beta \wedge \xi \restriction \beta = \xi_0 \restriction \beta\} \\ &= N_{\nu_0 \restriction \beta} \times N_{\eta_0 \restriction \beta} \times N_{\xi_0 \restriction \beta}. \end{aligned}$$

is the desired neighborhood. Indeed, if $(\nu, \eta, \xi) \in U$ and ν codes a strategy σ, then τ beats σ on the structures $\mathcal{A}_\eta, \mathcal{A}_\xi$, since the first α moves are exactly as in the corresponding game of the triple (ν_0, η_0, ξ_0).

Let us now turn to D. The proof is similar. Assume that $(\nu_0, \eta_0, \xi_0) \notin D$ and ν_0 codes strategy τ_0 of player **I**. Then there is a strategy of **II**, which beats τ_0. Let $\beta < \kappa$ be, as before, an ordinal such that all moves have occurred before β and the relations of the substructures generated by the moves are decided by $\eta_0 \restriction \beta, \xi_0 \restriction \beta$ as well as the strategy τ_0. Unlike for player **I**, the win of **II** is determined always only in the end of the game, so β can be $\geqslant \lambda$. This is why we made the assumption $\lambda < \kappa$, by which we can always have $\beta < \kappa$ and so

$$\begin{aligned} U &= \{(\nu, \eta, \xi) \mid \nu \restriction \beta = \nu_0 \restriction \beta \wedge \eta \restriction \beta = \eta_0 \restriction \beta \wedge \xi \restriction \beta = \xi_0 \restriction \beta\} \\ &= N_{\nu_0 \restriction \beta} \times N_{\eta_0 \restriction \beta} \times N_{\xi_0 \restriction \beta}. \end{aligned}$$

is an open neighborhood of (ν_0, η_0, ξ_0) in the complement of D. □

Let us list some theorems concerning Ehrenfeucht-Fraïssé games which we will use in the proofs.

4.8 Definition. Let T be a theory and $\mathcal{A}$ a model of T of size κ. The $L_{\infty\kappa}$-*Scott height* of $\mathcal{A}$ is

$$\sup\{\alpha \mid \exists \mathcal{B} \models T(\mathcal{A} \not\cong \mathcal{B} \wedge \mathbf{II} \uparrow \mathrm{EF}^\kappa_{t_\alpha}(\mathcal{A}, \mathcal{B}))\},$$

if the supremum exists and ∞ otherwise, where t_α is as in Example 4.2 and the subsequent Fact.

Remark. Sometimes the Scott height is defined in terms of quantifier ranks, but this gives an equivalent definition by Theorem 4.10 below.

4.9 Definition. The *quantifier rank* $R(\varphi)$ of a formula $\varphi \in L_{\infty\infty}$ is an ordinal defined by induction on the length of φ as follows. If φ quantifier free, then $R(\varphi) = 0$. If $\varphi = \exists \bar{x} \psi(\bar{x})$, then $R(\varphi) = R(\psi(\bar{x})) + 1$. If $\varphi = \neg\psi$, then $R(\varphi) = R(\psi)$. If $\varphi = \bigwedge_{\alpha<\lambda} \psi_\alpha$, then $R(\varphi) = \sup\{R(\psi_\alpha \mid \alpha < \lambda)\}$

4.10 Theorem. *Models $\mathcal{A}$ and $\mathcal{B}$ satisfy the same $L_{\infty\kappa}$-sentences of quantifier rank $< \alpha$ if and only if* $\mathbf{II} \uparrow \mathrm{EF}^\kappa_{t_\alpha}(\mathcal{A}, \mathcal{B})$. □

The following theorem is a well known generalization of a theorem of Karp [27]:

4.11 Theorem. *Models $\mathcal{A}$ and $\mathcal{B}$ are $L_{\infty\kappa}$-equivalent if and only if* $\mathbf{II} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B})$. □

4.12 Remark. Models $\mathcal{A}$ and $\mathcal{B}$ of size κ are $L_{\kappa^+\kappa}$-equivalent if and only if they are $L_{\infty\kappa}$-equivalent. For an extensive and detailed survey on this and related topics, see [51].

Coding Models

There are various degrees of generality to which the content of this text is applicable. Many of the results generalize to vocabularies with infinitary relations or to uncountable vocabularies, but not all. We find it reasonable though to fix the used vocabulary to make the presentation clearer.

Models can be coded to models with just one binary predicate. Function symbols often make situations unnecessarily complicated from the point of view of this paper.

Thus our approach is, without great loss of generality, to fix our attention to models with finitary relation symbols of all finite arities.

Let us fix L to be the countable relational vocabulary consisting of the relations P_n, $n < \omega$, $L = \{P_n \mid n < \omega\}$, where each P_n is an n-ary relation: the interpretation of P_n is a set consisting of n-tuples. We can assume without loss of generality that the domain of each L-structure of size κ is κ, i.e. $\operatorname{dom} \mathcal{A} = \kappa$. If we restrict our attention to these models, then the set of all L-models has the same cardinality as κ^κ.

We will next present the way we code the structures and the isomorphisms between them into the elements of κ^κ (or equivalently – as will be seen – to 2^κ).

4.13 Definition. Let π be a bijection $\pi\colon \kappa^{<\omega} \to \kappa$. If $\eta \in \kappa^\kappa$, define the structure $\mathcal{A}_\eta$ to have $\operatorname{dom}(\mathcal{A}_\eta) = \kappa$ and if $(a_1, \dots a_n) \in \operatorname{dom}(\mathcal{A}_\eta)^n$, then

$$(a_1, \dots, a_n) \in P_n^{\mathcal{A}_\eta} \iff \eta(\pi(a_1, \dots, a_n)) > 0.$$

In that way the rule $\eta \mapsto \mathcal{A}_\eta$ defines a surjective (onto) function from κ^κ to the set of all L-structures with domain κ. We say that η *codes* $\mathcal{A}_\eta$.

Remark. Define the equivalence relation on κ^κ by $\eta \sim \xi \iff \operatorname{sprt} \eta = \operatorname{sprt} \xi$, where sprt means support, see section *Functions* on page 60. Now we have $\eta \sim \xi \iff \mathcal{A}_\eta = \mathcal{A}_\xi$, i.e. the identity map $\kappa \to \kappa$ is an isomorphism between $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ when $\eta \sim \xi$ and vice versa. On the other hand $\kappa^\kappa / \sim \cong 2^\kappa$, so the coding can be seen also as a bijection between models and the space 2^κ.

The distinction will make little difference, but it is convenient to work with both spaces depending on context. To illustrate the insignificance of the choice between κ^κ and 2^κ, note that $\sim$ is a closed equivalence relation and identity on 2^κ is bireducible with $\sim$ on κ^κ (see page 60).

Coding Partial Isomorphisms

Let $\xi, \eta \in \kappa^\kappa$ and let p be a bijection $\kappa \to \kappa \times \kappa$. Let $\nu \in \kappa^\alpha$, $\alpha \leqslant \kappa$. The idea is that for $\beta < \alpha$, $p_1(\nu(\beta))$ is the image of β under a partial isomorphism and $p_2(\nu(\beta))$ is the inverse image of β. That is, for a $\nu \in \kappa^\alpha$, define a relation $F_\nu \subset \kappa \times \kappa$:

$$(\beta, \gamma) \in F_\nu \iff \big(\beta < \alpha \land p_1(\nu(\beta)) = \gamma\big) \lor \big(\gamma < \alpha \land p_2(\nu(\gamma)) = \beta\big)$$

If ν happens to be such that F_ν is a partial isomorphism $\mathcal{A}_\xi \to \mathcal{A}_\eta$, then we say that ν *codes a partial isomorphism between* $\mathcal{A}_\xi$ and $\mathcal{A}_\eta$, this isomorphism being determined by F_ν. If $\alpha = \kappa$ and ν codes a partial isomorphism, then F_ν is an isomorphism and we say that ν *codes an isomorphism.*

4.14 Theorem. *The set*

$$C = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \textit{ codes an isomorphism between } \mathcal{A}_\eta \textit{ and } \mathcal{A}_\xi\}$$

is a closed set.

Proof. Suppose that $(\nu, \eta, \xi) \notin C$ i.e. ν does not code an isomorphism $\mathcal{A}_\eta \cong \mathcal{A}_\xi$. Then (at least) one of the following holds:

1. F_ν is not a function,
2. F_ν is not one-to-one,
3. F_ν does not preserve relations of $\mathcal{A}_\eta$, $\mathcal{A}_\xi$.

(Note that F_ν is always onto if it is a function and $\operatorname{dom}\nu = \kappa$.) If (1), (2) or (3) holds for ν, then respectively (1), (2) or (3) holds for any triple (ν', η', ξ') where $\nu' \in N_{\nu\restriction\gamma}$, $\eta' \in N_{\eta\restriction\gamma}$ and $\xi' \in N_{\xi\restriction\gamma}$, so it is sufficient to check that (1), (2) or (3) holds for $\nu\restriction\gamma$ for some $\gamma < \kappa$, because

Let us check the above in the case that (3) holds. The other cases are left to the reader. Suppose (3) holds. There is $(a_0, \dots, a_{n-1}) \in (\operatorname{dom}\mathcal{A}_\eta)^n = \kappa^n$ such that $(a_0, \dots, a_{n-1}) \in P_n$ and $(a_0, \dots, a_{n-1}) \in P_n^{\mathcal{A}_\eta}$ and $(F_\nu(a_0), \dots, F_\nu(a_{n-1})) \notin P_n^{\mathcal{A}_\xi}$. Let β be greater than

$$\max(\{\pi(a_0, \dots, a_{n-1}), \pi(F_\nu(a_0), \dots, F_\nu(a_{n-1}))\} \cup \{a_0, \dots a_{n-1}, F_\nu(a_0), \dots, F_\nu(a_{n-1})\})$$

Then it is easy to verify that any $(\eta', \xi', \nu') \in N_{\eta\restriction\beta} \times N_{\xi\restriction\beta} \times N_{\nu\restriction\beta}$ satisfies (3) as well. □

4.15 Corollary. *The set* $\{(\eta, \xi) \in (\kappa^\kappa)^2 \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$ *is* Σ^1_1.

Proof. It is the projection of the set C of Theorem 4.14. □

4.2.3 Generalized Borel Sets

4.16 Definition. We have already discussed Δ^1_1-sets which generalize Borel subsets of Polish space in one way. Let us see how else can we generalize usual Borel sets to our setting.

- [9, 36] The collection of λ-*Borel* subsets of κ^κ is the smallest set, which contains the basic open sets of κ^κ and is closed under complementation and under taking intersections of size λ. Since we consider only κ-Borel sets, we write Borel = κ-Borel.
- The collection $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- [9, 36] The collection of *Borel** subsets of κ^κ. A set A is Borel* if there exists a $\kappa^+\kappa$-tree t in which each increasing sequence of limit order type has a unique supremum and a function

  $$h\colon \{\text{branches of } t\} \to \{\text{basic open sets of } \kappa^\kappa\}$$

  such that $\eta \in A \iff$ player **II** has a winning strategy in the game $G(t, h, \eta)$. The game $G(t, h, \eta)$ is defined as follows. At the first round player **I** picks a minimal element of the tree, on successive rounds he picks an immediate successor of the last move played by player **II** and if there is no last move, he chooses an immediate successor of the supremum of all previous moves. Player **II** always picks an immediate successor of the Player **I**'s choice.

The game ends when the players cannot go up the tree anymore, i.e. have chosen a branch b. Player **II** wins, if $\eta \in h(b)$. Otherwise **I** wins.

A *dual* of a Borel* set B is the set

$$B^d = \{\xi \mid \mathbf{I} \uparrow G(t,h,\xi)\}$$

where t and h satisfy the equation $B = \{\xi \mid \mathbf{II} \uparrow G(t,h,\xi)\}$. The dual is not unique.

Remark. Suppose that t is a $\kappa^+\kappa$ tree and $h\colon \{\text{branches of } t\} \to \text{Borel}^*$ is a labeling function taking values in Borel* sets instead of basic open sets. Then $\{\eta \mid \mathbf{II} \uparrow G(t,h,\eta)\}$ is a Borel* set.

Thus if we change the basic open sets to Borel* sets in the definition of Borel*, we get Borel*.

4.17 Remark. Blackwell [2] defined Borel* sets in the case $\kappa = \omega$ and showed that in fact Borel=Borel*. When κ is uncountable it is not the case. But it is easily seen that if t is a $\kappa^+\omega$-tree, then the Borel* set coded by t (with some labeling h) is a Borel set, and vice versa: each Borel set is a Borel* set coded by a $\kappa^+\omega$-tree. We will use this characterization of Borel.

It was first explicitly proved in [36] that these are indeed generalizations:

4.18 Theorem ([36], $\kappa^{<\kappa} = \kappa$)**.** *Borel* $\subset \Delta^1_1 \subset$ *Borel** $\subset \Sigma^1_1$*,*

Proof. (Sketch) If A is Borel*, then it is Σ^1_1, intuitively, because $\eta \in A$ if and only if *there exists* a winning strategy of player **II** in $G(t,h,\eta)$ where (t,h) is a tree that codes A (here one needs the assumption $\kappa^{<\kappa} = \kappa$ to be able to code the strategies into the elements of κ^κ). By Remark 4.17 above if A is Borel, then there is also such a tree. Since Borel $\subset$ Borel* by Remark 4.17 and Borel is closed under taking complements, Borel sets are Δ^1_1.

The fact that Δ^1_1-sets are Borel* is a more complicated issue; it follows from a separation theorem proved in [36]. The separation theorem says that any two disjoint Σ^1_1-sets can be separated by Borel* sets. It is proved in [36] for $\kappa = \omega_1$, but the proof generalizes to any κ (with $\kappa^{<\kappa} = \kappa$). □

Additionally we have the following results:

4.19 Theorem. *1.* Borel $\subsetneq \Delta^1_1$.

2. $\Delta^1_1 \subsetneq \Sigma^1_1$.

3. If $V = L$*, then* Borel$^* = \Sigma^1_1$.

4. $\Delta^1_1 \subsetneq$ Borel* *holds if* $V = L$*, and also in every* $\mathbb{P}$*-generic extension starting from a ground model with* $\kappa^{<\kappa} = \kappa$*, where*

$$\mathbb{P} = \{p \mid p \textit{ is a function}, |p| < \kappa, \operatorname{dom} p \subset \kappa \times \kappa^+, \operatorname{ran} p \subset \{0,1\}\}.$$

Proof. (Sketch)

1. The following universal Borel set is not Borel itself, but is Δ^1_1:

$$B = \{(\eta,\xi) \in 2^\kappa \times 2^\kappa \mid \eta \text{ is in the set coded by } (t_\xi, h_\xi)\},$$

where $\xi \mapsto (t_\xi, h_\xi)$ is a continuous coding of ($\kappa^+\omega$-tree, labeling)-pairs in such a way that for all $\kappa^+\omega$-trees $t \subset \kappa^{<\omega}$ and labelings h there is ξ with $(t_\xi, h_\xi) = (t, h)$. It is not Borel since if it were, then the diagonal's complement

$$D = \{\eta \mid (\eta, \eta) \notin B\}$$

would be a Borel set which it is not, since it cannot be coded by any (t_ξ, h_ξ). On the other hand its complement $C = (2^\kappa)^2 \setminus B$ is Σ_1^1, because $(\eta, \xi) \in C$ if and only if *there exists* a winning strategy of player **I** in the Borel-game $G(t_\xi, h_\xi, \eta)$ and the latter can be coded to a Borel set. It is left to the reader to verify that when $\kappa > \omega$, then the set

$$F = \{(\eta, \xi, \nu) \mid \nu \text{ codes a w.s. for } \mathbf{I} \text{ in } G(t_\xi, h_\xi, \eta)\}$$

is closed.

The existence of an isomorphism relation which is Δ_1^1 but not Borel follows from Theorems 4.72 and 4.73.

2. Similarly as above (and similarly as in the case $\kappa = \omega$), take a universal Σ_1^1-set $A \subset 2^\kappa \times 2^\kappa$ with the property that if $B \subset 2^\kappa$ is any Σ_1^1-set, then there is $\eta \in 2^\kappa$ such that $B \times \{\eta\} \subset A$. This set can be constructed as in the case $\kappa = \omega$, see [25]. The diagonal $\{\eta \mid (\eta, \eta) \in A\}$ is Σ_1^1 but not Π_1^1.

3. Suppose $V = L$ and $A \subset 2^\kappa$ is Σ_1^1. There exists a formula $\varphi(x, \xi)$ with parameter $\xi \in 2^\kappa$ which is Σ_1 in the Levy hierarchy (see [25]) and for all $\eta \in 2^\kappa$ we have

$$\eta \in A \iff L \models \varphi(\eta, \xi)$$

Now we have that $\eta \in A$ if and only if the set

$$\big\{\alpha < \kappa \mid \exists\beta\big(\eta\restriction\alpha, \xi\restriction\alpha \in L_\beta,\ L_\beta \models \big(\mathrm{ZF}^- \wedge (\alpha \text{ is a cardinal}) \wedge \varphi(\eta\restriction\alpha, \xi\restriction\alpha)\big)\big)\big\}$$

contains an ω-cub set.

But the ω-cub filter is Borel* so A is also Borel*.

4. The first part follows from clauses (2) and (3) of this Theorem and the second part from clauses (1), (6) and (7) of Theorem 4.52 on page 91, see especially the proof of (7). □

Open Problem. Is it consistent that Borel* is a proper subclass of Σ_1^1, or even equals Δ_1^1? Is it consistent that all the inclusions are proper at the same time: $\Delta_1^1 \subsetneq \text{Borel}^* \subsetneq \Sigma_1^1$?

4.20 Theorem. *For a set $S \subset \kappa^\kappa$ the following are equivalent.*

1. *S is Σ_1^1,*
2. *S is a projection of a Borel set,*
3. *S is a projection of a Σ_1^1-set,*
4. *S is a continuous image of a closed set.*

Proof. Let us go in the order.

(1) ⇒ (2) Closed sets are Borel.

(2) ⇒ (3) The same proof as in the standard case $\kappa = \omega$ gives that Borel sets are Σ_1^1 (see for instance [25]).

(3) ⇒ (4) Let $A \subset \kappa^\kappa \times \kappa^\kappa$ be a Σ_1^1-set which is the projection of A, $S = \mathrm{pr}_0 A$. Then let $C \subset \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ be a closed set such that $\mathrm{pr}_1 C = A$. Here $\mathrm{pr}_0 \colon \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa$ and $\mathrm{pr}_1 \colon \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa \times \kappa^\kappa$ are the obvious projections. Let $f \colon \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa$ be a homeomorphism. Then S is the image of the closed set $f[C]$ under the continuous map $\mathrm{pr}_0 \circ \mathrm{pr}_1 \circ f^{-1}$.

(4) ⇒ (1) The image of a closed set under a continuous map f is the projection of the graph of f restricted to that closed set. It is a basic topological fact that a graph of a continuous partial function with closed domain is closed (provided the range is Hausdorff). □

4.21 Theorem ([36]). *Borel* sets are closed under unions and intersections of size κ.* □

4.22 Definition. A Borel* set B is *determined* if there exists a tree t and a labeling function h such that the corresponding game $G(t, h, \eta)$ is determined for all $\eta \in \kappa^\kappa$ and

$$B = \{\eta \mid \mathbf{II} \text{ has a winning strategy in } G(t, h, \eta)\}.$$

4.23 Theorem ([36]). *Δ_1^1-sets are exactly the determined Borel* sets.* □

4.3 Borel Sets, Δ_1^1-sets and Infinitary Logic

4.3.1 The Language $L_{\kappa^+\kappa}$ and Borel Sets

The interest in the class of Borel sets is explained by the fact that the Borel sets are relatively simple yet at the same time this class includes many interesting definable sets. Below we prove Vaught's theorem (Theorem 4.25), which equates "invariant" Borel sets with those definable in the infinitary language $L_{\kappa^+\kappa}$. Recall that models $\mathcal{A}$ and $\mathcal{B}$ of size κ are $L_{\kappa^+\kappa}$-equivalent if and only if they are $L_{\infty\kappa}$-equivalent. Vaught proved his theorem for the case $\kappa = \omega_1$ assuming CH in [52], but the proof works for arbitrary κ assuming $\kappa^{<\kappa} = \kappa$.

4.24 Definition. Denote by S_κ the set of all permutations of κ. If $u \in \kappa^{<\kappa}$, denote

$$\bar{u} = \{p \in S_\kappa \mid p^{-1} \restriction \operatorname{dom} u = u\}.$$

Note that $\bar{\varnothing} = S_\kappa$ and if $u \in \kappa^\alpha$ is not injective, then $\bar{u} = \varnothing$.

A permutation $p \colon \kappa \to \kappa$ acts on 2^κ by

$$p\eta = \xi \iff p \colon \mathcal{A}_\eta \to \mathcal{A}_\xi \text{ is an isomorphism.}$$

The map $\eta \mapsto p\eta$ is well defined for every p and it is easy to check that it defines an action of the permutation group S_κ on the space 2^κ. We say that a set $A \subset 2^\kappa$ is *closed under permutations* if it is a union of orbits of this action.

4.25 Theorem ([52], $\kappa^{<\kappa}=\kappa$). *A set $B\subset\kappa^\kappa$ is Borel and closed under permutations if and only if there is a sentence φ in $L_{\kappa^+\kappa}$ such that $B=\{\eta\mid\mathcal{A}_\eta\models\varphi\}$.*

Proof. Let φ be a sentence in $L_{\kappa^+\kappa}$. Then $\{\eta\in 2^\kappa\mid\mathcal{A}_\eta\models\varphi\}$ is closed under permutations, because if $\eta=p\xi$, then $\mathcal{A}_\eta\cong\mathcal{A}_\xi$ and $\mathcal{A}_\eta\models\varphi\iff\mathcal{A}_\xi\models\varphi$ for every sentence φ. If φ is a formula with parameters $(a_i)_{i<\alpha}\in\kappa^\alpha$, one easily verifies by induction on the complexity of φ that the set

$$\{\eta\in 2^\kappa\mid\mathcal{A}_\eta\models\varphi((a_i)_{i<\alpha})\}$$

is Borel. This of course implies that for every sentence φ the set $\{\eta\mid\mathcal{A}_\eta\models\varphi\}$ is Borel.

The converse is less trivial. Note that the set of permutations $S_\kappa\subset\kappa^\kappa$ is Borel, since

$$S_\kappa=\bigcap_{\beta<\kappa}\bigcup_{\alpha<\kappa}\underbrace{\{\eta\mid\eta(\alpha)=\beta\}}_{\text{open}}\cap\bigcap_{\alpha<\beta<\kappa}\underbrace{\{\eta\mid\eta(\alpha)\neq\eta(\beta)\}}_{\text{open}}. \qquad (\cdot)$$

For a set $A\subset\kappa^\kappa$ and $u\in\kappa^{<\kappa}$, define

$$A^{*u}=\big\{\eta\in 2^\kappa\mid\{p\in\bar{u}\mid p\eta\in A\}\text{ is co-meager in }\bar{u}\big\}.$$

From now on in this section we will write "$\{p\in\bar{u}\mid p\eta\in A\}$ is co-meager", when we really mean "co-meager in $\bar{u}$".

Let us show that the set

$$Z=\{A\subset 2^\kappa\mid A\text{ is Borel},A^{*u}\text{ is }L_{\kappa^+\kappa}\text{-definable for all }u\in\kappa^{<\kappa}\}$$

contains all the basic open sets, is closed under intersections of size κ and under complementation in the three steps (a),(b) and (c) below. This implies that Z is the collection of all Borel sets. We will additionally keep track of the fact that the formula, which defines A^{*u} depends only on A and $\operatorname{dom}u$, i.e. for each $\beta<\kappa$ and Borel set A there exists $\varphi=\varphi_\beta^A$ such that for all $u\in\kappa^\beta$ we have $A^{*u}=\{\eta\mid\mathcal{A}_\eta\models\varphi((u_i)_{i<\beta})\}$. Setting $u=\varnothing$, we have the intended result, because $A^{*\varnothing}=A$ for all A which are closed under permutations and φ is a sentence (with no parameters).

If A is fixed we denote $\varphi_\beta^A=\varphi_\beta$.

(a) Assume $q\in 2^{<\kappa}$ and let N_q be the corresponding basic open set. Let us show that $N_q\in Z$. Let $u\in\kappa^\beta$ be arbitrary. We have to find $\varphi_\beta^{N_q}$. Let θ be a quantifier free formula with α parameters such that:

$$N_q=\{\eta\in 2^\kappa\mid\mathcal{A}_\eta\models\theta((\gamma)_{\gamma<\alpha})\}.$$

Here $(\gamma)_{\gamma<\alpha}$ denotes both an initial segment of κ as well as an α-tuple of the structure. Suppose $\alpha\leqslant\beta$. We have $p\in\bar{u}\Rightarrow u\subset p^{-1}$, so

$$\begin{aligned}\eta\in N_q^{*u}&\iff\{p\in\bar{u}\mid p\eta\in N_q\}\text{ is co-meager}\\&\iff\{p\in\bar{u}\mid\mathcal{A}_{p\eta}\models\theta((\gamma)_{\gamma<\alpha})\}\text{ is co-meager}\\&\iff\{p\in\bar{u}\mid\mathcal{A}_\eta\models\theta((p^{-1}(\gamma))_{\gamma<\alpha})\}\text{ is co-meager}\\&\iff\{p\in\bar{u}\mid\underbrace{\mathcal{A}_\eta\models\theta((u_\gamma)_{\gamma<\alpha})}_{\text{independent of }p}\}\text{ is co-meager}\\&\iff\mathcal{A}_\eta\models\theta((u_\gamma)_{\gamma<\alpha}).\end{aligned}$$

Then $\varphi_\beta = \theta$.

Assume then that $\alpha > \beta$. By the above, we still have

$$\eta \in N_q^{*u} \iff E = \{p \in \bar{u} \mid \mathcal{A}_\eta \models \theta((p^{-1}(\gamma))_{\gamma<\alpha})\} \text{ is co-meager}$$

Assume that $w = (w_\gamma)_{\gamma<\alpha} \in \kappa^\alpha$ is an arbitrary sequence with no repetition and such that $u \subset w$. Since $\bar{w}$ is an open subset of $\bar{u}$ and E is co-meager, there is $p \in \bar{w} \cap E$. Because $p \in E$, we have $\mathcal{A}_\eta \models \theta((p^{-1}(\gamma))_{\gamma<\alpha})$. On the other hand $p \in \bar{w}$, so we have $w \subset p^{-1}$, i.e. $w_\gamma = w(\gamma) = p^{-1}(\gamma)$ for $\gamma < \alpha$. Hence

$$\mathcal{A}_\eta \models \theta((w_\gamma)_{\gamma<\alpha}). \tag{$\star$}$$

On the other hand, if for every injective $w \in \kappa^\alpha$, $w \supset u$, we have $(\star)$, then in fact $E = \bar{u}$ and is trivially co-meager. Therefore we have an equivalence:

$$\eta \in N_q^{*u} \iff (\forall w \supset u)(w \in \kappa^\alpha \wedge w \text{ inj.} \Rightarrow \mathcal{A}_\eta \models \theta((w_\gamma)_{\gamma<\alpha})).$$

But the latter can be expressed in the language $L_{\kappa^+\kappa}$ by the formula $\varphi_\beta((w_i)_{i<\beta})$:

$$\bigwedge_{i<j<\beta} (w_i \neq w_j) \wedge \Big(\forall_{\beta\leqslant i<\alpha} w_i\Big)\Big(\bigwedge_{i<j<\alpha} (w_i \neq w_j) \to \theta((w_i)_{i<\alpha})\Big)$$

θ was defined to be a formula defining N_q with parameters. It is clear thus that θ is independent of u. Furthermore the formulas constructed above from θ depend only on $\beta = \operatorname{dom} u$ and on θ. Hence the formulas defining N_q^{*u} and N_q^{*v} for $\operatorname{dom} u = \operatorname{dom} v$ are the same modulo parameters.

(*b*) For each $i < \kappa$ let $A_i \in Z$. We want to show that $\bigcap_{i<\kappa} A_i \in Z$. Assume that $u \in \kappa^{<\kappa}$ is arbitrary. It suffices to show that

$$\bigcap_{i<\kappa}(A_i^{*u}) = \Big(\bigcap_{i<\kappa} A_i\Big)^{*u},$$

because then $\varphi_\beta^{\bigcap_i A_i}$ is just the κ-conjunction of the formulas $\varphi_\beta^{A_i}$ which exist by the induction hypothesis. Clearly the resulting formula depends again only on $\operatorname{dom} u$ if the previous did. Note that a κ-intersection of co-meager sets is co-meager. Now

$$\begin{aligned}
& \eta \in \bigcap_{i<\kappa}(A_i^{*u}) \\
\iff\ & (\forall i < \kappa)(\{p \in \bar{u} \mid p\eta \in A_i\} \text{ is co-meager}) \\
\iff\ & (\forall i < \kappa)(\forall i < \kappa)(\{p \in \bar{u} \mid p\eta \in A_i\} \text{ is co-meager}) \\
\iff\ & \bigcap_{i<\kappa}\{p \in \bar{u} \mid p\eta \in A_i\} \text{ is co-meager} \\
\iff\ & \{p \in \bar{u} \mid p\eta \in \bigcap_{i<\kappa} A_i\} \text{ is co-meager} \\
\iff\ & \eta \in \Big(\bigcap_{i<\kappa} A_i\Big)^{*u}.
\end{aligned}$$

(c) Assume that $A \in Z$ i.e. that A^{*u} is definable for any u. Let $\varphi_{\operatorname{dom} u}$ be the formula, which defines A^{*u}. Let now $u \in \kappa^{<\kappa}$ be arbitrary and let us show that $(A^c)^{*u}$ is definable. We will show that

$$(A^c)^{*u} = \bigcap_{v \supset u} (A^{*v})^c$$

i.e. for all η

$$\eta \in (A^c)^{*u} \iff \forall v \supset u(\eta \notin A^{*v}). \tag{4.1}$$

Granted this, one can write the formula "$\forall v \supset u \neg\varphi_{\operatorname{dom} u}((v_i)_{i<\operatorname{dom} v})$", which is not of course the real $\varphi^{A^c}_\beta$ which we will write in the end of the proof.

To prove (4.1) we have to show first that for all $\eta \in \kappa^\kappa$ the set $B = \{p \in \bar{u} \mid p\eta \in A\}$ has the Property of Baire (P.B.), see Section 4.4.3.

The set of all permutations $S_\kappa \subset \kappa^\kappa$ is Borel by $(\cdot)$ on page 71. The set $\bar{u}$ is an intersection of S_κ with an open set. Again the set $\{p \in \bar{u} \mid p\eta \in A\}$ is the intersection of $\bar{u}$ and the inverse image of A under the continuous map $(p \mapsto p\eta)$, so is Borel and so has the Property of Baire.

We can now turn to proving the equivalence (4.1). First "$\Leftarrow$":

$$\begin{aligned}
\eta \notin (A^c)^{*u} \quad &\Rightarrow \quad B = \{p \in \bar{u} \mid p\eta \in A\} \text{ is not meager in } \bar{u}\\
&\Rightarrow \quad \text{By P.B. of } B \text{ there is a non-empty open } U \text{ such that } U \setminus B \text{ is meager}\\
&\Rightarrow \quad \text{There is non-empty } \bar{v} \subset \bar{u} \text{ such that } \bar{v} \setminus B \text{ is meager.}\\
&\Rightarrow \quad \text{There exists } \bar{v} \subset \bar{u} \text{ such that } \{p \in \bar{v} \mid p\eta \in A\} = \bar{v} \cap B \text{ is co-meager}\\
&\Rightarrow \quad \exists v \supset u(\eta \in A^{*v}).
\end{aligned}$$

And then the other direction "$\Rightarrow$":

$$\begin{aligned}
\eta \in (A^c)^{*u} \quad &\Rightarrow \quad \{p \in \bar{u} \mid p\eta \in A\} \text{ is meager}\\
&\Rightarrow \quad \text{for all } \bar{v} \subset \bar{u} \text{ the set } \{p \in \bar{v} \mid p\eta \in A\} \text{ is meager.}\\
&\Rightarrow \quad \forall \bar{v} \subset \bar{u}(\eta \notin A^{*v}).
\end{aligned}$$

Let us now write the formula $\psi = \varphi^{A^c}_\beta$ such that

$$\forall \bar{v} \subset \bar{u}(\eta \notin A^{*v}) \iff \mathcal{A}_\eta \models \psi((u_i)_{i<\beta}),$$

where $\beta = \operatorname{dom} u$: let $\psi((u_i)_{i<\beta})$ be

$$\bigwedge_{\beta \leqslant \gamma < \kappa} \bigforall_{i<\gamma} x_i \left(\Big[\bigwedge_{j<\beta} (x_j = u_j) \wedge \bigwedge_{i<j<\gamma} (x_i \neq x_j) \Big] \to \neg\varphi_\gamma((x_i)_{i<\gamma}) \right)$$

One can easily see, that this is equivalent to $\forall v \supset u\big(\neg\varphi_{\operatorname{dom} v}((v_i)_{i<\operatorname{dom} v})\big)$ and that ψ depends only on $\operatorname{dom} u$ modulo parameters. □

4.26 Remark. If $\kappa^{<\kappa} > \kappa$, then the direction from right to left of the above theorem does not in general hold. Let $\langle \kappa, \lessdot, A \rangle$ be a model with domain κ, $A \subset \kappa$ and $\lessdot$ a well ordering of κ of order type κ. Väänänen and Shelah have shown [46, Corollary 17] that if $\kappa = \lambda^+$, $\kappa^{<\kappa} > \kappa$, $\lambda^{<\lambda} = \lambda$ and a forcing axiom holds (and $\omega_1^L = \omega_1$ if $\lambda = \omega$) then there is a sentence of $L_{\kappa\kappa}$ defining the set

$$\mathrm{STAT} = \{\langle \kappa, <, A\rangle \mid A \text{ is stationary}\}.$$

If now STAT is Borel, then so would be the set CUB defined in Section 4.4.3, but by Theorem 4.52(6), page 91, this set cannot be Borel since Borel sets have the Property of Baire by Theorem 4.48 on page 91.

Open Problem. Does the direction left to right of Theorem 4.25 hold without the assumption $\kappa^{<\kappa} = \kappa$?

4.3.2 The Language $M_{\kappa^+\kappa}$ and Δ_1^1-sets

In this section we will present a theorem similar to Theorem 4.25. It is also a generalization of the known result which follows from [36] and [50]:

4.27 Theorem ([36, 50]:). *Let $\mathcal{A}$ be a model of size ω_1. Then the isomorphism type $I = \{\eta \mid \mathcal{A}_\eta \cong \mathcal{A}\}$ is Δ_1^1 if and only if there is a sentence φ in $M_{\kappa^+\kappa}$ such that $I = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ and $2^\kappa \setminus I = \{\eta \mid \mathcal{A}_\eta \models \sim \varphi\}$, where $\sim\theta$ is the dual of θ.*

The idea of the proof of the following Theorem is due to Sam Coskey and Philipp Schlicht:

4.28 Theorem ($\kappa^{<\kappa} = \kappa$). *A set $D \subset 2^\kappa$ is Δ_1^1 and closed under permutations if and only if there is a sentence φ in $M_{\kappa^+\kappa}$ such that $D = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ and $\kappa^\kappa \setminus D = \{\eta \mid \mathcal{A}_\eta \models \sim \varphi\}$, where $\sim\theta$ is the dual of θ.*

We have to define these concepts before the proof.

4.29 Definition (Karttunen [28]). Let λ and κ be cardinals. The language $M_{\lambda\kappa}$ is then defined to be the set of pairs $(t, \mathscr{L})$ of a tree t and a labeling function $\mathscr{L}$. The tree t is a $\lambda\kappa$-tree where the limits of increasing sequences of t exist and are unique. The labeling $\mathscr{L}$ is a function satisfying the following conditions:

1. $\mathscr{L}\colon t \to a \cup \bar{a} \cup \{\bigwedge, \bigvee\} \cup \{\exists x_i \mid i < \kappa\} \cup \{\forall x_i \mid i < \kappa\}$ where a is the set of atomic formulas and $\bar{a}$ is the set of negated atomic formulas.
2. If $x \in t$ has no successors, then $\mathscr{L}(t) \in a \cup \bar{a}$.
3. If $x \in t$ has exactly one immediate successor then $\mathscr{L}(t)$ is either $\exists x_i$ or $\forall x_i$ for some $i < \kappa$.
4. Otherwise $\mathscr{L}(t) \in \{\bigvee, \bigwedge\}$.
5. If $x < y$, $\mathscr{L}(x) \in \{\exists x_i, \forall x_i\}$ and $\mathscr{L}(y) \in \{\exists x_j, \forall x_j\}$, then $i \neq j$.

4.30 Definition. Truth for $M_{\lambda\kappa}$ is defined in terms of a semantic game. Let $(t, \mathscr{L})$ be the pair which corresponds to a particular sentence φ and let $\mathcal{A}$ be a model. The semantic game $S(\varphi, \mathcal{A}) = S(t, \mathscr{L}, \mathcal{A})$ for $M_{\lambda\kappa}$ is played by players **I** and **II** as follows. At the first move the players are at the root and later in the game at some other element of t. Let us suppose that they are at the element $x \in t$. If $\mathscr{L}(x) = \bigvee$, then Player **II** chooses a successor of x and the players move to that chosen element. If $\mathscr{L}(x) = \bigwedge$, then player **I** chooses a successor of x and the players move to that chosen element. If $\mathscr{L}(x) = \forall x_i$ then player **I** picks an element $a_i \in \mathcal{A}$ and if $\mathscr{L}(x) = \exists x_i$ then player **II** picks an element a_i and they move to the immediate successor of x. If they come to a limit, they move to the unique supremum. If x is a maximal element of t, then they plug the elements a_i in place of the corresponding free variables in the atomic formula $\mathscr{L}(x)$. Player **II** wins if this atomic formula is true in $\mathcal{A}$ with these interpretations. Otherwise player **I** wins.

We define $\mathcal{A} \models \varphi$ if and only if **II** has a winning strategy in the semantic game.

Given a sentence φ, the *dual* sentence $\sim \varphi$ is defined by modifying the labeling function as follows. The atomic formulas are replaced by their negations, the symbols $\bigvee$ and $\bigwedge$ switch places and the quantifiers $\forall$ and $\exists$ switch places. A sentence $\varphi \in M_{\lambda\kappa}$ is *determined* if for all models $\mathcal{A}$ either $\mathcal{A} \models \varphi$ or $\mathcal{A} \models \sim \varphi$.

Now the statement of Theorem 4.28 makes sense. Theorem 4.28 concerns a sentence φ whose dual defines the complement of the set defined by φ among the models of size κ, so it is determined in that model class. Before the proof let us recall a separation theorem for $M_{\kappa^+\kappa}$, Theorem 3.9 from [48]:

4.31 Theorem. *Assume $\kappa^{<\kappa} = \lambda$ and let $\exists R\varphi$ and $\exists S\psi$ be two Σ^1_1 sentences where φ and ψ are in $M_{\kappa^+\kappa}$ and $\exists R$ and $\exists S$ are second order quantifiers. If $\exists R\varphi \wedge \exists S\psi$ does not have a model, then there is a sentence $\theta \in M_{\lambda^+\lambda}$ such that for all models $\mathcal{A}$*

$$\mathcal{A} \models \exists R\varphi \Rightarrow \mathcal{A} \models \theta \text{ and } \mathcal{A} \models \exists S\psi \Rightarrow \mathcal{A} \models \sim \theta \qquad \square$$

4.32 Definition. For a tree t, let σt be the tree of downward closed linear subsets of t ordered by inclusion.

Proof of Theorem 4.28. Let us first show that if φ is an arbitrary sentence of $M_{\kappa^+\kappa}$, then $D_\varphi = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ is Σ^1_1. The proof has the same idea as the proof of Theorem 4.18 that Borel* $\subset \Sigma^1_1$. Note that this implies that if $\sim \varphi$ defines the complement of D_φ in 2^κ, then D_φ is Δ^1_1.

A strategy in the semantic game $S(\varphi, \mathcal{A}_\eta) = S(t, \mathscr{L}, \mathcal{A}_\eta)$ is a function

$$\upsilon \colon \sigma t \times (\operatorname{dom} \mathcal{A}_\eta)^{<\kappa} \to t \cup (t \times \operatorname{dom} \mathcal{A}_\eta).$$

This is because the previous moves always form an initial segment of a branch of the tree together with the sequence of constants picked by the players from $\operatorname{dom} \mathcal{A}_\eta$ at the quantifier moves, and a move consists either of going to some node of the tree or going to a node of the tree together with choosing an element from $\operatorname{dom} \mathcal{A}_\eta$. By the convention that $\operatorname{dom} \mathcal{A}_\eta = \kappa$, a strategy becomes a function

$$\upsilon \colon \sigma t \times \kappa^{<\kappa} \to t \cup (t \times \kappa),$$

Because t is a $\kappa^+\kappa$-tree, there are fewer than κ moves in a play (there are no branches of length κ and the players go up the tree on each move). Let

$$f \colon \sigma t \times \kappa^{<\kappa} \to \kappa$$

be any bijection and let

$$g \colon t \cup (t \times \kappa) \to \kappa$$

be another bijection. Let F be the bijection

$$F\colon (t\cup(t\times\kappa))^{\sigma t\times\kappa^{<\kappa}}\to\kappa^\kappa$$

defined by $F(\upsilon)=g\circ\upsilon\circ f^{-1}$. Let

$$C=\{(\eta,\xi)\mid F^{-1}(\xi) \text{ is a winning strategy of } \mathbf{II} \text{ in } S(t,\mathscr{L},\mathcal{A}_\eta)\}.$$

Clearly D_φ is the projection of C. Let us show that C is closed. Consider an element (η,ξ) in the complement of C. We shall show that there is an open neighborhood of (η,ξ) outside C. Denote $\upsilon=F^{-1}(\xi)$. Since υ is not a winning strategy there is a strategy τ of **I** that beats υ. There are $\alpha+1<\kappa$ moves in the play $\tau*\upsilon$ (by definition all branches have successor order type). Assume that $b=(x_i)_{i\leqslant\alpha}$ is the chosen branch of the tree and $(c_i)_{i<\alpha}$ the constants picked by the players. Let $\beta<\kappa$ be an ordinal with the properties $\{f((x_i)_{i<\gamma},(c_i)_{i<\gamma})\mid\gamma\leqslant\alpha+1\}\subset\beta$ and

$$\eta'\in N_{\eta\restriction\beta}\to\mathcal{A}_{\eta'}\not\models\mathscr{L}(x_\alpha)((c_i)_{i<\alpha}). \tag{$\star$}$$

Such β exists, since $|\{f((x_i)_{i<\gamma},(c_i)_{i<\gamma})\mid\gamma\leqslant\alpha+1\}|<\kappa$ and $\mathscr{L}(x_\alpha)$ is a (possibly negated) atomic formula which is not true in $\mathcal{A}_\eta$, because **II** lost the game $\tau*\upsilon$ and because already a fragment of size $<\kappa$ of $\mathcal{A}_\eta$ decides this. Now if $(\eta',\xi')\in N_{\eta\restriction\beta}\times N_{\xi\restriction\beta}$ and $\upsilon'=F^{-1}(\xi')$, then $\upsilon*\tau$ is the same play as $\tau*\upsilon'$. So $\mathcal{A}_{\eta'}\not\models\mathscr{L}(x_\alpha)((c_i)_{i<\alpha})$ by $(\star)$ and (η',ξ') is not in C and

$$N_{\eta\restriction\beta}\times N_{\xi\restriction\beta}$$

is the intended open neighborhood of (η,ξ) outside C. This completes the "if"-part of the proof.

Now for a given $A\in\Delta_1^1$ which is closed under permutations we want to find a sentence $\varphi\in M_{\kappa^+\kappa}$ such that $A=\{\eta\mid\mathcal{A}_\eta\models\varphi\}$ and $2^\kappa\setminus A=\{\eta\mid\mathcal{A}_\eta\models\sim\varphi\}$. By our assumption $\kappa^{<\kappa}=\kappa$ and Theorems 4.23 and 4.31, it is enough to show that for a given Borel* set B which is closed under permutations, there is a sentence $\exists R\psi$ which is Σ_1^1 over $M_{\kappa^+\kappa}$ (as in the formulation of Theorem 4.31), such that $B=\{\eta\mid\mathcal{A}_\eta\models\exists R\psi\}$.

The sentence "R is a well ordering of the universe of order type κ", is definable by the formula $\theta=\theta(R)$ of $L_{\kappa^+\kappa}\subset M_{\kappa^+\kappa}$:

$$\begin{aligned}
&\text{"}R \text{ is a linear ordering on the universe"}\\
\wedge\quad &\Big(\forall_{i<\omega} x_i\Big)\Big(\bigvee_{i<\omega}\neg R(x_{i+1},x_i)\Big)\\
\wedge\quad &\forall x\bigvee_{\alpha<\kappa}\exists_{i<\alpha} y_i\left[\Big(\forall y(R(y,x)\to\bigvee_{i<\alpha}y_i=y)\Big)\right]
\end{aligned}\tag{4.2}$$

(We assume $\kappa>\omega$, so the infinite quantification is allowed. The second row says that there are no descending sequences of length ω and the third row says that the initial segments are of size less than κ. This ensures that $\theta(R)$ says that R is a well ordering of order type κ).

Let t and h be the tree and the labeling function corresponding to B. Define the tree $t^\star$ as follows.

1. Assume that b is a branch of t with $h(b)=N_{\xi\restriction\alpha}$ for some $\xi\in\kappa^\kappa$ and $\alpha<\kappa$. Then attach a sequence of order type α^* on top of b where

$$\alpha^*=\bigcup_{s\in\pi^{-1}[\alpha]}\operatorname{ran}s,$$

where π is the bijection $\kappa^{<\omega} \to \kappa$ used in the coding, see Definition 4.13 on page 66.

2. Do this to each branch of t and add a root r to the resulting tree.

After doing this, the resulting tree is $t^\star$. Clearly it is a $\kappa^+\kappa$-tree, because t is. Next, define the labeling function $\mathscr{L}$. If $x \in t$ then either $\mathscr{L}(x) = \bigwedge$ or $\mathscr{L}(x) = \bigvee$ depending on whether it is player **I**'s move or player **II**'s move: formally let $n < \omega$ be such that $\mathrm{OTP}(\{y \in t^\star \mid y \leqslant x\}) = \alpha + n$ where α is a limit ordinal or 0; then if n is odd, put $\mathscr{L}(x) = \bigwedge$ and otherwise $\mathscr{L}(x) = \bigvee$. If $x = r$ is the root, then $\mathscr{L}(x) = \bigwedge$. Otherwise, if x is not maximal, define

$$\beta = \mathrm{OTP}\{y \in t^\star \setminus (t \cup \{r\}) \mid y \leqslant x\}$$

and set $\mathscr{L}(x) = \exists x_\beta$.

Next we will define the labeling of the maximal nodes of $t^\star$. By definition these should be atomic formulas or negated atomic formulas, but it is clear that they can be replaced without loss of generality by any formula of $M_{\kappa^+\kappa}$; this fact will make the proof simpler. Assume that x is maximal in $t^\star$. $\mathscr{L}(x)$ will depend only on $h(b)$ where b is the unique branch of t leading to x. Let us define $\mathscr{L}(x)$ to be the formula of the form $\theta \wedge \Theta_b((x_i)_{i<\alpha^*})$, where θ is defined above and Θ_b is defined below. The idea is that

$$\mathcal{A}_\eta \models \Theta_b((a_\gamma)_{\gamma<\alpha^*})\} \iff \eta \in h(b) \text{ and } \forall\gamma < \alpha^*(a_\gamma = \gamma).$$

Let us define such a Θ_b. Suppose that ξ and α are such that $h(b) = N_{\xi\restriction\alpha}$. Define for $s \in \pi^{-1}[\alpha]$ the formula A^s_b as follows:

$$A^s_b = \begin{cases} P_{\mathrm{dom}\, s}, & \text{if } \mathcal{A}_\xi \models P_{\mathrm{dom}\, s}((s(i))_{i\in\mathrm{dom}\, s}) \\ \neg P_{\mathrm{dom}\, s}, & \text{if } \mathcal{A}_\xi \not\models P_{\mathrm{dom}\, s}((s(i))_{i\in\mathrm{dom}\, s}) \end{cases}$$

Then define

$$\begin{aligned} \psi_0((x_i)_{i<\alpha^*}) &= \bigwedge_{i<\alpha^*} \Big[\forall y(R(y,x_i) \leftrightarrow \bigvee_{j<i}(y = x_j))\Big] \\ \psi_1((x_i)_{i<\alpha^*}) &= \bigwedge_{s\in\pi^{-1}[\alpha]} A^s_b((x_{s(i)})_{i\in\mathrm{dom}\, s}), \\ \Theta_b &= \psi_0 \wedge \psi_1. \end{aligned}$$

The disjunction over the empty set is considered false.

Claim 1. Suppose for all η, R is the standard order relation on κ. Then

$$(\mathcal{A}_\eta, R) \models \Theta_b((a_\gamma)_{\gamma<\alpha^*}) \iff \eta \in h(b) \wedge \forall\gamma < \alpha^*(\alpha_\gamma = \gamma).$$

Proof of Claim 1. Suppose $\mathcal{A}_\eta \models \Theta((a_\gamma)_{\gamma<\alpha^*})$. Then by $\mathcal{A}_\eta \models \psi_0((a_\gamma)_{\gamma<\alpha^*})$ we have that $(a_\gamma)_{\gamma<\alpha^*}$ is an initial segment of $\mathrm{dom}\,\mathcal{A}_\eta$ with respect to R. But $(\mathrm{dom}\,\mathcal{A}_\eta, R) = (\kappa, <)$, so $\forall\gamma < \alpha^*(\alpha_\gamma = \gamma)$. Assume that $\beta < \alpha$ and $\eta(\beta) = 1$ and denote $s = \pi^{-1}(\beta)$. Then $\mathcal{A}_\eta \models P_{\mathrm{dom}\, s}((s(i))_{i\in\mathrm{dom}\, s})$. Since Θ is true in $\mathcal{A}_\eta$ as well, we must have $A^s_b = P_{\mathrm{dom}\, s}$ which by definition means that $\mathcal{A}_\xi \models P_{\operatorname{dom} s}((s(i))_{i\in \operatorname{dom} s})$ and hence $\xi(\beta) = \xi(\pi(s)) = 1$. In the same way one shows that if $\eta(\beta) = 0$, then $\xi(\beta) = 0$ for all $\beta < \alpha$. Hence $\eta\restriction\alpha = \xi\restriction\alpha$.

