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Noncommutative Quantum Field Theory: Problem of Time and Some Applications

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Abstract

In this thesis the current status and some open problems of noncommutative quantum field theory are reviewed. The introduction aims to put these theories in their proper context as a part of the larger program to model the properties of quantized space-time. Throughout the thesis, special focus is put on the role of noncommutative time and how its nonlocal nature presents us with problems.

Applications in scalar field theories as well as in gauge field theories are presented. The infinite nonlocality of space-time introduced by the noncommutative coordinate operators leads to interesting structure and new physics. High energy and low energy scales are mixed, causality and unitarity are threatened and in gauge theory the tools for model building are drastically reduced. As a case study in noncommutative gauge theory, the Dirac quantization condition of magnetic monopoles is examined with the conclusion that, at least in perturbation theory, it cannot be fulfilled in noncommutative space.

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Contents

List of publications **vii**

1 Introduction **1**

1.1 The need for quantum gravity 1

1.2 Theories of quantum gravity 2

1.3 Problem of time 5

1.4 Experimental searches 6

2 Noncommutative space-time **8**

2.1 Quantum space and quantum time 8

2.2 Different motivations for different models 9

2.3 Weyl quantization of space-time 12

2.4 Twisted symmetry 15

2.5 Dual algebra and noncommutative fields 18

3 Causality, unitarity and noncommutative time **21**

3.1 Causality 21

3.1.1 Solutions of Tomonaga-Schwinger equation 25

3.2 Unitarity 28

3.2.1 String theory and unitarity 28

3.2.2 Unitarity in covariant formalism 29

3.2.3 Unitarity in Hamiltonian formalism 29

3.2.4 Unitarity in Heisenberg picture 33

4 Effects of quantization of space **37**

4.1 UV/IR mixing 37

4.2 Spherical coordinates in quantum space 40

4.2.1 Using a quadratic twist element 44

5 Noncommutative gauge theory and magnetic monopoles **47**

5.1 Gauge theory in quantum space 47

5.1.1 Charge quantization and no-go theorems 47

5.1.2 Seiberg-Witten map 49

5.1.3 Gauge invariant observables 51

5.2 Magnetic monopoles 52

5.2.1 Wu-Yang approach 52

5.2.2 Wu-Yang procedure in noncommutative space 54

5.2.3 Deformed gauge transformations and Maxwell's equations . 55

5.2.4 The Dirac quantization condition in quantum space 57

5.2.5 Discussion and comparison to earlier results 60

5.3 A noncommutative particle source 62

6 Conclusions **64**

List of publications

I. *"Noncommutative Quantum Field Theory: A Confrontation of Symmetries,"*
M. Chaichian, K. Nishijima, T. Salminen and A. Tureanu,
JHEP **0806**, 078 (2008), [arXiv:0805.3500 [hep-th]].

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Chapter 1

Introduction

Noncommutative quantum field theory is an approach to describe physics in quantum space-time. It is a part of the larger program to combine quantum mechanics with general relativity that has so far been unsuccessful in leading to a consistent theory. The object of this chapter is to give on overview of the program of quantum gravity and to fit noncommutative quantum field theory in the appropriate context. As the problems connected with the special role of time constitute a major part of this thesis, the problem of time in quantum gravity in general is briefly commented on.

1.1 The need for quantum gravity

We live in quantum space-time. To believe this we only need to accept the basic principles of general relativity and quantum mechanics. Combined with the standard model of particle physics these two theories constitute the simplest and most accurate description we have of the world around us. They seem to be able to explain *all* of the experiments done to date, with the exception of the dark matter observations by a particle theory[1].

Yet most physicists share the view that a more fundamental theory is needed – a theory that in appropriate limits would give both general relativity and quantum mechanics, and would provide us with a consistent description of physics in

[1] Of course the mechanism for the accelerated expansion of the universe is still much debated, but at least in principle the dark energy explanation given by the cosmological constant in general relativity is able to account for it.

the regime where neither theory can be neglected, a theory of *quantum gravity*. An intuitive way to understand the discrepancy of the two theories is to consider Einstein's equations[2]

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu} \,. \tag{1.1}$$

The essential meaning of (1.1) is that the form of the gravitational field, i.e. that of the curved space-time, is given by the energy-momentum content of that space-time. Quantum mechanics tells us that the energy-momentum content is quantized, so then must be the left hand side of the equation. Hence, *we live in quantum space-time*.

The scale on which the quantum effects of gravity become important is extremely small. Combining the fundamental constants in a meaningful way to give a unit of length we have

$$\lambda_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}\mathrm{cm} \,. \tag{1.2}$$

λ_{Pl} is called the *Planck length*, and the corresponding energy scale the *Planck scale*. It is the small value of the Planck length that has allowed us to explain so much of nature without the need for a theory of quantum gravity – it has not so far been possible to probe such scales by experiment. Intuitively, as further explained in section 2.1, it can be thought of as the radius of the smallest volume of space-time we can observe. In some models it is thus the scale of "chunks of space", the building blocks space-time is made of. Similar intuitive explanations why the Planck length is thought to give a fundamental length scale can be found in [1].

1.2 Theories of quantum gravity

Schematically, the idea of noncommutative field theory[3] is to introduce fuzziness of space-time in terms of space-time uncertainty relations of the form

$$\Delta x \Delta y \sim \theta \sim \lambda_{Pl}^2 \,. \tag{1.3}$$

This is done by imposing commutation relations for coordinates just as is done in the quantization of phase space in quantum mechanics. This uncertainty can be

[2]The speed of light c and the Planck constant $\hbar$ will be, as usual, set to 1 for the entire thesis. Factors of c and $\hbar$ will be reinserted if needed.

[3]The structure and motivation of noncommutative field theories is presented in detail in chapter 2.

interpreted as a minimum measurable *quantum of area*. There does not appear a minimum length scale as such, just as there is no minimum scale in quantum mechanics. One direction can be infinitely well localized as long as the conjugate coordinate in infinitely nonlocal.

The clearest, in principle testable, experimental signature of these models is the breaking of Lorentz invariance. Using astrophysical as well as accelerator experiments one can place constraints on the scale of Lorentz invariance violation, thus constraining the value of the parameter θ in (1.3). Noncommutative field theories fit integrally in the grander scheme of quantizing gravity and have been connected with many other approaches including string theory and loop quantum gravity. The approaches to quantum gravity are numerous; an overview of the achievements and problems of current models can be found in [2].

String theory. By far the most effort in quantum gravity research to date has been put to the study of string theory [3]. By replacing points as the basic building blocks of space-time by one-dimensional strings, these theories aim to regulate the divergences in quantum field theory, as well as to describe a much fuller phenomenology with the help of the added degrees of freedom. String theory implies the existence of some exotic new physics such as supersymmetry and extra dimensions, which we hold the hope of finding already with the large hadron collider (LHC).

Apart from being the most studied field in quantum gravity, string theory is most important to the study of noncommutative field theory since it was shown in [4] (see also [5]) that in the low-energy limit of type IIB