Assume then that $a_\gamma = \gamma$ for all $\gamma < \alpha^*$ and that $\eta \in N_{\xi\restriction\alpha}$. Then $\mathcal{A}_\eta$ trivially satisfies ψ_0. Suppose that $s \in \pi^{-1}[\alpha]$ is such that $\mathcal{A}_\xi \models P_{\operatorname{dom} s}((s(i))_{i\in \operatorname{dom} s})$. Then $\xi(\pi(s)) = 1$ and since $\pi(s) < \alpha$, also $\eta(\pi(s)) = 1$, so $\mathcal{A}_\eta \models P_{\operatorname{dom} s}((s(i))_{i\in \operatorname{dom} s})$. Similarly one shows that if

$$\mathcal{A}_\xi \not\models P_{\operatorname{dom} s}((s(i))_{i\in \operatorname{dom} s}),$$

then $\mathcal{A}_\eta \not\models P_{\operatorname{dom} s}((s(i))_{i\in \operatorname{dom} s})$. This shows that $\mathcal{A}_\eta \models A^s_b((s(i))_{i\in \operatorname{dom} s})$ for all s. Hence $\mathcal{A}_\eta$ satisfies ψ_1, so we have $\mathcal{A}_\eta \models \Theta$. $\square_{\text{Claim 1}}$

Claim 2. t, h, $t^\star$ and $\mathscr{L}$ are such that for all $\eta \in \kappa^\kappa$

$$\mathbf{II} \uparrow G(t,h,\eta) \iff \exists R \subset (\operatorname{dom}\mathcal{A}_\eta)^2 \;\; \mathbf{II} \uparrow S(t^\star, \mathscr{L}, \mathcal{A}_\eta).$$

Proof of Claim 2. Suppose σ is a winning strategy of **II** in $G(t,h,\eta)$. Let R be the well ordering of $\operatorname{dom}\mathcal{A}_\eta$ such that $(\operatorname{dom}\mathcal{A}_\eta, R) = (\kappa, <)$. Consider the game $S(t^\star, \mathscr{L}, \mathcal{A}_\eta)$. On the first move the players are at the root and player **I** chooses where to go next. They go to to a minimal element of t. From here on **II** uses σ as long as they are in t. Let us see what happens if they got to a maximal element of t, i.e. they picked a branch b from t. Since σ is a winning strategy of **II** in $G(t,h,\eta)$, we have $\eta \in h(b)$ and $h(b) = N_{\xi\restriction\alpha}$ for some ξ and α. For the next α moves the players climb up the tower defined in item (1) of the definition of $t^\star$. All labels are of the form $\exists x_\beta$, so player **II** has to pick constants from $\mathcal{A}_\eta$. She picks them as follows: for the variable x_β she picks $\beta \in \kappa = \operatorname{dom}\mathcal{A}_\eta$. She wins now if $\mathcal{A}_\eta \models \Theta((\beta)_{\beta<\alpha^*})$ and $\mathcal{A}_\eta \models \theta$. But $\eta \in h(b)$, so by Claim 1 the former holds and the latter holds because we chose R to be a well ordering of order type κ.

Let us assume that there is no winning strategy of **II** in $G(t,h,\eta)$. Let R be an arbitrary relation on $\operatorname{dom}\mathcal{A}_\eta$. Here we shall finally use the fact that B is closed under permutations. Suppose R is not a well ordering of the universe of order type κ. Then after the players reached the final node of $t^\star$, player **I** chooses to go to θ and player **II** loses. So we can assume that R is a well ordering of the universe of order type κ. Let $p\colon \kappa \to \kappa$ be a bijection such that $p(\alpha)$ is the α^{th} element of κ with respect to R. Now p is a permutation and $\{\eta \mid \mathcal{A}_{p\eta} \in B\} = B$ since B is closed under permutations. So by our assumption that $\eta \notin B$ (i.e. $\mathbf{II} \not\uparrow G(t,h,\eta)$), we also have $p\eta \notin B$, i.e. player **II** has no winning strategy in $G(t,h,p\eta)$ either.

Suppose σ is any strategy of **II** in $S(t^\star, \mathscr{L}, \mathcal{A}_\eta)$. Player **I** imagines that σ is a strategy in $G(t,h,p\eta)$ and picks a strategy τ that beats it. In the game $S(t^\star, \mathscr{L}, \mathcal{A}_\eta)$, as long as the players are still in t, player **I** uses τ that would beat σ if they were playing $G(t,h,p\eta)$ instead of $S(t^\star, \mathscr{L}, \eta)$. Suppose they picked a branch b of t. Now $p\eta \notin h(b)$. If **II** wants to satisfy ψ_0 of the definition of Θ_b, she is forced to pick the constants $(a_i)_{i<\alpha^*}$ such that a_i is the i^{th} element of $\operatorname{dom}\mathcal{A}_\eta$ with respect to R. Suppose that $\mathcal{A}_\eta \models \psi_1((a_i)_{i<\alpha^*})$ (recall $\Theta_b = \psi_0 \wedge \psi_1$). But then $\mathcal{A}_{p\eta} \models \psi_1((\gamma)_{\gamma<\alpha^*})$ and also $\mathcal{A}_{p\eta} \models \psi_0((\gamma)_{\gamma<\alpha^*})$, so by Claim 1 we should have $p\eta \in h(b)$ which is a contradiction. $\square_{\text{Claim 2}}$

$\square_{\text{Theorem 4.28}}$

4.4 Generalizing Classical Descriptive Set Theory

4.4.1 Simple Generalizations

The Identity Relation

Denote by id the equivalence relation $\{(\eta,\xi)\in(2^\kappa)^2\mid \eta=\xi\}$. If we want to emphasize the set on which the identity relation lies, we denote it by id_X if the set is X. With respect to our choice of topology, the natural generalization of the equivalence relation

$$E_0=\{(\eta,\xi)\in 2^\omega\times 2^\omega\mid \exists n<\omega\forall m>n(\eta(m)=\xi(m))\}$$

is equivalence modulo sets of size $<\kappa$:

$$E_0^{<\kappa}=\{(\eta,\xi)\in 2^\kappa\times 2^\kappa\mid \exists\alpha<\kappa\forall\beta>\alpha(\eta(\beta)=\xi(\beta))\},$$

although the equivalences modulo sets of size $<\lambda$ for $\lambda<\kappa$ can also be studied:

$$E_0^{<\lambda}=\{(\eta,\xi)\in 2^\kappa\times 2^\kappa\mid \exists A\subset\kappa[|A|<\lambda\wedge\forall\beta\notin A(\eta(\beta)=\xi(\beta))]\},$$

but for $\lambda<\kappa$ these turn out to be bireducible with id (see below). Similarly one can define $E_0^{<\lambda}$ on κ^κ instead of 2^κ.

It makes no difference whether we define these relations on 2^κ or κ^κ since they become bireducible to each other:

4.33 Theorem. *Let $\lambda\leqslant\kappa$ be a cardinal and let $E_0^{<\lambda}(P)$ denote the equivalence relation $E_0^{<\lambda}$ on $P\in\{2^\kappa,\kappa^\kappa\}$ (notation defined above). Then*

$$E_0^{<\lambda}(2^\kappa)\leqslant_c E_0^{<\lambda}(\kappa^\kappa) \text{ and } E_0^{<\lambda}(\kappa^\kappa)\leqslant_c E_0^{<\lambda}(2^\kappa).$$

Note that when $\lambda=1$, we have $E_0^{<1}(P)=\mathrm{id}_P$.

Proof. In this proof we think of functions $\eta,\xi\in\kappa^\kappa$ as graphs $\eta=\{(\alpha,\eta(\alpha))\mid\alpha<\kappa\}$. Fix a bijection $h\colon\kappa\to\kappa\times\kappa$. Let $f\colon 2^\kappa\to\kappa^\kappa$ be the inclusion, $f(\eta)(\alpha)=\eta(\alpha)$. Then f is easily seen to be a continuous reduction $E_0^{<\lambda}(2^\kappa)\leqslant_c E_0^{<\lambda}(\kappa^\kappa)$. Define $g\colon\kappa^\kappa\to 2^\kappa$ as follows. For $\eta\in\kappa^\kappa$ let $g(\eta)(\alpha)=1$ if $h(\alpha)\in\eta$ and $g(\eta)(\alpha)=0$ otherwise. Let us show that g is a continuous reduction $E_0^{<\lambda}(\kappa^\kappa)\leqslant_c E_0^{<\lambda}(2^\kappa)$. Suppose $\eta,\xi\in\kappa$ are $E_0^{<\lambda}(\kappa^\kappa)$-equivalent. Then clearly $|\eta\triangle\xi|<\lambda$. On the other hand

$$I=\{\alpha\mid g(\eta)(\alpha)\neq g(\xi)(\alpha)\}=\{\alpha\mid h(\alpha)\in\eta\triangle\xi\}$$

and because h is a bijection, we have that $|I|<\lambda$.

Suppose η and ξ are not $E_0^{<\lambda}(\kappa^\kappa)$-equivalent. But then $|\eta\triangle\xi|\geqslant\lambda$ and the argument above shows that also $|I|\geqslant\lambda$, so $g(\eta)(\alpha)$ is not $E_0^{<\lambda}(2^\kappa)$-equivalent to $g(\xi)(\alpha)$.

g is easily seen to be continuous. □

We will need the following Lemma which is a straightforward generalization from the case $\kappa=\omega$:

4.34 Lemma. *Borel functions are continuous on a co-meager set.*

Proof. For each $\eta \in \kappa^{<\kappa}$ let V_η be an open subset of κ^κ such that $V_\eta \triangle f^{-1}N_\eta$ is meager. Let

$$D = \kappa^\kappa \setminus \bigcup_{\eta \in \kappa^{<\kappa}} V_\eta \triangle f^{-1}N_\eta.$$

Then D is as intended. Clearly it is co-meager, since we took away only a κ-union of meager sets. Let $\xi \in \kappa^{<\kappa}$ be arbitrary. The set $D \cap f^{-1}N_\xi$ is open in D since $D \cap f^{-1}N_\xi = D \cap V_\xi$ and so $f \restriction D$ is continuous. □

4.35 Theorem ($\kappa^{<\kappa} = \kappa$)**.** *$E_0^{<\lambda}$ is an equivalence relation on 2^κ for all $\lambda \leqslant \kappa$ and*

1. *$E_0^{<\lambda}$ is Borel.*
2. *$E_0^{<\kappa} \not\leqslant_B \mathrm{id}$.*
3. *If $\lambda \leqslant \kappa$, then $\mathrm{id} \leqslant_c E_0^{<\lambda}$.*
4. *If $\lambda < \kappa$, then $E_0^{<\lambda} \leqslant_c \mathrm{id}$.*

Proof. $E_0^{<\lambda}$ is clearly reflexive and symmetric. Suppose $\eta E_0^{<\lambda} \xi$ and $\xi E_0^{<\lambda} \zeta$. Denote $\eta = \eta^{-1}\{1\}$ and similarly for η, ζ. Then $|\eta \triangle \xi| < \lambda$ and $|\xi \triangle \zeta| < \lambda$; but $\eta \triangle \zeta \subset (\eta \triangle \xi) \cup (\xi \triangle \zeta)$. Thus $E_0^{<\lambda}$ is indeed an equivalence relation.

1. $E_0^{<\lambda} = \bigcup_{A \in [\kappa]^{<\lambda}} \bigcap_{\alpha \notin A} \underbrace{\{(\eta, \xi) \mid \eta(\alpha) = \xi(\alpha)\}}_{\text{open}}.$

2. Assume there were a Borel reduction $f\colon 2^\kappa \to 2^\kappa$ witnessing $E_0 \leqslant_B \mathrm{id}$. By Lemma 4.34 there are dense open sets $(D_i)_{i<\kappa}$ such that $f \restriction \bigcap_{i<\kappa} D_i$ is continuous. If $p, q \in 2^\alpha$ for some α and $\xi \in N_p$, let us denote $\xi^{(p/q)} = q^\frown(\xi \restriction (\kappa \setminus \alpha))$, and if $A \subset N_p$, denote

$$A^{(p/q)} = \{\eta^{(p/q)} \mid \eta \in A\}.$$

Let C is be the collection of sets, each of which is of the form

$$\bigcup_{q \in 2^\alpha} [D_i \cap N_p]^{(p/q)}$$

for some $\alpha < \kappa$ and some $p \in 2^\alpha$. It is easy to see that each such set is dense and open, so C is a collection of dense open sets. By the assumption $\kappa^{<\kappa} = \kappa$, C has size κ. Also C contains the sets D_i for all $i < \kappa$, (taking $\alpha = 0$). Denote $D = \bigcap_{i<\kappa} D_i$. Let $\eta \in \bigcap C$, $\xi = f(\eta)$ and $\xi' \neq \xi$, $\xi' \in \mathrm{ran}(f \restriction D)$. Now ξ and ξ' have disjoint open neighborhoods V and V' respectively. Let α and $p, q \in 2^\alpha$ be such that $\eta \in N_p$ and such that $D \cap N_p \subset f^{-1}[V]$ and $D \cap N_q \subset f^{-1}[V']$. These p and q exist by the continuity of f on D. Since $\eta \in \bigcap C$ and $\eta \in N_p$, we have

$$\eta \in [D_i \cap N_q]^{(q/p)}$$

for all $i < \kappa$, which is equivalent to

$$\eta^{(p/q)} \in [D_i \cap N_q]$$

for all $i < \kappa$, i.e. $\eta^{(p/q)}$ is in $D \cap N_q$. On the other hand (since $D_i \in C$ for all $i < \kappa$ and because $\eta \in N_p$), we have $\eta \in D \cap N_p$. This implies that $f(\eta) \in V$ and $f(\eta^{(p/q)}) \in V'$ which is a contradiction, because V and V' are disjoint and $(\eta, \eta^{p/q}) \in E_0$.

3. Let $(A_i)_{i<\kappa}$ be a partition of κ into pieces of size κ: if $i \neq j$ then $A_i \cap A_j = \varnothing$, $\bigcup_{i<\kappa} A_i = \kappa$ and $|A_i| = \kappa$. Obtain such a collection for instance by taking a bijection $h\colon \kappa \to \kappa \times \kappa$ and defining $A_i = h^{-1}[\kappa \times \{i\}]$. Let $f\colon 2^\kappa \to 2^\kappa$ be defined by $f(\eta)(\alpha) = \eta(i) \iff \alpha \in A_i$. Now if $\eta = \xi$, then clearly $f(\eta) = f(\xi)$ and so $f(\eta) E_0^{<\lambda} f(\xi)$. If $\eta \neq \xi$, then there exists i such that $\eta(i) \neq \xi(i)$ and we have that

$$A_i \subset \{\alpha \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\}$$

and A_i is of size $\kappa \geqslant \lambda$.

4. Let $P = \kappa^{<\kappa} \setminus \kappa^{<\lambda}$. Let $f\colon P \to \kappa$ be a bijection. It induces a bijection $g\colon 2^P \to 2^\kappa$. Let us construct a map $h\colon 2^\kappa \to 2^P$ such that $g \circ h$ is a reduction $E_0^{<\lambda} \to \mathrm{id}_{2^\kappa}$. Let us denote by $E^{<\lambda}(\alpha)$ the equivalence relation on 2^α such that two subsets X, Y of α are $E^{<\lambda}(\alpha)$-equivalent if and only if $|X \triangle Y| < \lambda$.

   For each α in $\lambda < \alpha < \kappa$ let h_α be any reduction of $E^{<\lambda}(\alpha)$ to id_{2^α}. This exists because both equivalence relations have 2^α many classes. Now reduce $E_0^{<\lambda}$ to $\mathrm{id}_{\kappa^{<\kappa}}$ by $f(A) = (h_\alpha(A \cap \alpha) \mid \lambda \leqslant \alpha < \kappa)$. If A, B are $E_0^{<\lambda}$-equivalent, then $f(A) = f(B)$. Otherwise $f_\alpha(A \cap \alpha)$ differs from $f_\alpha(B \cap \alpha)$ for large enough $\alpha < \kappa$ because λ is less than κ and κ is regular. Continuity of h is easy to check. □

4.4.2 On the Silver Dichotomy

To begin with, let us define the Silver Dichotomy and the Perfect Set Property:

4.36 Definition. Let $\mathcal{C} \in \{\text{Borel}, \Delta_1^1, \text{Borel}^*, \Sigma_1^1, \Pi_1^1\}$.

By *the Silver Dichotomy*, or more specifically, *κ-SD for $\mathcal{C}$* we mean the statement that there are no equivalence relations E in the class $\mathcal{C}$ such that $E \subset 2^\kappa \times 2^\kappa$ and E has more than κ equivalence classes such that $\mathrm{id} \not\leqslant_B E$, $\mathrm{id} = \mathrm{id}_{2^\kappa}$.

Similarly the *Perfect Set Property* , or *κ-PSP for $\mathcal{C}$*, means that each member A of $\mathcal{C}$ has either size $\leqslant \kappa$ or there is a Borel injection $2^\kappa \to A$. Using Lemma 4.34 it is not hard to see that this definition is equivalent to the game definition given in [36].

The Silver Dichotomy for Isomorphism Relations

Although the Silver Dichotomy for Borel sets is not provable from ZFC for $\kappa > \omega$ (see Theorem 4.44 on page 89), it holds when the equivalence relation is an isomorphism relation, if $\kappa > \omega$ is an inaccessible cardinal:

4.37 Theorem. *Assume that κ is inaccessible. If the number of equivalence classes of $\cong_T$ is greater than κ, then* $\mathrm{id} \leqslant_c \cong_T$.

Proof. Suppose that there are more than κ equivalence classes of $\cong_T$. We will show that then $\mathrm{id}_{2^\kappa} \leqslant_c \cong_T$. If T is not classifiable, then as was done in [41], we can construct a tree $t(S)$ for each $S \subset S^\kappa_\omega$ and Ehrenfeucht-Mostowski-type models $M(t(S))$ over these trees such that if $S \triangle S'$ is stationary, then $M(t(S)) \not\cong M(t(S'))$. Now it is easy to construct a reduction $f\colon \mathrm{id}_{2^\kappa} \leqslant_c E_{S^\kappa_\omega}$ (see notation defined in Section 4.2.1), so then $\eta \mapsto M(t(f(\eta)))$ is a reduction $\mathrm{id} \leqslant_c \cong_T$.

Assume now that T is classifiable. By $\lambda(T)$ we denote the least cardinal in which T is stable. By [40] Theorem XIII.4.8 (this is also mentioned in [12] Theorem 2.5), assuming that $\cong_T$ has more than κ equivalence classes, it has depth at least 2 and so there are: a $\lambda(T)^+$-saturated model $\mathcal{B} \models T$, $|\mathcal{B}| = \lambda(T)$, and a $\lambda(T)^+$-saturated elementary submodel $\mathcal{A} \preccurlyeq \mathcal{B}$ and $a \notin \mathcal{B}$ such that $\mathrm{tp}(a/\mathcal{B})$ is orthogonal to $\mathcal{A}$. Let $f\colon \kappa \to \kappa$ be strictly increasing and such that for all $\alpha < \kappa$, $f(\alpha) = \mu^+$, for some μ with the properties $\lambda(T) < \mu < \kappa$, $\mathrm{cf}(\mu) = \mu$ and $\mu^{2^\omega} = \mu$. For each $\eta \in 2^\kappa$ with $\eta^{-1}\{1\}$ is unbounded we will construct a model $\mathcal{A}_\eta$. As above, it will be enough to show that $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$ whenever $\eta^{-1}\{1\} \mathbin{\triangle} \xi^{-1}\{1\}$ is λ-stationary where $\lambda = \lambda(T)^+$. Fix $\eta \in 2^\kappa$ and let $\lambda = \lambda(T)^+$.

For each $\alpha \in \eta^{-1}\{1\}$ choose $\mathcal{B}_\alpha \supset \mathcal{A}$ such that

1. $\exists \pi_\alpha\colon \mathcal{B} \cong \mathcal{B}_\alpha$, $\pi_\alpha \restriction \mathcal{A} = \mathrm{id}_{\mathcal{A}}$.
2. $\mathcal{B}_\alpha \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_\beta \mid \beta \in \eta^{-1}\{1\}, \beta \neq \alpha\}$

Note that 2 implies that if $\alpha \neq \beta$, then $\mathcal{B}_\alpha \cap \mathcal{B}_\beta = \mathcal{A}$. For each $\alpha \in \eta^{-1}\{1\}$ and $i < f(\alpha)$ choose tuples a_i^α with the properties

3. $\mathrm{tp}(a_i^\alpha/\mathcal{B}_\alpha) = \pi_\alpha(\mathrm{tp}(a/\mathcal{B}))$
4. $a_i^\alpha \downarrow_{\mathcal{B}_\alpha} \bigcup\{a_j^\alpha \mid j < f(\alpha), j \neq i\}$

Let $\mathcal{A}_\eta$ be F^s_λ-primary over

$$S_\eta = \bigcup\{B_\alpha \mid a < \eta^{-1}\{1\}\} \cup \bigcup\{a_i^\alpha \mid \alpha < \eta^{-1}\{1\}, i < f(\alpha)\}.$$

It remains to show that if $S^\kappa_\lambda \cap \eta^{-1}\{1\} \mathbin{\triangle} \xi^{-1}\{1\}$ is stationary, then $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$. Without loss of generality we may assume that $S^\kappa_\lambda \cap \eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$ is stationary. Let us make a counter assumption, namely that there is an isomorphism $F\colon \mathcal{A}_\eta \to \mathcal{A}_\xi$.

Without loss of generality there exist singletons b_i^η and sets B_i^η, $i < \kappa$ of size $< \lambda$ such that $\mathcal{A}_\eta = S_\eta \cup \bigcup_{i<\kappa} b_i^\eta$ and $(S_\eta, (b_i^\eta, B_i^\eta)_{i<\kappa})$ is an F^s_λ-construction.

Let us find an ordinal $\alpha < \kappa$ and sets $C \subset \mathcal{A}_\eta$ and $D \subset \mathcal{A}_\xi$ with the properties listed below:

(a) $\alpha \in \eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$

(b) $D = F[C]$

(c) $\forall \beta \in (\alpha+1) \cap \eta^{-1}\{1\}(\mathcal{B}_\beta \subset C)$ and $\forall \beta \in (\alpha+1) \cap \xi^{-1}\{1\}(\mathcal{B}_\beta \subset D)$,

(d) for all $i < f(\alpha)$, $\forall \beta \in \alpha \cap \eta^{-1}\{1\}(a_i^\beta \in C)$ and $\forall \beta \in \alpha \cap \xi^{-1}\{1\}(a_i^\beta \in D)$,

(e) $|C| = |D| < f(\alpha)$,

(f) For all β, if $\mathcal{B}_\beta \cap C \setminus \mathcal{A} \neq \varnothing$, then $\mathcal{B}_\beta \subset C$ and if $\mathcal{B}_\beta \cap D \setminus \mathcal{A} \neq \varnothing$, then $\mathcal{B}_\beta \subset D$,

(g) C and D are λ-saturated,

(h) if $b_i^\eta \in C$, then $B_i^\eta \subset [S_\eta \cup \bigcup\{b_i^\eta \mid j < i\}] \cap C$ and if $b_i^\xi \in D$, then $B_i^\xi \subset [S_\xi \cup \bigcup\{b_i^\xi \mid j < i\}] \cap D$.

This is possible, because $\eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$ is stationary and we can close under the properties (b)–(h).

Now $\mathcal{A}_\eta$ is F^s_λ-primary over $C \cup S_\eta$ and $\mathcal{A}_\xi$ is F^s_λ-primary over $D \cup S_\eta$ and thus $\mathcal{A}_\eta$ is F^s_λ-atomic over $C \cup S_\eta$ and $\mathcal{A}_\xi$ is F^s_λ-atomic over $D \cup S_\xi$. Let

$$I_\alpha = \{a^\alpha_i \mid i < f(\alpha)\}.$$

Now $|I_\alpha \setminus C| = f(\alpha)$, because $|C| < f(\alpha)$, and so $I_\alpha \setminus C \neq \varnothing$. Let $c \in I_\alpha \setminus C$ and let $A \subset S_\xi \setminus D$ and $B \subset D$ be such that $\mathrm{tp}(F(c)/A \cup B) \vdash \mathrm{tp}(F(c)/D \cup S_\xi)$ and $|A \cup B| < \lambda$. Since $\alpha \notin \xi^{-1}\{1\}$, we can find (just take disjoint copies) a sequence $(A_i)_{i<f(\alpha)^+}$ such that $A_i \subset I_\alpha \cap \mathcal{A}_\xi$, $\mathrm{tp}(A_i/D) = \mathrm{tp}(A/D)$ and $A_i \downarrow_D \bigcup\{A_j \mid j \neq i, j < f(\alpha)^+\}$

Now we can find $(d_i)_{i<f(\alpha)^+}$, such that

$$\mathrm{tp}(d_i{}^\frown A_i{}^\frown B_i/\varnothing) = \mathrm{tp}(F(c){}^\frown A{}^\frown B/\varnothing).$$

Then it is a Morley sequence over D and for all $i < f(\alpha)^+$,

$$\mathrm{tp}(d_i/D) = \mathrm{tp}(F(c)/D),$$

which implies

$$\mathrm{tp}(F^{-1}(d_i)/C) = \mathrm{tp}(c/C),$$

for some i, since for some i we have $c = a^\alpha_i$. Since by (c), $\mathcal{B}_\alpha \subset C$, the above implies that

$$\mathrm{tp}(F^{-1}(d_i)/\mathcal{B}_\alpha) = \mathrm{tp}(a^\alpha_i/\mathcal{B}_\alpha)$$

which by the definition of a^α_i, item 3 implies

$$\mathrm{tp}(F^{-1}(d_i)/\mathcal{B}_\alpha) = \pi_\alpha(\mathrm{tp}(a/\mathcal{B})).$$

Thus the sequence $(F^{-1}(d_i))_{i<f(\alpha)^+}$ witnesses that the dimension of $\pi_\alpha(\mathrm{tp}(a/\mathcal{B}))$ in $\mathcal{A}_\eta$ is greater than $f(\alpha)$. Denote that sequence by J. Since $\pi_\alpha(\mathrm{tp}(a/\mathcal{B}))$ is orthogonal to $\mathcal{A}$, we can find $J' \subset J$ such that $|J'| = f(\alpha)^+$ and J' is a Morley sequence over S_η. Since $f(\alpha)^+ > \lambda$, this contradicts Theorem 4.9(2) of Chapter IV of [40]. □

Open Problem. Under what conditions on κ does the conclusion of Theorem 4.37 hold?

Theories Bireducible With id

4.38 Theorem. *Assume $\kappa^{<\kappa} = \kappa = \aleph_\alpha > \omega$, κ is not weakly inaccessible and $\lambda = |\alpha + \omega|$. Then the following are equivalent.*

1. *There is $\gamma < \omega_1$ such that $\beth_\gamma(\lambda) \geqslant \kappa$.*
2. *There is a complete countable T such that $\mathrm{id} \leqslant_B \cong_T$ and $\cong_T \leqslant_B \mathrm{id}$.*

Proof. (2)⇒(1): Suppose that (1) is not true. Notice that then $\kappa > 2^\omega$. Then every shallow classifiable theory has $< \kappa$ many models of power κ (see [12], item 6. of the Theorem which is on the first page of the article) and thus $\mathrm{id} \not\leqslant_B \cong_T$. On the other hand if T is not classifiable and shallow, $\cong_T$ is not Borel by Theorem 4.72 and thus it is not Borel reducible to id by Fact 4.78.

(1)⇒(2): Since $\mathrm{cf}(\kappa) > \omega$, (1) implies that there is $\alpha = \beta + 1 < \omega_1$ such that $\beth_\alpha(\lambda) = \kappa$. But then there is an L^*-theory T^* which has exactly κ many models in cardinality κ (up to isomorphism, use [12], Theorem 6.1 items 2. and 8.). But then it has exactly κ many models of cardinality $\leqslant \kappa$, let $\mathcal{A}_i$, $i < \kappa$, list these. Such a theory must be classifiable and shallow. Let L be the vocabulary we get from L^* by adding one binary relation symbol E. Let $\mathcal{A}$ be an L-structure in which E is an equivalence relation with infinitely many equivalence classes such that for every equivalence class a/E, $(\mathcal{A}\restriction a/E)\restriction L^*$ is a model of T^*. Let $T = Th(\mathcal{A})$.

We show first that identity on $\{\eta \in 2^\kappa \mid \eta(0) = 1\}$ reduces to $\cong_T$. For all $\eta \in 2^\kappa$, let $\mathcal{B}_\eta$ be a model of T of power κ such that if $\eta(i) = 0$, then the number of equivalence classes isomorphic to $\mathcal{B}_i$ is countable and otherwise the number is κ. Clearly we can code $\mathcal{B}_\eta$ as $\xi_\eta \in 2^\kappa$ so that $\eta \mapsto \xi_\eta$ is the required Borel reduction.

We show then that $\cong_T$ Borel reduces to identity on

$$X = \{\eta \colon \kappa \to (\kappa + 1)\}.$$

Since T^* is classifiable and shallow, for all $\delta, i < \kappa$ the set

$$\{\eta \in X \mid (\mathcal{A}_\eta \restriction \delta/E)\restriction L^* \cong \mathcal{A}_i\}$$

is Borel. But then for all cardinals $\theta \leqslant \kappa$ and $i < \kappa$, the set

$$\{\eta \in X \mid \mathrm{card}(\{\delta/E \mid \ \delta < \kappa,\ (\mathcal{A}_\eta \restriction \delta/E)\restriction L^* \cong \mathcal{A}_i\}) = \theta\}$$

is Borel. But then $\eta \mapsto \xi_\eta$ is the required reduction when

$$\xi_\eta(i) = |\{\delta/E \mid \ \delta < \kappa,\ (\mathcal{A}_\eta \restriction \delta/E)\restriction L^* \cong \mathcal{A}_i\}|. \qquad \square$$

In the above it was assumed that κ is not inaccessible. If κ is inaccessible, then (2) of the above theorem always holds:

4.39 Theorem. *Suppose κ is inaccessible and $\kappa^{<\kappa} = \kappa$. Then there is a theory T such that $\cong_T$ is bireducible with* id_{2^κ}.

Proof. Let $\mathcal{M}$ be the model with domain $M = \mathrm{dom}\,\mathcal{M} = \omega \cup (\omega \times \omega)$ and a binary relation R which is interpreted

$$R^{\mathcal{M}} = \{(a, (b, c)) \in M^2 \mid a \in \omega, (b, c) \in \omega \times \omega, a = b\}.$$

Then our intended theory is the complete first-order theory of this structure $T = \mathrm{Th}(\mathcal{M})$.

Let $\hat{C} = \{\aleph_\beta \mid \beta \leqslant \kappa\}$ and $C = \omega \cup \hat{C}$.

Let $\mathcal{A}$ be a model of T of size κ and let $f_{\mathcal{A}} \colon \hat{C} \to C$ be a function such that

$$f_{\mathcal{A}}(\aleph_\beta) = \mathrm{card}(\{x \in A \mid \mathrm{card}(\{(a, b) \in A \mid R(x, (a, b))\}) = \aleph_\beta\}), \qquad (*)$$

i.e. $f_{\mathcal{A}}(\aleph_\beta)$ equals the number of elements which are R-related to exactly $\aleph_\beta$ elements. Clearly $\mathcal{A} \cong \mathcal{B}$ is equivalent to $f_{\mathcal{A}} = f_{\mathcal{B}}$.

Let $g_0 \colon \hat{\mu} \to \hat{C}$ and $g_1 \colon \mu \to C$ be bijections. Let us define the function F by

$$F(\xi) = g_1^{-1} \circ f_{\mathcal{A}_\xi} \circ g_0.$$

Now F is a reduction $\cong_T \leqslant \mathrm{id}_{\kappa^\kappa}$. By Theorem 4.33, page 79, $\mathrm{id}_{\kappa^\kappa}$ is continuously bireducible with id_{2^κ}. Let us show that F is Borel. In order to do it, we will use the easy direction (right to left) of Theorem 4.25 on page 71. Because every basic open set in κ^κ is an intersection of the sets of the form

$$U_{\gamma\delta} = \{\eta \in \kappa^\kappa \mid \eta(\gamma) = \delta\},$$

it is enough to show that $F^{-1}[U_{\gamma\delta}]$ is Borel for any $\gamma, \delta \in \kappa$.

$\eta \in F^{-1}[U_{\gamma\delta}]$ is equivalent to

($\star$) *there exists exactly $g_1(\delta)$ elements in $F^{-1}(\eta)$ which are R-related to exactly $g_0(\gamma)$ elements.*

We can express ($\star$) in $L_{\kappa^+\kappa}$. First, let us define the formula φ_λ for $\lambda < \kappa$ which says that the variable x is R-related to exactly λ elements:

$$\varphi_\lambda(x) : \exists_{i<\lambda} y_i \Big[\Big(\bigwedge_{j_0<j_1<\lambda} \neg y_{j_0} = y_{j_1}\Big) \wedge \bigwedge_{i<\lambda} R(x, y_i) \wedge \forall z\Big(R(x,z) \to \bigvee_{i<\lambda} z = y_i\Big)\Big].$$

Then one can write the formula which says that there are exactly $\nu < \kappa$ such x_k that satisfy φ_λ:

$$\psi_{\lambda\nu} : \exists_{k<\nu} x_k \Big[\Big(\bigwedge_{i<j<\nu} \neg x_i = x_j\Big) \wedge \bigwedge_{k<\nu} \varphi_\lambda(x_k) \wedge \forall z\Big(\varphi_\lambda(z) \to \bigvee_{k<\nu} (z = x_k)\Big)\Big]$$

For the cases $\gamma = \kappa$, $\delta = \kappa$, define

$$\varphi_\kappa(x_k) : \bigwedge_{\beta<\kappa} \forall_{i<\beta} y_i \Big[\exists y_\beta \Big[\Big(\bigwedge_{i<\beta} (y_\beta \neq y_i)\Big) \wedge R(x_k, y_\beta)\Big]\Big]$$

and

$$\psi_{\kappa\lambda} : \bigwedge_{\beta<\kappa} \forall_{k<\beta} x_k \Big[\exists x_\beta \Big[\Big(\bigwedge_{k<\beta} (x_\beta \neq x_k)\Big) \wedge \varphi_\lambda(x_\beta)\Big]\Big]$$

Note that the last formulas say "for all $\beta < \kappa$ there exist more than β...", but it is equivalent to "there exist exactly κ..." in our class of models, because the models are all of size κ.

Thus $\psi_{g_0(\gamma), g_1(\delta)}$ is defined for all $\gamma \leqslant \kappa$ and $\delta \leqslant \kappa$. By the direction right to left of Theorem 4.25 this implies that the sets $F^{-1}U_{\gamma\delta}$ are Borel. This proves $\cong_T \leqslant_B \mathrm{id}_{2^\kappa}$.

Since κ is inaccessible, the other direction follows from Theorem 4.37, page 81. On the other hand one easily constructs such a reduction from scratch. Let us do it for the sake of completeness.

Let us show that $\mathrm{id} \leqslant_c \cong_T$. Let us modify the setting a little; let $C_{<\kappa} = \{\lambda < \kappa \mid \lambda \text{ is a cardinal}\}$ and $C^\omega_{<\kappa} = C_{<\kappa} \setminus \omega$ and let

$$h_0 \colon \kappa \to C^\omega_{<\kappa}$$

and

$$h_1 \colon \kappa \to C_{<\kappa}$$

be increasing bijections. Suppose $\eta \in \kappa^\kappa$ and define $f_\eta \colon C^\omega_{<\kappa} \to C_{<\kappa}$ by

$$f_\eta(\lambda) = [(h_1 \circ \eta \circ h_0^{-1})(\lambda)]^+$$

(recall that κ is inaccessible). Let us now build the model $\mathcal{M}_\eta$:

$$\operatorname{dom}\mathcal{M}_\eta = \bigcup_{\lambda\in C^\omega_{<\kappa}} \{(\lambda, f_\eta(\lambda))\}\times[f_\eta(\lambda)\cup f_\eta(\lambda)\times\lambda]$$

(that is, formally $\operatorname{dom}\mathcal{M}_\eta$ consists of pairs and triples the first projection being a pair of the form $(\lambda, f_\eta(\lambda)))$ and for all $x, y\in\operatorname{dom}\mathcal{M}_\eta$:

$$R(x,y) \iff \exists\lambda\exists\alpha\exists\beta\big(x = ((\lambda, f_\eta(\lambda)),\alpha)\wedge y = ((\lambda, f_\eta(\lambda)),\alpha,\beta)\big).$$

Denote the mapping $\eta\mapsto\mathcal{M}_\eta$ by G, i.e. $G(\eta)=\mathcal{M}_\eta$. Clearly $\mathcal{M}_\eta\models T$. Let us show that

$$\mathcal{M}_\eta\cong\mathcal{M}_\xi \iff \mathcal{M}_\eta=\mathcal{M}_\xi \iff \eta=\xi.$$

The implications from right to left are evident. Suppose $h\colon\mathcal{M}_\eta\to\mathcal{M}_\xi$ is an isomorphism. Since it preserves relations, the restrictions send bijectively the λ-levels to some other λ'-levels:

$$h\restriction\{(\lambda, f_\eta(\lambda))\}\times[\{\alpha\}\cup\{\beta\}\times\lambda]\to\{(\lambda', f_\eta(\lambda'))\}\times[\{\alpha'\}\cup\{\beta'\}\times\lambda']$$

is a bijection which implies $\lambda=\lambda'$. Further, by bijectivity, the map $\alpha\mapsto\alpha'$ induced by these restrictions is also bijective (by preservation of relations, pairs are sent to pairs), so this map $\alpha\mapsto\alpha'$ is a bijection between $f_\eta(\lambda)$ and $f_\xi(\lambda)$, thus they are the same cardinal for all λ, i.e. $f_\eta=f_\xi$.

For a model of the form $\mathcal{M}_\eta$ and $\alpha<\kappa$, let

$$\mathcal{M}_{\eta\restriction\alpha} = \bigcup_{\substack{\lambda\in C^\omega_{<\kappa}\\ \lambda<h_0(\alpha)}} \{(\lambda, f_\eta(\lambda))\}\times[f_\eta(\lambda)\cup f_\eta(\lambda)\times\lambda]$$

equipped with the relation $R^{\mathcal{M}_{\eta\restriction\alpha}} = R^{\mathcal{M}}\cap(\operatorname{dom}\mathcal{M}_{\eta\restriction\alpha})^2$.

Let us fix a well ordering of $\operatorname{dom}\mathcal{A}$ for each model $\mathcal{A}\in\operatorname{ran}G$ as follows. If $x,y\in\operatorname{dom}\mathcal{M}_\eta$, then

$$\begin{aligned} x\lessdot y \iff &\quad \operatorname{pr}_1(x)<\operatorname{pr}_1(y)\\ &\text{or}\quad \operatorname{pr}_1(x)=\operatorname{pr}_1(y)\wedge\operatorname{pr}_2(x)<\operatorname{pr}_2(y)\\ &\text{or}\quad \operatorname{pr}_1(x)=\operatorname{pr}_1(y)\wedge\operatorname{pr}_2(x)=\operatorname{pr}_2(y)\wedge\operatorname{pr}_3(x)<\operatorname{pr}_3(y)\end{aligned}$$

Note that in the last case it might happen that there is no third projection of x, in that case define $\operatorname{pr}_3(x)$ to be -1. (If $\operatorname{pr}_3(y)$ were also undefined, then we had $x=y$.) The initial segments with respect to $\lessdot$ are of size less than κ, because $f_\eta(\lambda)$ and λ are elements of $C_{<\kappa}$ and $\lessdot$ is clearly a well ordering. Moreover, since we added the $+$ in the definition of $f_\eta(\lambda)$, we have that $\forall\lambda\forall\eta(f_\eta(\lambda)>0)$, so we get the following:

($\star\star$) Suppose x is the γ^{th} element of the model with respect to $\lessdot$. Then $\operatorname{pr}_1(x)\leqslant\gamma$. Hence for any η

$$\begin{aligned} &\mathcal{M}_\eta\cap\{x\in\operatorname{dom}\mathcal{M}_\eta\mid \operatorname{OTP}_{\lessdot}(x)<\gamma\}\\ \subset\ &\mathcal{M}_{\eta\restriction(\gamma+1)}\end{aligned}$$

Note also that if $\mathcal{M}_{\eta\restriction\alpha} = \mathcal{M}_{\xi\restriction\alpha}$, then the identity map id: $\mathcal{M}_{\eta\restriction\alpha} = \mathcal{M}_{\xi\restriction\alpha}$ preserves $\lessdot$.

Recall the coding $\eta \mapsto \mathcal{A}_\eta$ of the Definition 4.13. In the definition it is assumed that $\operatorname{dom}\mathcal{A} = \kappa$, but instead of that we can use the well-ordering $\lessdot$. More precisely, for a given model $\mathcal{A}$, let $c(\mathcal{A})$ denote some η such that there is an isomorphism $f\colon \mathcal{A}_\eta \cong \mathcal{A}$ which preserves the ordering of the domain: $f(\alpha)$ is the α^{th} element of $\operatorname{dom}\mathcal{A}$ with respect to $\lessdot$. In our present case, $c\colon \operatorname{ran} G \to \kappa^\kappa$.

Let us show that the map $F = c \circ G\colon \eta \mapsto c(\mathcal{M}_\eta)$ is continuous and therefore is the intended bijection. For that purpose let us equip $\operatorname{ran} G$ with a topology τ. We will then show that G is continuous with respect to that topology and then show that also c is continuous.

Let τ be the topology on $\operatorname{ran} G$ generated by

$$U_p = \{\mathcal{M}_\eta \mid p \subset \eta\}$$

for $p \in \kappa^{<\kappa}$. In fact τ is the topology co-induced by G, so it trivially makes G continuous:

$$G^{-1}U_p = N_p.$$

Let us show that

$$U_p = \{\mathcal{M} \in \operatorname{ran} G \mid \mathcal{M}_p \subset \mathcal{M}\}. \qquad (\star\star\star)$$

Suppose $\mathcal{M}_\eta \in U_p$ for some η. This is equivalent to that there is ξ with $p \subset \xi$ such that $\mathcal{M}_\eta = \mathcal{M}_\xi$. This in turn is equivalent with $p \subset \eta$, since necessarily $\eta = \xi$. So $\mathcal{M}_\eta \in U_p$ implies

$$\begin{aligned}
\mathcal{M}_p &= \mathcal{M}_{\eta\restriction\operatorname{dom} p} \\
&= \mathcal{M}_\eta \cap \bigcup_{\substack{\lambda \in C^\omega_{<\kappa} \\ \lambda < h_0(\operatorname{dom} p)}} \{\lambda\} \times [f_\eta(\lambda) \cup f_\eta(\lambda) \times \lambda] \\
&\subset \mathcal{M}_\eta.
\end{aligned}$$

Assume that $\mathcal{M} \in \operatorname{ran} G$, $\mathcal{M}_p \subset \mathcal{M}$ and that η is such that $\mathcal{M} = \mathcal{M}_\eta$. Let us assume that ξ is such that $p \subset \xi$ and let us show that $\xi \restriction \operatorname{dom} p \subset \eta$. Let $\lambda < h_0(\operatorname{dom} p)$. Then because $f_\xi(\lambda) > 0$, we have

$$(\lambda, f_\xi(\lambda), 0) \in \mathcal{M}_p.$$

By the assumption $\mathcal{M}_p \subset \mathcal{M}_\eta$, this implies $(\lambda, f_\xi(\lambda), 0) \in \mathcal{M}_\eta$. By definition, this can only happen if $f_\eta(\lambda) = f_\xi(\lambda)$. Thus for all $\lambda < h_0(\operatorname{dom} p)$, we have $f_\eta(\lambda) = f_\xi(\lambda)$. Recall that h_1 and h_0 are an increasing bijections, so

$$\begin{aligned}
& [\forall\lambda < h_0(\operatorname{dom} p)](f_\eta(\lambda) = f_\xi(\lambda)) \\
\iff & [\forall\lambda < h_0(\operatorname{dom} p)]((h_1 \circ \eta \circ h_0^{-1})(\lambda) = (h_1 \circ \xi \circ h_0^{-1})(\lambda)) \\
\iff & [\forall\alpha < \operatorname{dom} p]((h_1 \circ \eta)(\alpha) = (h_1 \circ \xi)(\alpha)) \\
\iff & [\forall\alpha < \operatorname{dom} p](\eta(\alpha) = \xi(\alpha)) \\
\iff & [\forall\alpha < \operatorname{dom} p](\eta(\alpha) = p(\alpha))
\end{aligned}$$

$\Rightarrow p \subset \eta$.

Consider now the coding $c\colon \operatorname{ran} G \to \kappa^\kappa$. Let $N_{\xi\restriction\alpha}$ be a basic open set of κ^κ. Let $\mathcal{M}$ be a model in $c^{-1}N_{\xi\restriction\alpha}$. Let us show that there is an open τ-neighborhood of $\mathcal{M}$ inside $c^{-1}N_{\xi\restriction\alpha}$.

We know that $\xi\restriction\alpha$ decides a segment of $\mathcal{M}$ that is below γ^{th} element with respect to $\lessdot$, for some γ. Denote that segment by $S\subset\mathcal{M}$. Let η be such that $\mathcal{M}=\mathcal{M}_\eta$. From $(\star\star)$ we have:

$$\begin{aligned} S &\subset \mathcal{M}_\eta\cap\{x\in\operatorname{dom}\mathcal{M}_\eta \mid \mathrm{OTP}_{\lessdot}(x)<\gamma\}\\ &\subset \mathcal{M}_{\eta\restriction(\gamma+1)}\end{aligned}$$

Let us show that $U_{\eta\restriction(\gamma+1)}$ is an open neighborhood of $\mathcal{M}$ inside $c^{-1}[N_{\xi\restriction\alpha}]$. Suppose $\mathcal{M}\in U_{\eta\restriction(\gamma+1)}$ and $c(\mathcal{M})=\zeta$. Then by $(\star\star\star)$ we have $\mathcal{M}_{\eta\restriction(\gamma+1)}\subset\mathcal{M}$. Let $S'\subset\mathcal{M}$ be the subset of $\mathcal{M}$ decided by $\zeta\restriction\alpha$. Thus

$$\{\mathrm{OTP}_{\lessdot}(x)\mid x\in S'\}=\{\mathrm{OTP}_{\lessdot}(x)\mid x\in S\},$$

but by the note after $(\star\star)$ we have $S=S'$ and since $S\subset\mathcal{M}_{\eta\restriction(\gamma+1)}$ and $\mathcal{M}_{\eta\restriction(\gamma+1)}=\mathcal{M}_{\zeta\restriction(\gamma+1)}$, the codings must coincide and we have $\zeta\restriction\alpha=\xi\restriction\alpha$, i.e. $c(\mathcal{M})\in N_{\xi\restriction\alpha}$. □

Failures of Silver's Dichotomy

There are well-known dichotomy theorems for Borel equivalence relations on 2^ω. Two of them are:

4.40 Theorem (Silver, [47]). *Let $E\subset 2^\omega\times 2^\omega$ be a Π^1_1 equivalence relation. If E has uncountably many equivalence classes, then $\mathrm{id}_{2^\omega}\leqslant_B E$.* □

4.41 Theorem (Generalized Glimm-Effros dichotomy, [11]). *Let $E\subset 2^\omega\times 2^\omega$ be a Borel equivalence relation. Then either $E\leqslant_B \mathrm{id}_{2^\omega}$ or else $E_0\leqslant_c E$.* □

As in the case $\kappa=\omega$ we have the following also for uncountable κ (see Definition 4.36, page 81):

4.42 Theorem. *If κ-SD for Π^1_1 holds, then the κ-PSP holds for Σ^1_1-sets. More generally, if $\mathcal{C}\in\{\text{Borel},\Delta^1_1,\text{Borel}^*,\Sigma^1_1,\Pi^1_1\}$, then κ-SD for $\mathcal{C}$ implies κ-PSP for $\mathcal{C}'$, where elements in $\mathcal{C}'$ are all the complements of those in $\mathcal{C}$.*

Proof. Let us prove this for $\mathcal{C}=\Pi^1_1$, the other cases are similar. Suppose we have a Σ^1_1-set A. Let

$$E=\{(\eta,\xi)\mid \eta=\xi \text{ or } ((\eta\notin A)\wedge(\xi\notin A))\}.$$

Now $E=\mathrm{id}\cup(2^\kappa\setminus A)^2$. Since A is Σ^1_1, $(2^\kappa\setminus A)^2$ is Π^1_1 and because id is Borel, also E is Π^1_1. Obviously $|A|$ is the number of equivalence classes of E provided A is infinite. Then suppose $|A|>\kappa$. Then there are more than κ equivalence classes of E, so by κ-SD for Π^1_1, there is a reduction $f\colon \mathrm{id}\leqslant E$. This reduction in fact witnesses the PSP of A. □

The idea of using Kurepa trees for this purpose arose already in the paper [36] by Mekler and Väänänen.

4.43 Definition. If $t\subset 2^{<\kappa}$ is a tree, a *path* through t is a branch of length κ. A *κ-Kurepa tree* is a tree $K\subset 2^{<\kappa}$ which satisfies the following:

(a) K has more than κ paths,

(b) K is downward closed,

(c) for all $\alpha < \kappa$, the levels are small: $|\{p \in K \mid \operatorname{dom} p = \alpha\}| \leqslant |\alpha + \omega|$.

4.44 Theorem. *Assume one of the following:*

1. *κ is regular but not strongly inaccessible and there exists a κ-Kurepa tree $K \subset 2^{<\kappa}$,*
2. *κ is regular (might be strongly inaccessible), $2^\kappa > \kappa^+$ and there exists a tree $K \subset 2^{<\kappa}$ with more than κ but less than 2^κ branches.*

Then the Silver Dichotomy for κ does not hold. In fact there an equivalence relation $E \subset 2^\kappa \times 2^\kappa$ which is the union of a closed and an open set, has more than κ equivalence classes but $\mathrm{id}_{2^\kappa} \not\leqslant_B E$.

Proof. Let us break the proof according to the assumptions (1) and (2). So first let us consider the case where κ is not strongly inaccessible and there is a κ-Kurepa tree.

(1): Let us carry out the proof in the case $\kappa = \omega_1$. It should be obvious then how to generalize it to any κ not strongly inaccessible. So let $K \subset 2^{<\omega_1}$ be an ω_1-Kurepa tree. Let P be the collection of all paths of K. For $b \in P$, denote $b = \{b_\alpha \mid \alpha < \omega_1\}$ where b_α is an element of K with domain α.

Let
$$C = \{\eta \in 2^{\omega_1} \mid \eta = \bigcup_{\alpha<\omega_1} b_\alpha, b \in P\}.$$
Clearly C is closed.

Let $E = \{(\eta,\xi) \mid (\eta \notin C \wedge \xi \notin C) \vee (\eta \in C \wedge \eta = \xi)\}$. In words, E is the equivalence relation whose equivalence classes are the complement of C and the singletons formed by the elements of C. E is the union of the open set $\{(\eta,\xi) \mid \eta \notin C \wedge \xi \notin C\}$ and the closed set $\{(\eta,\xi) \mid \eta \in C \wedge \eta = \xi\} = \{(\eta,\eta) \mid \eta \in C\}$. The number of equivalence classes equals the number of paths of K, so there are more than ω_1 of them by the definition of Kurepa tree.

Let us show that $\mathrm{id}_{2^{\omega_1}}$ is not embeddable to E. Suppose that $f\colon 2^{\omega_1} \to 2^{\omega_1}$ is a Borel reduction. We will show that then K must have a level of size $\geqslant \omega_1$ which contradicts the definition of Kurepa tree. By Lemma 4.34, page 79, there is a co-meager set D on which $f \restriction D$ is continuous. There is at most one $\eta \in 2^{\omega_1}$ whose image $f(\eta)$ is outside C, so without loss of generality $f[D] \subset C$. Let p be an arbitrary element of K such that $f^{-1}[N_p] \neq \varnothing$. By continuity there is a $q \in 2^{<\omega_1}$ with $f[N_q \cap D] \subset N_p$. Since D is co-meager, there are η and ξ such that $\eta \neq \xi$, $q \subset \eta$ and $q \subset \xi$. Let $\alpha_1 < \omega_1$ and p_0 and p_1 be extensions of p with the properties $p_0 \subset f(\eta)$, $p_1 \subset f(\xi)$, $\alpha_1 = \operatorname{dom} p_0 = \operatorname{dom} p_1$, $f^{-1}[N_{p_0}] \neq \varnothing \neq f^{-1}[N_{p_1}]$ and $N_{p_0} \cap N_{p_1} = \varnothing$. Note that p_0 and p_1 are in K. Then, again by continuity, there are q_0 and q_1 such that $f[N_{q_0} \cap D] \subset N_{p_0}$ and $f[N_{q_1} \cap D] \subset N_{p_1}$. Continue in the same manner to obtain α_n and $p_s \in K$ for each $n < \omega$ and $s \in 2^{<\omega}$ so that $s \subset s' \iff p_s \subset p_{s'}$ and $\alpha_n = \operatorname{dom} p_s \iff n = \operatorname{dom} s$. Let $\alpha = \sup_{n<\omega} \alpha_n$. Now clearly the α's level of K contains continuum many elements: by (b) in the definition of Kurepa tree it contains all the elements of the form $\bigcup_{n<\omega} p_{\eta\restriction n}$ for $\eta \in 2^\omega$ and $2^\omega \geqslant \omega_1$.

If κ is arbitrary regular not strongly inaccessible cardinal, then the proof is the same, only instead of ω steps one has to do λ steps where λ is the least cardinal satisfying $2^\lambda \geqslant \kappa$.

(2): The argument is even simpler. Define the equivalence relation E exactly as above. Now E is again closed and has as many equivalence classes as is the number of paths in K. Thus the number of equivalence classes is $> \kappa$ but id cannot be reduced to E since there are less than 2^κ equivalence classes. □

4.45 Remark. Some related results:

1. In L, the PSP fails for closed sets for all uncountable regular κ. This is because "weak Kurepa trees" exist (see the proof sketch of (3) below for the definition of "weak Kurepa tree").
2. (P. Schlicht) In Silver's model where an inaccessible κ is made into ω_2 by Levy collapsing each ordinal below to ω_1 with countable conditions, every Σ^1_1 subset X of 2^{ω_1} obeys the PSP.
3. Supercompactness does not imply the PSP for closed sets.

Sketch of a proof of item (3). Suppose κ is supercompact and by a reverse Easton iteration add to each inaccessible α a "weak Kurepa tree", i.e., a tree T_α with α^+ branches whose β^{th} level has size β for stationary many $\beta < \alpha$. The forcing at stage α is α-closed and the set of branches through T_κ is a closed set with no perfect subset. If $j\colon V \to M$ witnesses λ-supercompactness ($\lambda > \kappa$) and G is the generic then we can find G^* which is $j(P)$-generic over M containing $j[G]$: Up to λ we copy G, between λ and $j(\kappa)$ we build G^* using λ^+ closure of the forcing and of the model M, and at $j(\kappa)$ we form a master condition out of $j[G(\kappa)]$ and build a generic below it, again using λ^+ closure. □

4.46 Corollary. *The consistency of the Silver Dichotomy for Borel sets on ω_1 with CH implies the consistency of a strongly inaccessible cardinal. In fact, if there is no equivalence relation witnessing the failure of the Silver Dichotomy for ω_1, then ω_2 is inaccessible in L.*

Proof. By a result of Silver, if there are no ω_1-Kurepa trees, then ω_2 is inaccessible in L, see Exercise 27.5 in Part III of [25]. □

Open Problem. Is the Silver Dichotomy for uncountable κ consistent?

4.4.3 Regularity Properties and Definability of the CUB Filter

In the standard descriptive theory ($\kappa = \omega$), the notions of Borel, Δ^1_1 and Borel* coincide and one of the most important observations in the theory is that such sets have the Property of Baire and that the Σ^1_1-sets obey the Perfect Set Property. In the case $\kappa > \omega$ the situation is more complicated as the following shows. It was already pointed out in the previous section that Borel $\subsetneq \Delta^1_1$. In this section we focus on the cub filter

$$\text{CUB} = \{\eta \in 2^\kappa \mid \eta^{-1}\{1\} \text{ contains a cub}\}.$$

The set CUB is easily seen to be Σ^1_1: the set

$$\{(\eta, \xi) \mid (\eta^{-1}\{1\} \subset \xi^{-1}\{1\}) \wedge (\eta^{-1}\{1\} \text{ is cub})\}$$

is Borel. CUB (restricted to cofinality ω, see Definition 4.51 below) will serve (consistently) as a counterexample to $\Delta^1_1 = \text{Borel*}$, but we will show that it is also consistent that CUB is Δ^1_1. The latter implies that it is consistent that Δ^1_1-sets do not have the Property of Baire and we will also show that in a forcing extension of L, Δ^1_1-sets all have the Property of Baire.

4.47 Definition. A *nowhere dense set* is a subset of a set whose complement is dense and open. Let $X \subset \kappa^\kappa$. A subset $M \subset X$ is *κ-meager in X*, if $M \cap X$ is the union of no more than κ nowhere dense sets,

$$M = \bigcup_{i<\kappa} N_i.$$

We usually drop the prefix "κ-".

Clearly κ-meager sets form a κ-complete ideal. A *co-meager* set is a set whose complement is meager.

A subset $A \subset X$ *has the Property of Baire* or shorter *P.B.*, if there exists an open $U \subset X$ such that the symmetric difference $U \triangle A$ is meager.

Halko showed in [9] that

4.48 Theorem ([9]). *Borel sets have the Property of Baire.* □

(The same proof as when $\kappa = \omega$ works.) This is independent of the assumption $\kappa^{<\kappa} = \kappa$. Borel* sets do not in general have the Property of Baire.

4.49 Definition ([34, 36, 18]). A $\kappa^+\kappa$-tree t is a *$\kappa\lambda$-canary tree* if for all stationary $S \subset S^\kappa_\lambda$ it holds that if $\mathbb{P}$ does not add subsets of κ of size less than κ and $\mathbb{P}$ kills the stationarity of S, then $\mathbb{P}$ adds a κ-branch to t.

Remark. Hyttinen and Rautila [18] use the notation *κ-canary tree* for our *$\kappa^+\kappa$-canary tree.*

It was shown by Mekler and Shelah [34] and Hyttinen and Rautila [18] that it is consistent with ZFC+GCH that there is a $\kappa^+\kappa$-canary tree *and* it is consistent with ZFC+GCH that there are no $\kappa^+\kappa$-canary trees. The same proof as in [34, 18] gives the following:

4.50 Theorem. *Assume GCH and assume $\lambda < \kappa$ are regular cardinals. Let $\mathbb{P}$ be the forcing which adds κ^+ Cohen subsets of κ. Then in the forcing extension there are no $\kappa\lambda$-canary trees.* □

4.51 Definition. Suppose $X \subset \kappa$ is stationary. For each such X define the set

$$\text{CUB}(X) = \{\eta \in 2^\kappa \mid X \setminus \eta^{-1}\{1\} \text{ is non-stationary}\},$$

so CUB(X) is "cub in X".

4.52 Theorem. *In the following κ satisfies $\kappa^{<\kappa} = \kappa > \omega$ unless stated otherwise.*

1. $\text{CUB}(S^\kappa_\omega)$ *is Borel*.*
2. *For all regular $\lambda < \kappa$, $\text{CUB}(S^\kappa_\lambda)$ is not Δ^1_1 in the forcing extension after adding κ^+ Cohen subsets of κ.*
3. *If $V = L$, then for every stationary $S \subset \kappa$, the set $\text{CUB}(S)$ is not Δ^1_1.*
4. *Assume GCH and that κ is not a successor of a singular cardinal. For any stationary set $Z \subset \kappa$ there exists a forcing notion $\mathbb{P}$ which has the κ^+-c.c., does not add bounded subsets of κ and preserves GCH and stationary subsets of $\kappa \setminus Z$ such that $\text{CUB}(\kappa \setminus Z)$ is Δ^1_1 in the forcing extension.*

5. *Let the assumptions for κ be as in (4). For all regular $\lambda < \kappa$, $\mathrm{CUB}(S^\kappa_\lambda)$ is Δ^1_1 in a forcing extension as in (4).*

6. $\mathrm{CUB}(X)$ *does not have the Property of Baire for stationary $X \subset \kappa$. Here the assumption $\kappa^{<\kappa} = \kappa$ is not needed. (Proved by Halko and Shelah in [10] for $X = \kappa$)*

7. *It is consistent that all Δ^1_1-sets have the Property of Baire. (Independently known to P. Lücke and P. Schlicht.)*

Proof of Theorem 4.52.

Proof of item (1)**.** Let $t = [\kappa]^{<\omega}$ (increasing functions ordered by end extension) and for all branches $b \subset t$

$$h(b) = \{\xi \in 2^\kappa \mid \xi(\sup_{n<\omega} b(n)) \neq 0\}.$$

Now if $\kappa \setminus \xi^{-1}\{0\}$ contains an ω-cub set C, then player **II** has a winning strategy in $G(t, h, \xi)$: for her n^{th} move she picks an element $x \in t$ with domain $2n + 2$ such that $x(2n + 1)$ is in C. Suppose the players picked a branch b in this way. Then the condition $\xi(b(2n + 1)) \neq 0$ holds for all $n < \omega$ and because C is cub outside $\xi^{-1}\{0\}$, we have $\xi(\sup_{n<\omega} b(n)) \neq 0$.

Suppose on the contrary that $S = \xi^{-1}\{0\}$ is stationary. Let σ be any strategy of player **II**. Let C_σ be the set of ordinals closed under this strategy. It is a cub set, so there is an $\alpha \in C_\sigma \cap S$. Player **I** can now easily play towards this ordinal to force $\alpha = \sup_{n<\omega} b(n)$ and so $\xi(\sup_{n<\omega} b(b)) = 0$, so σ cannot be a winning strategy. $\square_{\text{item (1)}}$

Proof of item (2)**.** It is not hard to see that $\mathrm{CUB}^\kappa_\lambda$ is Δ^1_1 if and only if there exists a $\kappa\lambda$-canary tree. This fact is proved in detail in [36] in the case $\kappa = \omega_1$, $\lambda = \omega$ and the proof generalizes easily to any regular uncountable κ along with the assumption $\kappa^{<\kappa} = \kappa$. So the statement follows from Theorem 4.50. $\square_{\text{item (2)}}$

Proof of item (3)**.** Suppose that φ is Σ_1 and for simplicity assume that φ has no parameters. Then for $x \subset \kappa$ we have:

Claim. $\varphi(x)$ holds if and only if the set A of those α for which there exists $\beta > \alpha$ such that

$$L_\beta \models \big(\mathrm{ZF}^- \wedge (\omega < \alpha \text{ is regular}) \wedge ((S \cap \alpha) \text{ is stationary }) \wedge \varphi(x \cap \alpha)\big)$$

contains $C \cap S$ for some cub set C.

Proof of the Claim. "$\Rightarrow$". If $\varphi(x)$ holds then choose a continuous chain $(M_i \mid i < \kappa)$ of elementary submodels of some large ZF^- model L_θ so that x and S belong to M_0 and the intersection of each M_i with κ is an ordinal α_i less than κ. Let C be the set of α_i's, cub in κ. Then any α in $C \cap S$ belongs to A by condensation.

"$\Leftarrow$". If $\varphi(x)$ fails then let C be any cub in κ and let D be the cub of $\alpha < \kappa$ such that $H(\alpha)$ is the Skolem Hull in some large L_θ of α together with $\{\kappa, S, C\}$ contains no ordinals in the interval $[\alpha, \kappa)$. Let α be the least element of $S \cap \lim(D)$. Then α does not belong to A: If L_β satisfies $\varphi(x \cap \alpha)$ then β must be greater than $\bar\beta$ where $\overline{H(\alpha)} = L_{\bar\beta}$ is the transitive collapse of $H(\alpha)$, because $\varphi(x \cap \alpha)$ fails in $\overline{H(\alpha)}$. But as $\lim(D) \cap \alpha$ is an element of $L_{\bar\beta+2}$ and is disjoint from S, it follows that either α is singular in L_β or $S \cap \alpha$ is not stationary in $L_{\bar\beta+2}$ and hence not in L_β. Of course α does belong to C so we have shown that A does not contain $S \cap C$ for an arbitrary cub C in κ. $\square_{\text{Claim}}$

It follows from the above that any Σ_1 subset of 2^κ is Δ_1 over $(L_\kappa^+, \text{CUB}(S))$ and therefore if $\text{CUB}(S)$ were Δ_1 then any Σ_1 subset of 2^κ would be Δ_1, a contradiction. $\square_{\text{item (3)}}$

Proof of item (4)**.** If $X \subset 2^\kappa$ is Δ_1^1, then $\{\eta \in X \mid \eta^{-1}\{1\} \subset \kappa \setminus Z\}$ is Δ_1^1, so it is sufficient to show that we can force a set $E \subset Z$ which has the claimed property. So we force a set $E \subset Z$ such that E is stationary but $E \cap \alpha$ is non-stationary in α for all $\alpha < \kappa$ and $\kappa \setminus E$ is fat. A set is *fat* if its intersection with any cub set contains closed increasing sequences of all order types $< \kappa$.

This can be easily forced with

$$\mathbb{R} = \{p\colon \alpha \to 2 \mid \alpha < \kappa, p^{-1}\{1\} \cap \beta \subset Z \text{ is non-stationary in } \beta \text{ for all } \beta \leqslant \alpha\}$$

ordered by end-extension. It is easy to see that for any $\mathbb{R}$-generic G the set $E = (\cup G)^{-1}\{1\}$ satisfies the requirements. Also $\mathbb{R}$ does not add bounded subsets of κ and has the κ^+-c.c. and does not kill stationary sets.

Without loss of generality assume that such E exists in V and that $0 \in E$.

Next let $\mathbb{P}_0 = \{p\colon \alpha \to 2^{<\alpha} \mid \alpha < \kappa, p(\beta) \in 2^\beta, p(\beta)^{-1}\{1\} \subset E\}$. This forcing adds a $\diamondsuit_E$-sequence $\langle A_\alpha \mid \alpha \in E\rangle$ (if G is generic, set $A_\alpha = (\cup G)(\alpha)^{-1}\{1\}$) such that for all $B \subset E$ there is a stationary $S \subset E$ such that $A_\alpha = B \cap \alpha$ for all $\alpha \in S$. This forcing $\mathbb{P}_0$ is $< \kappa$-closed and clearly has the κ^+-c.c., so it is easily seen that it does not add bounded subsets of κ and does not kill stationary sets.

Let $\psi(G, \eta, S)$ be a formula with parameters $G \in (2^{<\kappa})^\kappa$ and $\eta \in 2^\kappa$ and a free variable $S \subset \kappa$ which says:

$$\forall \alpha < \kappa(\alpha \in S \iff G(\alpha)^{-1}\{1\} = \eta^{-1}\{1\} \cap \alpha).$$

If $\langle G(\alpha)^{-1}\{1\}\rangle_{\alpha<\kappa}$ happens to be a $\diamondsuit_E$-sequence, then S satisfying ψ is always stationary. Thus if G_0 is $\mathbb{P}_0$-generic over V and $\eta \in 2^E$, then $(\psi(G_0, \eta, S) \to (S \text{ is stationary}))^{V[G_0]}$.

For each $\eta \in 2^E$, let $\dot{S}_\eta$ be a nice $\mathbb{P}_0$-name for the set S such that $V[G_0] \models \psi(G_0, \eta, S)$ where G_0 is $\mathbb{P}_0$-generic over V. By the definitions, $\mathbb{P}_0 \Vdash$ "$\dot{S}_\eta \subset \check{E}$ is stationary" and if $\eta \neq \eta'$, then $\mathbb{P}_0 \Vdash$ "$\dot{S}_\eta \cap \dot{S}_{\eta'}$ is bounded".

Let us enumerate $E = \{\beta_i \mid i < \kappa\}$ such that $i < j \Rightarrow \beta_i < \beta_j$ and for $\eta \in 2^E$ and $\gamma \in \kappa$ define $\eta + \gamma$ to be the $\xi \in 2^E$ such that $\xi(\beta_i) = 1$ for all $i < \gamma$ and $\xi(\beta_{\gamma+j}) = \eta(\beta_j)$ for $j \geqslant 0$. Let

$$F_0 = \{\eta \in 2^E \mid \eta(0) = 0\}^V \qquad (*)$$

Now for all $\eta, \eta' \in F_0$ and $\alpha, \alpha' \in \kappa$, $\eta + \alpha = \eta' + \alpha'$ implies $\eta = \eta'$ and $\alpha = \alpha'$. Let us now define the formula $\varphi(G, \eta, X)$ with parameters $G \in (2^{<\kappa})^\kappa$, $\eta \in 2^\kappa$ and a free variable $X \subset \kappa \setminus E$ which says:

$$\begin{aligned}(\eta(0) = 0) \wedge \forall \alpha < \kappa \big[\ &(\alpha \in X \to \exists S(\psi(G, \eta + 2\alpha, S) \wedge S \text{ is non-stationary}))\\ &\wedge (\alpha \notin X \to \exists S(\psi(G, \eta{+}2\alpha{+}1, S) \wedge S \text{ is non-stationary}))\big].\end{aligned}$$

Now, we will construct an iterated forcing $\mathbb{P}_{\kappa^+}$, starting with $\mathbb{P}_0$, which kills the stationarity of $\dot{S}_\eta$ for suitable $\eta \in 2^E$, such that if G is $\mathbb{P}_{\kappa^+}$-generic, then for all $S \subset \kappa \setminus E$, S is stationary if and only if

$$\exists \eta \in 2^E(\varphi(G_0, \eta, S))$$

where $G_0 = G \restriction \{0\}$. In this model, for each $\eta \in F_0$, there will be a unique X such that $\varphi(G_0, \eta, X)$, so let us denote this X by X_η. It is easy to check that the mapping $\eta \mapsto X_\eta$ defined by φ is Σ^1_1 so in the result, also $\mathcal{S} = \{S \subset \kappa \setminus E \mid S \text{ is stationary}\}$ is Σ^1_1. Since cub and non-stationarity are also Σ^1_1, we get that $\mathcal{S}$ is Δ^1_1, as needed.

Let us show how to construct the iterated forcing. For $S \subset \kappa$, we denote by $T(S)$ the partial order of all closed increasing sequences contained in the complement of S. Clearly $T(S)$ is a forcing that kills the stationarity of S. If the complement of S is fat and S is non-reflecting, then $T(S)$ has all the nice properties we need, as the following claims show. Let $f\colon \kappa^+ \setminus \{0\} \to \kappa^+ \times \kappa^+$ be a bijection such that $f_1(\gamma) \leqslant \gamma$.

$\mathbb{P}_0$ is already defined and it has the κ^+-c.c. and it is $< \kappa$-closed. Suppose that $\mathbb{P}_i$ has been defined for $i < \alpha$ and σ_i has been defined for $i < \cup\alpha$ such that σ_i is a (nice) $\mathbb{P}_i$-name for a κ^+-c.c. partial order. Also suppose that for all $i < \cup\alpha$, $\{(\dot{S}_{ij}, \delta_{ij}) \mid j < \kappa^+\}$ is the list of all pairs $(\dot{S}, \delta)$ such that $\dot{S}$ is a nice $\mathbb{P}_i$-name for a subset of $\check{\kappa} \setminus \check{E}$ and $\delta < \kappa$, and suppose that

$$g_\alpha\colon \{\dot{S}_{f(i)} \mid i < \alpha\} \to F_0 \qquad (***)$$

is an injective function, where F_0 is defined at $(*)$.

If α is a limit, let $\mathbb{P}_\alpha$ consist of those $p\colon \alpha \to \bigcup_{i<\alpha} \operatorname{dom} \sigma_i$ with $|\operatorname{sprt}(p)| < \kappa$ (support, see page 60) such that for all $\gamma < \alpha$, $p \restriction \gamma \in \mathbb{P}_\gamma$ and let $g_\alpha = \bigcup_{i<\alpha} g_i$. Suppose α is a successor, $\alpha = \gamma + 1$. Let $\{(\dot{S}_{\gamma j}, \delta_{\gamma j}) \mid j < \kappa\}$ be the the list of pairs as defined above. Let $(\dot{S}, \delta) = (\dot{S}_{f(\gamma)}, \delta_{f(\gamma)})$ where f is the bijection defined above. If there exists $i < \gamma$ such that $\dot{S}_{f(i)} = \dot{S}_{f(\gamma)}$ (i.e. $\dot{S}_i$ has been already under focus), then let $g_\alpha = g_\gamma$. Otherwise let

$$g_\alpha = g_\gamma \cup \{(\dot{S}_{f(\gamma)}, \eta)\}.$$

where η is some element in $F_0 \setminus \operatorname{ran} g_\gamma$. Doing this, we want to make sure that in the end $\operatorname{ran} g_{\kappa^+} = F_0$. We omit the technical details needed to ensure that.

Denote $\eta = g(\dot{S}_{f(\gamma)})$. Let σ_γ be a $\mathbb{P}_\gamma$-name such that for all $\mathbb{P}_\gamma$-generic G_γ it holds that

$$\mathbb{P}_\gamma \Vdash \begin{cases} \sigma_\gamma = T(\dot{S}_{\eta+2\delta}), & \text{if } V[G_\gamma] \models [(\delta_{f(\gamma)} \in \dot{S}_{f(\gamma)}) \wedge (\dot{S}_{f(\gamma)} \text{ is stationary})] \\ \sigma_\gamma = T(\dot{S}_{\eta+2\delta+1}), & \text{if } V[G_\gamma] \models [(\delta_{f(\gamma)} \notin \dot{S}_{f(\gamma)}) \wedge (\dot{S}_{f(\gamma)} \text{ is stationary})] \\ \sigma_\gamma = \{\check{\varnothing}\}, & \text{otherwise.} \end{cases}$$

Now let $\mathbb{P}_\alpha$ be the collection of sequences $p = \langle \rho_i \rangle_{i \leqslant \gamma}$ such that $p \restriction \gamma = \langle \rho_i \rangle_{i<\gamma} \in \mathbb{P}_\gamma$, $\rho_\gamma \in \operatorname{dom} \sigma_\gamma$ and $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \rho_\gamma \in \sigma_\gamma$ with the ordering defined in the usual way.

Let G be $\mathbb{P}_{\kappa^+}$-generic. Let us now show that the extension $V[G]$ satisfies what we want, namely that $S \subset \kappa \setminus E$ is stationary if and only if there exists $\eta \in 2^E$ such that $S = X_\eta$ (Claims 3 and 4 below).

Claim 1. For $\alpha \leqslant \kappa^+$ the forcing $\mathbb{P}_\alpha$ does not add bounded subsets of κ and the suborder

$$\mathbb{Q}_\alpha = \{p \mid p \in \mathbb{P}_\alpha, p = \langle \check{\rho}_i \rangle_{i<\alpha} \text{ where } \rho_i \in V \text{ for } i < \alpha\}$$

is dense in $\mathbb{P}_\alpha$.

Proof of Claim 1. Let us show this by induction on $\alpha \leqslant \kappa^+$. For $\mathbb{P}_0$ this is already proved and the limit case is left to the reader. Suppose this is proved for all $\gamma < \alpha < \kappa^+$ and $\alpha = \beta+1$. Then suppose $p \in \mathbb{P}_\alpha$, $p = \langle \rho_i \rangle_{i<\alpha}$. Now $p \restriction \beta \Vdash \rho_\beta \in \sigma_\beta$. Since by the induction hypothesis $\mathbb{P}_\beta$ does not add bounded subsets of κ and $\mathbb{Q}_\beta$ is dense in $\mathbb{P}_\beta$, there exists a condition $r \in \mathbb{Q}_\beta$, $r > p \restriction \beta$ and a standard name $\check{q}$ such that $r \Vdash \check{q} = \rho_\beta$. Now $r^\frown(\check{q})$ is in $\mathbb{Q}_\alpha$, so it is dense in $\mathbb{P}_\alpha$. To show that $\mathbb{P}_\alpha$ does not add bounded sets, it is enough to show that $\mathbb{Q}_\alpha$ does not. Let us think of $\mathbb{Q}_\alpha$ as a suborder of the product $\prod_{i<\alpha} 2^{<\kappa}$. Assume that τ is a $\mathbb{Q}_\alpha$-name and $p \in \mathbb{Q}_\alpha$ forces that $|\tau| = \check{\lambda} < \check{\kappa}$ for some cardinal λ. Then let $\langle M_\delta \rangle_{\delta<\kappa}$ be a sequence of elementary submodels of $H(\kappa^+)$ such that for all δ, β

(a) $|M_\delta| < \kappa$

(b) $\delta < \beta \Rightarrow M_\delta \preceq M_\beta$,

(c) $M_\delta \cap \kappa \subset M_\delta$,

(d) if β is a limit ordinal, then $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$,

(e) if $\kappa = \lambda^+$, then $M_\delta^{<\lambda} \subset M_\delta$ and if κ is inaccessible, then $M_\delta^{|M_\delta|} \subset M_{\delta+1}$,

(f) $M_\alpha \in M_{\alpha+1}$,

(g) $\{p, \kappa, \mathbb{Q}_\alpha, \tau, \check{E}\} \subset M_0$.

This (especially (e)) is possible since κ is not a successor of a singular cardinal and GCH holds. Now the set $C = \{M_\delta \cap \kappa \mid \delta < \kappa\}$ is cub, so because $\kappa \setminus E$ is fat, there is a closed sequence s of length $\lambda+1$ in $C \setminus E$. Let $(\delta_i)_{i\leqslant\lambda}$ be the sequence such that $s = \langle M_{\delta_i} \cap \kappa \rangle_{i\leqslant\lambda}$. For $q \in \mathbb{Q}_\alpha$, let

$$m(q) = \inf_{\gamma \in \operatorname{sprt} q} \operatorname{ran} q(\gamma). \tag{$\star$}$$

Let $p_0 = p$ and for all $i < \gamma$ let $p_{i+1} \in M_{\delta_{i+1}} \setminus M_{\delta_i}$ be such that $p_i < p_{i+1}$, p_{i+1} decides $i+1$ first values of τ (think of τ as a name for a function $\lambda \to \kappa$ and that p_i decides the first i values of that function) and $m(p_{i+1}) \geqslant M_{\delta_i} \cap \kappa$. This p_{i+1} can be found because clearly $p_i \in M_{\delta_{i+1}}$ and $M_{\delta_{i+1}}$ is an elementary submodel. If i is a limit, $i < \lambda$, then let p_i be an upper bound of $\{p_j \mid j < i\}$ which can be found in $M_{\delta_{i+1}}$ by the assumptions (f), (e) and (b), and because $M_{\delta_i} \cap \kappa \notin E$. Finally let p_λ be an upper bound of $\langle p_i \rangle_{i<\lambda}$ which exists because for all $\alpha \in \bigcup_{i<\lambda} \operatorname{sprt} p_i$ $\sup_{i<\lambda} \operatorname{ran} p_i(\alpha) = M_{\delta_\lambda} \cap \kappa$ is not in E and the forcing is closed under such sequences. So p_λ decides the whole τ. This completes the proof of the claim. $\square_{\text{Claim 1}}$

So for simplicity, instead of $\mathbb{P}_{\kappa^+}$ let us work with $\mathbb{Q}_{\kappa^+}$.

Claim 2. Let G be $\mathbb{P}_{\kappa^+}$-generic over V. Suppose $S \subset \kappa$, $S \in V[G]$ and $\dot{S}$ is a nice name for a subset of κ such that $\dot{S}_G = S$. Then let γ be the smallest ordinal with $S \in V[G_\gamma]$. If $(S \subset \kappa \setminus E \text{ is stationary})^{V[G_\gamma]}$, then S is stationary in $V[G]$. If $\dot{S} = \dot{S}_\eta$ for some $\eta \in V$ and $V[G_\gamma] \models \sigma_\gamma \neq T((\dot{S}_\eta)_{G_\gamma \restriction \{0\}})$ for all $\gamma < \kappa^+$, then S is stationary in $V[G]$.

Proof of Claim 2. Recall, σ_γ is as in the construction of $\mathbb{P}_{\kappa^+}$. Suppose first that $S \subset \kappa \setminus E$ is a stationary set in $V[G_\gamma]$ for some $\gamma < \kappa^+$. Let us show that S is stationary in $V[G]$. Note that $V[G] = V[G_\gamma][G^\gamma]$ where $G^\gamma = G \restriction \{\alpha \mid \alpha \geqslant \gamma\}$. Let us show this in the case $\gamma = 0$ and

$S \in V$, the other cases being similar. Let $\dot{C}$ be a name and p a condition which forces that $\dot{C}$ is cub. Let us show that then $p \Vdash \check{S} \cap \dot{C} \neq \check{\varnothing}$. For $q \in \mathbb{Q}_{\kappa^+}$ let $m(q)$ be defined as in $(\star)$ above.

Like in the proof of Claim 1, construct a continuous increasing sequence $\langle M_\alpha \rangle_{\alpha<\kappa}$ of elementary submodels of $H(\kappa^{++})$ such that $\{p, \kappa, \mathbb{P}_{\kappa^+}, \check{S}, \dot{C}\} \subset M_0$ and $M_\alpha \cap \kappa$ is an ordinal. Since $\{M_\alpha \cap \kappa \mid \alpha < \kappa, M_\alpha \cap \kappa = \alpha\}$ is cub, there exists $\alpha \in S$ such that $M_\alpha \cap \kappa = \alpha$ and because E does not reflect to α there exists a cub sequence

$$c \subset \{M_\beta \cap \kappa \mid \beta < \alpha, M_\beta \cap \kappa = \beta\} \setminus E,$$

$c = \langle c_i \rangle_{i<\mathrm{cf}(\alpha)}$. Now, similarly as in the proof of Claim 1, we can choose an increasing $\langle p_i \rangle_{i \leqslant \mathrm{cf}(\alpha)}$ such that $p_0 = p$, $p_i \in \mathbb{Q}_{\kappa^+}$ for all i, $p_{i+1} \Vdash \check{\beta} \in \dot{C}$ for some $c_i \leqslant \beta \leqslant c_{i+1}$, $p_{i+1} \in M_{c_{i+1}} \setminus M_{c_i}$ and $m(p_{i+1}) \geqslant c_i$. If i is a limit, let p_i be again an upper bound of $\{p_j \mid j < i\}$ in M_{c_i}. Since the limits are not in E, the upper bounds exist. Finally $p_{\mathrm{cf}(\alpha)} \Vdash \alpha \in \dot{C}$, which implies $p_{\mathrm{cf}(\alpha)} \Vdash \check{S} \cap \dot{C} \neq \varnothing$, because α was chosen from S.

Assume then that $\dot{S} = \dot{S}_\eta$ for some $\eta \in V$ such that

$$V[G_\gamma] \models \sigma_\gamma \neq T((\dot{S}_\eta)_{G_\gamma \restriction \{0\}})$$

for all $\gamma < \kappa^+$. To prove that $(\dot{S}_\eta)_G$ is stationary in $V[G]$, we carry the same argument as the above, a little modified. Let us work in $V[G_0]$ and let p_0 force that

$$\forall \gamma < \kappa^+ (\sigma_\gamma \neq T(S_\eta)).$$

(This p_0 exists for example because there is at most one γ such that $\sigma_\gamma = T(S_\eta)$) Build the sequences c, $\langle M_{c_i} \rangle_{i<\mathrm{cf}(\alpha)}$ and $\langle p_i \rangle_{i<\mathrm{cf}(\alpha)}$ in the same fashion as above, except that assume additionally that the functions g_{κ^+} and f, defined along with $\mathbb{P}_{\kappa^+}$, are in M_{c_0}.

At the successor steps one has to choose p_{i+1} such that for each $\gamma \in \operatorname{sprt} p_i$, p_{i+1} decides σ_γ. This is possible, since there are only three choices for σ_γ, namely $\{\varnothing\}$, $T(S_{\xi+2\alpha+1})$ or $T(S_{\xi+2\alpha})$ where ξ and α are justified by the functions g_{κ^+} and f. For all $\gamma \in \operatorname{sprt} p_i$ let us denote by ξ_γ the function such that $p_{i+1} \restriction \gamma \Vdash \sigma_\gamma = T(S_{\xi_\gamma})$. Clearly $\eta \neq \xi_\gamma$ for all $\gamma \in \operatorname{sprt} p_i$. Further demand that $m(p_{i+1}) > \sup(S_\eta \cap S_{\xi_\gamma})$ for all $\gamma \in \operatorname{sprt} p_i$. It is possible to find such p_{i+1} from M_{i+1} because M_{i+1} is an elementary submodel and such can be found in $H(\kappa^{++})$ since $\xi_\gamma \neq \eta$ and by the definitions $S_\eta \cap S_{\xi_\gamma}$ is bounded. $\square_{\text{Claim 2}}$

Claim 3. In $V[G]$ the following holds: if $S \subset \kappa \setminus E$ is stationary, then there exists $\eta \in 2^E$ with $\eta(0) = 0$ such that $S = X_\eta$.

Proof of Claim 3. Recall the function g_{κ^+} from the construction of $\mathbb{P}_{\kappa^+}$ (defined at $(***)$ and the paragraph below that). Let $\eta = g_{\kappa^+}(\dot{S})$ where $\dot{S}$ is a nice name $\dot{S} \in V$ such that $\dot{S}_G = S$. If $\alpha \in S$, then there is the smallest γ such that $\dot{S} = S_{f(\gamma)}$ and $\alpha = \delta_{f(\gamma)}$ (where f is as in the definition of $\mathbb{P}_{\kappa^+}$). This stage γ is the only stage where it is possible that $V[G_\gamma] \models \sigma_\gamma = T(S_{\eta+2\alpha+1})$, but since $V[G_\gamma] \models \check{\alpha} \in \dot{S}$, by the definition of $\mathbb{P}_{\kappa^+}$ it is not the case, so the stationarity of $S_{\eta+2\alpha+1}$ has not been killed by Claim 2. On the other hand the stationarity of $S_{\eta+2\alpha}$ is killed at this level γ of the construction, so $\alpha \in X_\eta$ by the definitions of φ and X_η. Similarly if $\alpha \notin S$, we conclude that $\alpha \notin X_\eta$. $\square_{\text{Claim 3}}$

Claim 4. In $V[G]$ the following holds: if $S \subset \kappa \setminus E$ is not stationary, then for all $\eta \in 2^E$ with $\eta(0) = 0$ we have $S \neq X_\eta$.

Proof of Claim 4. It is sufficient to show that X_η is stationary for all $\eta \in 2^E$ with $\eta(0) = 0$. Suppose first that $\eta \in F_0 \subset V$. Then since g_{κ^+} is a surjection onto F_0 (see $(***)$), there exists a name $\dot{S}$ such that $S = \dot{S}_G$ is stationary, $S \subset \kappa \setminus E$ and $g_{\kappa^+}(S) = \eta$. Now the same argument as in the proof of Claim 3 implies that $X_\eta = S$, so X_η is stationary by Claim 2.

If $\eta \notin F_0$, then by the definition of $\eta \mapsto X_\eta$ it is sufficient to show that the $\diamondsuit$-sequence added by $\mathbb{P}_0$ guesses in $V[G]$ every new set on a stationary set.

Suppose that τ and $\dot{C}$ are nice $\mathbb{P}_{\kappa^+}$-names for subsets of $\check{\kappa}$ and let p be a condition forcing that $\dot{C}$ is cub. We want to find γ and $q > p$ such that

$$q \Vdash ((\cup \dot{G}_0)(\check{\gamma})^{-1}\{1\} = \tau \cap \check{\gamma}) \wedge (\check{\gamma} \in \dot{C})$$

where $\dot{G}_0 = \dot{G} \restriction \{0\}$ is the name for the $\mathbb{P}_0$-generic. To do that let $p_0 \geqslant p$ be such that $p_0 \Vdash \tau \notin \mathcal{P}(\check{\kappa})^V$.

Similarly as in the proofs above define a suitable sequence $\langle M_i \rangle_{i<\lambda}$ of elementary submodels, of length $\lambda < \kappa$, where λ is a cofinality of a point in E, such that $\sup_{i<\lambda}(M_i \cap \kappa) = \alpha \in E$ and $M_i \cap \kappa \notin E$ for all $i < \lambda$. Assume also that $p_0 \in M_0$. Suppose $p_i \in M_i$ is defined. Let $p_{i+1} > p_i$ be an element of $M_{i+1} \setminus M_i$ satisfying the following:

1. p_{i+1} decides σ_β for all $\beta \in \operatorname{sprt} p_i$,
2. for all $\beta \in \operatorname{sprt} p_i$ there is $\beta' \in M_{i+1}$ such that $p_{i+1} \Vdash \beta' \in \tau \triangle \xi_\beta$, where ξ_β is defined as in the proof of Claim 2 and p_{i+1} decides what it is,
3. p_{i+1} decides τ up to $M_i \cap \kappa$,
4. $p_{i+1} \Vdash \delta \in \dot{C}$ for some $\delta \in M_{i+1} \setminus M_i$,
5. $m(p_{i+1}) > M_i \cap \kappa$, ($m(p)$ is defined at $(\star)$),

Item (1) is possible for the same reason as in the proof of Claim 2 and (2) is possible since $p_i \Vdash \forall \eta \in \mathcal{P}(\check{\kappa})^V (\tau \neq S_{\check{\eta}})$.

Since $M_i \cap \kappa \notin E$ for $i < \lambda$, this ensures that the sequence $p_0 \leqslant p_1 \leqslant \dots$ closes under limits $< \lambda$. Let $p_\lambda = \bigcup_{i<\lambda} p_i$ and let us define $q \supset p_\lambda$ as follows: $\operatorname{sprt} q = \operatorname{sprt} p_\lambda$, for $\delta \in \operatorname{sprt} p_\lambda \setminus \{0\}$ let $\operatorname{dom} q = \alpha + 1$, $p_\lambda(\delta) \subset q(\delta)$, $q(\alpha) = 1$ and $q(0)(\alpha) = \tau \cap \gamma$ (τ means here what have been decided by $\{p_i \mid i < \lambda\}$). Now q is a condition in the forcing notion.

Now certainly, if $q \in G$, then in the extension $\tau_G \cap \alpha = (\cup G_0)(\alpha)^{-1}\{1\}$ and $\alpha \in C$, so we finish. □ Claim 4

□ item (4)

Proof of item (5). If $\kappa = \lambda^+$, this follows from the result of Mekler and Shelah [34] and Hyttinen and Rautila [18] that the existence of a $\kappa\lambda$-canary tree is consistent. For arbitrary $\lambda < \kappa$ the result follows from the item (4) of this theorem proved above (take $Z = \kappa \setminus S^\kappa_\lambda$).

□ item (5)

Proof of item (6). For $X = \kappa$ this was proved by Halko and Shelah in [10], Theorem 4.2. For X any stationary subset of κ the proof is similar. It is sufficient to show that $2^\kappa \setminus \mathrm{CUB}(X)$ is not meager in any open set. Suppose U is an open set and $(D_\alpha)_{\alpha<\kappa}$ is a set of dense open sets and let us show that

$$(2^\kappa \setminus \mathrm{CUB}(X)) \cap U \cap \bigcap_{\alpha<\kappa} D_\alpha \neq \varnothing.$$

Let $p \in 2^{<\kappa}$ be such that $N_p \subset U$. Let $p_0 \geqslant p$ be such that $p_0 \in D_0$. Suppose p_β are defined for $\beta < \alpha+1$. Let $p_{\alpha+1}$ be such that $p_{\alpha+1} \geqslant p_\alpha$, $p_{\alpha+1} \in D_{\alpha+1}$. Suppose p_β is defined for $\beta < \alpha$ and α is a limit ordinal. Let p_α be any element of $2^{<\kappa}$ such that $p_\alpha > \bigcup_{\beta<\alpha} p_\beta$, $p_\alpha(\sup_{\beta<\alpha} \mathrm{dom}\, p_\beta) = 0$ and $p_\alpha \in D_\alpha$. Let $\eta = \bigcup_{\alpha<\kappa} p_\alpha$. The complement of $\eta^{-1}\{1\}$ contains a cub, so $X \setminus \eta^{-1}\{1\}$ is stationary whence $\eta \notin \mathrm{CUB}(X)$ and so $\eta \in 2^\kappa \setminus \mathrm{CUB}(X)$. Also clearly $\eta \in U \cap \bigcap_{\alpha<\kappa} D_\alpha$.
$\square_{\text{item (6)}}$

Proof of item (7)**.** Our proof is different from that given by Lücke and Schlicht. Suppose $\kappa^{<\kappa} = \kappa > \omega$. We will show that in a generic extension of V all Δ^1_1-sets have the Property of Baire. Let

$$\mathbb{P} = \{p \mid p \text{ is a function}, |p| < \kappa, \mathrm{dom}\, p \subset \kappa \times \kappa^+, \mathrm{ran}\, p \subset \{0,1\}\}$$

with the ordering $p < q \iff p \subset q$ and let G be $\mathbb{P}$-generic over V. Suppose that $X \subset 2^\kappa$ is a Δ^1_1-set in $V[G]$. It is sufficient to show that for every $r \in 2^{<\kappa}$ there is $q \supset r$ such that either $N_q \setminus X$ or $N_q \cap X$ is co-meager. So let $r \in 2^{<\kappa}$ be arbitrary.

Now suppose that $\langle p_i \rangle_{i<\kappa}$ and $\langle q_i \rangle_{i<\kappa}$ are sequences in $V[G]$ such that $p_i, q_i \in (2^{<\kappa})^2$ for all $i < \kappa$ and X is the projection of

$$C_0 = (2^\kappa)^2 \setminus \bigcup_{i<\kappa} N_{p_i}$$

and $2^\kappa \setminus X$ is the projection of

$$C_1 = (2^\kappa)^2 \setminus \bigcup_{i<\kappa} N_{q_i}.$$

(By N_{p_i} we mean $N_{p_i^1} \times N_{p_i^2}$ where $p_i = (p_i^1, p_i^2)$.) Since these sequences have size κ, there exists $\alpha_1 < \kappa^+$ such that they are already in $V[G_{\alpha_1}]$ where $G_{\alpha_1} = \{p \in G \mid \mathrm{dom}\, p \subset \kappa \times \alpha_1\}$. More generally, for $E \subset \mathbb{P}$ and $A \subset \kappa^+$, we will denote $E_A = \{p \in E \mid \mathrm{dom}\, p \subset \kappa \times A\}$ and if $p \in \mathbb{P}$, similarly $p_A = p \restriction (\kappa \times A)$.

Let $\alpha_2 \geqslant \alpha_1$ be such that $r \in G_{\{\alpha_2\}}$ (identifying $\kappa \times \{\alpha_2\}$ with κ). This is possible since G is generic. Let $x = G_{\{\alpha_2\}}$. In $V[G]$, $x \in X$ or $x \in 2^\kappa \setminus X$, so there are $\alpha_3 > \alpha_2$, $p \in G_{\alpha_3}$, $p_{\{\alpha_2\}} \supset r$ and a name τ such that p forces that $(x, \tau) \notin N_{p_i}$ for all $i < \kappa$ or $(x, \tau) \notin N_{q_i}$ for all $i < \kappa$. Without loss of generality assume that p forces $(x, \tau) \notin N_{p_i}$ for all $i < \kappa$. Also assume that τ is a $\mathbb{P}_{\alpha_3}$-name and that $\alpha_3 = \alpha_2 + 2$.

By working in $V[G_{\alpha_2}]$ we may assume that $\alpha_2 = 0$. For all $q \in \mathbb{P}_{\{1\}}$, $p_{\{1\}} \subseteq q$ and $i < \kappa$, let $D_{i,q}$ be the set of all $s \in \mathbb{P}_{\{0\}}$ such that $p_{\{0\}} \subseteq s$, $\mathrm{dom}(s) \geqslant \mathrm{dom}(p_i^1)$ and there is $q' \in \mathbb{P}_{\{1\}}$ such that $q \subseteq q'$ and $s \cup q'$ decides $\tau \restriction \mathrm{dom}(p_i^2)$. Clearly each $D_{i,q}$ is dense above $p_{\{0\}}$ in $\mathbb{P}_{\{0\}}$ and so it suffices to show that if $y \in 2^\kappa$ is such that for all $i < \kappa$ and q as above there is $\alpha < \kappa$ such that $y \restriction \alpha \in D_{i,q}$, then $y \in X$. So let y be such. Then we can find $z \in 2^\kappa$ such that for all $i < \kappa$ and q as above there are $\alpha, \beta < \kappa$ such that $\alpha \geqslant \mathrm{dom}(p_i^1)$ and $y \restriction \alpha \cup z \restriction \beta$ decides $t = \tau \restriction \mathrm{dom}(p_i^2)$. By the choice of p, $(y \restriction \mathrm{dom}(p_i^1), t) \neq p_i$. Letting τ^* be the function decided by y and z, $(y, \tau^*) \in C_0$ and so $y \in X$.
$\square_{\text{item (7)}}$
$\square_{\text{Theorem 4.52}}$

Remark ($\mathrm{cf}(\kappa) = \kappa > \omega$). There are some more results and strengthenings of the results in Theorem 4.52:

1. (Independently known by S. Coskey and P. Schlicht) If $V = L$ then there is a Δ_1^1 well-order of $\mathcal{P}(\kappa)$ and this implies that there is a Δ_1^1-set without the Baire Property.

2. Suppose that $\omega < \kappa < \lambda$, κ regular and λ inaccessible. Then after turning λ into κ^+ by collapsing each ordinal less than λ to κ using conditions of size $< \kappa$, the Baire Property holds for Δ_1^1 subsets of κ^κ.

4.53 Corollary. *For a regular $\lambda < \kappa$ let NS_λ denote the equivalence relation on 2^κ such that $\eta \mathrm{NS}_\lambda \xi$ if and only if $\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is not λ-stationary. Then NS_λ is not Borel and it is not Δ_1^1 in L or in the forcing extensions after adding κ^+ Cohen subsets of κ.*

Proof. Define a map $f\colon 2^\kappa \to (2^\kappa)^2$ by $\eta \mapsto (\varnothing, \kappa \setminus \eta)$. Suppose for a contradiction that NS_λ is Borel. Then

$$\mathrm{NS}_\varnothing = \mathrm{NS}_\lambda \cap \underbrace{\{(\varnothing, \eta) \mid \eta \in 2^\kappa\}}_{\text{closed}}$$

is Borel, and further $f^{-1}[\mathrm{NS}_\varnothing]$ is Borel by continuity of f. But $f^{-1}[\mathrm{NS}_\varnothing]$ equals CUB which is not Borel by Theorem 4.52 (6) and Theorem 4.48. Similarly, using items (2) and (3) of Theorem 4.52, one can show that NS_λ is not Δ_1^1 under the stated assumptions. □

4.4.4 Equivalence Modulo the Non-stationary Ideal

In this section we will investigate the relations defined as follows:

4.54 Definition. For $X \subset \kappa$, we denote by E_X the relation

$$E_X = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}) \cap X \text{ is not stationary}\}.$$

The set X consists usually of ordinals of fixed cofinality, i.e. $X \subset S^\kappa_\mu$ for some μ. These relations are easily seen to be Σ_1^1. If $X \subset S^\kappa_\omega$, then it is in fact Borel*. To see this use the same argument as in the proof of Theorem 4.52 (1) that the $\mathrm{CUB}^\kappa_\omega$-set is Borel*.

An Antichain

4.55 Theorem. *Assume GCH, $\kappa^{<\kappa} = \kappa$ is uncountable and $\mu < \kappa$ is a regular cardinal such that if $\kappa = \lambda^+$, then $\mu \leqslant \mathrm{cf}(\lambda)$. Then in a cofinality and GCH preserving forcing extension, there are stationary sets $K(A) \subset S^\kappa_\mu$ for each $A \subset \kappa$ such that $E_{K(A)} \not\leqslant_B E_{K(B)}$ if and only if $A \not\subset B$.*

Remark. Compare to Theorems 5.11 and 5.12 on page 146.

Proof. In this proof we identify functions $\eta \in 2^{\leqslant\kappa}$ with the sets $\eta^{-1}\{1\}$: for example we write $\eta \cap \xi$ to mean $\eta^{-1}\{1\} \cap \xi^{-1}\{1\}$.

The embedding will look as follows. Let $(S_i)_{i<\kappa}$ be pairwise disjoint stationary subsets of

$$\lim S^\kappa_\mu = \{\alpha \in S^\kappa_\mu \mid \alpha \text{ is a limit of ordinals in } S^\kappa_\mu\}.$$

Let

$$K(A) = E_{\bigcup\limits_{\alpha \in A} S_\alpha}. \tag{$*$}$$

If $X_1 \subset X_2 \subset \kappa$, then $E_{X_1} \leqslant_B E_{X_2}$, because $f(\eta) = \eta \cap X_1$ is a reduction. This guarantees that

$$A_1 \subset A_2 \Rightarrow K(A_1) \leqslant_B K(A_2).$$

Now suppose that for all $\alpha < \kappa$ we have killed (by forcing) all reductions from $K(\alpha) = E_{S_\alpha}$ to $K(\kappa \setminus \alpha) = E_{\bigcup_{\beta \neq \alpha} S_\beta}$ for all $\alpha < \kappa$. Then if $K(A_1) \leqslant_B K(A_2)$ it follows that $A_1 \subset A_2$: Otherwise choose $\alpha \in A_1 \setminus A_2$ and we have:

$$K(\alpha) \leqslant_B K(A_1) \leqslant_B K(A_2) \leqslant_B K(\kappa \setminus \alpha),$$

contradiction. So we have:

$$A_1 \subset A_2 \iff K(A_1) \leqslant_B K(A_2).$$

It is easy to obtain an antichain of length κ in $\mathcal{P}(\kappa)$ and so the result follows.

Suppose that $f\colon E_X \leqslant_B E_Y$ is a Borel reduction. Then $g\colon 2^\kappa \to 2^\kappa$ defined by $g(\eta) = f(\eta) \triangle f(0)$ is a Borel function with the following property:

$$\eta \cap X \text{ is stationary} \iff g(\eta) \cap Y \text{ is stationary.}$$

The function g is Borel, so by Lemma 4.34, page 79, there are dense open sets D_i for $i < \kappa$ such that $g \restriction D$ is continuous where $D = \bigcap_{i<\kappa} D_i$. Note that D_i are open so for each i we can write $D_i = \bigcup_{j<\kappa} N_{p(i,j)}$, where $(p(i,j))_{j<\kappa}$ is a suitable collection of elements of $2^{<\kappa}$.

Next define $Q_g\colon 2^{<\kappa} \times 2^{<\kappa} \to \{0,1\}$ by $Q_g(p,q) = 1 \iff N_p \cap D \subset g^{-1}[N_q]$ and $R_g\colon \kappa \times \kappa \to 2^{<\kappa}$ by $R_g(i,j) = p(i,j)$ where $p(i,j)$ are as above.

For any $Q\colon 2^{<\kappa} \times 2^{<\kappa} \to \{0,1\}$ define $Q^*\colon 2^\kappa \to 2^\kappa$ by

$$Q^*(\eta) = \begin{cases} \xi, & \text{s.t. } \forall \alpha < \kappa \exists \beta < \kappa Q(\eta \restriction \beta, \xi \restriction \alpha) = 1 \text{ if such exists,} \\ 0, & \text{otherwise.} \end{cases}$$

And for any $R\colon \kappa \times \kappa \to 2^{<\kappa}$ define

$$R^* = \bigcap_{i<\kappa} \bigcup_{j<\kappa} N_{R(i,j)}.$$

Now clearly $R_g^* = D$ and $Q_g^* \restriction D = g \restriction D$, i.e. (Q, D) *codes* $g \restriction D$ in this sense. Thus we have shown that if there is a reduction $E_X \leqslant_B E_Y$, then there is a pair (Q, R) which satisfies the following conditions:

1. $Q\colon (2^{<\kappa})^2 \to \{0,1\}$ is a function.
2. $Q(\varnothing, \varnothing) = 1$,
3. If $Q(p,q) = 1$ and $p' > p$, then $Q(p', q) = 1$,
4. If $Q(p,q) = 1$ and $q' < q$, then $Q(p, q') = 1$

5. Suppose $Q(p,q) = 1$ and $\alpha > \operatorname{dom} q$. There exist $q' > q$ and $p' > p$ such that $\operatorname{dom} q' = \alpha$ and $Q(p', q') = 1$,
6. If $Q(p,q) = Q(p,q') = 1$, then $q \leqslant q'$ or $q' < q$,
7. $R\colon \kappa \times \kappa \to 2^{<\kappa}$ is a function.
8. For each $i \in \kappa$ the set $\bigcup_{j<\kappa} N_{R(i,j)}$ is dense.
9. For all $\eta \in R^*$, $\eta \cap X$ is stationary if and only if $Q^*(\eta \cap X) \cap Y$ is stationary.

Let us call a pair (Q, R) which satisfies (1)–(9) *a code for a reduction (from E_X to E_Y)*. Note that it is not the same as the Borel code for the graph of a reduction function as a set. Thus we have shown that if $E_X \leqslant_B E_Y$, then there exists a code for a reduction from E_X to E_Y. We will now prove the following lemma which is stated in a general enough form so we can use it also in the next section:

4.56 Lemma (GCH). *Suppose μ_1 and μ_2 are regular cardinals less than κ such that if $\kappa = \lambda^+$, then $\mu_2 \leqslant \operatorname{cf}(\lambda)$, and suppose X is a stationary subset of $S^\kappa_{\mu_1}$, Y is a subset of $S^\kappa_{\mu_2}$, $X \cap Y = \varnothing$ (relevant if $\mu_1 = \mu_2$) and if $\mu_1 < \mu_2$ then $\alpha \cap X$ is not stationary in α for all $\alpha \in Y$. Suppose that (Q, R) is an arbitrary pair. Denote by φ the statement "(Q, R) is not a code for a reduction from E_X to E_Y". Then there is a κ^+-c.c. $<\kappa$-closed forcing $\mathbb{R}$ such that $\mathbb{R} \Vdash \varphi$.*

Remark. Clearly if $\mu_1 = \mu_2 = \omega$, then the condition $\mu_2 \leqslant \operatorname{cf}(\lambda)$ is of course true. We need this assumption in order to have $\nu^{<\mu_2} < \kappa$ for all $\nu < \kappa$.

Proof of Lemma 4.56. We will show that one of the following holds:

1. φ already holds, i.e. $\{\varnothing\} \Vdash \varphi$,
2. $\mathbb{P} = 2^{<\kappa} = \{p\colon \alpha \to 2 \mid \alpha < \kappa\} \Vdash \varphi$,
3. $\mathbb{R} \Vdash \varphi$,

where
$$\mathbb{R} = \{(p,q) \mid p, q \in 2^\alpha, \alpha < \kappa, X \cap p \cap q = \varnothing, q \text{ is } \mu_1\text{-closed}\}$$
Above "q is μ_1-closed" means "$q^{-1}\{1\}$ is μ_1-closed" etc., and we will use this abbreviation below. Assuming that (1) and (2) do not hold, we will show that (3) holds.

Since (2) does not hold, there is a $p \in \mathbb{P}$ which forces $\neg\varphi$ and so $\mathbb{P}_p = \{q \in \mathbb{P} \mid q > p\} \Vdash \neg\varphi$. But $\mathbb{P}_p \cong \mathbb{P}$, so in fact $\mathbb{P} \Vdash \neg\varphi$, because φ has only standard names as parameters (names for elements in V, such as Q, R, X and Y). Let G be any $\mathbb{P}$-generic and let us denote the set $G^{-1}\{1\}$ also by G. Let us show that $G \cap X$ is stationary. Suppose that $\dot{C}$ is a name and $r \in \mathbb{P}$ is a condition which forces that $\dot{C}$ is cub. For an arbitrary q_0, let us find a $q > q_0$ which forces $\dot{C} \cap \dot{G} \cap \check{X} \neq \varnothing$. Make a counter assumption: no such $q > q_0$ exists. Let $q_1 > q_0$ and $\alpha_1 > \operatorname{dom} q_0$ be such that $q_1 \Vdash \check{\alpha}_1 \in \dot{C}$, $\operatorname{dom} q_1 > \alpha_1$ is a successor and $q_1(\max \operatorname{dom} q_1) = 1$. Then by induction on $i < \kappa$ let q_{i+1} and $\alpha_{i+1} > \operatorname{dom} q_i$ be such that $q_{i+1} \Vdash \check{\alpha}_{i+1} \in \dot{C}$, $\operatorname{dom} q_{i+1} > \alpha_{i+1}$ is a successor and $q_{i+1}(\max \operatorname{dom} q_{i+1}) = 1$. If j is a limit ordinal, let $q_j = \bigcup_{i<j} q_i \cup \{(\sup_{i<j} \operatorname{dom} q_i, 1)\}$ and $\alpha_j = \sup_{i<j} \alpha_i$. We claim that for some $i < \kappa$, the condition q_i is as needed, i.e.
$$q_i \Vdash \dot{G} \cap \check{X} \cap \dot{C} \neq \varnothing.$$

Clearly for limit ordinals j, we have $\alpha_j = \max \operatorname{dom} q_j$ and $q_j(\alpha_j) = 1$ and $\{\alpha_j \mid j \text{ limit}\}$ is cub. Since X is stationary, there exists a limit j_0 such that $\alpha_{j_0} \in X$. Because q_0 forces that $\dot{C}$ is cub, $q_j > q_i > q_0$ for all $i < j$, $q_i \Vdash \check{\alpha}_i \in \dot{C}$ and $\alpha_j = \sup_{i<j} \alpha_i$, we have $q_j \Vdash \alpha_j \in \dot{C} \cap \check{X}$. On the other hand $q_j(\alpha_j) = 1$, so $q_j \Vdash \alpha_j \in G$ so we finish.

So now we have in $V[G]$ that $G \cap X$ is stationary, $G \in R^*$ (since R^* is co-meager) and Q is a code for a reduction, so Q^* has the property (9) and $Q^*(G \cap X) \cap Y$ is stationary. Denote $Z = Q^*(G \cap X) \cap Y$. We will now construct a forcing $\mathbb{Q}$ in $V[G]$ such that

$$V[G] \models (\mathbb{Q} \Vdash \text{"}G \cap X \text{ is not stationary, but } Z \text{ is stationary"}).$$

Then $V[G] \models (\mathbb{Q} \Vdash \varphi)$ and hence $\mathbb{P} * \mathbb{Q} \Vdash \varphi$. On the other hand $\mathbb{Q}$ will be chosen such that $\mathbb{P} * \mathbb{Q}$ and $\mathbb{R}$ give the same generic extensions. So let

$$\mathbb{Q} = \{q \colon \alpha \to 2 \mid X \cap G \cap q = \varnothing, q \text{ is } \mu_1\text{-closed}\}, \tag{$**$}$$

Clearly $\mathbb{Q}$ kills the stationarity of $G \cap X$. Let us show that it preserves the stationarity of Z. For that purpose it is sufficient to show that for any nice $\mathbb{Q}$-name $\dot{C}$ for a subset of κ and any $p \in \mathbb{Q}$, if $p \Vdash$ "$\dot{C}$ is μ_2-cub", then $p \Vdash (\dot{C} \cap \check{Z} \neq \check{\varnothing})$.

So suppose $\dot{C}$ is a nice name for a subset of κ and $p \in \mathbb{Q}$ is such that

$$p \Vdash \text{"} \dot{C} \text{ is cub"}$$

Let $\lambda > \kappa$ be a sufficiently large regular cardinal and let N be an elementary submodel of $\langle H(\lambda), p, \dot{C}, \mathbb{Q}, \kappa\rangle$ which has the following properties:

- $|N| = \mu_2$
- $N^{<\mu_2} \subset N$
- $\alpha = \sup(N \cap \kappa) \in Z$ (This is possible because Z is stationary).

Here we use the hypothesis that μ_2 is at most $\operatorname{cf}(\lambda)$ when $\kappa = \lambda^+$. Now by the assumption of the theorem, $\alpha \setminus X$ contains a μ_1-closed unbounded sequence of length μ_2, $\langle \alpha_i \rangle_{i<\mu_2}$. Let $\langle D_i \rangle_{i<\mu_2}$ list all the dense subsets of $\mathbb{Q}^N$ in N. Let $q_0 \geqslant p$, $q_0 \in \mathbb{Q}^N$ be arbitrary and suppose $q_i \in \mathbb{Q}^N$ is defined for all $i < \gamma$. If $\gamma = \beta + 1$, then define q_γ to be an extension of q_β such that $q_\gamma \in D_\beta$ and $\operatorname{dom} q_\gamma = \alpha_i$ for some $\alpha_i > \operatorname{dom} q_\beta$. To do that, for instance, choose $\alpha_i > \operatorname{dom} q_\beta$ and define $q' \supset q_\beta$ by $\operatorname{dom} q' = \alpha_i$, $q(\delta) = 0$ for all $\delta \in \operatorname{dom} q' \setminus \operatorname{dom} q_\beta$ and then q' to q_β in D_β. If γ is a limit ordinal with $\operatorname{cf}(\gamma) \neq \mu_1$, then let $q_\gamma = \bigcup_{i<\gamma} q_i$. If $\operatorname{cf}(\gamma) = \mu_1$, let

$$q_\gamma = \Big(\bigcup_{i<\gamma} q_i\Big)^\frown \langle \sup_{i<\gamma} \operatorname{dom} q_i, 1\rangle$$

Since N is closed under taking sequences of length less than μ_2, $q_\gamma \in N$. Since we required elements of $\mathbb{Q}$ to be μ_1-closed but not γ-closed if $\operatorname{cf}(\gamma) \neq \mu_1$, $q_\gamma \in \mathbb{Q}$ when $\operatorname{cf}(\gamma) \neq \mu_1$. When $\operatorname{cf}(\gamma) = \mu_1$, the limit $\sup_{i<\gamma} \operatorname{dom} q_i$ coincides with a limit of a subsequence of $\langle \alpha_i \rangle_{i<\mu_2}$ of length μ_1, i.e. the limit is α_β for some β since this sequence is μ_1-closed. So by definition $\sup_{i<\gamma} \operatorname{dom} q_i \notin X$ and again $q_\gamma \in \mathbb{Q}$.

Then $q = \bigcup_{\gamma<\mu} q_\gamma$ is a $\mathbb{Q}^N$-generic over N. Since $X \cap Y = \varnothing$, also $(X \cap G) \cap Z = \varnothing$ and $\alpha \notin X \cap G$. Hence $q^\frown(\alpha, 1)$ is in $\mathbb{Q}$. We claim that $q \Vdash (\dot{C} \cap \check{Z} \neq \varnothing)$.

Because $p \Vdash$ "$\dot{C}$ is unbounded", also $N \models (p \Vdash$ "$\dot{C}$ is unbounded") by elementarity. Assuming that λ is chosen large enough, we may conclude that for all $\mathbb{Q}^N$-generic g over N, $N[g] \models$ "$\dot{C}_g$ is unbounded", thus in particular $N[g] \models$ "$\dot{C}_g$ is unbounded in κ". Let G_1 be $\mathbb{Q}$-generic over $V[G]$ with $q \in G_1$. Then $\dot{C}_{G_1} \supset \dot{C}_q$ which is unbounded in α by the above, since $\sup(\kappa \cap N) = \alpha$. Because $\dot{C}_{G_1}$ is μ_2-cub, α is in $\dot{C}_{G_1}$.

Thus $\mathbb{P} * \mathbb{Q} \Vdash \varphi$. It follows straightforwardly from the definition of iterated forcing that $\mathbb{R}$ is isomorphic to a dense suborder of $\mathbb{P} * \dot{\mathbb{Q}}$ where $\dot{Q}$ is a $\mathbb{P}$-name for a partial order such that $\dot{\mathbb{Q}}_G$ equals $\mathbb{Q}$ as defined in $(**)$ for any $\mathbb{P}$-generic G.

Now it remains to show that $\mathbb{R}$ has the κ^+-c.c. and is $< \kappa$-closed. Since $\mathbb{R}$ is a suborder of $\mathbb{P} \times \mathbb{P}$, which has size κ, it trivially has the κ^+-c.c. Suppose $(p_i, q_i)_{i<\gamma}$ is an increasing sequence, $\gamma < \kappa$. Then the pair

$$(p,q) = \Big\langle \Big(\bigcup_{i<\gamma} p_i\Big)^\frown \langle \alpha, 0\rangle, \Big(\bigcup_{i<\gamma} q_i\Big)^\frown \langle \alpha, 1\rangle \Big\rangle$$

is an upper bound. $\square_{\text{Lemma 4.56}}$

Remark. Note that the forcing used in the previous proof is equivalent to κ-Cohen forcing.

4.57 Corollary (GCH)**.** *Let $K\colon A \mapsto E_{\bigcup_{\alpha\in A} S_\alpha}$ be as in the beginning of the proof. For each pair (Q,R) and each α there is a $<\kappa$-closed, κ^+-c.c. forcing $\mathbb{R}(Q,R,\alpha)$ such that*

$$\mathbb{R}(Q,R,\alpha) \Vdash \text{“}(Q,R) \text{ is not a code for a reduction from } K(\{\alpha\}) \text{ to } K(\kappa \setminus \{\alpha\})\text{”}$$

Proof. By the above lemma one of the choices $\mathbb{R} = \{\varnothing\}$, $\mathbb{R} = 2^{<\kappa}$ or

$$\mathbb{R} = \{(p,q) \mid p,q \in 2^\beta, \beta < \kappa, S_\alpha \cap p \cap q = \varnothing, q \text{ is } \mu\text{-closed}\}$$

suffices. $\square$

Start with a model satisfying GCH. Let $h\colon \kappa^+ \to \kappa^+ \times \kappa \times \kappa^+$ be a bijection such that $h_3(\alpha) < \alpha$ for $\alpha > 0$ and $h_3(0) = 0$. Let $\mathbb{P}_0 = \{\varnothing\}$. For each $\alpha < \kappa$, let $\{\sigma_{\beta\alpha 0} \mid \beta < \kappa^+\}$ be the list of all $\mathbb{P}_0$-names for codes for a reduction from $K(\{\alpha\})$ to $K(\kappa \setminus \{\alpha\})$. Suppose $\mathbb{P}_i$ and $\{\sigma_{\beta\alpha i} \mid \beta < \kappa^+\}$ are defined for all $i < \gamma$ and $\alpha < \kappa$, where $\gamma < \kappa^+$ is a successor $\gamma = \beta + 1$, $\mathbb{P}_i$ is $<\kappa$-closed and has the κ^+-c.c.

Consider $\sigma_{h(\beta)}$. By the above corollary, the following holds:

$$\begin{aligned}\mathbb{P}_\beta \quad \Vdash \quad & \big[\exists \mathbb{R} \in \mathcal{P}(2^{<\kappa} \times 2^{<\kappa})(\mathbb{R} \text{ is } <\kappa\text{-closed, } \kappa^+\text{-c.c. p.o. and} \\ & \mathbb{R} \Vdash \text{“}\, \sigma_{h(\beta)} \text{ is not a code for a reduction.”})\big]\end{aligned}$$

So there is a $\mathbb{P}_\beta$-name ρ_β such that $\mathbb{P}_\beta$ forces that ρ_β is as $\mathbb{R}$ above. Define

$$\mathbb{P}_\gamma = \{(p_i)_{i<\gamma} \mid ((p_i)_{i<\beta} \in \mathbb{P}_\beta) \wedge ((p_i)_{i<\beta} \Vdash p_\beta \in \rho_\beta)\}.$$

And if $p = (p_i)_{i<\gamma} \in \mathbb{P}_\gamma$ and $p' = (p'_i)_{i<\gamma} \in \mathbb{P}_\gamma$, then

$$p \leqslant_{\mathbb{P}_\gamma} p' \iff [(p_i)_{i<\beta} \leqslant_{\mathbb{P}_\beta} (p'_i)_{i<\beta}] \wedge [(p'_i)_{i<\beta} \Vdash (p_\beta \leqslant_{\rho_\beta} p'_\beta)]$$

If γ is a limit, $\gamma \leqslant \kappa^+$, let

$$\mathbb{P}_\gamma = \{(p_i)_{i<\gamma} \mid \forall\beta(\beta < \gamma \to (p_i)_{i<\beta} \in \mathbb{P}_\beta) \wedge (|\operatorname{sprt}(p_i)_{i<\gamma}| < \kappa)\},$$

where sprt means support, see page 60. For every α, let $\{\sigma_{\beta\alpha\gamma} \mid \beta < \kappa^+\}$ list all $\mathbb{P}_\beta$-names for codes for a reduction. It is easily seen that $\mathbb{P}_\gamma$ is $< \kappa$-closed and has the κ^+-c.c. for all $\gamma \leqslant \kappa^+$

We claim that $\mathbb{P}_{\kappa^+}$ forces that for all α, $K(\{\alpha\}) \not\leqslant_B K(\kappa \setminus \{\alpha\})$ which suffices by the discussion in the beginning of the proof, see $(*)$ for the notation.

Let G be $\mathbb{P}_{\kappa^+}$-generic and let $G_\gamma =$ "$G \cap \mathbb{P}_\gamma$" for every $\gamma < \kappa$. Then G_γ is $\mathbb{P}_\gamma$-generic.

Suppose that in $V[G]$, $f\colon 2^\kappa \to 2^\kappa$ is a reduction $K(\{\alpha\}) \leqslant_B K(\kappa \setminus \{\alpha\})$ and (Q, R) is the corresponding code for a reduction. By [32] Theorem VIII.5.14, there is a $\delta < \kappa^+$ such that $(Q, R) \in V[G_\delta]$. Let δ_0 be the smallest such δ.

Now there exists $\sigma_{\gamma\alpha\delta_0}$, a $\mathbb{P}_{\delta_0}$-name for (Q, R). By the definition of h, there exists a $\delta > \delta_0$ with $h(\delta) = (\gamma, \alpha, \delta_0)$. Thus

$$\mathbb{P}_{\delta+1} \Vdash \text{"}\sigma_{\gamma\alpha\delta_0} \text{ is not a code for a reduction"},$$

i.e. $V[G_{\delta+1}] \models (Q, R)$ is not a code for a reduction. Now one of the items (1)–(9) fails for (Q, R) in $V[G_{\delta+1}]$. We want to show that then one of them fails in $V[G]$. The conditions (1)–(8) are absolute, so if one of them fails in $V[G_{\delta+1}]$, then we are done. Suppose (1)–(8) hold but (9) fails. Then there is an $\eta \in R^*$ such that $Q^*(\eta \cap S_{\{\alpha\}}) \cap S_{\kappa\setminus\alpha}$ is stationary but $\eta \cap S_{\{\alpha\}}$ is not or vice versa. In $V[G_{\delta+1}]$ define

$$\mathbb{P}^{\delta+1} = \{(p_i)_{i<\kappa^+} \in \mathbb{P}_{\kappa^+} \mid (p_i)_{i<\delta+1} \in G_{\delta+1}\}.$$

Then $\mathbb{P}^{\delta+1}$ is $< \kappa$-closed. Thus it does not kill stationarity of any set. So if $G^{\delta+1}$ is $\mathbb{P}_{\delta+1}$-generic over $V[G_{\delta+1}]$, then in $V[G_{\delta+1}][G^{\delta+1}]$, (Q, R) is not a code for a reduction. Now it remains to show that $V[G] = V[G_{\delta+1}][G^{\delta+1}]$ for some $G^{\delta+1}$. In fact putting $G^{\delta+1} = G$ we get $\mathbb{P}^{\delta+1}$-generic over $V[G_{\delta+1}]$ and of course $V[G_{\delta+1}][G] = V[G]$ (since $G_{\delta+1} \subset G$). $\square_{\text{Theorem 4.55}}$

Remark. The forcing constructed in the proof of Theorem 4.55 above, combined with the forcing in the proof of item (4) of Theorem 4.52, page 91, gives that for $\kappa^{<\kappa} = \kappa > \omega_1$ not successor of a singular cardinal, we have in a forcing extension that $\langle \mathcal{P}(\kappa), \subset\rangle$ embeds into $\langle \mathcal{E}^{\Delta^1_1}, \leqslant_B\rangle$, i.e. the partial order of Δ^1_1-equivalence relations under Borel reducibility.

Reducibility Between Different Cofinalities

Recall the notation defined in Section 4.2.1. In this section we will prove the following two theorems:

4.58 Theorem. *Suppose that κ is a weakly compact cardinal and that $V = L$. Then*

(A) $E_{S^\kappa_\lambda} \leqslant_c E_{\operatorname{reg}(\kappa)}$ for any regular $\lambda < \kappa$, where $\operatorname{reg}(\kappa) = \{\lambda < \kappa \mid \lambda$ is regular$\}$,

(B) In a forcing extension $E_{S^{\omega_2}_\omega} \leqslant_c E_{S^{\omega_2}_{\omega_1}}$. Similarly for λ, λ^+ and λ^{++} instead of ω, ω_1 and ω_2 for any regular $\lambda < \kappa$.

4.59 Theorem. *For a cardinal κ which is a successor of a regular cardinal or κ inaccessible, there is a cofinality-preserving forcing extension in which for all regular $\lambda < \kappa$, the relations $E_{S^\kappa_\lambda}$ are $\leqslant_B$-incomparable with each other.*

Let us begin by proving the latter.

Proof of Theorem 4.59. Let us show that there is a forcing extension of L in which $E_{S^{\omega_2}_{\omega_1}}$ and $E_{S^{\omega_2}_{\omega}}$ are incomparable. The general case is similar.

We shall use Lemma 4.56 with $\mu_1 = \omega$ and $\mu_2 = \omega_1$ and vice versa, and then a similar iteration as in the end of the proof of Theorem 4.55. First we force, like in the proof of Theorem 4.52 (4), a stationary set $S \subset S^{\omega_2}_{\omega}$ such that for all $\alpha \in S^{\omega_2}_{\omega_1}$, $\alpha \cap S$ is non-stationary in α. Also for all $\alpha \in S^{\omega_2}_{\omega}$, $\alpha \cap S^{\omega_2}_{\omega_1}$ is non-stationary.

By Lemma 4.56, for each code for a reduction from E_S to $E_{S^{\omega_2}_{\omega_1}}$ there is a $< \omega_2$-closed ω_3-c.c. forcing which kills it. Similarly for each code for a reduction from $E_{S^{\omega_2}_{\omega_1}}$ to $E_{S^{\omega_2}_{\omega}}$. Making an ω_3-long iteration, similarly as in the end of the proof of Theorem 4.55, we can kill all codes for reductions from E_S to $E_{S^{\omega_2}_{\omega_1}}$ and from $E_{S^{\omega_2}_{\omega_1}}$ to $E_{S^{\omega_2}_{\omega}}$. Thus, in the extension there are no reductions from $E_{S^{\omega_2}_{\omega_1}}$ to $E_{S^{\omega_2}_{\omega}}$ and no reductions from $E_{S^{\omega_2}_{\omega}}$ to $E_{S^{\omega_2}_{\omega_1}}$. (Suppose there is one of a latter kind, $f\colon 2^{\omega_2} \to 2^{\omega_2}$. Then $g(\eta) = f(\eta \cap S)$ is a reduction from E_S to $E_{S^{\omega_2}_{\omega_1}}$.) $\square_{\text{Theorem 4.59}}$

4.60 Definition. Let X, Y be subsets of κ and suppose Y consists of ordinals of uncountable cofinality. We say that X *$\diamondsuit$-reflects to* Y if there exists a sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ such that

1. $D_\alpha \subset \alpha$ is stationary in α,
2. if $Z \subset X$ is stationary, then $\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}$ is stationary.

4.61 Theorem. *If X $\diamondsuit$-reflects to Y, then $E_X \leqslant_c E_Y$.*

Proof. Let $\langle D_\alpha \rangle_{\alpha \in Y}$ be the sequence of Definition 4.60. For a set $A \subset \kappa$ define

$$f(A) = \{\alpha \in Y | A \cap X \cap D_\alpha \text{ is stationary in } \alpha\}. \qquad (i)$$

We claim that f is a continuous reduction. Clearly f is continuous. Assume that $(A \triangle B) \cap X$ is non-stationary. Then there is a cub set $C \subset \kappa \setminus [(A \triangle B) \cap X]$. Now $A \cap X \cap C = B \cap X \cap C$ (ii). The set $C' = \{\alpha < \kappa \mid C \cap \alpha \text{ is unbounded in } \alpha\}$ is also cub and if $\alpha \in Y \cap C'$, we have that $D_\alpha \cap C$ is stationary in α. Therefore for $\alpha \in Y \cap C'$ (iii) we have the following equivalences:

$$\begin{aligned}
\alpha \in f(A) \quad &\iff \quad A \cap X \cap D_\alpha \text{ is stationary}\\
&\overset{(iii)}{\iff} \quad A \cap X \cap C \cap D_\alpha \text{ is stationary}\\
&\overset{(ii)}{\iff} \quad B \cap X \cap C \cap D_\alpha \text{ is stationary}\\
&\overset{(iii)}{\iff} \quad B \cap X \cap D_\alpha \text{ is stationary}\\
&\overset{(i)}{\iff} \quad \alpha \in f(B)
\end{aligned}$$

Thus $(f(A) \triangle f(B)) \cap Y \subset \kappa \setminus C'$ and is non-stationary.

Suppose $A \triangle B$ is stationary. Then either $A \setminus B$ or $B \setminus A$ is stationary. Without loss of generality suppose the former. Then

$$S = \{\alpha \in Y \mid (A \setminus B) \cap X \cap \alpha = D_\alpha\}$$

is stationary by the definition of the sequence $\langle D_\alpha \rangle_{\alpha \in Y}$. Thus for $\alpha \in S$ we have that $A \cap X \cap D_\alpha = A \cap X \cap (A \setminus B) \cap X \cap \alpha = (A \setminus B) \cap X \cap \alpha$ is stationary in α and $B \cap X \cap D_\alpha =$

$B \cap X \cap (A \setminus B) \cap X \cap \alpha = \varnothing$ is not stationary in α. Therefore $(f(A) \triangle f(B)) \cap Y$ is stationary (as it contains S). □

Fact (Π^1_1-reflection). *Assume that κ is weakly compact. If R is any binary predicate on V_κ and $\forall A\varphi$ is some Π^1_1-sentence where φ is a first-order sentence in the language of set theory together with predicates $\{R, A\}$ such that $(V_\kappa, R) \models \forall A\varphi$, then there exists stationary many $\alpha < \kappa$ such that $(V_\alpha, R \cap V_\alpha) \models \forall A\varphi$.*

We say that X *strongly reflects to* Y if for all stationary $Z \subset X$ there exist stationary many $\alpha \in Y$ with $X \cap \alpha$ stationary in α.

4.62 Theorem. *Suppose $V = L$, κ is weakly compact and that $X \subset \kappa$ and $Y \subset \operatorname{reg}\kappa$. If X strongly reflects to Y, then X $\diamondsuit$-reflects to Y.*

Proof. Define D_α by induction on $\alpha \in Y$. For the purpose of the proof also define C_α for each α as follows. Suppose (D_β, C_β) is defined for all $\beta < \alpha$. Let (D, C) be the L-least[1] pair such that

1. C is cub subset of α.
2. D is a stationary subset of $X \cap \alpha$
3. for all $\beta \in Y \cap C$, $D \cap \beta \neq D_\beta$

If there is no such pair then set $D = C = \varnothing$. Then let $D_\alpha = D$ and $C_\alpha = C$. We claim that the sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ is as needed. To show this, let us make a counter assumption: there is a stationary subset Z of X and a cub subset C of κ such that

$$C \cap Y \subset \{\alpha \in Y \mid D_\alpha \neq Z \cap \alpha\}. \qquad (\star)$$

Let (Z, C) be the L-least such pair. Let $\lambda > \kappa$ be regular and let M be an elementary submodel of L_λ such that

1. $|M| < \kappa$,
2. $\alpha = M \cap \kappa \in Y \cap C$,
3. $Z \cap \alpha$ is stationary in α,
4. $\{Z, C, X, Y, \kappa\} \subset M$

(2) and (3) are possible by the definition of strong reflection. Let $\bar{M}$ be the Mostowski collapse of M and let $G \colon M \to \bar{M}$ be the Mostowski isomorphism. Then $\bar{M} = L_\gamma$ for some $\gamma > \alpha$. Since $\kappa \cap M = \alpha$, we have

$$G(Z) = Z \cap \alpha,\ G(C) = C \cap \alpha,\ G(X) = X \cap \alpha,\ G(Y) = Y \cap \alpha \text{ and } G(\kappa) = \alpha,\ (\star\star).$$

Note that by the definability of the canonical ordering of L, the sequence $\langle D_\beta \rangle_{\beta < \kappa}$ is definable. Let $\varphi(x, y, \alpha)$ be the formula which says

[1] The least in the canonical definable ordering on L, see [32].

"(x, y) is the L-least pair such that x is contained in $X \cap \alpha$, x is stationary in α, y is cub in α and $x \cap \beta \neq D_\beta$ for all $\beta \in y \cap Y \cap \alpha$."

By the assumption,

$$L \models \varphi(Z, C, \kappa), \text{ so } M \models \varphi(Z, C, \kappa) \text{ and } L_\gamma \models \varphi(G(Z), G(C), G(\kappa)).$$

Let us show that this implies $L \models \varphi(G(Z), G(C), G(\kappa))$, i.e. $L \models \varphi(Z \cap \alpha, C \cap \alpha, \alpha)$. This will be a contradiction because then $D_\alpha = Z \cap \alpha$ which contradicts the assumptions (2) and ($\star$) above.

By the relative absoluteness of being the L-least, the relativised formula with parameters $\varphi^{L_\gamma}(G(Z), G(C), G(\kappa))$ says

"$(G(Z), G(C))$ is the L-least pair such that $G(Z)$ is contained in $G(X)$, $G(Z)$ is $(\text{stationary})^{L_\gamma}$ in $G(\kappa)$, $G(C)$ is cub in $G(\kappa)$ and $G(Z) \cap \beta \neq D_\beta^{L_\gamma}$ for all $\beta \in G(C) \cap G(Y) \cap G(\kappa)$."

Written out this is equivalent to

"$(Z \cap \alpha, C \cap \alpha)$ is the L-least pair such that $Z \cap \alpha$ is contained in $X \cap \alpha$, $Z \cap \alpha$ is $(\text{stationary})^{L_\gamma}$ in α, $C \cap \alpha$ is cub in α and $Z \cap \beta \neq D_\beta^{L_\gamma}$ for all $\beta \in C \cap Y \cap \alpha$."

Note that this is true in L. Since $Z \cap \alpha$ is stationary in α also in L by (3), it remains to show by induction on $\beta \in \alpha \cap Y$ that $Z \cap \alpha$ $D_\beta^{L_\gamma} = D_\beta^L$ and $C_\beta^{L_\gamma} = C_\beta^L$ and we are done. Suppose we have proved this for $\delta \in \beta \cap Y$ and $\beta \in \alpha \cap Y$. Then $(D_\beta^{L_\gamma}, C_\beta^{L_\gamma})$ is

(a) $(\text{the least } L\text{-pair})^{L_\gamma}$ such that

(b) $(C_\beta \text{ is a cub subset of } \beta)^{L_\gamma}$,

(c) $(D_\beta \text{ is a stationary subset of } \beta)^{L_\gamma}$

(d) and for all $\delta \in Y \cap \beta$, $(D_\beta \cap \delta \neq D_\delta)^{L_\gamma}$.

(e) Or there is no such pair and $D_\beta = \varnothing$.

The L-order is absolute as explained above, so (a) is equivalent to $(\text{the least } L\text{-pair})^L$. Being a cub subset of α is also absolute for L_γ so (b) is equivalent to $(C_\beta \text{ is a cub subset of } \alpha)^L$. All subsets of β in L are elements of $L_{|\beta|^+}$ (see [32]), and since α is regular and $\beta < \alpha \leqslant \gamma$, we have $\mathcal{P}(\beta) \subset L_\gamma$. Thus

$$(D_\beta \text{ is stationary subset of } \beta)^{L_\gamma} \iff (D_\beta \text{ is stationary subset of } \beta)^L.$$

Finally the statement of (d), $(D_\beta \cap \delta \neq D_\delta)^{L_\gamma}$ is equivalent to $D_\beta \cap \delta \neq D_\delta^{L_\gamma}$ as it is defining D_β, but by the induction hypothesis $D_\delta^{L_\gamma} = D_\delta^L$, so we are done. For (e), the fact that

$$\mathcal{P}(\beta) \subset L_{|\beta|^+} \subset L_\alpha \subset L_\gamma$$

as above implies that if there is no such pair in L_γ, then there is no such pair in L. □

Proof of Theorem 4.58. In the case (A) we will show that S^κ_λ strongly reflects to $\operatorname{reg}(\kappa)$ in L which suffices by Theorems 4.61 and 4.62. For (B) we will assume that κ is a weakly compact cardinal in L and then collapse it to ω_2 to get a $\diamondsuit$-sequence which witnesses that $S^{\omega_2}_\omega$ $\diamondsuit$-reflects to $S^{\omega_2}_{\omega_1}$ which is sufficient by Theorem 4.61. In the following we assume: $V = L$ and κ is weakly compact.

(A): Let us use Π^1_1-reflection. Let $X \subset S^\kappa_\lambda$. We want to show that the set

$$\{\lambda \in \operatorname{reg}(\kappa) \mid X \cap \lambda \text{ is stationary in } \lambda\}$$

is stationary. Let $C \subset \kappa$ be cub. The sentence

$$\text{“}(X \text{ is stationary in } \kappa) \wedge (C \text{ is cub in } \kappa) \wedge (\kappa \text{ is regular})\text{”}$$

is a Π^1_1-property of (V_κ, X, C). By Π^1_1-reflection we get $\delta < \kappa$ such that $(V_\delta, X\cap\delta, C\cap\delta)$ satisfies it. But then δ is regular, $X \cap \delta$ is stationary and δ belongs to C.

(B): Let κ be weakly compact and let us Levy-collapse κ to ω_2 with the following forcing:

$$\mathbb{P} = \{f\colon \operatorname{reg}\kappa \to \kappa^{<\omega_1} \mid \operatorname{ran}(f(\mu)) \subset \mu,\ |\{\mu \mid f(\mu) \neq \varnothing\}| \leqslant \omega\}.$$

Order $\mathbb{P}$ by $f < g$ if and only if $f(\mu) \subset g(\mu)$ for all $\mu \in \operatorname{reg}(\kappa)$. For all μ put $\mathbb{P}_\mu = \{f \in \mathbb{P} \mid \operatorname{sprt} f \subset \mu\}$ and $\mathbb{P}^\mu = \{f \in \mathbb{P} \mid \operatorname{sprt} f \subset \kappa \setminus \mu\}$, where sprt means support, see page 60.

Claim 1. For all regular μ, $\omega < \mu \leqslant \kappa$, $\mathbb{P}_\mu$ satisfies the following:

(a) If $\mu > \omega_1$, then $\mathbb{P}_\mu$ has the μ-c.c.,

(b) $\mathbb{P}_\mu$ and $\mathbb{P}^\mu$ are $< \omega_1$-closed,

(c) $\mathbb{P} = \mathbb{P}_\kappa \Vdash \omega_2 = \check\kappa$,

(d) If $\mu < \kappa$, then $\mathbb{P} \Vdash \operatorname{cf}(\check\mu) = \omega_1$,

(e) if $p \in \mathbb{P}$, σ a name and $p \Vdash$ “σ is cub in ω_2”, then there is cub $E \subset \kappa$ such that $p \Vdash \check E \subset \sigma$.

Proof. Standard (see for instance [25]). □

We want to show that in the generic extension $S^{\omega_2}_\omega$ $\diamondsuit$-reflects to $S^{\omega_2}_{\omega_1}$. It is sufficient to show that $S^{\omega_2}_\omega$ $\diamondsuit$-reflects to some stationary $Y \subset S^{\omega_2}_{\omega_1}$ by letting $D_\alpha = \alpha$ for $\alpha \notin Y$. In our case $Y = \{\mu \in V[G] \mid (\mu \in \operatorname{reg}(\kappa))^V\}$. By (d) of Claim 1, $Y \subset S^{\omega_2}_{\omega_1}$, $(\operatorname{reg}(\kappa))^V$ is stationary in V (for instance by Π^1_1-reflection) and by (e) it remains stationary in $V[G]$.

It is easy to see that $\mathbb{P} \cong \mathbb{P}_\mu \times \mathbb{P}^\mu$. Let G be a $\mathbb{P}$-generic over (the ground model) V. Define

$$G_\mu = G \cap \mathbb{P}_\mu.$$

and

$$G^\mu = G \cap \mathbb{P}^\mu.$$

Then G_μ is $\mathbb{P}_\mu$-generic over V.

Also G^μ is $\mathbb{P}^\mu$-generic over $V[G_\mu]$ and $V[G] = V[G_\mu][G^\mu]$.

Let

$$E = \{p \in \mathbb{P} \mid (p > q) \wedge (p_\mu \Vdash p^\mu \in \dot{D})\}$$

Then E is dense above q: if $p > q$ is arbitrary element of $\mathbb{P}$, then $q \Vdash \exists p' > \check{p}^\mu(p' \in \dot{D})$. Thus there exists $q' > q$ with $q' > p_\mu$, $q' \in \mathbb{P}_\mu$ and $p' > p, p' \in \mathbb{P}^\mu$ such that $q' \Vdash p' \in \dot{D}$ and so $(q' \restriction \mu) \cup (p' \restriction (\kappa \setminus \mu))$ is above p and in E. So there is $p \in G \cap E$. But then $p_\mu \in G_\mu$ and $p^\mu \in G^\mu$ and $p_\mu \Vdash p^\mu \in \dot{D}$, so $G^\mu \cap D \neq \varnothing$. Since D was arbitrary, this shows that G^μ is $\mathbb{P}^\mu$-generic over $V[G_\mu]$. Clearly $V[G]$ contains both G_μ and G^μ. On the other hand, $G = G_\mu \cup G^\mu$, so $G \in V[G_\mu][G^\mu]$. By the minimality of forcing extensions, we get $V[G] = V[G_\mu][G^\mu]$.

For each $\mu \in \operatorname{reg}(\kappa) \setminus \{\omega, \omega_1\}$ let

$$k_\mu \colon \mu^+ \to \{\sigma \mid \sigma \text{ is a nice } \mathbb{P}_\mu \text{ name for a subset of } \mu\}$$

be a bijection. A nice $\mathbb{P}_\mu$ name for a subset of $\check{\mu}$ is of the form

$$\bigcup \{\{\check{\alpha}\} \times A_\alpha \mid \alpha \in B\},$$

where $B \subset \check{\mu}$ and for each $\alpha \in B$, A_α is an antichain in $\mathbb{P}_\mu$. By (a) there are no antichains of length μ in $\mathbb{P}_\mu$ and $|\mathbb{P}_\mu| = \mu$, so there are at most $\mu^{<\mu} = \mu$ antichains and there are μ^+ subsets $B \subset \mu$, so there indeed exists such a bijection k_μ (these cardinality facts hold because $V = L$ and μ is regular). Note that if σ is a nice $\mathbb{P}_\mu$-name for a subset of $\check{\mu}$, then $\sigma \subset V_\mu$.

Let us define

$$D_\mu = \begin{cases} \Big[k_\mu\Big([(\cup G)(\mu^+)](0)\Big)\Big]_G & \text{if it is stationary} \\ \mu & \text{otherwise.} \end{cases}$$

Now D_μ is defined for all $\mu \in Y$, recall $Y = \{\mu \in V[G] \mid (\mu \in \operatorname{reg} \kappa)^V\}$. We claim that $\langle D_\mu \rangle_{\mu \in Y}$ is the needed $\diamondsuit$-sequence. Suppose it is not. Then there is a stationary set $S \subset S^{\omega_2}_\omega$ and a cub $C \subset \omega_2$ such that for all $\alpha \in C \cap Y$, $D_\alpha \neq S \cap \alpha$. By (e) there is a cub set $C_0 \subset C$ such that $C_0 \in V$. Let $\dot{S}$ be a nice name for S and p' such that p' forces that $\dot{S}$ is stationary. Let us show that

$$H = \{q \geqslant p' \mid q \Vdash D_\mu = \dot{S} \cap \check{\mu} \text{ for some } \mu \in C_0\}$$

is dense above p' which is obviously a contradiction. For that purpose let $p > p'$ be arbitrary and let us show that there is $q > p$ in H. Let us now use Π^1_1-reflection. First let us redefine $\mathbb{P}$. Let $\mathbb{P}^* = \{q \mid \exists r \in \mathbb{P}(r \restriction \operatorname{sprt} r = q)\}$. Clearly $\mathbb{P}^* \cong \mathbb{P}$ but the advantage is that $\mathbb{P}^* \subset V_\kappa$ and $\mathbb{P}^*_\mu = \mathbb{P}^* \cap V_\mu$ where $\mathbb{P}^*_\mu$ is defined as $\mathbb{P}_\mu$. One easily verifies that all the above things (concerning $\mathbb{P}_\mu$, $\mathbb{P}^\mu$ etc.) translate between $\mathbb{P}$ and $\mathbb{P}^*$. From now on denote $\mathbb{P}^*$ by $\mathbb{P}$. Let

$$R = (\mathbb{P} \times \{0\}) \cup (\dot{S} \times \{1\}) \cup (C_0 \times \{2\}) \cup (\{p\} \times \{3\})$$

Then $(V_\kappa, R) \models \forall A\varphi$, where φ says: "(if A is closed unbounded and $r > p$ arbitrary, then there exist $q > r$ and α such that $\alpha \in A$ and $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$)." So basically $\forall A\varphi$ says "$p \Vdash$ ($\dot{S}$ is stationary)". It follows from (e) that it is enough to quantify over cub sets in V. Let us explain why such a formula can be written for (V_κ, R). The sets (classes from the viewpoint of V_κ) $\mathbb{P}$, $\dot{S}$ and C_0 are coded into R, so we can use them as parameters. That $r > p$ and $q > r$ and A is closed and unbounded is expressible in first-order as well as $\alpha \in A$. How do we express $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$? The definition of $\check{\alpha}$ is recursive in α:

$$\check{\alpha} = \{(\check{\beta}, 1_{\mathbb{P}}) \mid \beta < \alpha\}$$

and is absolute for V_κ. Then $q \Vdash_{\mathbb{P}} \check\alpha \in \dot S$ is equivalent to saying that for each $q' > q$ there exists $q'' > q'$ with $(\check\alpha, q'') \in \dot S$ and this is expressible in first-order (as we have taken R as a parameter).

By Π^1_1-reflection there is $\mu \in C_0$ such that $p \in \mathbb{P}_\mu$ and $(V_\mu, R) \models \forall A\varphi$. Note that we may require that μ is regular, i.e. $(\check\mu_G \in Y)^{V[G]}$ and such that $\alpha \in S \cap \mu$ implies $(\check\alpha, \check p) \in \dot S$ for some $p \in \mathbb{P}_\mu$. Let $\dot S_\mu = \dot S \cap V_\mu$.

Thus $p \Vdash_{\mathbb{P}_\mu}$ "$\dot S_\mu$ is stationary". Define q as follows: $\operatorname{dom} q = \operatorname{dom} p \cup \{\mu^+\}$, $q \restriction \mu = p \restriction \mu$ and $q(\mu^+) = f$, $\operatorname{dom} f = \{0\}$ and $f(0) = k_\mu^{-1}(\dot S_\mu)$. Then $q \Vdash_{\mathbb{P}} \dot S_\mu = D_\mu$ provided that $q \Vdash_{\mathbb{P}}$ "$\dot S_\mu$ is stationary". The latter holds since $\mathbb{P}^\mu$ is $< \omega_1$-closed., and does not kill stationarity of $(\dot S_\mu)_{G_\mu}$ so $(\dot S_\mu)_{G_\mu}$ is stationary in $V[G]$ and by the assumption on μ, $(\dot S_\mu)_{G_\mu} = (\dot S_\mu)_G$. Finally, it remains to show that in $V[G]$, $(\dot S_\mu)_G = S \cap \mu$. But this again follows from the definition of μ.

Instead of collapsing κ to ω_2, we could do the same for λ^{++} for any regular $\lambda < \kappa$ and obtain a model in which $E_{S^{\lambda^{++}}_{\lambda}} \leqslant_c E_{S^{\lambda^{++}}_{\lambda^+}}$. □

Open Problem. Is it consistent that $S^{\omega_2}_{\omega_1}$ Borel reduces to $S^{\omega_2}_{\omega}$?

E_0 and $E_{S^\kappa_\lambda}$

In the Section 4.4.4 above, Theorem 4.59, we showed that the equivalence relations of the form $E_{S^\kappa_\lambda}$ can form an antichain with respect to $\leqslant_B$. We will show that under mild set theoretical assumptions, all of them are strictly above

$$E_0 = \{(\eta, \xi) \mid \eta^{-1}\{1\} \triangle \xi^{-1}\{1\} \text{ is bounded}\}.$$

4.63 Theorem. *Let κ be regular and $S \subset \kappa$ stationary and suppose that $\diamondsuit_\kappa(S)$ holds (i.e., $\diamondsuit_\kappa$ holds on the stationary set S). Then E_0 is Borel reducible to E_S.*

Proof. The proof uses similar ideas than the proof of Theorem 4.61. Suppose that the $\diamondsuit_\kappa(S)$ holds and let $\langle D_\alpha \rangle_{\alpha \in S}$ be the $\diamondsuit_\kappa(S)$-sequence. Define the reduction $f \colon 2^\kappa \to 2^\kappa$ by

$$f(X) = \{\alpha \in S \mid D_\alpha \text{ and } X \cap \alpha \text{ agree on a final segment of } \alpha\}$$

If X, Y are E_0-equivalent, then $f(X)$, $f(Y)$ are E_S-equivalent, because they are in fact even E_0-equivalent as is easy to check. If X, Y are not E_0-equivalent, then there is a club C of α where X, Y differ cofinally in α; it follows that $f(X)$, $f(Y)$ differ on a stationary subset of S, namely the elements α of $C \cap S$ where D_α equals $X \cap \alpha$. □

4.64 Corollary. *Suppose $\kappa = \lambda^+ = 2^\lambda$. Then E_0 is Borel reducible to E_S where $S \subset \kappa \setminus S^\kappa_{\mathrm{cf}(\lambda)}$ is stationary.*

Proof. Gregory proved in [8] that if $2^\mu = \mu^+ = \kappa$, μ is regular and $\lambda < \mu$, then $\diamondsuit_\kappa(S^\kappa_\lambda)$ holds. Shelah extended this result in [45] and proved that if $\kappa = \lambda^+ = 2^\lambda$ and $S \subset \kappa \setminus S^\kappa_{\mathrm{cf}(\lambda)}$, then $\diamondsuit_\kappa(S)$ holds. Now apply Theorem 4.63. □

4.65 Corollary (GCH)**.** *Let us assume that κ is a successor cardinal. Then in a cofinality and GCH preserving forcing extension, there is an embedding*

$$f \colon \langle \mathcal{P}(\kappa), \subset \rangle \to \langle \mathcal{E}^{\Sigma^1_1}, \leqslant_B \rangle,$$

where $\mathcal{E}^{\Sigma_1^1}$ is the set of Σ_1^1-equivalence relations (see Theorem 4.55) such that for all $A \in \mathcal{P}(\kappa)$, E_0 is strictly below $f(A)$. If κ is not the successor of an ω-cofinal cardinal, we may replace Σ_1^1 above by Borel.*

Proof. Suppose first that κ is not the successor of an ω-cofinal cardinal. By Theorem 4.55 there is a GCH and cofinality-preserving forcing extension such that there is an embedding

$$f\colon \langle \mathcal{P}(\kappa), \subset \rangle \to \langle \mathcal{E}^{\mathrm{Borel}^*}, \leqslant_B \rangle.$$

From the proof of Theorem 4.55 one sees that $f(A)$ is of the form E_S where $S \subset S^\kappa_\omega$. Now E_0 is reducible to such relations by Corollary 4.64, as GCH continues to hold in the extension.

So it suffices to show that $E_S \not\leqslant_B E_0$ for stationary $S \subset S^\kappa_\omega$. By the same argument as in Corollary 4.53 on page 99, E_S is not Borel and by Theorem 4.35 on page 80, E_0 is Borel, so by Fact 4.78 on page 119, $E_{S^\kappa_\lambda}$ is not reducible to E_0.

Suppose κ is the successor of an ω-cofinal ordinal and $\kappa > \omega_1$. Then, in the proof of Theorem 4.55 replace μ by ω_1 and get the same result as above but for relations of the form E_S where $S \subset S^\kappa_{\omega_1}$.

The remaining case is $\kappa = \omega_1$. Let $\{S_\alpha \mid \alpha < \omega_1\}$ be a set of pairwise disjoint stationary subsets of ω_1. Let $\mathbb{P}$ be the forcing given by the proof of Theorem 4.55 such that in the $\mathbb{P}$-generic extension the function $f\colon \langle \mathcal{P}(\omega_1), \subset \rangle \to \langle \mathcal{E}^{\mathrm{Borel}^*}, \leqslant_B \rangle$ given by $f(A) = E_{\bigcup_{\alpha \in A} S_\alpha}$ is an embedding. This forcing preserves stationary sets, so as in the proof of clause (4) of Theorem 4.52, we can first force a $\diamondsuit$-sequence which guesses each subset of $\bigcup_{\alpha<\omega_1} S_\alpha$ on a set S such that $S \cap S_\alpha$ is stationary for all α. Then by Corollary 4.64 E_0 is reducible to $E_{\bigcup_{\alpha \in A} S_\alpha}$ for all $A \subset \kappa$. □

Remark. The embeddings of Theorems 5.11 and 5.12 (page 146) are in contrast *strictly below* E_0.

4.5 Complexity of Isomorphism Relations

Let T be a countable complete theory. Let us turn to the question discussed in Section 4.1: "How is the set theoretic complexity of $\cong_T$ related to the stability theoretic properties of T?". The following theorems give some answers. As pointed out in Section 4.1, the assumption that κ is uncountable is crucial in the following theorems. For instance the theory of dense linear orderings without end points is unstable, but $\cong_T$ is an open set in case $\kappa = \omega$, while we show below that for unstable theories T the set $\cong_T$ cannot be even Δ_1^1 when $\kappa > \omega$. Another example introduced by Martin Koerwien in his Ph.D. thesis and in [29] shows that there are classifiable shallow theories whose isomorphism is not Borel when $\kappa = \omega$, although we prove below that the isomorphism of such theories is always Borel, when $\kappa^{<\kappa} = \kappa > 2^\omega$. This justifies in particular the motivation for studying the space κ^κ for model theoretic purpose: the set theoretic complexity of $\cong_T$ positively correlates with the model theoretic complexity of T.

The following stability theoretical notions will be used: stable, superstable, DOP, OTOP, shallow, $\lambda(T)$ and $\kappa(T)$. Classifiable means superstable with no DOP nor OTOP and $\lambda(T)$ is the least cardinal in which T is stable.

Recall that by $\cong_T^\kappa$ we denote the isomorphism relation of models of T whose size is κ. The main theme in this section is exposed in the following two theorems:

4.66 Theorem ($\kappa^{<\kappa} = \kappa$). *Assume that κ is a successor and let T be a complete countable theory. If $\cong_T^\kappa$ is Borel, then T is classifiable and shallow. If additionally $\kappa > 2^\omega$, then the converse holds: if T is classifiable and shallow, then $\cong_T^\kappa$ is Borel.*

4.67 Theorem ($\kappa^{<\kappa} = \kappa$). *Assume that for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and $\kappa > \omega_1$. Then in L and in the forcing extension after adding κ^+ Cohen subsets of κ we have: for any theory T, T is classifiable if and only if $\cong_T$ is Δ_1^1.*

The two theorems above are proved in many sub-theorems below. Our results are stronger than those given by 4.66 and 4.67 (for instance the cardinality assumption $\kappa > \omega_1$ is needed only in the case where T is superstable with DOP and the stable unsuperstable case is the only one for which Theorem 4.67 cannot be proved in ZFC). Theorem 4.66 follows from Theorems 4.71, 4.72. Theorem 4.67 follows from Theorems 4.73, 4.74, 4.75 and items (2) and (3) of Theorem 4.52.

4.5.1 Preliminary Results

The following Theorems 4.68 and 4.70 (page 115) will serve as bridges between the set theoretic complexity and the model theoretic complexity of an isomorphism relation.

4.68 Theorem ($\kappa^{<\kappa} = \kappa$). *For a theory T, the set $\cong_T$ is Borel if and only if the following holds: there exists a $\kappa^+\omega$-tree t such that for all models $\mathcal{A}$ and $\mathcal{B}$ of T, $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$.*

Proof. Recall that we assume $\operatorname{dom} \mathcal{A} = \kappa$ for all models in the discourse. First suppose that there exists a $\kappa^+\omega$-tree t such that for all models $\mathcal{A}$ and $\mathcal{B}$ of T, $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Let us show that there exists a $\kappa^+\omega$-tree u which constitutes a Borel code for $\cong_T$ (see Remark 4.17 on page 68).

Let u be the tree of sequences of the form

$$\langle (p_0, A_0), f_0, (p_1, A_1), f_1, \dots, (p_n, A_n), f_n \rangle$$

such that for all $i \leqslant n$

1. (p_i, A_i) is a move of player $\mathbf{I}$ in EF_t^κ, i.e. $p_i \in t$ and $A_i \subset \kappa$ with $|A_i| < \kappa$,
2. f_i is a move of player $\mathbf{II}$ in EF_t^κ, i.e. it is a partial function $\kappa \to \kappa$ with $|\operatorname{dom} f_i|, |\operatorname{ran} f_i| < \kappa$ and $A_i \subset \operatorname{dom} f_i \cap \operatorname{ran} f_i$
3. $\langle (p_0, A_0), f_0, (p_1, A_1), f_1, \dots, (p_n, A_n), f_n \rangle$ is a valid position of the game, i.e. $(p_i)_{i \leqslant n}$ is an initial segment of a branch in t and $A_i \subset A_j$ and $f_i \subset f_j$ whenever $i < j \leqslant n$.

Order u by end extension. The tree u is a $\kappa^+\omega$-tree (because t is and by (3)).

Let us now define the function

$$h\colon \{\text{branches of } u\} \to \{\text{basic open sets of } (\kappa^\kappa)^2\}.$$

Let $b \subset u$ be a branch,

$$b = \{\varnothing, \langle (p_0, A_0) \rangle, \langle (p_0, A_0), f_0 \rangle, \dots, \langle (p_0, A_0), f_0, \dots, (p_k, A_k), f_k \rangle\}.$$

It corresponds to a unique EF-game between some two structures with domains κ. In this game the players have chosen some set $A_k = \bigcup_{i\leqslant k} A_i \subset \kappa$ and some partial function $f_k = \bigcup_{i\leqslant k} f_i\colon \kappa \to \kappa$. Let $h(b)$ be the set of all pairs $(\eta,\xi) \in (\kappa^\kappa)^2$ such that $f_\kappa\colon \mathcal{A}_\eta \restriction A_\kappa \cong \mathcal{A}_\xi \restriction A_\kappa$ is a partial isomorphism. This is clearly an open set:

$$(\eta,\xi) \in h(b) \Rightarrow N_{\eta\restriction((\sup A_\kappa)+1)} \times N_{\xi\restriction((\sup A_\kappa)+1)} \subset h(b).$$

Finally we claim that $\mathcal{A}_\eta \cong \mathcal{A}_\xi \iff \mathbf{II} \uparrow G(u,h,(\eta,\xi))$. Here G is the game as in Definition 4.16 of Borel* sets, page 67 but played on the product $\kappa^\kappa \times \kappa^\kappa$. Assume $\mathcal{A}_\eta \cong \mathcal{A}_\xi$. Then $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}_\eta,\mathcal{A}_\xi)$. Let υ denote the winning strategy. In the game $G(u,h,(\eta,\xi))$, let us define a winning strategy for player **II** as follows. By definition, at a particular move, say n, **I** chooses a sequence

$$\langle (p_0,A_0), f_0, \dots (p_n,A_n)\rangle.$$

Next **II** extends it according to υ to

$$\langle (p_0,A_0), f_0, \dots (p_n,A_n), f_n\rangle,$$

where $f_n = \upsilon((p_0,A_0),\dots,(p_n,A_n))$. Since υ was a winning strategy, it is clear that $f_\kappa = \bigcup_{i<\kappa} f_i$ is going to be a isomorphism between $\mathcal{A}_\eta \restriction A_\kappa$ and $\mathcal{A}_\xi \restriction A_\kappa$, so $(\eta,\xi) \in h(b)$.

Assume that $\mathcal{A}_\eta \ncong \mathcal{A}_\xi$. Then by the assumption there is no winning strategy of **II**, so player **I** can play in such a way that $f_\kappa = \bigcup_{i\leqslant\kappa} f_i$ is not an isomorphism between $\mathcal{A}_\eta \restriction \cup A_i$ and $\mathcal{A}_\xi \restriction \cup A_i$, so (η,ξ) is not in $h(b)$. This completes the proof of the direction "⇐".

Let us prove "⇒". Suppose $\cong_T$ is Borel and let us show that there is a tree as in the statement of the theorem. We want to use Theorem 4.25 and formalize the statement "$\cong_T$ is definable in $L_{\kappa^+\kappa}$" by considering the space consisting of pairs of models.

Denote the vocabulary of $\mathcal{A}$ and $\mathcal{B}$ as usual by L. Let P be a unary relation symbol not in L. We will now discuss two distinct vocabularies, L and $L \cup \{P\}$ at the same time, so we have to introduce two distinct codings. Fix an $\eta \in 2^\kappa$. Let $\mathcal{A}_\eta$ denote the L-structure as defined in Definition 4.13 of our usual coding. Let $\rho\colon \kappa \cup \kappa^{<\omega} \to \kappa$ be a bijection and define $\mathcal{A}^\eta$ to be the model with $\operatorname{dom}\mathcal{A}^\eta = \kappa$ and if $a \in \operatorname{dom}\mathcal{A}^\eta$, then $\mathcal{A}^\eta \models P(a) \iff \eta(\rho(a)) = 1$ such that if $(a_1,\dots,a_n) \in (\operatorname{dom}\mathcal{A}^\eta)^n$, then $\mathcal{A}^\eta \models P_n(a_1,\dots,a_n) \iff \eta(\rho(a_1,\dots,a_n)) = 1$. Note that we are making a distinction here between κ and $\kappa^{\{0\}}$.

Claim 1. The set $W = \{\eta \in 2^\kappa \mid \kappa = |P^{\mathcal{A}^\eta}| = |\kappa \setminus P^{\mathcal{A}^\eta}|\}$ is Borel.

Proof of Claim 1. Let us show that the complement is Borel. By symmetry it is sufficient to show that

$$B = \{\eta \mid \kappa > |P^{\mathcal{A}^\eta}|\}$$

is Borel. Let $I \subset \kappa$ be a subset of size $< \kappa$. For $\beta \notin I$ define $U(I,\beta)$ to be the set

$$U(I,\beta) = \{\eta \mid \eta(\rho(\beta)) = 0\}.$$

Clearly $U(I,\beta)$ is open for all I, β. Now

$$B = \bigcup_{I\in[\kappa]^{<\kappa}} \bigcap_{\beta\notin I} U(I,\beta).$$

By the assumption $\kappa^{<\kappa} = \kappa$, this is Borel (in fact a union of closed sets). $\square_{\text{Claim 1}}$

Define a mapping $h\colon W \to (2^\kappa)^2$ as follows. Suppose $\xi \in W$. Let

$$r_1\colon \kappa \to P^{\mathcal{A}^\xi}$$

and

$$r_2\colon \kappa \to \kappa \setminus P^{\mathcal{A}^\xi}$$

be the order preserving bijections (note $P^{\mathcal{A}^\eta} \subset \kappa = \operatorname{dom} \mathcal{A}^\eta$).

Let η_1 be such that r_1 is an isomorphism

$$\mathcal{A}_{\eta_1} \to (\mathcal{A}^\xi \cap P^{\mathcal{A}^\xi}) \restriction L$$

and η_2 such that r_2 is an isomorphism

$$\mathcal{A}_{\eta_2} \to (\mathcal{A}^\xi \setminus P^{\mathcal{A}^\xi}) \restriction L.$$

Clearly η_1 and η_2 are unique, so we can define $h(\xi) = (\eta_1, \eta_2)$.

Claim 2. h is continuous.

Proof of Claim 2. Let $U = N_p \times N_q$ be a basic open set of $(2^\kappa)^2$, $p, q \in 2^{<\kappa}$ and let $\xi \in h^{-1}[U]$. Let $P^{\mathcal{A}^\xi} = \{\beta_i \mid i < \kappa\}$ be an enumeration such that $\beta_i < \beta_j \iff i < j$ and similarly $\kappa \setminus P^{\mathcal{A}^\xi} = \{\gamma_i \mid i < \kappa\}$. Let $\alpha = \max\{\beta_{\operatorname{dom} p}, \gamma_{\operatorname{dom} q}\} + 1$. Then $N_{\xi\restriction\alpha} \subset h^{-1}[U]$. Thus arbitrary ξ in $h^{-1}[U]$ have an open neighborhood in $h^{-1}[U]$, so it is open. $\square_{\text{Claim 2}}$

Recall our assumption that $E = \{(\eta, \xi) \in 2^\kappa \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$ is Borel. Since h is continuous and in particular Borel, this implies that

$$E' = \{\eta \mid \mathcal{A}_{h_1(\eta)} \cong \mathcal{A}_{h_2(\eta)}\} = h^{-1}E$$

is Borel in W. Because W is itself Borel, E' is Borel in 2^κ. Additionally, E' is closed under permutations: if $\mathcal{A}^\eta$ is isomorphic to $\mathcal{A}^\xi$, then $\mathcal{A}^\eta \cap P^{\mathcal{A}^\eta}$ is isomorphic to $\mathcal{A}^\xi \cap P^{\mathcal{A}^\xi}$ and $\mathcal{A}^\eta \setminus P^{\mathcal{A}^\eta}$ is isomorphic to $\mathcal{A}^\xi \setminus P^{\mathcal{A}^\xi}$, so if $\mathcal{A}^\eta \in E'$, then also $\mathcal{A}^\xi \in E'$ (and note that since $\eta \in W$, also $\xi \in W$). By Theorem 4.25 (page 71), there is a sentence θ of $L_{\kappa^+\kappa}$ over $L \cup \{P\}$ that defines E'. Thus by Theorem 4.10 (page 65) and Remark 4.12 (page 65) there is a $\kappa^+\omega$-tree t such that

$$\text{if } \eta \in E' \text{ and } \xi \notin E', \text{ then } \mathbf{II} \not\uparrow \mathrm{EF}^\kappa_t(\mathcal{A}^\eta, \mathcal{A}^\xi). \qquad \odot$$

We claim that t is as needed, i.e. for all models $\mathcal{A}, \mathcal{B}$ of T

$$\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B}).$$

Suppose not. Then there are models $\mathcal{A} \ncong \mathcal{B}$ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$. Let η and ξ be such that $\mathcal{A}_{h_1(\eta)} = \mathcal{A}_{h_2(\eta)} = \mathcal{A}_{h_1(\xi)} = \mathcal{A}$ and $\mathcal{A}_{h_2(\xi)} = \mathcal{B}$. Clearly $\eta \in E'$, but $\xi \notin E'$, so by $\odot$ there is no winning strategy of $\mathbf{II}$ in $\mathrm{EF}^\kappa_t(\mathcal{A}^\eta, \mathcal{A}^\xi)$ which is clearly a contradiction, because $\mathbf{II}$ can apply her winning strategies in $\mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$ and $\mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{A})$ to win in $\mathrm{EF}^\kappa_t(\mathcal{A}^\eta, \mathcal{A}^\xi)$. $\square_{\text{Theorem 4.68}}$

We will use the following lemma from [36]:

4.69 Lemma. *If $t \subset (\kappa^{<\kappa})^2$ is a tree and $\xi \in \kappa^\kappa$, denote*

$$t(\xi) = \{p \in \kappa^{<\kappa} \mid (p, \xi \restriction \operatorname{dom} p) \in t\}$$

Similarly if $t \in (\kappa^{<\kappa})^3$, then

$$t(\eta, \xi) = \{p \in \kappa^{<\kappa} \mid (p, \eta \restriction \operatorname{dom} p, \xi \restriction \operatorname{dom} p) \in t\}.$$

Assume that Z is Σ^1_1. Then Z is Δ^1_1 if and only if for every tree $t \subset (\kappa^{<\kappa})^2$ such that

$$t(\xi) \text{ has a } \kappa\text{-branch} \iff \xi \in Z$$

there exists a $\kappa^+\kappa$-tree t' such that $\xi \in Z \iff t(\xi) \not\leqslant t'$. (Recall that $t \leqslant t'$ when there exists a strictly order preserving map $t \to t'$)

4.70 Theorem. *Let T be a theory and assume that for every $\kappa^+\kappa$-tree t there exist $(\eta, \xi) \in (2^\kappa)^2$ such that $\mathcal{A}_\eta, \mathcal{A}_\xi \models T$, $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$ but $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}_\eta, \mathcal{A}_\xi)$. Then $\cong_T$ is not Δ^1_1.*

Proof. Let us abbreviate some statements:

$A(t)$: $t \subset (\kappa^{<\kappa})^3$ is a tree and for all $(\eta, \xi) \in (\kappa^\kappa)^2$,

$$(\eta, \xi) \in \cong_T \iff t(\eta, \xi) \text{ contains a } \kappa\text{-branch} .$$

$B(t, t')$: $t \subset (\kappa^{<\kappa})^3$ is a $\kappa^+\kappa$-tree and for all $(\eta, \xi) \in \kappa^\kappa$,

$$(\eta, \xi) \in \cong_T \iff t(\eta, \xi) \not\leqslant t'.$$

Now Lemma 4.69 implies that if $\cong_T$ is Δ^1_1, then $\forall t[A(t) \to \exists t' B(t, t')]$. We will show that $\exists t[A(t) \wedge \forall t' \neg B(t, t')]$, which by Lemma 4.69 suffices to prove the theorem. Let us define t. In the following, ν_α, η_α and ξ_α stand respectively for $\nu \restriction \alpha$, $\eta \restriction \alpha$ and $\xi \restriction \alpha$.

$$t = \{(\nu_\alpha, \eta_\alpha, \xi_\alpha) \mid \alpha < \kappa \text{ and } \nu \text{ codes an isomorphism between } \mathcal{A}_\eta \text{ and } \mathcal{A}_\xi\}.$$

Using Theorem 4.14 it is easy to see that t satisfies $A(t)$. Assume now that t' is an arbitrary $\kappa^+\kappa$-tree. We will show that $B(t, t')$ does not hold. For that purpose let $u = \omega \times t'$ be the tree defined by the set $\{(n, s) \mid n \in \omega, s \in t'\}$ and the ordering

$$(n_0, s_0) <_u (n_1, s_1) \iff \big(s_0 <_{t'} s_1 \vee (s_0 = s_1 \wedge n_0 <_\omega n_1)\big). \tag{1}$$

This tree u is still a $\kappa^+\kappa$-tree, so by the assumption of the theorem there is a pair (ξ_1, ξ_2) such that $\mathcal{A}_{\xi_1}$ and $\mathcal{A}_{\xi_2}$ are non-isomorphic, but $\mathbf{II} \uparrow \mathrm{EF}^\kappa_u(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$.

It is now sufficient to show that $t(\xi_1, \xi_2) \not\leqslant t'$.

Claim 1. There is no order preserving function

$$\sigma t' \to t',$$

where $\sigma t'$ is defined in Definition 4.32.

Proof of Claim 1. Assume $g\colon \sigma t' \to t'$, is order preserving. Define $x_0 = g(\varnothing)$ and

$$x_\alpha = g(\{y \in t' \mid \exists\beta < \alpha(y \leqslant x_\beta)\}) \text{ for } 0 < \alpha < \kappa$$

Then $(x_\alpha)_{\alpha<\kappa}$ contradicts the assumption that t' is a $\kappa^+\kappa$-tree. $\square_{\text{Claim 1}}$

Claim 2. There is an order preserving function

$$\sigma t' \to t(\xi_1, \xi_2).$$

Proof of Claim 2. The idea is that players **I** and **II** play an EF-game for each branch of the tree t' and **II** uses her winning strategy in $\text{EF}^\kappa_u(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$to embed that branch into the tree of partial isomorphisms. A problem is that the winning strategy gives arbitrary partial isomorphisms while we are interested in those which are coded by functions defined on page 67. Now the tree u of (1) above becomes useful.

Let σ be a winning strategy of player **II** in $\text{EF}^\kappa_u(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$. Let us define $g\colon \sigma t' \to t(\xi_1, \xi_2)$ recursively. Recall the function π from Definition 4.13 and define

$$C = \{\alpha \mid \pi[\alpha^{<\omega}] = \alpha\}.$$

Clearly C is cub. If $s \subset t'$ is an element of $\sigma t'$, then we assume that g is defined for all $s' <_{\sigma t'} s$ and that EF^κ_u is played up to $(0, \sup s) \in u$. If s does not contain its supremum, then put $g(s) = \bigcup_{s'<s} g(s')$. Otherwise let them continue playing the game for ω more moves; at the n^{th} of these moves player **I** picks $(n, \sup s)$ from u and a $\beta < \kappa$ where β is an element of C above

$$\max\{\operatorname{ran} f_{n-1}, \operatorname{dom} f_{n-1}\}$$

where f_{n-1} is the previous move by **II**. (If $n = 0$, it does not matter what **I** does.) In that way the function $f = \bigcup_{n<\omega} f_n$ is a partial isomorphism such that $\operatorname{dom} f = \operatorname{ran} f = \alpha$ for some ordinal α. It is straightforward to check that such an f is coded by some $\nu_\alpha\colon \alpha \to \kappa$. It is an isomorphism between $\mathcal{A}_{\xi_1} \cap \alpha$ and $\mathcal{A}_{\xi_2} \cap \alpha$ and since α is in C, there are ξ'_1 and ξ'_2 such that $\xi_1 \restriction \alpha \subset \xi'_1$, $\xi_2 \restriction \alpha \subset \xi'_2$ and there is an isomorphism $\mathcal{A}_{\xi'_1} \cong \mathcal{A}_{\xi'_2}$ coded by some ν such that $\nu_\alpha = \nu \restriction \alpha$. Thus $\nu_\alpha \in t(\xi_1, \xi_2)$ is suitable for setting $g(s) = \nu_\alpha$. $\square_{\text{Claim 2}}$

$\square_{\text{Theorem 4.70}}$

4.5.2 Classifiable

Throughout this section κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa > \omega$.

4.71 Theorem ($\kappa > 2^\omega$)**.** *If the theory T is classifiable and shallow, then $\cong_T$ is Borel.*

Proof. If T is classifiable and shallow, then from [40, Theorem XIII.1.5 and Claim XIII.1.3] it follows that the models of T are characterized by a fragment of $L_{\kappa^+\kappa}$ which consists of formulas of bounded quantifier rank (the bound depends on depth of T). By the standard argument this implies that the game EF^κ_t characterized models of T of size κ up to isomorphism, where t is some $\kappa^+\omega$-tree (in fact a tree of descending sequences of an ordinal $\alpha < \kappa^+$). Hence by Theorem 4.68 the isomorphism relation of T is Borel. $\square$

4.72 Theorem. *If the theory T is classifiable but not shallow, then $\cong_T$ is not Borel. If κ is not weakly inaccessible and T is not classifiable, then $\cong_T$ is not Borel.*

Proof. If T is classifiable but not shallow, then by [40] XIII.1.8, the $L_{\infty\kappa}$-Scott heights of models of T of size κ are not bounded by any ordinal $< \kappa^+$ (see Definition 4.8 on page 65). Because any $\kappa^+\omega$-tree can be embedded into $t_\alpha = \{\text{decreasing sequences of } \alpha\}$ for some α (see Fact 4.3 on page 61), this implies that for any $\kappa^+\omega$-tree t there exists a pair of models $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \not\cong \mathcal{B}$ but $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$. Theorem 4.68 now implies that the isomorphism relation is not Borel.

If T is not classifiable κ is not weakly inaccessible, then by [41] Theorem 0.2 (Main Conclusion), there are non-isomorphic models of T of size κ which are $L_{\infty\kappa}$-equivalent, so the same argument as above, using Theorem 4.68, gives that $\cong_T$ is not Borel. □

4.73 Theorem. *If the theory T is classifiable, then $\cong_T$ is Δ^1_1.*

Proof. Shelah's theorem [40, Theorem XIII.1.1] says that if a theory T is classifiable, then any two models that are $L_{\infty\kappa}$-equivalent are isomorphic. But $L_{\infty\kappa}$ equivalence is equivalent to $\mathrm{EF}^\kappa_\omega$-equivalence (see Theorem 4.11 on page 65). So in order to prove the theorem it is sufficient to show that if for any two models $\mathcal{A}$, $\mathcal{B}$ of the theory T it holds that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \cong \mathcal{B}$, then the isomorphism relation is Δ^1_1. The game $\mathrm{EF}^\kappa_\omega$ is a closed game of length ω and so determined. Hence we have $\mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \not\cong \mathcal{B}$. By Theorem 4.7 the set

$$\{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \text{ codes a winning strategy for } \mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}_\eta, \mathcal{A}_\xi))\}$$

is closed and thus $\{(\eta, \xi) \mid \mathcal{A}_\eta \not\cong \mathcal{A}_\xi\}$ is Σ^1_1, which further implies that $\cong_T$ is Δ^1_1 by Corollary 4.15. □

4.5.3 Unclassifiable

The Unstable, DOP and OTOP Cases

As before, κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa > \omega$.

4.74 Theorem. *1. If T is unstable then $\cong_T$ is not Δ^1_1.*

2. If T is stable with OTOP, then $\cong_T$ is not Δ^1_1.

3. If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is not Δ^1_1.

4. If T is stable with DOP and $\lambda = \mathrm{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geqslant \omega_1, \kappa > \lambda^+$ and for all $\xi < \kappa$, $\xi^\lambda < \kappa$, then $\cong_T$ is not Δ^1_1. (Note that $\kappa(T) \in \{\omega, \omega_1\}$.)

Proof. For a model $\mathcal{A}$ of size κ of a theory T let us denote by

$$E(\mathcal{A})$$

the following property: for every $\kappa^+\kappa$-tree t there is a model $\mathcal{B}$ of T of cardinality κ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$ and $\mathcal{A} \not\cong \mathcal{B}$.

For (3) we need a result by Hyttinen and Tuuri, Theorem 6.2. from [23]:

Fact (Superstable with DOP). *Let T be a superstable theory with DOP and $\kappa^{<\kappa} = \kappa > \omega_1$. Then there exists a model $\mathcal{A}$ of T of cardinality κ with the property $E(\mathcal{A})$.*

For (4) we will need a result by Hyttinen and Shelah from [21]:

Fact (Stable with DOP). *Let T be a stable theory with DOP and $\lambda = \mathrm{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geqslant \omega_1$, $\kappa^{<\kappa} = \kappa > \lambda^+$ and for all $\xi < \kappa$, $\xi^\lambda < \kappa$. Then there is a model $\mathcal{A}$ of T of power κ with the property $E(\mathcal{A})$.*

For (1) a result by Hyttinen and Tuuri Theorem 4.9 from [23]:

Fact (Unstable). *Let T be an unstable theory. Then there exists a model $\mathcal{A}$ of T of cardinality κ with the property $E(\mathcal{A})$.*

And for (2) another result by Hyttinen and Tuuri, Theorem 6.6 in [23]:

Fact (Stable with OTOP). *Suppose T is a stable theory with OTOP. Then there exists a model $\mathcal{A}$ of T of cardinality κ with the property $E(\mathcal{A})$.*

Now (1), (2) and (4) follow immediately from Theorem 4.70. □

Stable Unsuperstable

We assume $\kappa^{<\kappa} = \kappa > \omega$ in all theorems below.

4.75 Theorem. *Assume that for all $\lambda < \kappa$, $\lambda^\omega < \kappa$.*

1. *If T is stable unsuperstable, then $\cong_T$ is not Borel.*
2. *If κ is as above and T is stable unsuperstable, then $\cong_T$ is not Δ_1^1 in the forcing extension after adding κ^+ Cohen subsets of κ, or if $V = L$.*

Proof. By Theorem 4.90 on page 137 the relation $E_{S^\kappa_\omega}$ can be reduced to $\cong_T$. The theorem follows now from Corollary 4.53 on page 99. □

On the other hand, stable unsuperstable theories sometimes behave nicely to some extent:

4.76 Lemma. *Assume that T is a theory and t a $\kappa^+\kappa$-tree such that if $\mathcal{A}$ and $\mathcal{B}$ are models of T, then $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$. Then $\cong$ of T is Borel*.*

Proof. Similar to the proof of Theorem 4.68. □

4.77 Theorem. *Assume $\kappa \in I[\kappa]$ and $\kappa = \lambda^+$ ("$\kappa \in I[\kappa]$" is known as the Approachability Property and follows from $\lambda^{<\lambda} = \lambda$, see Section 5.3 on page 144 of this thesis). Then there exists an unsuperstable theory T whose isomorphism relation is Borel*.*

Proof. In [19] and [20] Hyttinen and Shelah show the following (Theorem 1.1 of [20], but the proof is essentially in [19]):

> Suppose $T = ((\omega^\omega, E_i)_{i<\omega})$, where $\eta E_i \xi$ if and only if for all $j \leqslant i$, $\eta(j) = \xi(j)$. If $\kappa \in I[\kappa]$, $\kappa = \lambda^+$ and $\mathcal{A}$ and $\mathcal{B}$ are models of T of cardinality κ, then $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}^\kappa_{\lambda\cdot\omega+2}(\mathcal{A}, \mathcal{B})$, where $+$ and $\cdot$ denote the ordinal sum and product, i.e. $\lambda \cdot \omega + 2$ is just an ordinal.

So taking the tree t to be $\lambda \cdot \omega + 2$ the claim follows from Lemma 4.76. □

Open Problem. If $\kappa = 2^\omega$, is the isomorphism relation of all classifiable and shallow theories Borel on structures of size κ?

Open Problem. We proved that if $\kappa > 2^\omega$ the isomorphism relation of a theory T is Borel if and only if T is classifiable and shallow. Is there a connection between the depth of a shallow theory and the Borel degree of its isomorphism relation? Is one monotone in the other?

Open Problem. Can it be proved in ZFC that if T is stable unsuperstable then $\cong_T$ is not Δ^1_1?

4.6 Reductions

Recall that in Section 4.5 we obtained a provable characterization of theories which are both classifiable and shallow in terms of the definability of their isomorphism relations. Without the shallowness condition we obtained only a consistency result. In this section we improve this to a provable characterization by analyzing isomorphism relations in terms of Borel reducibility.

Recall the definition of a reduction, section *Reductions* page 60, and recall that if $X \subset \kappa$ is a stationary subset, we denote by E_X the equivalence relation defined by

$$\forall \eta, \xi \in 2^\kappa (\eta E_X \xi \iff (\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}) \cap X \text{ is non-stationary}),$$

and by S^κ_λ we mean the ordinals of cofinality λ that are less than κ.

The equivalence relations E_X are Σ^1_1 ($\eta E_X \xi$ if and only if *there exists* a cub subset of $\kappa \setminus (X \cap (\eta \triangle \xi))$).

Simple conclusions can readily be made from the following observation that roughly speaking, the set theoretic complexity of a relation does not decrease under reductions:

4.78 Fact. *If E_1 is a Borel (or Δ^1_1) equivalence relation and E_0 is an equivalence relation with $E_0 \leqslant_B E_1$, then E_0 is Borel (respectively Δ^1_1 if E_1 is Δ^1_1).* □

The main theorem of this section is:

4.79 Theorem. *Suppose $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Let T be a first-order theory. Then T is classifiable if and only if for all regular $\mu < \kappa$, $E_{S^\kappa_\mu} \not\leqslant_B \cong^\kappa_T$.*

4.6.1 Classifiable Theories

The following follows from [40] Theorem XIII.1.1 (see also the proof of Theorem 4.73 above):

4.80 Theorem ([40])**.** *If a first-order theory T is classifiable and $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic models of T of size κ, then $\mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B})$.* □

4.81 Theorem ($\kappa^{<\kappa} = \kappa$)**.** *If a first-order theory T is classifiable, then for all $\lambda < \kappa$*

$$E_{S^\kappa_\lambda} \not\leqslant_B \cong^\kappa_T .$$

Proof. Let $\mathrm{NS} \in \{E_{S^\kappa_\lambda} \mid \lambda \in \mathrm{reg}(\kappa)\}$.

Suppose $r\colon 2^\kappa \to 2^\kappa$ is a Borel function such that

$$\forall \eta, \xi \in 2^\kappa (\mathcal{A}_{r(\eta)} \models T \wedge \mathcal{A}_{r(\xi)} \models T \wedge (\eta \,\mathrm{NS}\, \xi \iff \mathcal{A}_{r(\eta)} \cong \mathcal{A}_{r(\xi)})). \qquad (\nabla)$$

By Lemma 4.34, page 79, let D be an intersection of κ-many dense open sets such that $R = r \restriction D$ is continuous. D can be coded into a function $v\colon \kappa \times \kappa \to \kappa^{<\kappa}$ such that $D = \bigcap_{i<\kappa} \bigcup_{j<\kappa} N_{v(i,j)}$. Since R is continuous, it can also be coded into a single function $u\colon \kappa^{<\kappa} \times \kappa^{<\kappa} \to \{0,1\}$ such that

$$R(\eta) = \xi \iff (\forall \alpha < \kappa)(\exists \beta < \kappa)[u(\eta \restriction \beta, \xi \restriction \alpha) = 1].$$

(For example define $u(p,q) = 1$ if $D \cap N_p \subset R^{-1}[N_q]$.) Let

$$\varphi(\eta, \xi, u, v) = (\forall \alpha < \kappa)(\exists \beta < \kappa)[u(\eta \restriction \beta, \xi \restriction \alpha) = 1] \wedge (\forall i < \kappa)(\exists j < \kappa)[\eta \in N_{v(i,j)}].$$

It is a formula of set theory with parameters u and v. It is easily seen that φ is absolute for transitive elementary submodels M of $H(\kappa^+)$ containing κ, u and v with $(\kappa^{<\kappa})^M = \kappa^{<\kappa}$.

Let $\mathbb{P} = 2^{<\kappa}$ be the Cohen forcing. Suppose $M \preccurlyeq H(\kappa^+)$ is a model as above, i.e. transitive, $\kappa, u, v \in M$ and $(\kappa^{<\kappa})^M = \kappa^{<\kappa}$. Note that then $\mathbb{P} \cup \{\mathbb{P}\} \subset M$. Then, if G is $\mathbb{P}$-generic over M, then $\cup G \in D$ and there is ξ such that $\varphi(\cup G, \xi, u, v)$. By the definition of φ and u, an initial segment of ξ can be read from an initial segment of $\cup G$. That is why there is a nice $\mathbb{P}$-name τ for a function (see [32]) such that

$$\varphi(\cup G, \tau_G, u, v)$$

whenever G is $\mathbb{P}$-generic over M.

Now since the game $\mathrm{EF}^\kappa_\omega$ is determined on all structures, (at least) one of the following holds:

1. there is p such that $p \Vdash \mathbf{II} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}_\tau, \mathcal{A}_{r(\bar{0})})$
2. there is p such that $p \Vdash \mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}_\tau, \mathcal{A}_{r(\bar{0})})$

where $\bar{0}$ is the constant function with value 0. Let us show that both of them lead to a contradiction.

Assume (1). Fix a nice $\mathbb{P}$-name σ such that

$$p \Vdash \text{“}\sigma \text{ is a winning strategy of } \mathbf{II} \text{ in } \mathrm{EF}^\kappa_\omega(\mathcal{A}_\tau, \mathcal{A}_{r(\bar{0})})\text{”}$$

A strategy is a subset of $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa}$ (see Definition 4.6 on page 63), and the forcing does not add elements to that set, so the nice name can be chosen such that all names in $\mathrm{dom}\,\sigma$ are standard names for elements that are in $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa} \in H(\kappa^+)$.

Let M be an elementary submodel of $H(\kappa^+)$ of size κ such that

$$\{u, v, \sigma, r(\bar{0}), \tau, \mathbb{P}\} \cup (\kappa + 1) \cup M^{<\kappa} \subset M.$$

Listing all dense subsets of $\mathbb{P}$ in M, it is easy to find a $\mathbb{P}$-generic G over M which contains p and such that $(\cup G)^{-1}\{1\}$ contains a cub. Now in V, $\cup G \not\!\mathrm{NS}\, \bar{0}$. Since $\varphi(\cup G, \tau_G, u, v)$ holds, we have by (∇):

$$\mathcal{A}_{\tau_G} \not\cong \mathcal{A}_{r(\bar{0})}. \qquad (i)$$

Let us show that σ_G is a winning strategy of player **II** in $\mathrm{EF}^\kappa_\omega(\mathcal{A}_{\tau_G}, \mathcal{A}_{r(\bar{0})})$ (in V) which by Theorem 4.80 above is a contradiction with (1).

Let μ be any strategy of player **I** in $\mathrm{EF}^\kappa_\omega(\mathcal{A}_{\tau_G}, \mathcal{A}_{r(\bar{0})})$ and let us show that σ_G beats it. Consider the play $\sigma_G * \mu$ and assume for a contradiction that it is a win for **I**. This play is well defined, since the moves made by μ are in the domain of σ_G by the note after the definition of σ, and because $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa} \subset M$.

The play consists of ω moves and is a countable sequence in the set $([\kappa]^{<\kappa}) \times \kappa^{<\kappa}$. Since $\mathbb{P}$ is $< \kappa$ closed, there is $q_0 \in \mathbb{P}$ which decides $\sigma_G * \mu$ (i.e. $\sigma_{G_0} * \mu = \sigma_{G_1} * \mu$ whenever $q_0 \in G_0 \cap G_1$). Assume that G' is a $\mathbb{P}$-generic over V with $q_0 \in G'$. Then

$$(\sigma_{G'} * \mu)^{V[G']} = (\sigma_G * \mu)^{V[G']} = (\sigma_G * \mu)^V$$

(again, because $\mathbb{P}$ does not add elements of $\kappa^{<\kappa}$) and so

$$(\sigma_{G'} * \mu \text{ is a win for } \mathbf{I})^{V[G']}$$

But $q_0 \Vdash$ "$\sigma * \mu$ is a win for **II**", because q_0 extends p and by the choice of σ.

The case (2) is similar, just instead of choosing $\cup G$ such that $(\cup G)^{-1}\{1\}$ contains a cub, choose G such that $(\cup G)^{-1}\{0\}$ contains a cub. Then we should have $\mathcal{A}_{\tau_G} \cong \mathcal{A}_{r(\bar{0})}$ which contradicts (2) by the same absoluteness argument as above. □

4.6.2 Unstable and Superstable Theories

In this section we use Shelah's ideas on how to prove non-structure theorems using Ehrenfeucht-Mostowski models, see [41]. We use the definition of Ehrenfeucht-Mostowski models from [23, Definition 4.2.].

4.82 Definition. In the following discussion of linear orderings we use the following concepts.

- *Coinitiality* or *reverse cofinality* of a linear order η, denoted $\mathrm{cf}^*(\eta)$ is the smallest ordinal α such that there is a map $f\colon \alpha \to \eta$ which is strictly decreasing and $\operatorname{ran} f$ has no (strict) lower bound in η.
- If $\eta = \langle \eta, < \rangle$ is a linear ordering, by η^* we denote its mirror image: $\eta^* = \langle \eta, <^* \rangle$ where $x <^* y \iff y < x$.
- Suppose λ is a cardinal. We say that an ordering η is λ-dense if for all subsets A and B of η with the properties $\forall a \in A \forall b \in B (a < b)$ and $|A| < \lambda$ and $|B| < \lambda$ there is $x \in \eta$ such that $a < x < b$ for all $a \in A$, $b \in B$. Dense means ω-dense.

4.83 Theorem. *Suppose that $\kappa = \lambda^+ = 2^\lambda$ such that $\lambda^{<\lambda} = \lambda$. If T is unstable or superstable with OTOP, then $E_{S^\kappa_\lambda} \leqslant_c \cong_T$. If additionally $\lambda \geqslant 2^\omega$, then $E_{S^\kappa_\lambda} \leqslant_c \cong_T$ holds also for superstable T with DOP.*

Proof. We will carry out the proof for the case where T is unstable and shall make remarks on how certain steps of the proof should be modified in order this to work for superstable theories with DOP or OTOP. First for each $S \subset S^\kappa_\lambda$, let us construct the linear orders $\Phi(S)$ which will serve a fundamental role in the construction. The following claim is a special case of Lemma 7.17 in [14]:

Claim 1. For each cardinal μ of uncountable cofinality there exists a linear ordering $\eta = \eta_\mu$ which satisfies:

1. $\eta \cong \eta + \eta$,
2. for all $\alpha \leqslant \mu$, $\eta \cong \eta \cdot \alpha + \eta$,
3. $\eta \cong \eta \cdot \mu + \eta \cdot \omega_1^*$,
4. η is dense,
5. $|\eta| = \mu$,
6. $\mathrm{cf}^*(\eta) = \omega$.

Proof of Claim 1. Essentially the same as in [14]. $\square_{\text{Claim 1}}$

For a set $S \subset S^\kappa_\lambda$, define the linear order $\Phi(S)$ as follows:

$$\Phi(S) = \sum_{i<\kappa} \tau(i, S),$$

where $\tau(i, S) = \eta_\lambda$ if $i \notin S$ and $\tau(i, S) = \eta_\lambda \cdot \omega_1^*$, if $i \in S$. Note that $\Phi(S)$ is dense. For $\alpha < \beta < \kappa$ define

$$\Phi(S, \alpha, \beta) = \sum_{\alpha \leqslant i < \beta} \tau(i, S).$$

(These definitions are also as in [14] although the idea dates back to J. Conway's Ph.D. thesis from the 1960's; they are first referred to in [37]). From now on denote $\eta = \eta_\lambda$.

Claim 2. If $\alpha \notin S$, then for all $\beta \geqslant \alpha$ we have $\Phi(S, \alpha, \beta + 1) \cong \eta$ and if $\alpha \in S$, then for all $\beta \geqslant \alpha$ we have $\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \omega_1^*$.

Proof of Claim 2. Let us begin by showing the first part, i.e. assume that $\alpha \notin S$. This is also like in [14]. We prove the statement by induction on $\mathrm{OTP}(\beta \setminus \alpha)$. If $\beta = \alpha$, then $\Phi(S, \alpha, \alpha + 1) = \eta$ by the definition of Φ. If $\beta = \gamma + 1$ is a successor, then $\beta \notin S$, because S contains only limit ordinals, so $\tau(\beta, S) = \eta$ and

$$\Phi(S, \alpha, \beta + 1) = \Phi(S, \alpha, \gamma + 1 + 1) = \Phi(S, \alpha, \gamma + 1) + \eta$$

which by the induction hypothesis and by 1 is isomorphic to η. If $\beta \notin S$ is a limit ordinal, then choose a continuous cofinal sequence $s\colon \mathrm{cf}(\beta) \to \beta$ such that $s(\gamma) \notin S$ for all $\gamma < \mathrm{cf}(\beta)$. This is possible since S contains only ordinals of cofinality λ. By the induction hypothesis $\Phi(S, \alpha, s(0) + 1) \cong \eta$,

$$\Phi(S, s(\gamma) + 1, s(\gamma + 1) + 1) \cong \eta$$

for all successor ordinals $\gamma < \mathrm{cf}(\beta)$,

$$\Phi(S, s(\gamma), s(\gamma + 1) + 1) \cong \eta$$

for all limit ordinals $\gamma < \mathrm{cf}(\beta)$ and so now

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \mathrm{cf}(\beta) + \eta$$

which is isomorphic to η by 2. If $\beta \in S$, then $\mathrm{cf}(\beta) = \lambda$ and we can again choose a cofinal sequence $s\colon \lambda \to \beta$ such that $s(\alpha)$ is not in S for all $\alpha < \lambda$. By the induction hypothesis. as above,

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \lambda + \tau(\beta, S)$$

and since $\beta \in S$ we have $\tau(\beta, S) = \eta \cdot \omega_1^*$, so we have

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \lambda + \eta \cdot \omega_1^*$$

which by 3 is isomorphic to η.

Suppose $\alpha \in S$. Then $\alpha + 1 \notin S$, so by the previous part we have

$$\Phi(S, \alpha, \beta + 1) \cong \tau(\alpha, S) + \Phi(S, \alpha + 1, \beta + 1) = \eta \cdot \omega_1^* + \eta = \eta \cdot \omega_1^*.$$

$\square_{\text{Claim 2}}$

This gives us a way to show that the isomorphism type of $\Phi(S)$ depends only on the $E_{S^\kappa_\lambda}$-equivalence class of S:

Claim 3. If $S, S' \subset S^\kappa_\lambda$ and $S \triangle S'$ is non-stationary, then $\Phi(S) \cong \Phi(S')$.

Proof of Claim 3. Let C be a cub set outside $S \triangle S'$. Enumerate it $C = \{\alpha_i \mid i < \kappa\}$ where $(\alpha_i)_{i<\kappa}$ is an increasing and continuous sequence. Now $\Phi(S) = \bigcup_{i<\kappa} \Phi(S, \alpha_i, \alpha_{i+1})$ and $\Phi(S') = \bigcup_{i<\kappa} \Phi(S', \alpha_i, \alpha_{i+1})$. Note that by the definitions these are disjoint unions, so it is enough to show that for all $i < \kappa$ the orders $\Phi(S, \alpha_i, \alpha_{i+1})$ and $\Phi(S', \alpha_i, \alpha_{i+1})$ are isomorphic. But for all $i < \kappa$ $\alpha_i \in S \iff \alpha_i \in S'$, so by Claim 2 either

$$\Phi(S, \alpha_i, \alpha_{i+1}) \cong \eta \cong \Phi(S', \alpha_i, \alpha_{i+1})$$

(if $\alpha_i \notin S$) or

$$\Phi(S, \alpha_i, \alpha_{i+1}) \cong \eta \cdot \omega_1^* \cong \Phi(S', \alpha_i, \alpha_{i+1})$$

(if $\alpha_i \in S$). $\square_{\text{Claim 3}}$

4.84 Definition. K^λ_{tr} is the set of L-models $\mathcal{A}$ where $L = \{<, \lessdot, (P_\alpha)_{\alpha\leqslant\lambda}, h\}$, with the properties

- $\mathrm{dom}\,\mathcal{A} \subset I^{\leqslant\lambda}$ for some linear order I.
- $\forall x, y \in A(x < y \iff x \subset y)$.
- $\forall x \in A(P_\alpha(x) \iff \mathrm{length}(x) = \alpha)$.
- $\forall x, y \in A[x \lessdot y \iff \exists z \in A((x, y \in \mathrm{Succ}(z)) \wedge (I \models x < y))]$
- $h(x, y)$ is the maximal common initial segment of x and y.

For each S, define the tree $T(S) \in K^{\lambda}_{tr}$ by

$$T(S) = \Phi(S)^{<\lambda} \cup \{\eta\colon \lambda \to \Phi(S) \mid \eta \text{ increasing and } \mathrm{cf}^*(\Phi(S) \setminus \{x \mid (\exists y \in \operatorname{ran}\eta)(x < y)\}) = \omega_1\}.$$

The relations $<$, $\lessdot$, P_n and h are interpreted in the natural way.

Clearly an isomorphism between $\Phi(S)$ and $\Phi(S')$ induces an isomorphism between $T(S)$ and $T(S')$, thus $T(S) \cong T(S')$ if $S \bigtriangleup S'$ is non-stationary.

Claim 4. Suppose T is unstable in the vocabulary v. Let T_1 be T with Skolem functions in the Skolemized vocabulary $v_1 \supset v$. Then there is a function $\mathcal{P}(S^{\kappa}_{\lambda}) \to \{\mathcal{A}^1 \mid \mathcal{A}^1 \models T_1, |\mathcal{A}^1| = \kappa\}$, $S \mapsto \mathcal{A}^1(S)$ which has following properties:

(a) There is a mapping $T(S) \to (\operatorname{dom}\mathcal{A}^1(S))^n$ for some $n < \omega$, $\eta \mapsto a_\eta$, such that $\mathcal{A}^1(S)$ is the Skolem hull of $\{a_\eta \mid \eta \in T(S)\}$, i.e. $\{a_\eta \mid \eta \in T(S)\}$ is the skeleton of $\mathcal{A}^1(S)$. Denote the skeleton of $\mathcal{A}$ by $\mathrm{Sk}(\mathcal{A})$.

(b) $\mathcal{A}(S) = \mathcal{A}^1(S) \restriction v$ is a model of T.

(c) $\mathrm{Sk}(\mathcal{A}^1(S))$ is indiscernible in $\mathcal{A}^1(S)$, i.e. if $\bar\eta, \bar\xi \in T(S)$ and $\mathrm{tp}_{\text{q.f.}}(\bar\eta/\varnothing) = \mathrm{tp}_{\text{q.f.}}(\bar\xi/\varnothing)$, where $\mathrm{tp}_{\text{q.f.}}$ is the quantifier free type, then $\mathrm{tp}(a_{\bar\eta}/\varnothing) = \mathrm{tp}(a_{\bar\xi}/\varnothing)$ where $a_{\bar\eta} = (a_{\eta_1}, \dots, a_{\eta_{\text{length}\,\bar\eta}})$. This assignment of types in $\mathcal{A}^1(S)$ to q.f.-types in $T(S)$ is independent of S.

(d) There is a formula $\varphi \in L_{\omega\omega}(v)$ such that for all $\eta, \nu \in T(S)$ and $\alpha < \lambda$, if $T(S) \models P_\lambda(\eta) \wedge P_\alpha(\nu)$, then $T(S) \models \eta > \nu$ if and only if $\mathcal{A}(S) \models \varphi(a_\eta, a_\nu)$.

Proof of Claim 4. The following is known:

(F1) Suppose that T is a complete unstable theory. Then for each linear order η, T has an Ehrenfeucht-Mostowski model $\mathcal{A}$ of vocabulary v_1, where $|v_1| = |T| + \omega$ and order is definable by a first-order formula, such that the template (assignment of types) is independent of η.[2]

It is not hard to see that for every tree $t \in K^{\omega}_{tr}$ we can define a linear order $L(t)$ satisfying the following conditions:

1. $\operatorname{dom}(L(t)) = (\operatorname{dom} t \times \{0\}) \cup (\operatorname{dom} t \times \{1\})$,

2. for all $a \in t$, $(a, 0) <_{L(t)} (a, 1)$,

3. if $a, b \in t$, then $a <_t b \iff [(a,0) <_{L(t)} (b,0)] \wedge [(b,1) <_{L(t)} (a,1)]$,

4. if $a, b \in t$, then

$$(a \not\leq b) \wedge (b \not\leq a) \iff [(b,1) <_{L(t)} (a,0)] \vee [(a,1) <_{L(t)} (b,0)].$$

[2] This is from [42]; there is a sketch of the proof also in [23, Theorem 4.7].

Now for every $S \subset \kappa$, by (F1), there is an Ehrenfeucht-Mostowski model $\mathcal{A}^1(S)$ for the linear order $L(T(S))$ where order is definable by the formula ψ which is in $L_{\infty\omega}$. Suppose $\bar{\eta} = (\eta_0, \dots, \eta_n)$ and $\bar{\xi} = (\xi_0, \dots, \xi_n)$ are sequences in $T(S)$ that have the same quantifier free type. Then the sequences

$$\langle(\eta_0, 0), (\eta_0, 1), (\eta_1, 0), (\eta_1, 1), \dots, (\eta_n, 0), (\eta_n, 1)\rangle$$

and

$$\langle(\xi_0, 0), (\xi_0, 1), (\xi_1, 0), (\xi_1, 1), \dots, (\xi_n, 0), (\xi_n, 1)\rangle$$

have the same quantifier free type in $L(T(S))$. Now let the canonical skeleton of $\mathcal{A}^1(S)$ given by (F1) be $\{a_x \mid x \in L(T(S))\}$. Define the $T(S)$-skeleton of $\mathcal{A}^1(S)$ to be the set

$$\{a_{(\eta,0)}{}^\frown a_{(\eta,1)} \mid \eta \in T(S)\}.$$

Let us denote $b_\eta = a_{(\eta,0)}{}^\frown a_{(\eta,1)}$. This guarantees that (a), (b) and (c) are satisfied.

For (d) suppose that the order $L(T(S))$ is definable in $\mathcal{A}(S)$ by the formula $\psi(\bar{u}, \bar{c})$, i.e. $\mathcal{A}(S) \models \psi(a_x, a_y) \iff x < y$ for $x, y \in L(T(S))$. Let $\varphi(x_0, x_1, y_0, y_1)$ be the formula

$$\psi(x_0, y_0) \wedge \psi(y_1, x_1).$$

Suppose $\eta, \nu \in T(S)$ are such that $T(S) \models P_\lambda(\eta) \wedge P_\alpha(\nu)$. Then

$$\varphi((a_\nu, 0), (a_\nu, 1), (a_\eta, 0), (a_\eta, 1))$$

holds in $\mathcal{A}(S)$ if and only if $\nu <_{T(S)} \eta$. $\square_{\text{Claim 4}}$

Claim 5. Suppose $S \mapsto \mathcal{A}(S)$ is a function as described in Claim 4 with the identical notation. Suppose further that $S, S' \subset S^\kappa_\lambda$. Then $S \triangle S'$ is non-stationary if and only if $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Proof of Claim 5. Suppose $S \triangle S'$ is non-stationary. Then by Claim 3 $T(S) \cong T(S')$ which implies $L(T(S)) \cong L(T(S'))$ (defined in the proof of Claim 4) which in turn implies $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Let us now show that if $S \triangle S'$ is stationary, then $\mathcal{A}(S) \not\cong \mathcal{A}(S')$. Let us make a counter assumption, namely that there is an isomorphism

$$f \colon \mathcal{A}(S) \cong \mathcal{A}(S')$$

and that $S \triangle S'$ is stationary, and let us deduce a contradiction. Without loss of generality we may assume that $S \setminus S'$ is stationary. Denote

$$X_0 = S \setminus S'$$

For all $\alpha < \kappa$ define $T^\alpha(S)$ and $T^\alpha(S')$ by

$$T^\alpha(S) = \{\eta \in T(S) \mid \operatorname{ran}\eta \subset \Phi(S, 0, \beta + 1) \text{ for some } \beta < \alpha\}$$

and

$$T^\alpha(S') = \{\eta \in T(S) \mid \operatorname{ran}\eta \subset \Phi(S', 0, \beta + 1) \text{ for some } \beta < \alpha\}.$$

Then we have:

(i) if $\alpha < \beta$, then $T^\alpha(S) \subset T^\beta(S)$

(ii) if γ is a limit ordinal, then $T^\gamma(S) = \bigcup_{\alpha<\gamma} T^\alpha(S)$

The same of course holds for S'. Note that if $\alpha \in S \setminus S'$, then there is $\eta \in T^\alpha(S)$ cofinal in $\Phi(S,0,\alpha)$ but there is no such $\eta \in T^\alpha(S')$ by definition of Φ: a cofinal function η is added only if $\mathrm{cf}^*(\Phi(S',\alpha,\kappa)) = \omega_1$ which it is not if $\alpha \notin S'$. This is the key to achieving the contradiction.

But the clauses (i),(ii) are not sufficient to carry out the following argument, because we would like to have $|T^\alpha(S)| < \kappa$. That is why we want to define a different kind of filtration for $T(S)$, $T(S')$.

For all $\alpha \in X_0$ fix a function

$$\eta^\alpha_\lambda \in T(S) \tag{1}$$

such that $\operatorname{dom} \eta^\alpha_\lambda = \lambda$, for all $\beta < \lambda$, $\eta^\alpha_\lambda \restriction \beta \in T^\alpha(S)$ and $\eta^\alpha_\lambda \notin T^\alpha(S)$.

For arbitrary $A \subset T(S) \cup T(S')$ let $\mathrm{cl}_{\mathrm{Sk}}(A)$ be the set $X \subset \mathcal{A}(S) \cup \mathcal{A}(S')$ such that $X \cap \mathcal{A}(S)$ is the Skolem closure of $\{a_\eta \mid \eta \in A \cap T(S)\}$ and $X \cap \mathcal{A}(S')$ the Skolem closure of $\{a_\eta \mid \eta \in A \cap T(S')\}$. The following is easily verified:

There exists a λ-cub set C and a set $K^\alpha \subset T^\alpha(S) \cup T^\alpha(S')$ for each $\alpha \in C$ such that

(i') If $\alpha < \beta$, then $K^\alpha \subset K^\beta$

(ii') If γ is a limit ordinal in C, then $K^\gamma = \bigcup_{\alpha \in C\cap\gamma} K^\alpha$

(iii) for all $\beta < \alpha$, $\eta^\beta_\lambda \in K^\alpha$. (see (1) above)

(iv) $|K^\alpha| = \lambda$.

(v) $\mathrm{cl}_{\mathrm{Sk}}(K^\alpha)$ is closed under $f \cup f^{-1}$.

(vi) $\{\eta \in T^\alpha(S) \cup T^\alpha(S') \mid \operatorname{dom} \eta < \lambda\} \subset K^\alpha$.

(vii) K^α is downward closed.

Denote $K^\kappa = \bigcup_{\alpha<\kappa} K^\alpha$. Clearly K^κ is closed under $f \cup f^{-1}$ and so f is an isomorphism between $\mathcal{A}(S) \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$ and $\mathcal{A}(S') \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$. We will derive a contradiction from this, i.e. we will actually show that $\mathcal{A}(S) \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$ and $\mathcal{A}(S') \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$ cannot be isomorphic by f. Clauses (iii), (v), (vi) and (vii) guarantee that all elements we are going to deal with will be in K^κ.

Let

$$X_1 = X_0 \cap C.$$

For $\alpha \in X_1$ let us use the following abbreviations:

- By $\mathcal{A}_\alpha(S)$ denote the Skolem closure of $\{a_\eta \mid \eta \in K^\alpha \cap T(S)\}$.
- By $\mathcal{A}_\alpha(S')$ denote the Skolem closure of $\{a_\eta \mid \eta \in K^\alpha \cap T(S')\}$.
- $K^\alpha(S) = K^\alpha \cap T(S)$.
- $K^\alpha(S') = K^\alpha \cap T(S')$.

In the following we will often deal with finite sequences. When defining such a sequence we will use a bar, but afterwards we will not use the bar in the notation (e.g. let $a = \bar{a}$ be a finite sequence...).

Suppose $\alpha \in X_1$. Choose

$$\xi^\alpha_\lambda = \bar{\xi}^\alpha_\lambda \in T(S') \tag{2}$$

to be such that for some (finite sequence of) terms $\pi = \bar{\pi}$ we have

$$\begin{aligned} & f(a_{\eta^\alpha_\lambda}) = \pi(a_{\xi^\alpha_\lambda}) \\ = \quad & \langle \pi_1(a_{\xi^\alpha_\lambda(1)}, \ldots, a_{\xi^\alpha_\lambda(\mathrm{length}(\bar{\xi}^\alpha_\lambda))}), \ldots \pi_{\mathrm{length}\,\bar{\pi}}(a_{\xi^\alpha_\lambda(1)}, \ldots, a_{\xi^\alpha_\lambda(\mathrm{length}(\bar{\xi}^\alpha_\lambda))})\rangle. \end{aligned}$$

Note that ξ^α_λ is in K^κ by the definition of K^α's.

$$\text{Let us denote by } \eta^\alpha_\beta, \text{ the element } \eta^\alpha_\lambda \restriction \beta. \tag{3}$$

Let

$$\xi^\alpha_* = \{\nu \in T(S') \mid \exists \xi \in \xi^\alpha_\lambda (\nu < \xi)\}.$$

Also note that $\xi^\alpha_* \subset K^\beta$ for some β.

Next define the function $g\colon X_1 \to \kappa$ as follows. Suppose $\alpha \in X_1$. Let $g(\alpha)$ be the smallest ordinal β such that $\xi^\alpha_* \cap K^\alpha(S') \subset K^\beta(S')$. We claim that $g(\alpha) < \alpha$. Clearly $g(\alpha) \leqslant \alpha$, so suppose that $g(\alpha) = \alpha$. Since ξ^α_λ is finite, there must be a $\xi^\alpha_\lambda(i) \in \xi^\alpha_\lambda$ such that for all $\beta < \alpha$ there exists γ such that $\xi^\alpha_\lambda(i) \restriction \gamma \in K^\alpha(S') \setminus K^\beta(S')$, i.e. $\xi^\alpha_\lambda(i)$ is cofinal in $\Phi(S', 0, \alpha)$ which it cannot be, because $\alpha \notin S'$.

Now by Fodor's lemma there exists a stationary set

$$X_2 \subset X_1$$

and γ_0 such that $g[X_2] = \{\gamma_0\}$.

Since there is only $< \kappa$ many finite sequences in $K_{\gamma_0}(S')$, there is a stationary set

$$X_3 \subset X_2$$

and a finite sequence $\xi = \bar{\xi} \in K^{\gamma_0}(S')$ such that for all $\alpha \in X_3$ we have $\xi^\alpha_* \cap K^{\gamma_0}(S') = \xi_*$ where ξ_* is the set

$$\xi_* = \{\nu \in T(S') \mid \nu \leqslant \zeta \text{ for some } \zeta \in \bar{\xi}\} \subset K^{\gamma_0}(S').$$

Let us fix a (finite sequence of) term(s) $\pi = \bar{\pi}$ such that the set

$$X_4 = \{\alpha \in X_3 \quad | \quad f(a_{\eta^\alpha_\lambda}) = \pi(a_{\xi^\alpha_\lambda})\}$$

is stationary (see (1)). Here $f(\bar{a})$ means $\langle f(a_1), \ldots, f(a_{\mathrm{length}\,\bar{a}})\rangle$ and $\bar{\pi}(\bar{b})$ means

$$\langle \pi_1(b_1, \ldots, b_{\mathrm{length}\,\bar{a}}), \ldots, \pi_{\mathrm{length}\,\pi}(b_1, \ldots, b_{\mathrm{length}\,\bar{a}})\rangle.$$

We can find such π because there are only countably many such finite sequences of terms.

We claim that in $T(S')$ there are at most λ many quantifier free types over ξ_*. All types from now on are quantifier free. Let us show that there are at most λ many 1-types; the general case is left to the reader. To see this, note that a type p over ξ_* is described by the triple

$$(\nu_p, \beta_p, m_p) \tag{$\star$}$$

defined as follows: if η satisfies p, then ν_p is the maximal element of ξ_* that is an initial segment of η, β_p is the level of η and m_p tells how many elements of $\xi_* \cap P_{\operatorname{dom}\nu_p+1}$ are there $\lessdot$-below $\eta(\operatorname{dom}\nu_p)$ (recall the vocabulary from Definition 4.84, page 123).

Since $\nu_p \in \xi_*$ and ξ_* is of size λ, $\beta_p \in (\lambda+1) \cup \{\infty\}$ and $m_p < \omega$, there can be at most λ such triples.

Recall the notations (1), (2) and (3) above.

We can pick ordinals $\alpha < \alpha'$, $\alpha, \alpha' \in X_4$, a term τ and an ordinal $\beta < \lambda$ such that

$$\eta_\beta^{\alpha'} \neq \eta_\beta^\alpha,$$

$$f(a_{\eta_\beta^\alpha}) = \tau(a_{\xi_\beta^\alpha}) \text{ and } f(a_{\eta_\beta^{\alpha'}}) = \tau(a_{\xi_\beta^{\alpha'}}) \text{ for some } \xi_\beta^\alpha, \xi_\beta^{\alpha'},$$

$$\operatorname{tp}(\xi_\lambda^\alpha/\xi_*) = \operatorname{tp}(\xi_\lambda^{\alpha'}/\xi_*)$$

and

$$\operatorname{tp}(\xi_\beta^\alpha/\xi_*) = \operatorname{tp}(\xi_\beta^{\alpha'}/\xi_*). \tag{4}$$

We claim that then in fact

$$\operatorname{tp}(\xi_\beta^\alpha/(\xi_* \cup \{\xi_\lambda^{\alpha'}\})) = \operatorname{tp}(\xi_\beta^{\alpha'}/(\xi_* \cup \{\xi_\lambda^{\alpha'}\})).$$

Let us show this. Denote

$$p = \operatorname{tp}(\xi_\beta^\alpha/(\xi_* \cup \{\xi_\lambda^{\alpha'}\}))$$

and

$$p' = \operatorname{tp}(\xi_\beta^{\alpha'}/(\xi_* \cup \{\xi_\lambda^{\alpha'}\})).$$

By the assumption (4) however $p \restriction \xi_* = p' \restriction \xi_*$, so because it is a tree, it sufficies to show that $p \restriction \{\xi_\lambda^{\alpha'}\} = p' \restriction \{\xi_\lambda^{\alpha'}\}$. Since α and α' are in X_3 and X_2, we have $\xi_*^{\alpha'} \cap K^{\alpha'}(S') = \xi_*^\alpha \cap K^\alpha(S') = \xi_* \subset K^{\gamma_0}(S')$. On the other hand $f \restriction \mathcal{A}_{\alpha'}(S)$ is an isomorphism between $\mathcal{A}_{\alpha'}(S)$ and $\mathcal{A}_{\alpha'}(S')$, because α and α' are in X_1, so $\xi_\beta^\alpha, \xi_\beta^{\alpha'} \in K^{\alpha'}(S')$. Thus ξ_β^α and $\xi_\beta^{\alpha'}$ are either both in ξ_* whence they are the same, or not whence they both are not below $\xi_\lambda^{\alpha'}$. From (4) it follows that ξ_β^α and $\xi_\beta^{\alpha'}$ are on the same level and if $\xi_\lambda^{\alpha'}$ is also on the same level, then the above also implies that they are both $\lessdot$-below $\xi_\lambda^{\alpha'}$. From (4) and the above we also have that $h(\xi_\beta^\alpha, \xi^{\alpha'}) = h(\xi_\beta^{\alpha'}, \xi^{\alpha'})$ (see Definition 4.84).

Now we have: ξ_λ^α and π are such that $f(a_{\eta_\lambda^\alpha}) = \pi(a_{\xi_\lambda^\alpha})$ and ξ_β^α and τ are such that $f(a_{\eta_\beta^\alpha}) = \tau(a_{\xi_\beta^\alpha})$. Similarly for α'. The formula φ is defined in Claim 4.

We know that

$$\mathcal{A}(S) \models \varphi(a_{\eta_\lambda^{\alpha'}}, a_{\eta_\beta^{\alpha'}})$$

and because f is isomorphism, this implies

$$\mathcal{A}(S') \models \varphi(f(a_{\eta_\lambda^{\alpha'}}), f(a_{\eta_\beta^{\alpha'}}))$$

which is equivalent to

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi_\lambda^{\alpha'}}), \tau(a_{\xi_\beta^{\alpha'}}))$$

(because α, α' are in X_4). Since $T(S')$ is indiscernible in $\mathcal{A}(S')$ and $\xi^{\alpha'}_\beta$ and ξ^α_β have the same type over over $(\xi_* \cup \{\xi^{\alpha'}_\lambda\})$, we have

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha'}_\beta})) \iff \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha}_\beta})) \tag{$*$}$$

and so we get

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha}_\beta}))$$

which is equivalent to

$$\mathcal{A}(S') \models \varphi(f(a_{\eta^{\alpha'}_\lambda}), f(a_{\eta^{\alpha}_\beta}))$$

and this in turn is equivalent to

$$\mathcal{A}(S) \models \varphi(a_{\eta^{\alpha'}_\lambda}, a_{\eta^{\alpha}_\beta})$$

The latter cannot be true, because the definition of β, α and α' implies that $\eta^{\alpha'}_\beta \neq \eta^\alpha_\beta$. $\square_{\text{Claim 5}}$

Thus, the above Claims 1 – 5 justify the embedding of $E_{S^\kappa_\lambda}$ into the isomorphism relation on the set of structures that are models for T for unstable T. This embedding combined with a suitable coding of models gives a continuous map.

DOP and OTOP cases. The above proof was based on the fact (F1) that for unstable theories there are Ehrenfeucht-Mostowski models for any linear order such that the order is definable by a first-order formula φ and is indiscernible relative to $L_{\omega\omega}$, (see (c) on page 124); it is used in $(*)$ above. For the OTOP case, we use instead the fact (F2):

(F2) Suppose that T is a theory with OTOP in a countable vocabulary v. Then for each dense linear order η we can find a model $\mathcal{A}$ of a countable vocabulary $v_1 \supset v$ such that $\mathcal{A}$ is an Ehrenfeucht-Mostowski model of T for η where order is definable by an $L_{\omega_1\omega}$-formula.[3]

Since the order $\Phi(S)$ is dense, it is easy to argue that if $T(S)$ is indiscernible relative to $L_{\omega\omega}$, then it is indiscernible relative to $L_{\infty\omega}$ (define this as in (c) on page 124 changing tp to $\text{tp}_{L_{\infty\omega}}$). Other parts of the proof remain unchanged, because although the formula φ is not first-order anymore, it is still in $L_{\infty\omega}$.

In the DOP case we have the following fact:

(F3) Let T be a countable superstable theory with DOP of vocabulary v. Then there exists a vocabulary $v_1 \supset v$, $|v_1| = \omega_1$, such that for every linear order η there exists a v_1-model $\mathcal{A}$ which is an Ehrenfeucht-Mostowski model of T for η where order is definable by an $L_{\omega_1\omega_1}$-formula.[4]

Now the problem is that φ is in $L_{\infty\omega_1}$. By (c) of Claim 4, $T(S)$ is indiscernible in $\mathcal{A}(S)$ relative to $L_{\omega\omega}$ and by the above relative to $L_{\infty\omega}$. If we could require $\Phi(S)$ to be ω_1-dense, we would similarly get indiscernible relative to $L_{\infty\omega_1}$. Let us show how to modify the proof in order to do that. Recall that in the DOP case,we assume $\lambda \geqslant 2^\omega$.

In Claim 1 (page 122), we have to replace clauses (3), (4) and (6) by (3'), (4') and (6'):

[3]Contained in the proof of [38, Theorem 2.5]; see also [23, Theorem 6.6].
[4]This is essentially from [43, Fact 2.5B]; a proof can be found also in [23, Theorem 6.1]

(3') $\eta \cong \eta \cdot \mu + \eta \cdot \omega^*$,

(4') η is ω_1-dense,

(6') $\mathrm{cf}^*(\eta) = \omega_1$.

The proof that such an η exists is exactly as the proof of Lemma 7.17 [14] except that instead of putting $\mu = (\omega_1)^V$ put $\mu = \omega$, build θ-many functions with domains being countable initial segments of ω_1 instead of finite initial segments of ω and instead of $\mathbb{Q}$ (the countable dense linear order) use an ω_1-saturated dense linear order – this order has size 2^ω and that is why the assumption $\lambda \geqslant 2^\omega$ is needed.

In the definition of $\Phi(S)$ (right after Claim 1), replace ω_1^* by ω^* and η by the new η satisfying (3'), (4') and (6') above. Note that $\Phi(S)$ becomes now ω_1-dense. In Claim 2 one has to replace ω_1^* by ω^*. The proof remains similar. In the proof of Claim 3 (page 123) one has to adjust the use of Claim 2. Then, in the definition of $T(S)$ replace ω_1 by ω.

Claim 4 for superstable T with DOP now follows with (c) and (d) modified: instead of indiscernible relative to $L_{\omega\omega}$, demand $L_{\infty\omega_1}$ and instead of $\varphi \in L_{\omega\omega}$ we have now $\varphi \in L_{\infty\omega_1}$. The proof is unchanged except that the language is replaced by $L_{\infty\omega_1}$ everywhere and fact (F1) replaced by (F3) above.

Everything else in the proof, in particular the proof of Claim 5, remains unchanged modulo some obvious things that are evident from the above explanation. $\square_{\text{Theorem 4.83}}$

4.6.3 Stable Unsuperstable Theories

In this section we provide a tree construction (Lemma 4.89) which is similar to Shelah's construction in [41] which he used to obtain (via Ehrenfeucht-Mostowski models) many pairwise non-isomorphic models. Then using a prime-model construction (proof of Theorem 4.90, page 137) we will obtain the needed result.

4.85 Definition. Let I be a tree of size κ. Suppose $(I_\alpha)_{\alpha<\kappa}$ is a collection of subsets of I such that

- For each $\alpha < \kappa$, I_α is a downward closed subset of I
- $\bigcup_{\alpha<\kappa} I_\alpha = I$
- If $\alpha < \beta < \kappa$, then $I_\alpha \subset I_\beta$
- If γ is a limit ordinal, then $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$
- For each $\alpha < \kappa$ the cardinality of I_α is less than κ.

Such a sequence $(I_\alpha)_{\alpha<\kappa}$ is called *κ-filtration* or just *filtration* of I.

4.86 Definition. Recall K_{tr}^λ from Definition 4.84 on page 123. Let $K_{tr*}^\lambda = \{A \restriction L^* \mid A \in K_{tr}^\lambda\}$, where L^* is the vocabulary $\{<\}$.

4.87 Definition. Suppose $t \in K^{\omega}_{tr*}$ is a tree of size κ (i.e. $t \subset \kappa^{\leqslant\omega}$) and let $\mathcal{I} = (I_\alpha)_{\alpha<\kappa}$ be a filtration of t. Define

$$S_{\mathcal{I}}(t) = \Big\{\alpha < \kappa \mid (\exists \eta \in t)\big[(\operatorname{dom}\eta = \omega) \wedge \forall n < \omega(\eta\restriction n \in I_\alpha) \wedge (\eta \notin I_\alpha)\big]\Big\}$$

By $S \sim_{\mathrm{NS}} S'$ we mean that $S \triangle S'$ is not ω-stationary

4.88 Lemma. *Suppose trees t_0 and t_1 are isomorphic, and $\mathcal{I} = (I_\alpha)_{\alpha<\kappa}$ and $\mathcal{J} = (J_\alpha)_{\alpha<\kappa}$ are κ-filtrations of t_0 and t_1 respectively. Then $S_{\mathcal{I}}(t_0) \sim_{\mathrm{NS}} S_{\mathcal{J}}(t_1)$.*

Proof. Let $f\colon t_0 \to t_1$ be an isomorphism. Then $f\mathcal{I} = (f[I_\alpha])_{\alpha<\kappa}$ is a filtration of t_1 and

$$\alpha \in S_{\mathcal{I}}(t_0) \iff \alpha \in S_{f\mathcal{I}}(t_1). \qquad (\star)$$

Define the set $C = \{\alpha \mid f[I_\alpha] = J_\alpha\}$. Let us show that it is cub. Let $\alpha \in \kappa$. Define $\alpha_0 = \alpha$ and by induction pick $(\alpha_n)_{n<\omega}$ such that $f[I_{\alpha_n}] \subset J_{\alpha_{n+1}}$ for odd n and $J_{\alpha_n} \subset f[I_{\alpha_{n+1}}]$ for even n. This is possible by the definition of a κ-filtration. Then $\alpha_\omega = \bigcup_{n<\omega}\alpha_n \in C$. Clearly C is closed and $C \subset \kappa \setminus S_{f\mathcal{I}}(t_1) \triangle S_{\mathcal{J}}(t_1)$, so now by $(\star)$

$$S_{\mathcal{I}}(t_0) = S_{f\mathcal{I}}(t_1) \sim_{\mathrm{NS}} S_{\mathcal{J}}(t_1).$$ □

4.89 Lemma. *Suppose for $\lambda < \kappa$, $\lambda^\omega < \kappa$ and $\kappa^{<\kappa} = \kappa$. There exists a function $J\colon \mathcal{P}(\kappa) \to K^{\omega}_{tr*}$ such that*

- $\forall S \subset \kappa(|J(S)| = \kappa)$.
- *If $S \subset \kappa$ and $\mathcal{I}$ is a κ filtration of $J(S)$, then $S_{\mathcal{I}}(J(S)) \sim_{\mathrm{NS}} S$.*
- *If $S_0 \sim_{\mathrm{NS}} S_1$, then $J(S_0) \cong J(S_1)$.*

Proof. Let $S \subset S^\kappa_\omega$ and let us define a preliminary tree $I(S)$ as follows. For each $\alpha \in S$ let C_α be the set of all strictly increasing cofinal functions $\eta\colon \omega \to \alpha$. Let $I(S) = \underline{[\kappa]}^{<\omega} \cup \bigcup_{\alpha\in S} C_\alpha$ where $\underline{[\kappa]}^{<\omega}$ is the set of strictly increasing functions from finite ordinals to κ.

For ordinals $\alpha < \beta \leqslant \kappa$ and $i < \omega$ we adopt the notation:

- $[\alpha, \beta] = \{\gamma \mid \alpha \leqslant \gamma \leqslant \beta\}$
- $[\alpha, \beta) = \{\gamma \mid \alpha \leqslant \gamma < \beta\}$
- $\tilde{f}(\alpha, \beta, i) = \bigcup_{i\leqslant j\leqslant\omega}\{\eta\colon [i, j) \to [\alpha, \beta) \mid \eta \text{ strictly increasing}\}$

For each $\alpha, \beta < \kappa$ let us define the sets $P^{\alpha,\beta}_\gamma$, for $\gamma < \kappa$ as follows. If $\alpha = \beta = \gamma = 0$, then $P^{0,0}_0 = I(S)$. Otherwise let $\{P^{\alpha,\beta}_\gamma \mid \gamma < \kappa\}$ enumerate all downward closed subsets of $\tilde{f}(\alpha, \beta, i)$ for all i, i.e.

$$\{P^{\alpha,\beta}_\gamma \mid \gamma < \kappa\} = \bigcup_{i<\omega} \mathcal{P}(\tilde{f}(\alpha, \beta, i)) \cap \{A \mid A \text{ is closed under inital segments}\}.$$

Define

$$\tilde{n}(P^{\alpha,\beta}_\gamma)$$

to be the natural number i such that $P_\gamma^{\alpha,\beta} \subset \tilde f(\alpha,\beta,i)$. The enumeration is possible, because by our assumption $\kappa^{<\kappa} = \kappa$ we have

$$\begin{aligned}\Big|\bigcup_{i<\omega}\mathcal{P}(\tilde f(\alpha,\beta,i))\Big| &\leqslant \omega\times|\mathcal{P}(\tilde f(0,\beta,0))|\\ &\leqslant \omega\times|\mathcal{P}(\beta^\omega)|\\ &= \omega\times 2^{\beta^\omega}\\ &\leqslant \omega\times\kappa\\ &= \kappa\end{aligned}$$

Let $S \subset \kappa$ be a set and define $J(S)$ to be the set of all $\eta\colon s \to \omega\times\kappa^4$ such that $s \leqslant \omega$ and the following conditions are met for all $i, j < s$:

1. η is strictly increasing with respect to the lexicographical order on $\omega\times\kappa^4$.
2. $\eta_1(i) \leqslant \eta_1(i+1) \leqslant \eta_1(i)+1$
3. $\eta_1(i) = 0 \to \eta_2(i) = \eta_3(i) = \eta_4(i) = 0$
4. $\eta_1(i) < \eta_1(i+1) \to \eta_2(i+1) \geqslant \eta_3(i)+\eta_4(i)$
5. $\eta_1(i) = \eta_1(i+1) \to (\forall k \in \{2,3,4\})(\eta_k(i) = \eta_k(i+1))$
6. if for some $k<\omega$, $[i,j) = \eta_1^{-1}\{k\}$, then
   $\eta_5\restriction[i,j) \in P_{\eta_4(i)}^{\eta_2(i),\eta_3(i)}$
7. if $s = \omega$, then either
   $(\exists m<\omega)(\forall k<\omega)(k>m \to \eta_1(k) = \eta_1(k+1))$
   or
   $\sup\operatorname{ran}\eta_5 \in S$.
8. Order $J(S)$ by inclusion.

Note that it follows from the definition of $P_\gamma^{\alpha,\beta}$ and the conditions (6) and (4) that for all $i < j < \operatorname{dom}\eta$, $\eta \in J(S)$:

9. $i < j \to \eta_5(i) < \eta_5(j)$.

For each $\alpha < \kappa$ let

$$J^\alpha(S) = \{\eta \in J(S) \mid \operatorname{ran}\eta \subset \omega\times(\beta+1)^4 \text{ for some } \beta < \alpha\}.$$

Then $(J^\alpha(S))_{\alpha<\kappa}$ is a κ-filtration of $J(S)$ (see Claim 2 below). For the first item of the lemma, clearly $|J(S)| = \kappa$.

Let us observe that if $\eta \in J(S)$ and $\operatorname{ran}\eta_1 = \omega$, then

$$\sup\operatorname{ran}\eta_4 \leqslant \sup\operatorname{ran}\eta_2 = \sup\operatorname{ran}\eta_3 = \sup\operatorname{ran}\eta_5 \tag{\#}$$

and if in addition to that, $\eta\restriction k \in J^\alpha(S)$ for all k and $\eta \notin J^\alpha(S)$ or if $\operatorname{ran}\eta_1 = \{0\}$, then

$$\sup\operatorname{ran}\eta_5 = \alpha. \qquad (\circledast)$$

To see (#) suppose $\operatorname{ran}\eta_1 = \omega$. By (9), $(\eta_5(i))_{i<\omega}$ is an increasing sequence. By (6) $\sup\operatorname{ran}\eta_3 \geqslant \sup\operatorname{ran}\eta_5 \geqslant \sup\operatorname{ran}\eta_2$. By (4), $\sup\operatorname{ran}\eta_2 \geqslant \sup\operatorname{ran}\eta_3$ and again by (4) $\sup\operatorname{ran}\eta_2 \geqslant \sup\operatorname{ran}\eta_4$. Inequality $\sup\operatorname{ran}\eta_5 \leqslant \alpha$ is an immediate consequence of the definition of $J^\alpha(S)$, so $(\circledast)$ follows now from the assumption that $\eta \notin J^\alpha(S)$.

Claim 1. Suppose $\xi \in J^\alpha(S)$ and $\eta \in J(S)$. Then if $\operatorname{dom}\xi < \omega$, $\xi \subsetneq \eta$ and $(\forall k \in \operatorname{dom}\eta \setminus \operatorname{dom}\xi)\big(\eta_1(k) = \xi_1(\max\operatorname{dom}\xi) \wedge \eta_1(k) > 0\big)$, then $\eta \in J^\alpha(S)$.

Proof of Claim 1. Suppose $\xi, \eta \in J^\alpha(S)$ are as in the assumption. Let us define $\beta_2 = \xi_2(\max\operatorname{dom}\xi)$, $\beta_3 = \xi_2(\max\operatorname{dom}\xi)$, and $\beta_4 = \xi_4(\max\operatorname{dom}\xi)$. Because $\xi \in J^\alpha(S)$, there is β such that $\beta_2, \beta_3, \beta_4 < \beta+1$ and $\beta < \alpha$. Now by (5) $\eta_2(k) = \beta_2$, $\eta_3(k) = \beta_3$ and $\eta_4(k) = \beta_4$, for all $k \in \operatorname{dom}\eta \setminus \operatorname{dom}\xi$. Then by (6) for all $k \in \operatorname{dom}\eta \setminus \operatorname{dom}\xi$ we have that $\beta_2 < \eta_5(k) < \beta_3 < \beta+1$. Since $\xi \in J^\alpha(S)$, also $\beta_4 < \beta+1$, so $\eta \in J^\alpha(S)$. $\square_{\text{Claim 1}}$

Claim 2. $|J(S)| = \kappa$, $(J^\alpha(S))_{\alpha<\kappa}$ is a κ-filtration of $J(S)$ and if $S \subset \kappa$ and $\mathcal{I}$ is a κ-filtration of $J(S)$, then $S_{\mathcal{I}}(J(S)) \sim_{\text{NS}} S$.

Proof of Claim 2. For all $\alpha \leqslant \kappa$, $J^\alpha(S) \subset (\omega \times \alpha^4)^{\leqslant\omega}$, so by the cardinality assumption of the lemma, the cardinality of $J^\alpha(S)$ is $< \kappa$ if $\alpha < \kappa$ ($J^\kappa(S) = J(S)$). Clearly $\alpha < \beta$ implies $J^\alpha(S) \subset J^\beta(S)$. Continuity is verified by

$$\begin{aligned}\bigcup_{\alpha<\gamma} J^\alpha(S) &= \{\eta \in J(S) \mid \exists\alpha < \gamma, \exists\beta < \alpha(\operatorname{ran}\eta \subset \omega \times (\beta+1)^4)\} \\ &= \{\eta \in J(S) \mid \exists\beta < \cup\gamma(\operatorname{ran}\eta \subset \omega \times (\beta+1)^4)\}\end{aligned}$$

which equals $J^\gamma(S)$ if γ is a limit ordinal. By Lemma 4.88 it is enough to show $S_{\mathcal{I}}(J(S)) \sim_{\text{NS}} S$ for $\mathcal{I} = (J^\alpha(S))_{\alpha<\kappa}$, and we will show that if $\mathcal{I} = (J^\alpha(S))_{\alpha<\kappa}$, then in fact $S_{\mathcal{I}}(J(S)) = S$.

Suppose $\alpha \in S_{\mathcal{I}}(J(S))$. Then there is $\eta \in J(S)$, $\operatorname{dom}\eta = \omega$, such that $\eta\restriction k \in J^\alpha(S)$ for all $k < \omega$ but $\eta \notin J^\alpha(S)$. Thus there is no $\beta < \alpha$ such that $\operatorname{ran}\eta \subset \omega \times (\beta+1)^4$ but on the other hand for all $k < \omega$ there is β such that $\operatorname{ran}\eta\restriction k \subset \omega \times (\beta+1)^4$. By (5) and (6) this implies that either $\operatorname{ran}\eta_1 = \omega$ or $\operatorname{ran}\eta_1 = \{0\}$. By $(\circledast)$ on page 133 it now follows that $\sup\operatorname{ran}\eta_5 = \alpha$ and by (7), $\alpha \in S$.

Suppose then that $\alpha \in S$. Let us show that $\alpha \in S_{\mathcal{I}}(J(S))$. Fix a function $\eta_\alpha\colon \omega \to \kappa$ with $\sup\operatorname{ran}\eta_\alpha = \alpha$. Then $\eta_\alpha \in I(S)$ and the function η such that $\eta(n) = (0,0,0,0,\eta_\alpha(n))$ is as required. (Recall that $P_0^{0,0} = I(S)$ in the definition of $J(S)$). $\square_{\text{Claim 2}}$

Claim 3. Suppose $S \sim_{\text{NS}} S'$. Then $J(S) \cong J(S')$.

Proof of Claim 3. Let $C \subset \kappa \setminus (S \triangle S')$ be the cub set which exists by the assumption. By induction on $i < \kappa$ we will define α_i and F_{α_i} such that

(a) If $i < j < \kappa$, then $\alpha_i < \alpha_j$ and $F_{\alpha_i} \subset F_{\alpha_j}$.

(b) If i is a successor, then α_i is a successor and if i is limit, then $\alpha_i \in C$.

(c) If γ is a limit ordinal, then $\alpha_\gamma = \sup_{i<\gamma} \alpha_i$,

(d) F_{α_i} is a partial isomorphism $J(S) \to J(S')$

(e) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0 and $n < \omega$ is even. Then $\operatorname{dom} F_{\alpha_i} = J^{\alpha_i}(S)$ (e1). If also $n > 0$ and $(\eta_k)_{k<\omega}$ is an increasing sequence in $J^{\alpha_i}(S)$ such that $\eta = \bigcup_{k<\omega} \eta_k \notin J(S)$, then $\bigcup_{k<\omega} F_{\alpha_i}(\eta_k) \notin J(S')$ (e2).

(f) If $i = \gamma + n$, where γ is a limit ordinal or 0 and $n < \omega$ is odd, then $\operatorname{ran} F_{\alpha_i} = J^{\alpha_i}(S')$ (f1). Further, if $(\eta_k)_{k<\omega}$ is an increasing sequence in $J^{\alpha_i}(S')$ such that $\eta = \bigcup_{k<\omega} \eta_k \notin J(S')$, then $\bigcup_{k<\omega} F^{-1}_{\alpha_i}(\eta_k) \notin J(S)$ (f3).

(g) If $\operatorname{dom}\xi < \omega$, $\xi \in \operatorname{dom} F_{\alpha_i}$, $\eta \restriction \operatorname{dom}\xi = \xi$ and $(\forall k \geqslant \operatorname{dom}\xi)\big(\eta_1(k) = \xi_1(\max\operatorname{dom}\xi) \wedge \eta_1(k) > 0\big)$, then $\eta \in \operatorname{dom} F_{\alpha_i}$. Similarly for $\operatorname{ran} F_{\alpha_i}$

(h) If $\xi \in \operatorname{dom} F_{\alpha_i}$ and $k < \operatorname{dom}\xi$, then $\xi \restriction k \in \operatorname{dom} F_{\alpha_i}$.

(i) For all $\eta \in \operatorname{dom} F_{\alpha_i}$, $\operatorname{dom}\eta = \operatorname{dom}(F_{\alpha_i}(\eta))$

The first step. The first step and the successor steps are similar, but the first step is easier. Thus we give it separately in order to simplify the readability. Let us start with $i = 0$. Let $\alpha_0 = \beta + 1$, for arbitrary $\beta \in C$. Let us denote by

$$\tilde{o}(\alpha)$$

the ordinal that is order isomorphic to $(\omega \times \alpha^4, <_{\text{lex}})$. Let γ be such that there is an isomorphism $h\colon P^{0,\tilde{o}(\alpha_0)}_\gamma \cong J^{\alpha_0}(S)$ and such that $\tilde{n}(P^{0,\alpha_0}_\gamma) = 0$. Such exists by (1). Suppose that $\eta \in J^{\alpha_0}(S)$. Note that because P^{0,α_0}_γ and $J^{\alpha_0}(S)$ are closed under initial segments and by the definitions of $\tilde{n}$ and $P^{\alpha,\beta}_\gamma$, we have $\operatorname{dom} h^{-1}(\eta) = \operatorname{dom}\eta$, Define $\xi = F_{\alpha_0}(\eta)$ such that $\operatorname{dom}\xi = \operatorname{dom}\eta$ and for all $k < \operatorname{dom}\xi$

- $\xi_1(k) = 1$
- $\xi_2(k) = 0$
- $\xi_3(k) = \tilde{o}(\alpha_0)$
- $\xi_4(k) = \gamma$
- $\xi_5(k) = h^{-1}(\eta)(k)$

Let us check that $\xi \in J(S')$. Conditions (1)-(5) and (7) are satisfied because ξ_k is constant for all $k \in \{1,2,3,4\}$, $\xi_1(i) \neq 0$ for all i and ξ_5 is increasing. For (6), if $\xi_1^{-1}\{k\}$ is empty, the condition is verified since each $P^{\alpha,\beta}_\gamma$ is closed under initial segments and contains the empty function. If it is non-empty, then $k = 1$ and in that case $\xi_1^{-1}\{k\} = [0,\omega)$ and by the argument above ($\operatorname{dom} h^{-1}(\eta) = \operatorname{dom}\eta = \operatorname{dom}\xi$) we have $\xi_5 = h^{-1}(\eta) \in P^{0,\tilde{o}(\alpha_0)}_\gamma = P^{\xi_2(0),\xi_3(0)}_{\xi_4(0)}$, so the condition is satisfied.

Let us check whether all the conditions (a)-(i) are met. In (a), (b), (c), (e2) and (f) there is nothing to check. (d) holds, because h is an isomorphism. (e1) and (i) are immediate from the definition. Both $J^{\alpha_0}(S)$ and $P_\gamma^{0,\tilde{o}(\alpha_0)}$ are closed under initial segments, so (h) follows, because $\operatorname{dom} F_{\alpha_0} = J^{\alpha_0}(S)$ and $\operatorname{ran} F_{\alpha_0} = \{1\} \times \{0\} \times \{\tilde{o}(\alpha_0)\} \times \{\gamma\} \times P_\gamma^{0,\alpha_0}$. Claim 1 implies (g) for $\operatorname{dom} F_{\alpha_0}$. Suppose $\xi \in \operatorname{ran} F_{\alpha_0}$ and $\eta \in J(S')$ are as in the assumption of (g). Then $\eta_1(i) = \xi_1(i) = 1$ for all $i < \operatorname{dom} \eta$. By (5) it follows that $\eta_2(i) = \xi_2(i) = 0$, $\eta_3(i) = \xi_3(i) = \tilde{o}(\alpha_0)$ and $\eta_4(i) = \xi_4(i) = \gamma$ for all $i < \operatorname{dom} \eta$, so by (6) $\eta_5 \in P_\gamma^{0,\tilde{o}(\alpha_0)}$ and since h is an isomorphism, $\eta \in \operatorname{ran} F_{\alpha_0}$.

Odd successor step. We want to handle odd case but not the even case first, because the most important case is the successor of a limit ordinal, see $(\iota\iota\iota)$ below. Except that, the even case is similar to the odd case.

Suppose that $j < \kappa$ is a successor ordinal. Then there exist β_j and n_j such that $j = \beta_j + n_j$ and β is a limit ordinal or 0. Suppose that n_j is odd and that α_l and F_{α_l} are defined for all $l < j$ such that the conditions (a)–(i) and (1)–(9) hold for $l < j$.

Let $\alpha_j = \beta + 1$ where β is such that $\beta \in C$, $\operatorname{ran} F_{\alpha_{j-1}} \subset J^\beta(S')$, $\beta > \alpha_{j-1}$. For convenience define $\xi(-1) = (0,0,0,0,0)$ for all $\xi \in J(S) \cup J(S')$. Suppose $\eta \in \operatorname{ran} F_{\alpha_{j-1}}$ has finite domain $\operatorname{dom} \eta = m < \omega$ and denote $\xi = F_{\alpha_{j-1}}^{-1}(\eta)$. Fix γ_η to be such that $\tilde{n}(P_{\gamma_\eta}^{\alpha,\beta}) = m$ and such that there is an isomorphism $h_\eta \colon P_{\gamma_\eta}^{\alpha,\beta} \to W$, where

$$W = \{\zeta \mid \operatorname{dom} \zeta = [m,s), m < s \leqslant \omega, \eta^\frown\langle m, \zeta(m)\rangle \notin \operatorname{ran} F_{\alpha_{j-1}}, \eta^\frown\zeta \in J^{\alpha_j}(S')\},$$

$\alpha = \xi_3(m-1) + \xi_4(m-1)$ and $\beta = \alpha + \tilde{o}(\alpha_j)$ (defined in the beginning of the First step).

We will define F_{α_j} so that its range is $J^{\alpha_j}(S')$ and instead of F_{α_j} we will define its inverse. So let $\eta \in J^{\alpha_j}(S')$. We have three cases:

(ι) $\eta \in \operatorname{ran} F_{\alpha_{j-1}}$,

$(\iota\iota)$ $\exists m < \operatorname{dom} \eta (\eta \restriction m \in \operatorname{ran} F_{\alpha_{j-1}} \wedge \eta \restriction (m+1) \notin F_{\alpha_{j-1}})$,

$(\iota\iota\iota)$ $\forall m < \operatorname{dom} \eta (\eta \restriction (m+1) \in \operatorname{ran} F_{\alpha_{j-1}} \wedge \eta \notin \operatorname{ran} F_{\alpha_{j-1}})$.

Let us define $\xi = F_{\alpha_j}^{-1}(\eta)$ such that $\operatorname{dom} \xi = \operatorname{dom} \eta$. If (ι) holds, define $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta)(n)$ for all $n < \operatorname{dom} \eta$. Clearly $\xi \in J(S)$ by the induction hypothesis. Suppose that $(\iota\iota)$ holds and let m witness this. For all $n < \operatorname{dom} \xi$ let

- If $n < m$, then $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta \restriction m)(n)$.
- Suppose $n \geqslant m$. Let
  - $\xi_1(n) = \xi_1(m-1) + 1$
  - $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$
  - $\xi_3(n) = \xi_2(m) + \tilde{o}(\alpha_j)$
  - $\xi_4(n) = \gamma_{\eta \restriction m}$
  - $\xi_5(n) = h_{\eta \restriction m}^{-1}(\eta)(n)$.

Next we should check that $\xi \in J(S)$; let us check items (1) and (6), the rest are left to the reader.

(1) By the induction hypothesis $\xi\restriction m$ is increasing. Next, $\xi_1(m)=\xi_1(m-1)+1$, so $\xi(m-1)<_{\text{lex}}\xi(m)$. If $m\leqslant n_1<n_2$, then $\xi_k(n_1)=\xi_k(n_2)$ for all $k\in\{1,2,3,4\}$ and ξ_5 is increasing.

(6) Suppose that $[i,j)=\xi_1^{-1}\{k\}$. Since $\xi_1\restriction[m,\omega)$ is constant, either $j<m$, when we are done by the induction hypothesis, or $i=m$ and $j=\omega$. In that case one verifies that $\eta\restriction[m,\omega)\in W=\operatorname{ran}h_{\eta\restriction m}$ and then, imitating the corresponding argument in the first step, that
$$\xi_5\restriction[m,\omega)=h^{-1}_{\eta\restriction m}(\eta\restriction[m,\omega))$$
and hence in $\operatorname{dom}h_{\eta\restriction m}=P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$.

Suppose finally that $(\iota\iota\iota)$ holds. Then $\operatorname{dom}\eta$ must be ω since otherwise the condition $(\iota\iota\iota)$ is simply contradictory (because $\eta\restriction(\operatorname{dom}\eta-1+1)=\eta$ (except for the case $\operatorname{dom}\eta=0$, but then condition (ι) holds and we are done)). By (g), we have $\operatorname{ran}\eta_1=\omega$, because otherwise we had $\eta\in\operatorname{ran}F_{\alpha_{j-1}}$. Let $F^{-1}_{\alpha_j}(\eta)=\xi=\bigcup_{n<\omega}F^{-1}_{\alpha_{j-1}}(\eta\restriction n)$.

Let us check that it is in $J(S)$. Conditions (1)–(6) are satisfied by ξ, because they are satisfied by all its initial segments. Let us check (7).

First of all ξ cannot be in $J^{\alpha_{j-1}}(S)$, since otherwise, by (d) and (i),
$$F_{\alpha_{j-1}}(\xi)=\bigcup_{n<\omega}F_{\alpha_{j-1}}(\xi\restriction n)=\bigcup_{n<\omega}\eta\restriction n=\eta$$
were again in $\operatorname{ran}F_{\alpha_{j-1}}$. If $j-1$ is a successor ordinal, then we are done: by (b) α_{j-1} is a successor and we assumed $\eta\in J(S')$, so by (e2) we have $\xi\in J(S)$. Thus we can assume that $j-1$ is a limit ordinal. Then by (b), α_{j-1} is a limit ordinal in C and by (a), (e) and (f), $\operatorname{ran}F_{\alpha_{j-1}}=J^{\alpha_{j-1}}(S')$ and $\operatorname{dom}F_{\alpha_{j-1}}=J^{\alpha_{j-1}}(S)$. This implies that $\operatorname{ran}\eta\not\subset\omega\times\beta^4$ for any $\beta<\alpha_{j-1}$ and by (⊛) on page 133 we must have $\sup\operatorname{ran}\eta_5=\alpha_{j-1}$ which gives $\alpha_{j-1}\in S'$ by (7). Since $\alpha_{j-1}\in C\subset\kappa\setminus S\bigtriangleup S'$, we have $\alpha_{j-1}\in S$. Again by (⊛) and that $\operatorname{dom}F_{\alpha_{j-1}}=J^{\alpha_{j-1}}(S)$ by (e1), we have $\sup\operatorname{ran}\xi_5=\alpha_{j-1}$, thus ξ satisfies the condition (7).

Let us check whether all the conditions (a)-(i) are met. (a), (b), (c) are common to the cases (ι), $(\iota\iota)$ and $(\iota\iota\iota)$ in the definition of $F^{-1}_{\alpha_j}$ and are easy to verify. Let us sketch a proof for (d); the rest is left to the reader.

(d) Let $\eta_1,\eta_2\in\operatorname{ran}F_{\alpha_j}$ and let us show that
$$\eta_1\subsetneq\eta_2\iff F^{-1}_{\alpha_j}(\eta_1)\subsetneq F^{-1}_{\alpha_j}(\eta_2).$$

The case where both η_1 and η_2 satisfy $(\iota\iota)$ is the interesting one (implies all the others).

So suppose $\eta_1,\eta_2\in(\iota\iota)$. Then there exist m_1 and m_2 as described in the statement of $(\iota\iota)$. Let us show that $m_1=m_2$. We have $\eta_1\restriction(m_1+1)=\eta_2\restriction(m_1+1)$ and $\eta_1\restriction(m_1+1)\notin\operatorname{ran}F_{\alpha_{j-1}}$, so $m_2\leqslant m_1$. If $m_2\leqslant m_1$, then $m_2<\operatorname{dom}\eta_1$, since $m_1<\operatorname{dom}\eta_1$. Thus if $m_2\leqslant m_1$, then $\eta_1\restriction(m_2+1)=\eta_2\restriction(m_2+1)\notin\operatorname{ran}F_{\alpha_{j-1}}$, which implies $m_2=m_1$. According to the definition of $F^{-1}_{\alpha_j}(\eta_i)(k)$ for $k<\operatorname{dom}\eta_1$, $F^{-1}_{\alpha_j}(\eta_i)(k)$ depends only on m_i and $\eta\restriction m_i$ for $i\in\{1,2\}$. Since $m_1=m_2$ and $\eta_1\restriction m_1=\eta_2\restriction m_2$, we have $F^{-1}_{\alpha_j}(\eta_1)(k)=F^{-1}_{\alpha_j}(\eta_2)(k)$ for all $k<\operatorname{dom}\eta_1$.

Let us now assume that $\eta_1\not\subset\eta_2$. Then take the smallest $n\in\operatorname{dom}\eta_1\cap\operatorname{dom}\eta_2$ such that $\eta_1(n)\neq\eta_2(n)$. It is now easy to show that $F^{-1}_{\alpha_j}(\eta_1)(n)\neq F^{-1}_{\alpha_j}(\eta_2)(n)$ by the construction.

Even successor step. Namely the one where $j = \beta + n$ and n is even. But this case goes exactly as the above completed step, except that we start with $\operatorname{dom} F_{\alpha_j} = J^{\alpha_j}(S)$ where α_j is big enough successor of an element of C such that $J^{\alpha_j}(S)$ contains $\operatorname{ran} F_{\alpha_{j-1}}$ and define $\xi = F_{\alpha_j}(\eta)$. Instead of (e) we use (f) as the induction hypothesis. This step is easier since one does not need to care about the successors of limit ordinals.

Limit step. Assume that j is a limit ordinal. Then let $\alpha_j = \bigcup_{i<j} \alpha_i$ and $F_{\alpha_j} = \bigcup_{i<j} F_{\alpha_i}$. Since α_i are successors of ordinals in C, $\alpha_j \in C$, so (b) is satisfied. Since each F_{α_i} is an isomorphism, also their union is, so (d) is satisfied. Because conditions (e), (f) and (i) hold for $i < j$, the conditions (e) and (i) hold for j. (f) is satisfied because the premise is not true. (a) and (c) are clearly satisfied. Also (g) and (h) are satisfied by Claim 1 since now $\operatorname{dom} F_{\alpha_j} = J^{\alpha_j}(S)$ and $\operatorname{ran} F_{\alpha_j} = J^{\alpha_j}(S')$ (this is because (a), (e) and (f) hold for $i < j$).

Finally $F = \bigcup_{i<\kappa} F_{\alpha_i}$ is an isomorphism between $J(S)$ and $J(S')$. $\square_{\text{Claim 3}}$

$\square_{\text{Lemma 4.89}}$

4.90 Theorem. *Suppose κ is such that $\kappa^{<\kappa} = \kappa$ and for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and that T is a stable unsuperstable theory. Then $E_{S^\kappa_\omega} \leqslant_c \cong_T$.*

Proof. For $\eta \in 2^\kappa$ let $J_\eta = J(\eta^{-1}\{1\})$ where the function J is as in Lemma 4.89 above. For notational convenience, we assume that J_η is a downward closed subtree of $\kappa^{\leqslant\omega}$. Since T is stable unsuperstable, for all η and $t \in J_\eta$, there are finite sequences $a_t = a_t^\eta$ in the monster model such that

1. If $\operatorname{dom}(t) = \omega$ and $n < \omega$ then
$$a_t \underset{\bigcup_{m<n} a_t\restriction m}{\not\downarrow} a_{t\restriction n}.$$

2. For all downward closed subtrees $X, Y \subset J_\eta$,
$$\bigcup_{t\in X} a_t \underset{\bigcup_{t\in X\cap Y} a_t}{\downarrow} \bigcup_{t\in Y} a_t$$

3. For all downward closed subtrees $X \subset J_\eta$ and $Y \subset J_{\eta'}$ the following holds: If $f\colon X \to Y$ is an isomorphism, then there is an automorphism F of the monster model such that for all $t \in X$, $F(a_t^\eta) = a_{f(t)}^{\eta'}$

Then we can find an F^f_ω-construction
$$(\bigcup_{t\in J_\eta} a_t, (b_i, B_i)_{i<\kappa})$$
(here $(t(b/C), D) \in F^f_\omega$ if $D \subset C$ is finite and $b \downarrow_D C$, see [40]) such that

($\star$) for all $\alpha < \kappa$, c and finite $B \subset \bigcup_{t\in J_\eta} a_t \cup \bigcup_{i<\alpha} b_i$ there is $\alpha < \beta < \kappa$ such that $B_\beta = B$ and
$$\operatorname{stp}(b_\beta/B) = \operatorname{stp}(c/B).$$

Then

$$M_\eta = \bigcup_{t\in J_\eta} a_t \cup \bigcup_{i<\kappa} b_i \models T.$$

Without loss of generality we may assume that the trees J_η and the F^f_ω-constructions for M_η are chosen coherently enough such that one can find a code ξ_η for (the isomorphism type of) M_η so that $\eta \mapsto \xi_\eta$ is continuous. Thus we are left to show that $\eta E_{S^\kappa_\omega} \eta' \iff M_\eta \cong M_{\eta'}$

"$\Rightarrow$" Assume $J_\eta \cong J_{\eta'}$. By (3) it is enough to show that F^f_ω-construction of length κ satisfying ($\star$) are unique up to isomorphism over $\bigcup_{t\in J_\eta} a_t$. But ($\star$) guarantees that the proof of the uniqueness of F-primary models from [40] works here.

"$\Leftarrow$" Suppose $F\colon M_\eta \to M_{\eta'}$ is an isomorphism and for a contradiction suppose $(\eta,\eta') \notin E_{S^\kappa_\omega}$. Let $(J^\alpha_\eta)_{\alpha<\kappa}$ be a filtration of J_η and $(J^\alpha_{\eta'})_{\alpha<\kappa}$ be a filtration of $J_{\eta'}$ (see Definition 4.85 above). For $\alpha < \kappa$, let

$$M^\alpha_\eta = \bigcup_{t\in J^\alpha_\eta} a_t \cup \bigcup_{i<\alpha} b_i$$

and similarly for η':

$$M^\alpha_{\eta'} = \bigcup_{t\in J^\alpha_{\eta'}} a_t \cup \bigcup_{i<\alpha} b_i.$$

Let C be the cub set of those $\alpha < \kappa$ such that $F \restriction M^\alpha_\eta$ is onto $M^\alpha_{\eta'}$ and for all $i < \alpha$, $B_i \subset M^\alpha_\eta$ and $B'_i \subset M^\alpha_{\eta'}$, where $(\bigcup_{t\in J_{\eta'}}, (b'_i, B'_i)_{i<b})$ is in the construction of $M_{\eta'}$. Then we can find $\alpha \in \lim C$ such that in J_η there is t^* satisfying (a)–(c) below, but in $J_{\eta'}$ there is no such t^*.:

(a) $\mathrm{dom}(t^*) = \omega$,

(b) $t^* \notin J^\alpha_\eta$,

(c) for all $\beta < \alpha$ there is $n < \omega$ such that $t^* \restriction n \in J^\alpha_\eta \setminus J^\beta_\eta$,

Note that

($\star\star$) if $\alpha \in C$ and $c \in M^\alpha_\eta$, there is a finite $D \subset \bigcup_{t\in J^\alpha_\eta} a_t$ such that $(t(c, \bigcup_{t\in J_\eta} a_t), D) \in F^f_\omega$,

Let $c = F(a_{t^*})$. By the construction we cat find finite $D \subset M^\alpha_{\eta'}$, and $X \subset J_{\eta'}$ such that

$$\Big(t(c, M^\alpha_{\eta'} \cup \bigcup_{t\in J_{\eta'}} a^{\eta'}_t), D \cup \bigcup_{t\in X} a^{\eta'}_t\Big) \in F^f_\omega.$$

But then there is $\beta \in C$, $\beta < \alpha$, such that $D \subset M^\beta_{\eta'}$ and if $u \leqslant t$ for some $t \in X$, then $u \in J^\beta_{\eta'}$ (since in $J_{\eta'}$ there is no element like t^* is in J_η). But then using ($\star\star$) and (2), it is easy to see that

$$c \downarrow_{M^\beta_{\eta'}} M^\alpha_{\eta'}.$$

On the other hand, using (1), (2), ($\star\star$) and the choice of t^* one can see that $a_{t^*} \not\downarrow_{M^\beta_\eta} M^a_\eta$, a contradiction. □

Open Problem. If $\kappa = \lambda^+$, λ regular and uncountable, does equality modulo λ-non-stationary ideal, $E_{S^\kappa_\lambda}$, Borel reduce to T for all stable unsuperstable T?

4.7 Further Research

In this section we merely list all the questions that also appear in the text:

Open Problem. Is it consistent that Borel* is a proper subclass of Σ^1_1, or even equals Δ^1_1? Is it consistent that all the inclusions are proper at the same time: $\Delta^1_1 \subsetneq \text{Borel}^* \subsetneq \Sigma^1_1$?

Open Problem. Does the direction left to right of Theorem 4.25 hold without the assumption $\kappa^{<\kappa} = \kappa$?

Open Problem. Under what conditions on κ does the conclusion of Theorem 4.37 hold?

Open Problem. Is the Silver Dichotomy for uncountable κ consistent?

Open Problem. Is it consistent that $S^{\omega_2}_{\omega_1}$ Borel reduces to $S^{\omega_2}_{\omega}$?

Open Problem. We proved that the isomorphism relation of a theory T is Borel if and only if T is classifiable and shallow. Is there a connection between the depth of a shallow theory and the Borel degree of its isomorphism relation? Is one monotone in the other?

Open Problem. Can it be proved in ZFC that if T is stable unsuperstable then $\cong_T$ is not Δ^1_1?

Open Problem. If $\kappa = \lambda^+$, λ regular and uncountable, does equality modulo λ-non-stationary ideal, $E_{S^\kappa_\lambda}$, Borel reduce to T for all stable unsuperstable T?

Open Problem. Let T_{dlo} be the theory of dense linear orderings without end points and T_{gr} the theory of random graphs. Does the isomorphism relation of T_{gr} Borel reduce to T_{dlo}, i.e. $\cong_{T_{\text{gr}}} \leqslant_B \cong_{T_{\text{dlo}}}$?

5

Borel Reductions on the Generalized Cantor Space

If your method does not solve the problem, change the problem.

Saharon Shelah

5.1 Introduction

It is shown that the partial order of Borel equivalence relations on the generalized Baire spaces (2^κ for $\kappa > \omega$) under Borel reducibility has high complexity already at low levels (below E_0).

This extends an answer stated in [6] to an open problem stated in [7] and in particular solves open problems 7 and 9 from [6].

The development of the theory of the generalized Baire and Cantor spaces dates back to 1990's when it A. Mekler and J. Väänänen published the paper *Trees and Π^1_1-Subsets of ${}^{\omega_1}\omega_1$* [36] and A. Halko published *Negligible subsets of the generalized Baire space $\omega_1^{\omega_1}$*. More recently equivalence relations and Borel reducibility on these spaces and their applications to model theory have been under focus, see my latest joint work with S. Friedman and T. Hyttinen [7].

Suppose κ is an infinite cardinal and let $\mathcal{E}^B_\kappa$ be the collection of all Borel equivalence relations on 2^κ. (For definitions in the case $\kappa > \omega$ see next section.) For equivalence relations E_0 and E_1 let us denote $E_0 \leqslant_B E_1$ if there exists a Borel function $f\colon 2^\kappa \to 2^\kappa$ such that $(\eta, \xi) \in E_0 \iff (f(\eta), f(\xi)) \in E_1$. The relation $\leqslant_B$ defines a quasiorder on $\mathcal{E}^B_\kappa$, i.e. it induces a partial order on $\mathcal{E}^B_\kappa / \sim_B$ where $\sim_B$ is the equivalence relation of bireducibility: $E_0 \sim_B E_1 \iff (E_0 \leqslant_B E_1) \wedge (E_1 \leqslant_B E_0)$.

In the case $\kappa = \omega$ there are many known results that describe the order $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$. Two of them are:

Theorem (Louveau-Velickovic [33]). *The partial order $\langle \mathcal{P}(\omega), \subset_* \rangle$ can be embedded into the partial order $\langle \mathcal{E}^B_\omega, \leqslant_B \rangle$, where $A \subset_* B$ if $A \setminus B$ is finite.*

Theorem (Adams-Kechris [1]). *The partial order $\langle \mathcal{B}, \subset \rangle$ can be embedded into the partial order $\langle \mathcal{E}^B_\omega, \leqslant_B \rangle$, where $\mathcal{B}$ is the collection of all Borel subsets of the real line $\mathbb{R}$. In fact, the embedding is into the suborder of $\langle \mathcal{E}^B_\omega, \leqslant_B \rangle$ consisting of the countable Borel equivalence relations, i.e., those Borel equivalence relations each of whose equivalence classes is countable.*

Our aim is to generalize these results to uncountable κ with $\kappa^{<\kappa} = \kappa$ and it is proved that $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\omega)} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$, where $A \subset_{\mathrm{NS}(\omega)} B$ means that $A \setminus B$ is not ω-stationary. This is proved in ZFC. However under mild additional assumptions on κ or on the underlying set theory, it is shown that $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$, where $A \subset_{\mathrm{NS}} B$ means that $A \setminus B$ is non-stationary and that $\langle \mathcal{P}(\kappa), \subset_* \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$, where $A \subset_* B$ means that $A \setminus B$ is bounded.

Assumption. Everywhere in this chapter it is assumed that κ is a cardinal which satisfies $|\kappa^\alpha| = \kappa$ for all $\alpha < \kappa$. This requirement is briefly denoted by $\kappa^{<\kappa} = \kappa$.

5.2 Background in Generalized Descriptive Set Theory

5.1 Definition. Consider the function space 2^κ (all functions from κ to $\{0,1\}$) equipped with the topology generated by the sets

$$N_p = \{\eta \in 2^\kappa \mid \eta \restriction \alpha = p\}$$

for $\alpha < \kappa$ and $p \in 2^\alpha$. Borel sets on this space are obtained by closing the topology under unions and intersections of length $\leqslant \kappa$, and complements.

An equivalence relation E on 2^κ is *Borel reducible* to an equivalence relation E' on 2^κ if there exists a Borel function $f\colon 2^\kappa \to 2^\kappa$ (inverse images of open sets are Borel) such that $\eta E \xi \iff f(\eta) E' f(\xi)$. This is denoted by $E \leqslant_B E'$.

The descriptive set theory of these spaces, of equivalence relations on them and of their reducibility properties for $\kappa > \omega$, has been developed at least in [7, 9, 36]. For $\kappa = \omega$ this is the field of standard descriptive set theory.

By id_X we denote the identity relation on X: $(\eta,\xi) \in \mathrm{id}_X \iff (\eta,\xi) \in X^2 \wedge \eta = \xi$ and by E_0 the equivalence relation on 2^κ (or on κ^κ as in the proof of Theorem 5.30) such that $(\eta,\xi) \in E_0 \iff \{\alpha \mid \eta(\alpha) \neq \xi(\alpha)\}$ is bounded.

Notation. Let $\mathcal{E}^B_\kappa$ denote the set of all Borel equivalence relations on 2^κ (i.e. equivalence relations $E \subset (2^\kappa)^2$ such that E is a Borel set). If $X, Y \subset \kappa$ and $X \setminus Y$ is non-stationary, let us denote it by $X \subset_{\mathrm{NS}} Y$. If $X \setminus Y$ is not λ-stationary for some regular $\lambda < \kappa$, it is denoted by $X \subset_{\mathrm{NS}(\lambda)} Y$.

The set of all ordinals below κ which have cofinality λ is denoted by S^κ_λ, and $\lim(\kappa)$ denotes the set of all limit ordinals below κ. Also $\operatorname{reg}\kappa$ denotes the set of regular cardinals below κ and

$$S^\kappa_{\geqslant\lambda} = \bigcup_{\substack{\mu \geqslant \lambda \\ \mu \in \operatorname{reg}\kappa}} S^\kappa_\mu,$$

$$S^\kappa_{\leqslant\lambda} = \bigcup_{\substack{\mu \leqslant \lambda \\ \mu \in \operatorname{reg}\kappa}} S^\kappa_\mu.$$

If $A \subset \alpha$ and α is an ordinal, then $\mathrm{OTP}(A)$ is the order type of A in the ordering induced on it by α.

For ordinals $\alpha < \beta$ let us adopt the following abbreviations:

- $(\alpha,\beta) = \{\gamma \mid \alpha < \gamma < \beta\}$,
- $[\alpha,\beta] = \{\gamma \mid \alpha \leqslant \gamma \leqslant \beta\}$,
- $(\alpha,\beta] = \{\gamma \mid \alpha < \gamma \leqslant \beta\}$,
- $[\alpha,\beta) = \{\gamma \mid \alpha \leqslant \gamma < \beta\}$.

If η and ξ are functions in 2^κ, then $\eta \triangle \xi$ is the function $\zeta \in 2^\kappa$ such that $\zeta(\alpha) = 1 \iff \eta(\alpha) \neq \xi(\alpha)$ for all $\alpha < \kappa$, and $\bar\eta = 1 - \eta$ is the function $\zeta \in 2^\kappa$ such that $\zeta(\alpha) = 1 - \eta(\alpha)$ for all $\alpha < \kappa$. If A and B are sets, then $A \triangle B$ is just the symmetric difference.

For any set X, 2^X denotes the set of all functions from X to $2 = \{0,1\}$. If $p \in 2^{[0,\alpha)}$ and $\eta \in 2^{[\alpha,\kappa)}$, then $p^\frown\eta \in 2^\kappa$ is the catenation: $(p^\frown\eta)(\beta) = p(\beta)$ for $\beta < \alpha$ and $(p^\frown\eta)(\beta) = \eta(\beta)$ for $\beta \geqslant \alpha$.

5.2 Definition. A *co-meager* subset of X is a set which contains an intersection of length $\leqslant \kappa$ of dense open subsets of X. Co-meager sets are always non-empty and form a filter on 2^κ, [36]. A set X has the *Property of Baire* if there exists an open set A such that $X \triangle A$ is meager, i.e. a complement of a co-meager set. As in standard descriptive set theory, Borel sets have the Property of Baire (proved in [9]). For a Borel function $f\colon 2^\kappa \to 2^\kappa$ denote by $C(f)$ one of the co-meager sets restricted to which f is continuous (such set is not unique, but we can always pick one using the Property of Baire of Borel sets, see [7]).

5.3 Lemma. *Let D be a co-meager set in 2^κ and let $p, q \in 2^\alpha$ for some $\alpha < \kappa$. Then there exists $\eta \in 2^{[\alpha,\kappa)}$ such that $p^\frown\eta \in D$ and $q^\frown\eta \in D$. Also there exists $\eta \in 2^{[\alpha,\kappa)}$ such that $p^\frown\bar{\eta} \in D$ and $q^\frown\eta \in D$ where $\bar{\eta} = 1 - \eta$.*

Proof. Let h be the homeomorphism $N_p \to N_q$ defined by $p^\frown\eta \mapsto q^\frown\eta$. Then $h[N_p \cap D]$ is co-meager in N_q, so $N_q \cap D \cap h[N_p \cap D]$ is non-empty. Pick η' from that intersection and let $\eta = \eta' \restriction [\alpha, \kappa)$. This will do. For the second part take for h the homeomorphism defined by $p^\frown\eta \mapsto q^\frown\bar{\eta}$. □

5.3 On Cub-games and GC_λ-characterization

The notion of cub-games is a useful way to treat certain properties of subsets of cardinals. They generalize closed unbounded sets and are related to combinatorial principles such as $\square_\kappa$. Under mild set theoretic assumptions, they give characterizations of CUB-filters in different cofinalities. Treatments of this subject can be found for example in [15, 16, 22].

5.4 Definition. Let $A \subset \kappa$. The game $\mathrm{GC}_\lambda(A)$ is played between players **I** and **II** as follows. There are λ moves and at the i:th move player **I** picks an ordinal α_i which is greater than any ordinal picked earlier in the game and then **II** picks an ordinal $\beta_i > \alpha_i$. Player **II** wins if $\sup_{i<\lambda} \alpha_i \in A$. Otherwise player **I** wins.

5.5 Definition. A set $C \subset \kappa$ is λ-*closed* for a regular cardinal $\lambda < \kappa$, if for all increasing sequences $\langle \alpha_i \in C \mid i < \lambda \rangle$, the limit $\sup_{i<\lambda} \alpha_i$ is in C. A set $C \subset \kappa$ is *closed* if it is λ-closed for all regular $\lambda < \kappa$. A set is λ-*cub* if it is λ-closed and unbounded and *cub*, if it is closed and unbounded. A set is λ-stationary, if it intersects all λ-cub sets and *stationary* if it intersects all cub sets.

5.6 Definition. We say that GC_λ-characterization holds for κ, if

$$\{A \subset \kappa \mid \mathbf{II} \text{ has a winning strategy in } \mathrm{GC}_\lambda(A)\} = \{A \subset \kappa \mid A \text{ contains a } \lambda\text{-cub set}\}$$

and we say that GC-characterization holds for κ if GC_λ-characterization holds for κ for all regular $\lambda < \kappa$.

5.7 Definition. Assume $\kappa = \lambda^+$ and $\mu \leqslant \lambda$ a regular uncountable cardinal. The *square principle on κ for μ*, denoted $\square^\kappa_\mu$, defined by Jensen in case $\lambda = \mu$, is the statement that there exists a sequence $\langle C_\alpha \mid \alpha \in S^\kappa_{\leqslant\mu} \rangle$ with the following properties:

1. $C_\alpha \subset \alpha$ is closed and unbounded in α,
2. if $\beta \in \lim C_\alpha$, then $C_\beta = \beta \cap C_\alpha$,

3. if $\mathrm{cf}(\alpha) < \mu$, then $|C_\alpha| < \mu$.

5.8 Remark. For $\omega < \mu < \lambda$ in the definition above, it was proved by Shelah in [44] that $\square_\mu^\kappa$ holds (can be proved in ZFC, for a proof see also [3, Lemma 7.7]). If $\mu = \lambda$, then $\square_\mu^\kappa = \square_\mu^{\mu^+}$ is denoted by $\square_\mu$ and can be easily forced or, on the other hand, it holds, if $V = L$. The failure of $\square_\mu$ implies that μ^+ is Mahlo in L, as pointed out by Jensen, see [25].

5.9 Definition. For $\kappa > \omega$, the set $I[\kappa]$ consists of those $S \subset \kappa$ that have the following property: there exists a cub set C and a sequence $\langle \mathcal{D}_\alpha \mid \alpha < \kappa \rangle$ such that

1. $\mathcal{D}_\alpha \subset \mathcal{P}(\alpha)$, $|\mathcal{D}_\alpha| < \kappa$,
2. $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$ for all $\alpha < \beta$,
3. for all $\alpha \in C \cap S$ there exists $E \subset \alpha$ unbounded in α and of order type $\mathrm{cf}(\alpha)$ such that for all $\beta < \alpha$, $E \cap \beta \in \mathcal{D}_\gamma$ for some $\gamma < \alpha$.

5.10 Remark. The following is known.

1. $I[\kappa]$ is a normal ideal and contains the non-stationary sets.
2. If $\lambda < \kappa$ is regular and $S_\lambda^\kappa \in I[\kappa]$, then GC_λ-characterization holds for κ.
3. If μ is regular and $\kappa = \mu^+$, then $S_{<\mu}^\kappa \in I[\kappa]$, [44]. This follows also from 4. and Remark 5.8
4. When $\lambda > \omega$, then $\square_\lambda^\kappa$ implies that $S_\lambda^\kappa \in I[\kappa]$ (take $\mathcal{D}_\alpha = \{C_\alpha \cap \beta \mid \beta < \alpha\}$).
5. $S_\omega^\kappa \in I[\kappa]$.
6. If $\kappa^{<\lambda} = \kappa = \lambda^+$, then GC_λ-characterization holds for κ if and only if $\kappa \in I[\kappa]$ if and only if $S_\lambda^\kappa \in I[\kappa]$, see [15, Corollary 2.4] and [44].
7. The existence of $\lambda < \kappa$ such that GC_λ-characterization does not hold for κ is equiconsistent with the existence of a Mahlo cardinal.[1] Briefly this is because the failure of the characterization implies the failure of $\square_\lambda$ which implies that λ^+ is Mahlo in L as discussed above. On the other hand, in the Mitchell model, obtained from $S_{\mathrm{in}} = \{\delta < \lambda \mid \delta \text{ is inaccessible}\}$ where $\lambda > \kappa$ is Mahlo, it holds that $S_{\mathrm{in}} \notin I[\kappa^+]$, [15, Lemma 2.6].
8. If κ is regular and for all regular $\mu < \kappa$ we have $\mu^{<\lambda} < \kappa$, then $\kappa \in I[\kappa]$.

Remark. As Remark 5.10 shows, the assumption that GC_λ-characterization holds for κ is quite weak. For instance GC_ω-characterization holds for all regular $\kappa > \omega$ and GCH implies that GC_λ-characterization holds for κ for all regular $\lambda < \kappa$.

[1] A good exposition of this result can be found in Lauri Tuomi's Master's thesis (University of Helsinki, 2009).

5.4 Main Results

Theorems 5.11 and 5.12 constitute the goal of this work. They are stated below but proved in the end of this section, starting at pages 153 and 156 respectively.

5.11 Theorem. *Assume that $\lambda < \kappa$ are regular and GC_λ-characterization holds for κ. Then the order $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\lambda)} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$ strictly between id_{2^κ} and E_0. More precisely there exists a one-to-one map $F\colon \mathcal{P}(\kappa) \to \mathcal{E}^B_\kappa$ such that for all $X, Y \in \mathcal{P}(\kappa)$ we have $\mathrm{id}_{2^\kappa} \lneq_B F(X) \lneq_B E_0$ and*

$$X \subset_{\mathrm{NS}(\lambda)} Y \iff F(X) \leqslant_B F(Y).$$

5.12 Theorem. *Assume either $\kappa = \omega_1$ or $\kappa = \lambda^+ > \omega_1$ and $\square_\lambda$. Then the partial order $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$.*

5.4.1 Corollaries

5.13 Corollary. *Assume that $\lambda < \kappa$ is regular. Additionally assume one of the following:*

1. *$\kappa = \mu^+$, μ is regular and $\lambda < \mu$,*
2. *$\kappa = \lambda^+$ and $\square_\lambda$ holds,*
3. *for all regular $\mu < \kappa$, $\mu^{<\lambda} < \kappa$ (e.g. κ is ω_1 or inaccessible).*

Then the partial order $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\lambda)} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$.

Proof. Any of the assumptions 1 – 3 is sufficient to obtain GC_λ-characterization for κ by Remarks 5.10 and 5.8, so the result follows from Theorem 5.11. □

5.14 Corollary. *The partial order $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\omega)} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$. In particular $\langle \mathcal{P}(\omega_1), \subset_{\mathrm{NS}} \rangle$ can be embedded into $\langle \mathcal{E}^B_{\omega_1}, \leqslant_B \rangle$ assuming CH.*

Proof. By Remark 5.10 GC_ω-characterization holds for κ for any regular $\kappa > \omega$, so the result follows from Theorem 5.11. □

5.15 Definition. Let $S \subset \kappa$. Then the combinatorial principle $\diamondsuit_\kappa(S)$ states that there exists a sequence $\langle D_\alpha \mid \alpha \in S \rangle$ such that for every $A \subset \kappa$ the set $\{\alpha \mid A \cap \alpha = D_\alpha\}$ is stationary.

5.16 Theorem (Shelah [45])**.** *If $\kappa = \lambda^+ = 2^\lambda$ and $S \subset \kappa \setminus S^\kappa_{\mathrm{cf}(\lambda)}$ is stationary, then $\diamondsuit_\kappa(S)$ holds.* □

5.17 Corollary.

1. *The ordering $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$.*
2. *Assume that $\kappa = \omega_1$ and $\diamondsuit_{\omega_1}$ holds or that κ is not a successor of an ω-cofinal cardinal. Then also the ordering $\langle \mathcal{P}(\kappa), \subset_* \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$, where $\subset_*$ is inclusion modulo bounded sets.*

Proof. For the first part it is sufficient to show that the partial order $\langle \mathcal{P}(\kappa), \subset\rangle$ can be embedded into $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\omega)}\rangle$. Let $G(A) = \bigcup_{i\in A} S_i$ where $\{S_i \subset S^\kappa_\omega \mid i < \kappa\}$ is a collection of disjoint stationary sets. Then $A \subset B \iff G(A) \subset_{\mathrm{NS}} G(B)$, so this proves the first part.

For the second part, let us show that if $\diamondsuit_\kappa(S^\kappa_\lambda)$ holds, then $\langle \mathcal{P}(\kappa), \subset_*\rangle$ can be embedded into $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\lambda)}\rangle$. Then the result follows. If $\kappa = \omega_1$ and $\diamondsuit_{\omega_1}$ holds, then it follows by Corollary 5.14. On the other hand, if κ is not a successor of an ω-cofinal cardinal, then from Theorem 5.16 it follows that $\diamondsuit_\kappa(S^\kappa_\omega)$ holds and the result follows again from Corollary 5.14.

Suppose that $\langle D_\alpha \mid \alpha \in S^\kappa_\lambda\rangle$ is a $\diamondsuit_\kappa(S^\kappa_\lambda)$-sequence. If $X, Y \subset \alpha$ for $\alpha \leqslant \kappa$, let $X \subset_* Y$ denote that there is $\beta < \alpha$ such that $X \setminus \beta \subset Y \setminus \beta$, i.e. X is a subset of Y on a final segment of α. Note that this coincides with the earlier defined $\subset_*$ when $\alpha = \kappa$. For $A \subset \kappa$ let

$$H(A) = \{\alpha < \kappa \mid D_\alpha \subset_* A \cap \alpha\}.$$

If $A \subset_* B$ then there is $\gamma < \kappa$ such that $A \setminus \gamma \subset B \setminus \gamma$ and if $\beta > \gamma$ is in $H(A)$, then $D_\beta \subset_* A \cap \beta$ and since $A \cap \beta \subset_* B \cap \beta$, we have $D_\beta \subset_* B \cap \beta$, so $H(A) \subset_* H(B)$ which finally implies $H(A) \subset_{\mathrm{NS}(\omega)} H(B)$.

Assume now that $A \not\subset_* B$ and let $C = A \setminus B$. Let S' be the stationary set such that for all $\alpha \in S'$, $C \cap \alpha = D_\alpha$. Let S be the λ-stationary set $S' \cap \{\alpha \mid C \text{ is unbounded below } \alpha\}$. S is stationary, because it the intersection of S' and a cub set. Now for all $\alpha \in S$ we have $D_\alpha = C \cap \alpha \subset A \cap \alpha$, so $S \subset H(A)$. On the other hand if $\alpha \in S$, then

$$D_\alpha \setminus (B \cap \alpha) = (C \cap \alpha) \setminus (B \cap \alpha) = ((A \setminus B) \cap \alpha) \setminus (B \cap \alpha) = C \cap \alpha$$

is unbounded in α, so $D_\alpha \not\subset_* B \cap \alpha$ and so $S \subset H(A) \setminus H(B)$, whence $H(A) \not\subset_{\mathrm{NS}(\lambda)} H(B)$. □

5.18 Corollary. *There are 2^κ equivalence relations between* id *and E_0 that form a linear order with respect to $\lneqq_B$.*

Proof. Let $K = \{\eta \in 2^\kappa \mid (\exists\beta)(\forall\gamma > \beta)(\eta(\gamma) = 0)\}$, let $f : K \to \kappa$ be a bijection and for $\eta, \xi \in 2^\kappa$ define $\eta \lessdot \xi$ if and only if

$$\eta(\min\{\alpha \mid \eta(\alpha) \neq \xi(\alpha)\}) < \xi(\min\{\alpha \mid \eta(\alpha) \neq \xi(\alpha)\}).$$

For $\eta \in 2^\kappa$ let $A_\eta = \{f(\xi) \mid \xi \lessdot \eta \wedge \xi \in K\}$. Clearly $A_\eta \subsetneq A_\xi$ if and only if $\eta \lessdot \xi$ and the latter is a linear order. The statement now follows from Corollary 5.17. □

5.4.2 Preparing for the Proofs

5.19 Definition. For each $S \subset \lim \kappa$ let us define equivalence relations E^*_S, E_S and $E^*_S(\alpha)$, $\alpha \leqslant \kappa$, on the space 2^κ as follows. Suppose $\eta, \xi \in 2^\delta$ for some $\delta \leqslant \kappa$ and let $\zeta = \eta \triangle \xi$. Let us define η and ξ to be $E^*_S(\delta)$-equivalent if and only if for all ordinals $\alpha \in S \cap \delta$ there exists $\beta < \alpha$ such that $\zeta(\gamma)$ has the same value for all $\gamma \in (\beta, \alpha)$. Let $E^*_S = E^*_S(\kappa)$ and $E_S = E^*_S \cap E_0$, where E_0 is the equivalence modulo bounded sets.

Remark. If $S = \varnothing$, then $E_S = E_\varnothing = E_0$. If $S = \lim \kappa$ or equivalently if $S = \lim_\omega \kappa = S^\kappa_\omega$ (ω-cofinal limit ordinals), then $E_S = E'_0$, where E'_0 is defined in [6].

5.20 Theorem. *For any $S \subset \lim \kappa$ the equivalence relations E_S and E^*_S are Borel.*

Proof. This is obvious by writing out the definitions:

$$\begin{aligned} E_S^* &= \bigcap_{\alpha\in S}\bigcup_{\beta<\alpha}\Big(\bigcap_{\beta<\gamma<\alpha}\{(\eta,\xi)\mid \eta(\gamma)\neq\xi(\gamma)\}\cup\bigcap_{\beta<\gamma<\alpha}\{(\eta,\xi)\mid\eta(\gamma)=\xi(\gamma)\}\Big),\\ E_0 &= \bigcup_{\alpha<\kappa}\bigcap_{\alpha<\beta<\kappa}\{(\eta,\xi)\mid \eta(\beta)=\xi(\beta)\}.\\ E_S &= E_S^*\cap E_0. \end{aligned}$$

□

The ideas of the following proofs are simple, but are repeated many times in this article in one way or another.

5.21 Theorem. *For all $S\subset\lim\kappa$, $E_S\not\leqslant_B \mathrm{id}_{2^\kappa}$ and $E_S^*\leqslant_B\mathrm{id}_{2^\kappa}$.*

Proof. For the first part suppose f is a Borel reduction from E_S to id_{2^κ}. Let η be a function such that η and $\bar\eta=1-\eta$ are both in $C(f)$ (see Definition 5.2, page 144). This is possible by Lemma 5.3, page 144. Then $(\eta,\bar\eta)\notin E_S$. Let α be so large that $f(\eta)\restriction\alpha\neq f(\bar\eta)\restriction\alpha$ and pick β so that

$$f[N_{\eta\restriction\beta}\cap C(f)]\subset N_{f(\eta)\restriction\alpha}$$

and

$$f[N_{\bar\eta\restriction\beta}\cap C(f)]\subset N_{f(\eta)\restriction\alpha}.$$

This is possible by the continuity of f on $C(f)$. By Lemma 5.3 pick now a $\zeta\in 2^{[\beta,\kappa)}$ so that $\eta\restriction\beta^\frown\zeta\in C(f)$ and $\bar\eta\restriction\beta^\frown\zeta\in C(f)$ which provides us with a contradiction, since

$$\big(\eta\restriction\beta^\frown\zeta,\bar\eta\restriction\beta^\frown\zeta\big)\in E_S,\text{ but } f(\eta\restriction\beta^\frown\zeta)\neq f(\bar\eta\restriction\beta^\frown\zeta)$$

To prove the second part it is sufficient to construct a reduction from E_S^* to $\mathrm{id}_{\kappa^\kappa}$, since $\mathrm{id}_{\kappa^\kappa}$ and id_{2^κ} are bireducible (see [7]). Let us define an equivalence relation $\sim$ on $2^{<\kappa}$ such that $p\sim q$ if and only if $\operatorname{dom}p=\operatorname{dom}q$ and $p\,\triangle\,q$ is eventually constant, i.e. for some $\alpha<\operatorname{dom}p$, $(p\,\triangle\,q)(\gamma)$ is the same for all $\gamma\in[\alpha,\operatorname{dom}p)$. Let $s\colon 2^{<\kappa}\to\kappa$ be a map such that $p\sim q\iff s(p)=s(q)$. Suppose $\eta\in 2^\kappa$ and let us define $\xi=f(\eta)$ as follows. Let β_γ denote the γ:th element of S and let $\xi(\gamma)=s(\eta\restriction\beta_\gamma)$. Now we have $\eta E_S^*\xi$ if and only if $\eta\restriction\beta_\gamma=\xi\restriction\beta_\gamma$ for all $\gamma\in\kappa$ if and only if $f(\eta)=f(\xi)$. □

5.22 Corollary. *Let $S\subset\kappa$. If $p\in 2^{<\kappa}$ and $C\subset N_p$ is any co-meager subset of N_p, then there is no continuous function $C\to 2^\kappa$ such that $(\eta,\xi)\in E_S\cap C^2\iff f(\eta)=f(\xi)$.*

Proof. Apply the same proof as for the first part of Theorem 5.21; take C instead of $C(f)$ and work inside N_p, e.g. instead of $\eta,\bar\eta$ take $p^\frown\eta,p^\frown\bar\eta$ for suitable $\eta\in 2^{[\operatorname{dom}p,\kappa)}$. □

5.23 Definition. A set $A\subset\kappa$ *does not reflect* to an ordinal α, if the set $\alpha\cap A$ is non-stationary in α, i.e. there exists a closed unbounded subset of α outside of $A\cap\alpha$.

5.24 Theorem. *If $\kappa=\lambda^+>\omega_1$ and $\square_\mu^\kappa$ holds, $\mu\leqslant\lambda$, then for every stationary $S\subset S^\kappa_\omega$, there exists a set $B^\mu_{\mathrm{nr}}(S)\subset S$ (nr for non-reflecting) such that $B^\mu_{\mathrm{nr}}(S)$ does not reflect to any $\alpha\in S^\kappa_{\leqslant\mu}\cap S^\kappa_{\geqslant\omega_1}$ and the sets $\lim C_\alpha$ witness that, where $\langle C_\alpha\mid\alpha\in S^\kappa_{\leqslant\mu}\rangle$ is the $\square_\lambda$-sequence, i.e. $\lim C_\alpha\subset\alpha\setminus B^\mu_{\mathrm{nr}}(S)$ for $\alpha\in S^\kappa_{\leqslant\mu}\cap S^\kappa_{\geqslant\omega_1}$. Since $\mathrm{cf}(\alpha)>\omega$, $\lim C_\alpha$ is cub in α.*

Proof. This is a well known argument and can be found in [25]. Let $g\colon S \to \kappa$ be the function defined by $g(\alpha) = \mathrm{OTP}(C_\alpha)$. By the definition of $\square_\mu$, $\mathrm{OTP}(C_\alpha) < \mu$ for $\alpha \in S^\kappa_\omega$, so for $\alpha > \mu$ we have $g(\alpha) < \alpha$. By Fodor's lemma there exists a stationary $B^\mu_{\mathrm{nr}}(S) \subset S$ such that $\mathrm{OTP}(C_\alpha) = \mathrm{OTP}(C_\beta)$ for all $\alpha, \beta \in B^\mu_{\mathrm{nr}}(\mu)$. If $\alpha \in \lim C_\beta$, then $C_\alpha = C_\beta \cap \alpha$ and therefore $\mathrm{OTP}(C_\alpha) < \mathrm{OTP}(C_\beta)$. Hence $\lim C_\beta \subset \beta \setminus B^\mu_{\mathrm{nr}}(S)$. □

5.25 Definition. Let E_i be equivalence relations on $2^{\kappa\times\{i\}}$ for all $i < \alpha$ where $\alpha < \kappa$. Let $E = \bigotimes_{i<\alpha} E_i$ be an equivalence relation on the space $2^{\kappa\times\alpha}$ such that $(\eta, \xi) \in E$ if and only if for all $i < \alpha$, $(\eta \restriction (\kappa \times \{i\}), \xi \restriction (\kappa \times \{i\})) \in E_i$.

Naturally, if $\alpha = 2$, we denote $\bigotimes_{i<2} E_i$ by just $E_0 \otimes E_1$ and we constantly identify $2^{\kappa\times\{i\}}$ with 2^κ.

5.26 Definition. Given equivalence relations E_i on $2^{\kappa\times\{i\}}$ for $i < \alpha < \kappa^+$, let $\bigoplus_{i\in I} E_i$ be an equivalence relation on $\bigcup_{i<\alpha} 2^{\kappa\times\{i\}}$ such that η and ξ are equivalent if and only if for some $i < \alpha$, $\eta, \xi \in 2^{\kappa\times\{i\}}$ and $(\eta, \xi) \in E_i$.

Intuitively the operation $\oplus$ is taking disjoint unions of the equivalence relations. As above, if say $\alpha = 2$, we denote $\bigoplus_{i<2} E_i$ by just $E_0 \otimes E_1$ and we identify $2^{\kappa\times\{i\}}$ with 2^κ.

5.27 Theorem. *Assume that $\lambda \in \operatorname{reg}\kappa$ and GC_λ-characterization holds for κ.*

1. *Suppose that $S_1, S_2 \subset S^\kappa_{\geqslant\lambda}$ and that $(S_2 \setminus S_1) \cap S^\kappa_\lambda$ is stationary. Then the following holds:*
   - *(a) $E_{S_1} \not\leqslant_B E_{S_2}$.*
   - *(b) If $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p, then there is no continuous function $C \to 2^\kappa$ such that $(\eta, \xi) \in E_{S_1} \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2}$.*
2. *Assume that $\kappa = \lambda^+ > \omega_1$, $\mu \in \operatorname{reg}(\kappa) \setminus \{\omega\}$ and $\square^\kappa_\mu$ holds. Let $S \subset S^\kappa_\omega$ be any stationary set and let $B^\mu_{\mathrm{nr}}(S)$ be the set defined by Theorem 5.24. Then the following holds:*
   - *(a) Suppose that $S_1, S_2 \subset S^\kappa_\mu$, $B \subset B^\mu_{\mathrm{nr}}(S)$ and let $S'_1 = S_1 \cup B$, $S'_2 = S_2 \cup B$. If $(S'_2 \setminus S'_1) \cap S^\kappa_\mu$ is stationary, then $E_{S'_1} \not\leqslant_B E_{S'_2}$.*
   - *(b) Let S_1, S_2, B, S'_1 and S'_2 be as above. If $(S'_2 \setminus S'_1) \cap S^\kappa_\mu$ is stationary, $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p, then there is no continuous function $C \to 2^\kappa$ such that $(\eta, \xi) \in E_{S'_1} \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S'_2}$.*
3. *Let $S_1, S_2, A_1, A_2 \subset S^\kappa_\omega$ be either such that $S_2 \setminus S_1$ and $A_2 \setminus S_1$ are stationary* or *such that $S_2 \setminus A_1$ and $A_2 \setminus A_1$ are stationary. Then the following holds:*
   - *(a) $E_{S_1} \otimes E_{A_1} \not\leqslant_B E_{S_2} \otimes E_{A_2}$.*
   - *(b) If $C \subset (2^\kappa)^2$ (we identify $2^{\kappa\times 2}$ with $(2^\kappa)^2$) is a set which is co-meager in some $N_r = \{\eta \in (2^\kappa)^2 \mid \eta \restriction \operatorname{dom} r = r\}$, $r \in (2^\alpha)^2$, $\alpha < \kappa$, then there is no continuous function f from $C \cap N_r$ to $(2^\kappa)^2$ such that $(\eta, \xi) \in (E_{S_1} \otimes E_{A_1}) \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2} \otimes E_{A_2}$.*
4. *Assume that $S_1, S_2, A_2 \subset \kappa$ are such that $A_2 \setminus S_1$ and $S_2 \setminus S_1$ are ω-stationary. Then*
   - *(a) $E_{S_1} \not\leqslant_B E_{S_2} \otimes E_{A_2}$.*

(b) *If $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p, then there is no continuous function $C \to (2^\kappa)^2$ such that $(\eta, \xi) \in E_{S_1} \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2} \otimes E_{A_2}$.*

5. *Assume that $S_1, A_1, S_2, A_2 \subset \kappa$ are such that $A_2 \setminus A_1$ is ω-stationary. Then*

   (a) *$E_{S_1} \otimes E_{A_1} \not\leqslant_B E_{S_2 \cup A_2}$.*

   (b) *If $p \in (2^{<\kappa})^2$ and $C \subset N_p$ is any co-meager subset of N_p, then there is no continuous function $C \to 2^\kappa$ such that $(\eta, \xi) \in (E_{S_1} \otimes E_{A_1}) \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2 \cup A_2}$.*

Proof. Item 1b of the theorem implies item 1a as well as all (b)-parts imply the corresponding (a)-parts, because if $f\colon 2^\kappa \to 2^\kappa$ is a Borel function, then it is continuous on the co-meager set $C(f)$ (see Definition 5.2). Let us start by proving 1b:

Assume that $S_2 \setminus S_1$ is λ-stationary, $p \in 2^{<\kappa}$, $C \subset N_p$ and assume that $f\colon C \to 2^\kappa$ is a continuous function as described in the Theorem. Let us derive a contradiction. Define a strategy for player **II** in the game $\mathrm{GC}_\lambda(\kappa \setminus (S_2 \setminus S_1))$ as follows.

Denote the i:th move of player **I** by α_i and the i:th move of player **II** by β_i. During the game, at the i:th move, $i < \lambda$, player **II** secretly defines functions $p_i^0, p_i^1, q_i^0, q_i^1 \in 2^{<\kappa}$ in such a way that for all i and all $j < i$ we have

(a) $\operatorname{dom} p_j^0 = \operatorname{dom} p_j^1 = \beta_j$ and $\alpha_j \leqslant \operatorname{dom} q_{j+1}^0 = \operatorname{dom} q_{j+1}^1 \leqslant \alpha_j$, and if j is a limit, then $\sup_{i<j} \alpha_i \leqslant \operatorname{dom} q_j^0 = \operatorname{dom} q_j^1 \leqslant \beta_j$,

(b) $p_j^0 \subset p_{j+1}^0$, $p_i^1 \subset p_{i+1}^1$, $q_i^0 \subset q_{i+1}^0$ and $q_i^1 \subset q_{i+1}^1$,

(c) $f[C \cap N_{p_i^0}] \subset N_{q_i^0}$ and $f[C \cap N_{p_i^1}] \subset N_{q_i^1}$.

Suppose it is i:th move and $i = \gamma + 2k$ for some $k < \omega$ and γ which is either 0 or a limit ordinal, and suppose that the players have picked the sequences $(\alpha_j)_{j \leqslant i}$ and $(\beta_j)_{j<i}$. Additionally **II** has secretly picked the sequences

$$(p_i^0)_{i<j}, (p_i^1)_{i<j}, (q_i^0)_{i<j}, (q_i^1)_{i<j}$$

which satisfy conditions (a)–(c). Assume first that i is a successor. If q_{i-1}^0 is not $E_{S_2}^*(\operatorname{dom} q_{i-1}^0)$-equivalent to q_{i-1}^1, then player **II** plays arbitrarily. Otherwise, to decide her next move, player **II** uses Lemma 5.3 (page 144) to find $\eta \in 2^{[\beta_{i-1},\kappa)}$ and $\xi = 1 - \eta$, such that $p_{i-1}^0{}^\frown\eta \in C$ and $p_{i-1}^1{}^\frown\xi \in C$. Then she finds $\beta_i' > \alpha_i$ such that $f(p_{i-1}^0{}^\frown\eta)(\delta) \neq f(p_{i-1}^1{}^\frown\xi)(\delta)$ for some $\delta \in [\alpha_i, \beta_i')$. This is possible since f is a reduction and $(q_{i-1}^0, q_{i-1}^1) \in E_{S_2}^*$. Then she picks $\beta_i > \beta_i'$ so that

$$f[C \cap N_{(p_{i-1}^0{}^\frown\eta)\restriction\beta_i}] \subset N_{f(p_{i-1}^0{}^\frown\eta)\restriction\beta_i'}$$

and

$$f[C \cap N_{(p_{i-1}^1{}^\frown\xi)\restriction\beta_i}] \subset N_{f(p_{i-1}^1{}^\frown\xi)\restriction\beta_i'}.$$

This choice is possible by the continuity of f. Then she (secretly) sets $p_i^0 = (p_{i-1}^0{}^\frown\eta) \restriction \beta_i$, $p_i^1 = (p_{i-1}^1{}^\frown\xi) \restriction \beta_i$, $q_i^0 = f(p_{i-1}^0{}^\frown\eta) \restriction \beta_i'$ and $q_i^1 = f(p_{i-1}^1{}^\frown\xi) \restriction \beta_i'$. Note that the new partial functions secretly picked by **II** satisfy conditions (a)–(c).

If i is a limit, then player **II** proceeds as above but instead of p_{i-1}^n she uses $\bigcup_{i'<i} p_{i'}^n$, $n \in \{0,1\}$, and instead of β_{i-1} she uses $\sup_{i'<i} \beta_{i'}$. If i is 0, then proceed in the same way assuming $p_{-1}^0 = p_{-1}^1 = q_{-1}^0 = q_{-1}^1 = \varnothing$ and $\alpha_{-1} = \beta_{-1} = 0$.

Suppose $i = \gamma + 2k + 1$ where γ is again a limit or zero and $k < \omega$. Then the moves go in the same way, except that she sets $\eta = \xi$ instead of $\eta = 1 - \xi$ and requires $f(p^0_{i-1}{}^\frown\eta)(\delta) = f(p^1_{i-1}{}^\frown\xi)(\delta)$ for some $\delta \in [\alpha_{i-1}, \beta'_i)$ instead of $f(p^0_{i-1}{}^\frown\eta)(\delta) \neq f(p^1_{i-1}{}^\frown\xi)(\delta)$ for some $\delta \in [\alpha_{i-1}, \beta'_i)$. Denote this strategy by σ.

Since $S_2 \setminus S_1$ is stationary and GC_λ-characterization holds for κ, player **I** is able play against this strategy such that $\sup_{i<\lambda} \alpha_i \in S_2 \setminus S_1$. Suppose they have played the game to the end, so that player **II** used σ, player **I** has won and they have picked the sequence $\langle \alpha_i, \beta_i \mid i < \lambda \rangle$. Let

$$\alpha_\lambda = \sup_{i<\lambda} \alpha_i = \sup_{i<\lambda} \beta_i = \sup_{i<\lambda} \operatorname{dom} p_i = \sup_{i<\lambda} \operatorname{dom} q_i$$

and

$$p^0_\lambda = \bigcup_{i<\lambda} p^0_i,\ p^1_\lambda = \bigcup_{i<\lambda} p^1_i,\ q^0_\lambda = \bigcup_{i<\lambda} q^0_i \text{ and } q^1_\lambda = \bigcup_{i<\lambda} q^1_i.$$

By continuity, p^0_λ, p^1_λ, q^0_λ and q^1_λ satisfy condition (c) above and $\operatorname{dom} p^0_\lambda = \operatorname{dom} p^1_\lambda = \operatorname{dom} q^0_\lambda = \operatorname{dom} q^1_\lambda = \sup_{i<\lambda} \alpha_i = \sup_{i<\lambda} \beta_i$, so α_λ is well defined.

On one hand q^0_λ and q^1_λ cannot be extended in an E_{S_2}-equivalent way, since either they cofinally get same and different values below $\alpha_\lambda \in S_2$, or they are not $E^*_{S_2}(\gamma)$-equivalent already for some $\gamma < \alpha_\lambda$. On the other hand p^0_λ and p^1_λ can be extended in an E_{S_1}-equivalent way, since α_λ is not in S_1 and for all $\gamma < \lambda$, $\sup_{i<\gamma} \alpha_\gamma$ is not μ-cofinal for any $\mu \geqslant \lambda$, so cannot be in S_1 either $(*)$.

Let $\eta, \xi \in 2^\kappa$ be extensions of p^0_λ and p^1_λ respectively such that $(\eta, \xi) \in E_{S_1} \cap C^2$. Now $f(\eta)$ and $f(\xi)$ cannot be E_{S_2}-equivalent, since by condition (c), they must extend q^0_λ and q^1_λ respectively.

Now let us prove 2b which implies 2a. Let $\langle C^\mu_\alpha \mid \alpha \in S^\kappa_{\leqslant\mu} \rangle$ be the $\square^\kappa_\mu$-sequence and denote by t^μ the function $\alpha \mapsto C^\mu_\alpha$.

Let player **II** define her strategy in the game $\mathrm{GC}(\kappa \setminus (S'_2 \setminus S'_1))$ exactly as in the proof of 1b. Note that $S'_2 \setminus S'_1 = S_2 \setminus S_1$ since $\mu > \omega$. Denote this strategy by σ. We know that, as above, Player **I** is able to beat σ. However, now it is not enough, because in order to be able to extend p^0_μ and p^1_μ in an $E_{S'_1}$-equivalent way, he needs to ensure that

$$S'_1 \cap \lim_\omega(\{\alpha_i \mid i < \mu\}) = \varnothing \tag{$**$}$$

where $\lim_\omega X$ is the set of ω-limits of elements of X, i.e. we cannot rely on the sentence followed by $(*)$ above. On the other hand $(**)$ is sufficient, because $S'_1 \subset S^\kappa_\mu \cup S^\kappa_\omega$.

Let us show that it is possible for player **I** to play against σ as required.

Let $\nu > \kappa$ be a sufficiently large cardinal and let M be an elementary submodel of $\langle H_\nu, \sigma, \kappa, t^\mu \rangle$ such that $|M| < \kappa$ and $\alpha = \kappa \cap M$ is an ordinal in $S'_2 \setminus S'_1$.

In the game, suppose that the sequence $d = \langle \alpha_j, \beta_j \mid j < i \rangle$ has been played before move i and suppose that this sequence is in M. Player **I** will now pick α_i to be the smallest element in C^μ_α which is above $\sup_{j<i} \beta_j$. Since $C^\mu_\alpha \cap \beta = C^\mu_\beta$ for any $\beta \in \lim C^\mu_\alpha$ and $C^\mu_\beta \in M$, this element is definable in M from the sequence d and t^μ. This guarantees that the sequence obtained on the following move is also in M. At limits the sequence is in M, because it is definable from t^μ and σ. Since $\mathrm{OTP}(C^\mu_\alpha) = \mu$, the game ends at α and player **I** wins. Also the requirement $(**)$ is satisfied because he picked elements only from C^μ_α and so $\lim_\omega\{\alpha_i \mid i < \mu\} \subset \lim_\omega(C^\mu_\alpha) \subset \alpha \setminus B$ which gives the result.

Next let us prove 3b which again implies 3a. The proofs of 4 and 5 are very similar to that of 3 and are left to the reader.

So, let S_1, A_1, S_2, A_2, C and r be as in the statement of 3 and suppose that there is a counter example f. Assume that $S_2 \setminus S_1$ and $A_2 \setminus S_1$ are stationary, the other case being symmetric. Let us define the property P:

P: There exist $p, p' \in (2^\alpha)^2$, $p = (p_1, p_2)$ and $p' = (p'_1, p'_2)$, such that

(a) $r \subset p \cap p'$,

(b) $p_2 = p'_2$, $(p_1, p'_1) \in E^*_{S_1}(\alpha + 1)$ (see Definition 5.19, page 147),

(c) for all $\eta \in C \cap N_p$ and $\eta' \in C \cap N_{p'}$, $\eta = (\eta_1, \eta_2)$, $\eta' = (\eta'_1, \eta'_2)$, if $\eta_2 = \eta'_2$ and $(\eta_1, \eta'_1) \in E^*_{S_1}$, then $f(\eta)_1 \triangle f(\eta')_1 \subset \operatorname{dom} p_1$ where $f(\eta) = (f(\eta)_1, f(\eta)_2)$.

We will show that both P and $\neg P$ lead to a contradiction. Assume first $\neg P$. Now the argument is similar to the proof of 1b. Player **II** defines her strategy in the same way but this time she chooses the elements p_i^n and q_i^n from $(2^\alpha)^2$ instead of 2^α so that $p_i^n = (p_{i,1}^n, p_{i,2}^n)$, $q_i^n = (q_{i,1}^n, q_{i,2}^n)$ and for all $i < \lambda$, $p_{i,2}^0 = p_{i,2}^1$. In building the strategy she looks only at $q_{i,1}^n$ and ignores $q_{i,2}^n$. In other words she pretends that the game is for E_{S_1} and E_{S_2} in the proof of 1. At the even moves she extends $p_{i,1}^0$ and $p_{i,1}^1$ by η and η' which witness the failure of item (c) (but not of (a) and (b)) of property P for p_i^0 and p_i^1. Then there is $\alpha \in f(\eta)_1 \triangle f(\eta')_1$, $\alpha > \operatorname{dom} p_{i,1}^0$. And then she chooses $q_{i,1}^0$ and $q_{i,1}^1$ to be initial segments of $f(\eta)_1$ and $f(\eta')_1$ respectively.

At the odd moves she just extends $p_{i,1}^0$ and $p_{i,1}^1$ in an E_{S_1}-equivalent way, so that she finds an $\alpha > \operatorname{dom} p_{i,1}^0$, $q_{i,1}^0$ and $q_{i,1}^1$ such that $q_{i,1}^0(\alpha) = q_{i,1}^1(\alpha)$ and $f[N_{p_i^0} \cap C] \subset N_{q_i^0}$.

As in the proof of 1, **I** responses by playing towards an ordinal in $S_2 \setminus S_1$. During the game they either hit a point at which $q_{i,2}^0$ and $q_{i,2}^1$ cannot be extended to be E_{A_2}-equivalent or else they play the game to the end whence $q_{\lambda,1}^0$ and $q_{\lambda,1}^1$ cannot be extended in a E_{S_2}-equivalent way but p_λ^0 and p_λ^1 can be extended to $E_{S_1} \otimes E_{A_1}$-equivalent way.

Assume that P holds. Fix p and p' which witness that. Now player **II** builds her strategy as if they were playing between E_{S_1} and E_{A_2}. This time she concentrates on $q_{i,2}^0$ and $q_{i,2}^1$ instead of $q_{i,1}^0$ and $q_{i,1}^1$. At the even moves she extends $p_{i,1}^0$ and $p_{i,1}^1$ by η and $\bar{\eta}$ respectively for some η. Also, as above, $p_{i,2}^0$ and $p_{i,2}^1$ are extended in the same way. By item (c) $f(\eta)_1 \triangle f(\eta')_1$ is bounded by $\operatorname{dom} p_{i,1}^0$, but $f(\eta)$ and $f(\eta')$ can't be $E_{S_2} \otimes E_{A_2}$-equivalent, because f is assumed to be a reduction. Hence there must exist $\alpha > \operatorname{dom} p_{i,1}^0$, $q_{i,2}^0$ and $q_{i,2}^0$ such that $q_{i,2}^0(\alpha) \neq q_{i,2}^1(\alpha)$. The rest of the argument goes similarly as above. □

5.28 Corollary. *If* GC_λ*-characterization holds for* κ *and* $S \subset \kappa$ *is* λ*-stationary, then* $E_0 \not\leqslant E_S$*. In particular, if* S *is* ω*-stationary, then* $E_0 \not\leqslant E_S$*.*

Proof. Follows from Theorem 5.27.1a by taking $S_1 = \varnothing$, since $E_\varnothing = E_0$ and GC_ω-characterization holds for κ. □

5.29 Corollary. *There is an antichain*[2] *of Borel equivalence relations on* 2^κ *of length* 2^κ*.*

[2]By an antichain I refer here to a family of pairwise incomparable elements unlike e.g. in forcing context.

Proof. Take disjoint ω-stationary sets S_i, $i < \kappa$. Let $f\colon \kappa \times 2 \to \kappa$ be a bijection. For each $\eta \in 2^\kappa$ let $A_\eta = \{(\alpha, n) \in \kappa \times 2 \mid (n = 0 \wedge \eta(\alpha) = 1) \vee (n = 1 \wedge \eta(\alpha) = 0)\}$. For each $\eta \neq \xi$ clearly $A_\eta \setminus A_\xi \neq \varnothing \neq A_\xi \setminus A_\eta$. Let

$$S_\eta = \bigcup_{i \in f[A_\eta]} S_i.$$

Now $\{E_{S_\eta} \mid \eta \in 2^\kappa\}$ is an antichain by Theorem 5.27.1b. □

Let us show that all these relations are below E_0. It is already shown that they are not above it (Corollary 5.28), provided GC_λ-characterization holds for κ. Again, similar ideas will be used in the proof of Theorems 5.11 and 5.12.

5.30 Theorem. *For all S, $E_S \leqslant_B E_0$.*

Proof. Let us show that E_S is reducible to E_0 on κ^κ which is in turn bireducible with E_0 on 2^κ (see [7]). Let us define an equivalence relation $\sim$ on $2^{<\kappa}$ as on page 148, such that $p \sim q$ if and only if $\operatorname{dom} p = \operatorname{dom} q$ and $p \triangle q$ is eventually constant, i.e. for some $\alpha < \operatorname{dom} p$, $(p \triangle q)(\gamma)$ is the same for all $\gamma \in [\alpha, \operatorname{dom} p)$. Let $s\colon 2^{<\kappa} \to \kappa$ be a map such that $p \sim q \iff s(p) = s(q)$. Let $\{A_i \mid i \in S\}$ be a partition of $\lim \kappa$ into disjoint unbounded sets. Suppose $\eta \in 2^\kappa$ and define $f(\eta) = \xi \in \kappa^\kappa$ as follows.

- If α is a successor, $\alpha = \beta + 1$, then $\xi(\alpha) = \eta(\beta)$.
- If α is a limit, then $\alpha \in A_i$ for some $i \in S$. Let $\xi(\alpha) = s(\eta \restriction i)$

Let us show that f is the desired reduction from E_S to E_0. Assume that η and ξ are E_S-equivalent. If α is a limit and $\alpha \in A_i$, then, since η and ξ are E_S-equivalent, we have $\eta \restriction i \sim \xi \restriction i$, so $s(\eta \restriction i) = s(\xi \restriction i)$ and so $f(\eta)(\alpha) = f(\xi)(\alpha)$. There is β such that $\eta(\gamma) = \xi(\gamma)$ for all $\gamma > \beta$. This implies that for all successors $\gamma > \beta$ we also have $f(\eta)(\gamma) = f(\xi)(\gamma)$. Hence $f(\eta)$ and $f(\xi)$ are E_0-equivalent. Assume now that η and ξ are not E_S-equivalent. Then there are two cases:

1. $\eta \triangle \xi$ is unbounded. Now $f(\eta)(\beta+1) = \eta(\beta)$ and $f(\xi)(\beta+1) = \xi(\beta)$ for all β, so we have

   $$\{\beta \mid \eta(\beta) \neq \xi(\beta)\} = \{\beta \mid f(\eta)(\beta+1) \neq \xi(\beta+1)\}.$$

   If the former is unbounded, then so is the latter.

2. For some $i \in S$, $\eta \restriction i \not\sim \xi \restriction i$. This implies that $f(\eta)(\alpha) \neq f(\xi)(\alpha)$ for all $\alpha \in A_i$. and we get that $\{\beta \mid f(\eta)(\beta) \neq \xi(\beta)\}$ is again unbounded.

It is easy to check that f is continuous. □

5.4.3 Proofs of the Main Theorems

Proof of Theorem 5.11. The subject of the proof is that for a regular $\lambda < \kappa$, if GC_λ-characterization holds for κ, then the order $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}(\lambda)} \rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$ strictly below E_0 and above id_{2^κ}.

Let $h\colon \omega \times \kappa \to \kappa$ be a bijection. Let $\tilde{h}\colon 2^{\omega \times \kappa} \to 2^\kappa$ be defined by $\tilde{h}(\eta)(\alpha) = \eta(h^{-1}(\alpha))$. We define the topology on $2^{\omega \times \kappa}$ to be generated by the sets $\{\tilde{h}^{-1} V \mid V \text{ is open in } 2^\kappa\}$. Then

$\tilde{h}$ is a homeomorphism between $2^{\omega\times\kappa}$ and 2^κ. If $g\colon \kappa\times\kappa\to\kappa$ is a bijection, we similarly get a topology onto $2^{\kappa\times\kappa}$ and a homeomorphism $\tilde{g}$ from $2^{\kappa\times\kappa}$ onto 2^κ. By combining these two we get a homeomorphism between $2^{\omega\times\kappa}\times 2^\kappa$ and 2^κ, and so without loss of generality we can consider equivalence relations on these spaces.

For a given equivalence relation E on 2^κ, let $\overline{E}$ be the equivalence relation on $2^{\omega\times\kappa}\times 2^\kappa$ defined by

$$((\eta,\xi),(\eta',\xi'))\in\overline{E} \iff \eta=\eta'\wedge(\xi,\xi')\in E.$$

Essentially $\overline{E}$ is the same as $\mathrm{id}\otimes E$, since $2^{\omega\times\kappa}\approx 2^\kappa$.

5.31 Remark. Corollary 5.22, Theorem 5.27 and Corollary 5.28 hold even if E_S is replaced everywhere by $\overline{E_S}$ for all $S\subset\kappa$.

Proof. Let us show this for Theorem 5.27.1. The proof goes exactly as the proof of Theorem 5.27.1, but player **I** now picks the functions p_k^n from $\bigcup_{\alpha<\kappa}2^{\omega\times\alpha}\times 2^\alpha$ instead of $2^{<\kappa}$, $p_k^n=(p_{k,1}^n,p_{k,2}^n)$, and requires that at each move $p_{k,1}^0=p_{k,1}^1$. Otherwise the argument proceeds in the same manner. Similarly for 5.27.2, 5.27.3, 5.27.4 and 5.27.5.

Modify the proof of the first part of Theorem 5.21 in a similar way to obtain the result for Corollary 5.22. Corollary 5.28 follows from the modified version of Theorem 5.27. □

For $S\subset\kappa$ let

$$G(S)=\overline{E_{S^\kappa_\lambda\setminus S}}.$$

Let us show that $G\colon\mathcal{P}(\kappa)\to\mathcal{E}^B_\kappa$ is the desired embedding. Without loss of generality let us assume that G is restricted to $\mathcal{P}(S^\kappa_\lambda)$, whence stationary is the same as λ-stationary and non-stationary is the same as not λ-stationary. For arbitrary $S_1,S_2\subset S^\kappa_\lambda$ we have to show:

1. If $S_2\setminus S_1$ is stationary, then $\overline{E_{S_1}}\not\leqslant_B\overline{E_{S_2}}$
2. If $S_2\setminus S_1$ is non-stationary, then $\overline{E_{S_1}}\leqslant_B\overline{E_{S_2}}$
3. $\mathrm{id}_{2^\kappa}\lneqq_B\overline{E_{S_1}}\lneqq_B E_0$.

If $\eta\in 2^{\omega\times\kappa}$, denote $\eta_i(\alpha)=\eta(i,\alpha)$ and $(\eta_i)_{i<\omega}=\eta$.

Claim 1. If $S_2\setminus S_1$ is stationary, then $\overline{E_{S_1}}\not\leqslant_B\overline{E_{S_2}}$. Also $E_0\not\leqslant\overline{E_S}$.

Proof. Follows from Theorem 5.27.1a and Remark 5.31. □

Claim 2. If $S_2\setminus S_1$ is non-stationary, then $\overline{E_{S_1}}\leqslant_B\overline{E_{S_2}}$.

Proof. Let us split this into two parts according to the stationarity of S_2. Assume first that S_2 is non-stationary. Let C be a cub set outside S_2. Let $f\colon 2^\kappa\to 2^{\omega\times\kappa}\times 2^\kappa$ be the function defined as follows. For $\eta\in 2^\kappa$ let $f(\eta)=\langle(\eta_i)_{i<\omega},\xi\rangle$ be such that $\eta_i(\alpha)=0$ for all $\alpha<\kappa$ and $i<\omega$ and $\xi(\alpha)=0$ for all $\alpha\notin C$. If $\alpha\in C$, then let $\xi(\alpha)=\eta(\mathrm{OTP}(\alpha\cap C))$. This is easily verified to be a reduction from E_0 to $\overline{E_{S_2}}$. By the following Claim 3, $\overline{E_{S_1}}\leqslant_B E_0$, so we are done.

Assume now that S_2 is stationary. Note that then S_1 is also stationary. Let C be a cub set such that $S_2 \cap C \subset S_1$. Assume that $\langle (\eta_i)_{i<\omega}, \xi\rangle \in 2^{\omega\times\kappa} \times 2^\kappa$ and let us define

$$f(\langle (\eta_i)_{i<\omega}, \xi\rangle) = \langle (\eta'_i)_{i<\omega}, \xi'\rangle \in 2^{\omega\times\kappa} \times 2^\kappa$$

as follows. For $i \geqslant 0$ let

$$\eta'_{i+1} = \eta_i.$$

For all $\alpha < \kappa$, let $\xi'(\alpha) = \xi(\min(C \setminus \alpha))$. Then let s be the function as defined in the proof of Theorem 5.21 (on page 148) and for all $\alpha < \kappa$ let $\beta(\alpha)$ be the α:th element of $S_1 \setminus S_2$. For all $\alpha < \kappa$, let

$$\eta'_0(\alpha) = s(\xi \restriction \beta(\alpha)).$$

Let us show that this defines a continuous reduction.

Suppose $\langle (\eta^0_i)_{i<\omega}, \xi^0\rangle$ and $\langle (\eta^1_i)_{i<\omega}, \xi^1\rangle$ are $\overline{E_{S_1}}$-equivalent. Denote their images under f by $\langle (\rho^0_i)_{i<\omega}, \zeta^0\rangle$ and $\langle (\rho^1_i)_{i<\omega}, \zeta^1\rangle$ respectively. Since $\eta^0_i = \eta^1_i$ for all $i < \omega$, we have $\rho^0_i = \rho^1_i$ for all $0 < i < \omega$. Since for all $\alpha \in S_1$ we have that $\xi^0 \restriction \alpha$ and $\xi^1 \restriction \alpha$ are $\sim$-equivalent (as in the definition of s), we have that $\rho^0_0(\beta) = \rho^1_0(\beta)$ for all $\beta < \kappa$.

Suppose now that $\alpha \in S_2$. The aim is to show that $\zeta^0 \restriction \alpha \sim \zeta^1 \restriction \alpha$. If $\alpha \notin C$, then there is $\beta < \alpha$ such that $C \cap (\beta, \alpha) = \varnothing$, because C is closed. This implies that for all $\beta < \gamma < \gamma' < \alpha$, $\min(C \setminus \gamma') = \min(C \setminus \gamma)$, so by the definition of f, $\zeta^0(\gamma) = \zeta^0(\gamma')$ and $\zeta^1(\gamma) = \zeta^1(\gamma')$. Now by fixing γ_0 between β and α we deduce that $\zeta^0 \restriction (\beta, \alpha)$ is constant and $\zeta^1 \restriction (\beta, \alpha)$ is constant, since for all $\gamma < \alpha$ we have $\zeta^0(\gamma) = \zeta^0(\gamma_0)$ and $\zeta^1(\gamma) = \zeta^1(\gamma_0) = \zeta^1(\gamma)$. Hence $(\zeta^0 \triangle \zeta^1) \restriction (\beta, \alpha)$ is constant which by the definition of $\sim$ implies that $\zeta^0 \restriction \alpha \sim \zeta^1 \restriction \alpha$.

If $\alpha \in C$, then, since α is also in S_2, we have by the definition of C that $\alpha \in S_1$. Thus, there is $\beta < \alpha$ such that $(\xi^0 \triangle \xi^1) \restriction (\beta, \alpha)$ is constant which implies that for some $k \in \{0, 1\}$ we have $(\zeta^0 \triangle \zeta^1)(\gamma) = k$ for all $\gamma \in (\beta, \alpha) \cap C$. But if $\gamma \in (\beta, \alpha) \setminus C$, then, again by the definition of f, we have $(\zeta^0 \triangle \zeta^1)(\gamma) = (\zeta^0 \triangle \zeta^1)(\gamma')$ for some $\gamma \in (\beta, \alpha) \cap C$, so $(\zeta^0 \triangle \zeta^1)(\gamma)$ also equals to k.

This shows that ζ^0 and ζ^1 are $E^*_{S_2}$-equivalent. It remains to show that they are E_0-equivalent. But since ξ^0 and ξ^1 are E_0-equivalent, the number $k \in \{0, 1\}$ referred above equals 0 for all α large enough and we are done.

Next let us show that if $\langle (\eta^0_i)_{i<\omega}, \xi^0\rangle$ and $\langle (\eta^1_i)_{i<\omega}, \xi^1\rangle$ are not $\overline{E_{S_1}}$-equivalent, then $\langle (\rho^0_i)_{i<\omega}, \zeta^0\rangle$ and $\langle (\rho^1_i)_{i<\omega}, \zeta^1\rangle$ are not $\overline{E_{S_2}}$-equivalent. This is just reversing implications of the above argument. If $\eta^0_i \neq \eta^1_i$ for some $i < \omega$, then $\rho^0_{i+1} \neq \rho^1_{i+1}$, so we can assume that $(\xi^0, \xi^1) \notin E_{S_1}$. If ξ^0 and ξ^1 are not $E^*_{S_1}$-equivalent, then $\rho^0(\alpha) \neq \rho^1(\alpha)$ for some $\alpha < \kappa$.

The remaining case is that ξ^0 and ξ^1 are $E^*_{S_1}$-equivalent but not E_0-equivalent. But then in fact $\xi^0 \triangle \xi^1$ is eventually equal to 1, since otherwise the sets

$$C_1 = \{\alpha \mid \{\beta < \alpha \mid (\xi^0 \triangle \xi^1)(\beta) = 1\} \text{ is unbounded in } \alpha\}$$

and

$$C_2 = \{\alpha \mid \{\beta < \alpha \mid (\xi^0 \triangle \xi^1)(\beta) = 0\} \text{ is unbounded in } \alpha\}$$

are both cub and by the stationarity of S_1, there exists a point $\alpha \in C_1 \cap C_2 \cap S_1$ which contradicts the fact that ξ_0 and ξ_1 are $E^*_{S_1}$-equivalent. So $\xi^0 \triangle \xi^1$ is eventually equal to 1 and this finally implies that also ζ^0 and ζ^1 cannot be E_0-equivalent. □

Claim 3. Let $S \subset S^\kappa_\lambda$. Then $\mathrm{id} \lneq_B \overline{E_S} <_B E_0$. If S is stationary, then also $E_0 \not\leqslant_B \overline{E_S}$.

Proof. If $\eta \in 2^\kappa$, let $\eta_0 = \eta$ and $\eta_i(\alpha) = \xi(\alpha) = 0$ for all $\alpha < \kappa$. Then $\eta \mapsto \langle (\eta_i)_{i<\omega}, \xi\rangle$ defines a reduction from id to $\overline{E_S}$. On the other hand $\overline{E_S}$ is not reducible to id by Remark 5.31.

Let $u\colon 2^{\omega\times\kappa} \to 2^\kappa$ be a reduction from $\mathrm{id}_{2^{\omega\times\kappa}}$ to E_0. Let $v\colon 2^\kappa \to 2^\kappa$ be a reduction from E_S to E_0 which exists by 5.30. Let $\{A, B\}$ be a partition of κ into two disjoint unbounded subsets. Let $(\eta, \eta') \in 2^{\omega\times\kappa} \times 2^\kappa$ and let us define $\xi = f(\eta, \eta') \in 2^\kappa$. If $\alpha \in A$, then let $\xi(\alpha) = u(\eta)(\mathrm{OTP}(\alpha \cap A))$. If $\alpha \in B$, then let $\xi(\alpha) = v(\eta')(\mathrm{OTP}(\alpha \cap B))$. (See page 143 for notation.)

Now if $((\eta_0, \eta_0'), (\eta_1, \eta_1')) \in (2^{\omega\times\kappa} \times 2^\kappa)^2$ are $\overline{E_S}$-equivalent, then both $u(\eta_0) \triangle u(\eta_1)$ and $v(\eta_0') \triangle v(\eta_1')$ are eventually equal to zero which clearly implies that $f(\eta_0, \eta_0') \triangle f(\eta_1, \eta_1')$ is eventually zero, and so $f(\eta_0, \eta_0')$ and $f(\eta_1, \eta_1')$ are E_0-equivalent. Similarly, if (η_0, η_0') and (η_1, η_1') are not $\overline{E_S}$-equivalent, then either $u(\eta_0) \triangle u(\eta_1)$ or $v(\eta_0') \triangle v(\eta_1')$ is not eventually zero, and so $f(\eta_0, \eta_0')$ and $f(\eta_1, \eta_1')$ are not E_0-equivalent.

If S is stationary, then $E_0 \not\leqslant_B \overline{E_S}$ by Corollary 5.28 and Remark 5.31. □

□

Proof of Theorem 5.12. Let us review the statement of the Theorem: assuming $\kappa = \omega_1$, or $\kappa = \lambda^+$ and $\square_\lambda$, the partial order $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}}\rangle$ can be embedded into $\langle \mathcal{E}^B_\kappa, \leqslant_B\rangle$.

If $\kappa = \omega_1$, then this is just the second part (a special case) of Corollary 5.17 on page 146 and follows from Theorem 5.11.

Recall Definition 5.26 on page 149. Let us see that if $\alpha < \kappa$, then $\bigcup_{i<\alpha} 2^{\kappa\times\{i\}}$ is homeomorphic to 2^κ and so the domains of the forthcoming equivalence relations can be thought without loss of generality to be 2^κ. So fix $\alpha < \kappa$. For all $\beta + 1 < \alpha$ let $\zeta_\beta\colon \beta + 1 \to 2$ be the function $\zeta_\beta(\gamma) = 0$ for all $\gamma < \beta$ and $\zeta_\beta(\beta) = 1$ and let $\zeta_\alpha\colon \alpha \to 2$ be the constant function with value 0. Clearly $(\zeta_\beta)_{\beta\leqslant\alpha}$ is a maximal antichain. By rearranging the indexation we can assume that $(\zeta_\beta)_{\beta<\alpha}$ is a maximal antichain. If $\eta \in 2^{\kappa\times\{i\}}$, $i < \alpha$, let $\xi = \eta + i$ be the function with $\mathrm{dom}\,\xi = [i + 1, \kappa)$ and $\xi(\gamma) = \eta(\mathrm{OTP}(\gamma \setminus i))$ and let

$$f(\eta) = \zeta_i{}^\frown(\eta + i).$$

Then f is a homeomorphism $\bigcup_{i<\alpha} 2^{\kappa\times\{i\}} \to 2^\kappa$.

Assume $S \subset \kappa$ and let us construct the equivalence relation H_S. Denote for short $r = \mathrm{reg}\,\kappa$, the set of regular cardinals below κ. Since κ is not inaccessible, $|r| < \kappa$. Let $\{K_\mu \subset S^\kappa_\omega \mid \mu \in r\}$ be a partition of S^κ_ω into disjoint stationary sets. For each $\mu \in r \setminus \{\omega\}$, let $A_\mu = B^\mu_{\mathrm{nr}}(K_\mu)$ be the set given by Theorem 5.24. Additionally let $\{A^0_\omega, A^1_\omega, A^2_\omega, A^3_\omega\}$ be a partition of K_ω into disjoint stationary sets.

Let

$$\begin{aligned} H_S \quad = \quad & (\mathrm{id}_{2^\kappa} \otimes E_{A^2_\omega \cup ((S\cap S^\kappa_\omega)\setminus A^0_\omega)} \otimes E_{A^0_\omega}) \\ & \oplus (\mathrm{id}_{2^\kappa} \otimes E_{A^3_\omega \cup ((S\cap S^\kappa_\omega)\setminus A^1_\omega)} \otimes E_{A^1_\omega}) \\ & \oplus \bigoplus_{\substack{\mu\in r\\ \mu>\omega}} (\mathrm{id}_{2^\kappa} \otimes E_{(S\cap S^\kappa_\mu)\cup A_\mu}). \end{aligned}$$

This might require a bit of explanation. H_S is a disjoint union of the equivalence relations listed in the equation. The final part of the equation lists all the relations obtained by splitting the set S into pieces of fixed uncountable cofinality and coupling them with the non-reflecting

ω-stationary sets A_μ. The operation $E \mapsto \mathrm{id}_{2^\kappa} \otimes E$ is the same as the operation $E \mapsto \overline{E}$ in the proof of Theorem 5.11 above after the identification $2^{\omega\times\kappa} \approx 2^\kappa$. The first two lines of the equation deal with the ω-cofinal part of S. It is trickier, because the "coding sets" A_μ also consist of ω-cofinal ordinals. The way we have built up the relations makes it possible to use Theorem 5.27 to prove that $S \mapsto H_{\kappa\setminus S}$ is the desired embedding.

In order to make the sequel a bit more readable, let us denote

$$\begin{aligned}\mathcal{B}^0_\omega(S) &= (\mathrm{id}_{2^\kappa} \otimes E_{A^2_\omega \cup ((S\cap S^\kappa_\omega)\setminus A^0_\omega)} \otimes E_{A^0_\omega}),\\ \mathcal{B}^1_\omega(S) &= (\mathrm{id}_{2^\kappa} \otimes E_{A^3_\omega \cup ((S\cap S^\kappa_\omega)\setminus A^1_\omega)} \otimes E_{A^1_\omega}),\\ \mathcal{B}_\mu(S) &= (\mathrm{id}_{2^\kappa} \otimes E_{(S\cap S^\kappa_\mu)\cup A_\mu}),\end{aligned}$$

for $\mu \in r \setminus \{\omega\}$. With this notation we have

$$H_S = \mathcal{B}^0_\omega(S) \oplus \mathcal{B}^1_\omega(S) \oplus \bigoplus_{\substack{\mu\in r\\ \mu>\omega}} \mathcal{B}_\mu(S).$$

Let us show that $S \mapsto H_{\kappa\setminus S}$ is an embedding from $\langle \mathcal{P}(\kappa), \subset_{\mathrm{NS}}\rangle$ into $\langle \mathcal{E}^B_\kappa, \leqslant_B\rangle$. Suppose $S_2 \setminus S_1$ is non-stationary. Then for each $\mu \in r \setminus \{\omega\}$ the set

$$\big((S^\kappa_\mu \cap S_2) \cup A_\mu\big) \setminus \big((S^\kappa_\mu \cap S_1) \cup A_\mu\big)$$

is non-stationary as well as are the sets

$$\big(A^2_\omega \cup ((S_2 \cap S^\kappa_\omega) \setminus A^0_\omega)\big) \setminus \big(A^2_\omega \cup ((S_1 \cap S^\kappa_\omega) \setminus A^0_\omega)\big)$$

and

$$\big(A^3_\omega \cup ((S_2 \cap S^\kappa_\omega) \setminus A^1_\omega)\big) \setminus \big(A^3_\omega \cup ((S_1 \cap S^\kappa_\omega) \setminus A^1_\omega)\big)$$

so by Claim 2 of the proof of Theorem 5.11 (page 154) we have for all $\mu \in r \setminus \{\omega\}$ that

$$(\mathrm{id}_{2^\kappa} \otimes E_{(S_1\cap S^\kappa_\mu)\cup A_\mu}) \leqslant_B (\mathrm{id}_{2^\kappa} \otimes E_{(S_2\cap S^\kappa_\mu)\cup A_\mu}),$$

$$(\mathrm{id}_{2^\kappa} \otimes E_{A^2_\omega \cup ((S_1\cap S^\kappa_\omega)\setminus A^0_\omega)}) \leqslant_B (\mathrm{id}_{2^\kappa} \otimes E_{A^2_\omega \cup ((S_2\cap S^\kappa_\omega)\setminus A^0_\omega)}),$$

and

$$(\mathrm{id}_{2^\kappa} \otimes E_{A^3_\omega \cup ((S_1\cap S^\kappa_\omega)\setminus A^1_\omega)}) \leqslant_B (\mathrm{id}_{2^\kappa} \otimes E_{A^3_\omega \cup ((S_2\cap S^\kappa_\omega)\setminus A^1_\omega)}).$$

Of course this implies that for all $\mu \in r \setminus \{\omega\}$

$$(\mathrm{id}_{2^\kappa} \otimes E_{A^2_\omega \cup ((S_1\cap S^\kappa_\omega)\setminus A^0_\omega)} \otimes E_{A^0_\omega}) \leqslant_B (\mathrm{id}_{2^\kappa} \otimes E_{A^2_\omega \cup ((S_2\cap S^\kappa_\omega)\setminus A^0_\omega)} \otimes E_{A^0_\omega})$$

and that

$$(\mathrm{id}_{2^\kappa} \otimes E_{A^3_\omega \cup ((S_1\cap S^\kappa_\omega)\setminus A^1_\omega)} \otimes E_{A^1_\omega}) \leqslant_B (\mathrm{id}_{2^\kappa} \otimes E_{A^3_\omega \cup ((S_2\cap S^\kappa_\omega)\setminus A^1_\omega)} \otimes E_{A^1_\omega})$$

which precisely means that $\mathcal{B}^0_\omega(S_1) \leqslant_B \mathcal{B}^0_\omega(S_2)$, $\mathcal{B}^1_\omega(S_1) \leqslant_B \mathcal{B}^1_\omega(S_2)$ and $\mathcal{B}_\mu(S_1) \leqslant_B \mathcal{B}_\mu(S_2)$ for all $\mu \in r \setminus \{\omega\}$. Combining these reductions we get a reduction from H_{S_1} to H_{S_2}.

Assume that $S_2 \setminus S_1$ is stationary. We want to show that $H_{S_1} \not\leqslant_B H_{S_2}$. H_{S_1} is a disjoint union the equivalence relations $\mathcal{B}^0_\omega(S_1)$, $\mathcal{B}^1_\omega(S_1)$ and $\mathcal{B}_\mu(S_1)$ for $\mu \in r \setminus \{\omega\}$. Let us call these equivalence relations *the building blocks of* H_{S_1} and similarly for H_{S_2}.

Each building block of H_{S_1} can be easily reduced to H_{S_1} via inclusion, so it is sufficient to show that there is one block that cannot be reduced to H_{S_2}. We will show that if μ_1 is the least cardinal such that $S^\kappa_{\mu_1} \cap (S_2 \setminus S_1)$ is stationary, then

- that building block is $\mathcal{B}_{\mu_1}(S_1)$, if $\mu_1 > \omega$.
- that building block is either $\mathcal{B}^0_\omega(S_1)$ or $\mathcal{B}^1_\omega(S_1)$, if $\mu_1 = \omega$.

Such a cardinal μ_1 exists because κ is not inaccessible and $|r| < \kappa$.

Suppose that f is a reduction from a building block of H_{S_1}, call it $\mathcal{B}$, to H_{S_2}. H_{S_2} is a disjoint union of less than κ building blocks whose domains' inverse images decompose $\operatorname{dom} f$ into less than κ disjoint pieces and one of them, say C, is not meager. By the Property of Baire one can find a basic open set U such that $C \cap U$ is co-meager in U. Let $C(f)$ be a co-meager set in which f is continuous. Now $f \restriction (U \cap C \cap C(f))$ is a continuous reduction from $\mathcal{B}$ restricted to $(U \cap C \cap C(f))^2$ to a building block of H_{S_2}. Thus it is sufficient to show that this correctly chosen building block of H_{S_1} is not reducible to any of the building blocks of H_{S_2} on any such $U \cap C \cap C(f)$. This will follow from Theorem 5.27 and Remark 5.31 once we go through all the possible cases. So the following Lemma concludes the proof.

5.32 Lemma. *Assume that $\mu_1 \in r$ is the least cardinal such that $(S_2 \setminus S_1) \cap S^\kappa_{\mu_1}$ is stationary. If $\mu_1 > \omega$, then*

(i) for all $\mu_2 > \omega$, $\mathcal{B}_{\mu_1}(S_1) \not\leqslant_B \mathcal{B}_{\mu_2}(S_2)$,

(ii) $\mathcal{B}_{\mu_1}(S_1) \not\leqslant_B \mathcal{B}^0_\omega(S_2)$,

(iii) $\mathcal{B}_{\mu_1}(S_1) \not\leqslant_B \mathcal{B}^1_\omega(S_2)$,

and if $\mu_1 = \omega$, then

(i) for all $\mu_2 > \omega$, $\mathcal{B}^0_\omega(S_1) \not\leqslant_B \mathcal{B}_{\mu_2}(S_2)$,*

(ii) for all $\mu_2 > \omega$, $\mathcal{B}^1_\omega(S_1) \not\leqslant_B \mathcal{B}_{\mu_2}(S_2)$,*

(iii) either*

$$\mathcal{B}^0_\omega(S_1) \not\leqslant_B \mathcal{B}^0_\omega(S_2) \text{ and } \mathcal{B}^0_\omega(S_1) \not\leqslant_B \mathcal{B}^1_\omega(S_2) \tag{1}$$

or

$$\mathcal{B}^1_\omega(S_1) \not\leqslant_B \mathcal{B}^0_\omega(S_2) \text{ and } \mathcal{B}^1_\omega(S_1) \not\leqslant_B \mathcal{B}^1_\omega(S_2). \tag{2}$$

Proof of the lemma. First we assume $\mu_1 > \omega$.

(i) There are two cases:

Case 1: $\mu_2 = \mu_1$. Denote $B = A_{\mu_1} = A_{\mu_2}$ and $S'_1 = (S_1 \cap S^\kappa_{\mu_1}) \cup B$ and $S'_2 = (S_2 \cap S^\kappa_{\mu_2}) \cup B$. Now $\mathcal{B}_{\mu_1}(S_1) = \mathrm{id} \otimes E_{S'_1}$ and $\mathcal{B}_{\mu_2}(S_2) = \mathrm{id} \otimes E_{S'_2}$. Since by definition $B = B^\mu_{\mathrm{nr}}(K_\mu)$ where $K_\mu \subset S^\kappa_\omega$ is stationary, and $(S_2 \setminus S_1) \cap S^\kappa_{\mu_1}$ is stationary, the sets S'_1 and S'_2 satisfy the assumptions of Theorem 5.27.2b, so the statement follows from Theorem 5.27.2b and Remark 5.31.

Case 2: $\mu_2 \neq \mu_1$. Let $S'_1 = (S_1 \cap S^\kappa_{\mu_1}) \cup A_{\mu_1}$ and $S'_2 = (S_2 \cap S^\kappa_{\mu_2}) \cup A_{\mu_2}$ whence $B_{\mu_1}(S_1) = \mathrm{id} \otimes E_{S'_1}$ and $B_{\mu_2}(S_2) = \mathrm{id} \otimes E_{S'_2}$. Now $S'_1 \subset S^\kappa_{\geqslant\omega}$ and $S'_2 \subset S^\kappa_{\geqslant\omega}$ and since $A_{\mu_1} \cap A_{\mu_2} = \varnothing$, the result follows from Theorem 5.27.1b and Remark 5.31.

(ii) Let $S_1' = (S_1 \cap S^\kappa_{\mu_1}) \cup A_{\mu_1}$, $S_2' = A^2_\omega \cup ((S_2 \cap S^\kappa_\omega) \setminus A^0_\omega)$, and $A_2' = A^0_\omega$. By definition,

$$B^0_\omega(S_2) = \mathrm{id}_{2^\kappa} \otimes E_{S_2'} \otimes E_{A_2'}$$

and $B_{\mu_1}(S_1) = E_{S_1'}$. Since $A_{\mu_1} \cap A^2_\omega = \varnothing$, $S_1' \cap S^\kappa_\omega = A_{\mu_1}$ and $A^2_\omega \subset S_2'$, we have that $S_2' \setminus S_1'$ is ω-stationary, because it contains A^2_ω. Also $A^0_\omega \setminus S_1' = A^0_\omega$, because $S_1' \cap A^0_\omega = \varnothing$, so $A_2' \setminus S_1'$ is ω-stationary. Now the result follows from Theorem 5.27.4b and Remark 5.31.

(iii) Similar to (ii).

Then we assume $\mu_1 = \omega$.

(i*) Let $S_1' = A^2_\omega \cup ((S_1 \cap S^\kappa_\omega) \setminus A^0_\omega)$, $A_1' = A^0_\omega$ $A_2' = A_{\mu_2}$ and $S_2' = (S_2 \cap S^\kappa_{\mu_2})$. Since $A^0_\omega \cap A_{\mu_2} = \varnothing$, we have that $A_2' \setminus A_1'$ is ω-stationary, so by Theorem 5.27.5 and Remark 5.31,

$$\mathrm{id} \otimes E_{S_1'} \otimes E_{A_1'} \not\leq_B \mathrm{id} \otimes E_{S_2' \cup A_2'},$$

which by definitions is exactly the subject of the proof.

(ii*) Similar to (i*).

(iii*) The situation is split into two cases, the latter of which is split into two subcases:

Case 1: $((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^2_\omega \cup A^0_\omega)$ *is stationary.* Let $S_1' = A^2_\omega \cup ((S_1 \cap S^\kappa_\omega) \setminus A^0_\omega)$, $A_1' = A^0_\omega$, $S_2' = A^2_\omega \cup ((S_2 \cap S^\kappa_\omega) \setminus A^0_\omega)$ and $A_2' = A^0_\omega$. Now $A_2' \setminus S_1'$ is obviously ω-stationary, since it is equal to A^0_ω. Also $S_2' \setminus S_1'$ is stationary, because it equals to $((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^2_\omega \cup A^0_\omega)$ which is stationary by the assumption. Now the first part of (1) follows from Theorem 5.27.3b and Remark 5.31, because $\mathcal{B}^0_\omega(S_1) = \mathrm{id} \otimes E_{S_1'} \otimes E_{A_1'}$ and $\mathcal{B}^0_\omega(S_2) = \mathrm{id} \otimes E_{S_2'} \otimes E_{A_2'}$. On the other hand let $S_2'' = A^3_\omega \cup ((S_2 \cap S^\kappa_\omega) \setminus A^1_\omega)$ and $A_2'' = A^1_\omega$. Now $S_2'' \setminus A_1'$ is stationary, because $A^3_\omega \subset S_2''$ but $A^3_\omega \cap A_1' = A^3_\omega \cap A^0_\omega = \varnothing$. Also $A_2'' \setminus A_1'$ is stationary since $A_2'' \cap A_1' = A^1_\omega \cap A^0_\omega = \varnothing$. Now also the second part of (1) follows from Theorem 5.27.3b and Remark 5.31, because $B^0_1(S_1) = \mathrm{id} \otimes E_{S_1'} \otimes E_{A_1'}$ and $B^1_1(S_2) = \mathrm{id} \otimes E_{S_2''} \otimes E_{A_2''}$.

Case 2: $((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^2_\omega \cup A^0_\omega)$ *is non-stationary.*

Case 2a: $((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^3_\omega \cup A^1_\omega)$ *is stationary.* Now (2) follows from Theorem 5.27.3b and Remark 5.31 in a similar way as (1) followed in Case 1.

Case 2b: $((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^3_\omega \cup A^1_\omega)$ *is non-stationary.* Now we have both:

$$((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^2_\omega \cup A^0_\omega) \text{ is non-stationary} \qquad (*)$$

and

$$((S_2 \setminus S_1) \cap S^\kappa_\omega) \setminus (A^3_\omega \cup A^1_\omega) \text{ is non-stationary.} \qquad (**)$$

Now from $(*)$ it follows that $S_2 \setminus S_1 \subset_{\mathrm{NS}(\omega)} A^2_\omega \cup A^0_\omega$. From $(**)$ it follows that $S_2 \setminus S_1 \subset_{\mathrm{NS}(\omega)} A^3_\omega \cup A^1_\omega$. This is a contradiction, because $S_2 \setminus S_1$ is ω-stationary and $(A^2_\omega \cup A^0_\omega) \cap (A^3_\omega \cup A^1_\omega) = \varnothing$. □

□

5.5 On Chains in $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$

There are chains of order type κ^+ in Borel equivalence relation on 2^κ:

5.33 Theorem. *Let $\kappa > \omega$. There are equivalence relations $R_i \in \mathcal{E}^B_\kappa$, for $i < \kappa^+$, such that $i < j \iff R_i \lneqq_B R_j \lneqq E_0$.*

5.34 Remark. In many cases there are κ^+-long chains in the power set of κ ordered by inclusion modulo the non-stationary ideal whence a weak version of this theorem could be proved using Theorem 5.12. Namely if the ideal I^κ_{NS} of non-stationary subsets of κ is *not* κ^+-saturated, then there are κ^+-long chains. In this case being *not* κ^+-saturation means that there exists a sequence $\langle A_i \mid i < \kappa^+ \rangle$ of subsets of κ such that A_i is stationary for all i but $A_i \cap A_j$ is non-stationary for all $i \neq j$. Now let f_α be a bijection from κ to α for all $\alpha < \kappa^+$ and let

$$B_\alpha = \mathop{\triangledown}_{i<\alpha} A_i = \{\alpha \mid \text{ for some } i < \alpha, \alpha \in A_{f_\alpha(i)}\}$$

It is not difficult to see that $\langle B_\alpha \mid \alpha < \kappa^+ \rangle$ is a chain. On the other hand the existence of such a chain implies that I^κ_{NS} is not κ^+-saturated.

By a theorem of Gitik and Shelah [25, Theorem 23.17], I^κ_{NS} is not κ^+-saturated for all $\kappa \geqslant \aleph_2$. By a result of Shelah [25, Theorem 38.1], it is consistent relative to the consistency of a Woodin cardinal that $I^{\aleph_1}_{\text{NS}}$ *is* $\aleph_2$-saturated in which case there are no chains of length ω_2 in $\langle \mathcal{P}(\omega_1), \subset_{\text{NS}} \rangle$. On the other hand in the model provided by Shelah, CH fails. According to Jech [5] it is an open question whether CH implies that $I^{\aleph_1}_{\text{NS}}$ is not $\aleph_2$-saturated.

However, as the following shows, it follows from ZFC that there are κ^+-long chains in $\langle \mathcal{E}^B_\kappa, \leqslant_B \rangle$ for any uncountable κ.

Proof of Theorem 5.33. By the proof of Corollary 5.29, page 152, one can find ω-stationary sets S_i for $i < \kappa^+$ such that $S_i \setminus S_j$ and $S_j \setminus S_i$ are stationary whenever $i \neq j$. For all $j \in [1, \kappa^+)$, let

$$R_j = \bigoplus_{i<j} E_{S_i},$$

where the operation $\oplus$ is from Definition 5.26, page 149.

Let us denote $P_A = \bigcup_{i \in A} 2^{\kappa \times \{i\}}$ for $A \subset \kappa^+$, i.e. for example $P_j = \bigcup_{i<j} 2^{\kappa \times \{i\}}$.

Let us show that

1. if $i < j$, then $R_i \leqslant_B R_j$,
2. if $i < j$, then $R_j \nleqslant_B R_i$,
3. for all $i < \kappa^+$, $R_i \lneqq_B E_0$.

Item 1 is simple: let $f \colon P_i \to P_j$ be the inclusion map (as $P_i \subset P_j$). Then f is clearly a reduction from R_i to R_j.

Suppose then that $i < j$ and that $i \leqslant k < j$. To prove 2 it is sufficient to show that there is no reduction from E_{S_k} to R_j. Let us assume that $f \colon 2^\kappa \to P_j$ is a Borel reduction from E_{S_k} to R_j. Now

$$2^\kappa = \bigcup_{\alpha<i} f^{-1}[P_{\{\alpha\}}],$$

so one of the sets $f^{-1}[P_{\{\alpha\}}]$ is not meager; let α_0 be an index witnessing this. Note that $\alpha_0 < k$, because $\alpha_0 < i \leqslant k$. Because f is a Borel function and Borel sets have the Property of Baire, we can find a $p \in 2^{<\kappa}$ such that $C = N_p \cap C(f) \cap f^{-1}[P_{\{j\}}]$ is co-meager in N_p. But now $f \restriction C$ is a continuous reduction from $E_{S_k} \cap C^2$ to E_{S_α} which contradicts Theorem 5.27.1b.

To prove 3 we will show first that $R_i \leqslant_B \bigoplus_{j<i} E_0$ and then that $\bigoplus_{j<i} E_0 \leqslant_B E_0$, after which we will show that $E_0 \not\leqslant_B R_i$ for all i.

Let f_j be a reduction from E_{S_j} to E_0 for all $j < i$ given by Claim 3 of the proof of Theorem 5.11. Then combine these reductions to get a reduction from R_i to $\bigoplus_{j<i} E_0$. To be more precise, for each $\eta \in P_{\{j\}}$ let $f(\eta)$ be ξ such that $\xi \in P_{\{j\}}$ and $\xi = f_j(\eta)$.

Let $\{A_k \mid k \leqslant i\}$ be a partition of κ into disjoint unbounded sets. Let $\eta \in P_i$. By definition, $\eta \in P_{\{k\}}$ for some $k < i$. Define $\xi = F(\eta)$ as follows. Let $f\colon A_i \to \kappa$ be a bijection.

- If $\alpha \in A_i$, then let $\xi(\alpha) = \eta(f(\alpha))$.
- If $\alpha \in A_j$ and $j \neq k$, then let $\xi(\alpha) = 0$.
- If $\alpha \in A_k$, then let $\xi(\alpha) = 1$.

It is easy to see that F is a continuous reduction.

Assume for a contradiction that $E_0 \leqslant_B R_i$ for some $i < \kappa^+$. Then by 1 and transitivity, $E_0 \leqslant_B R_j$ for all $j \in [i, \kappa^+)$. By the above also $R_j \leqslant_B E_0$ for all $j \in [i, \kappa^+)$ which, again by transitivity, implies that the relations R_j for $j \in [i, \kappa^+)$ are mutually bireducible to each other which contradicts 2. □

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List of Symbols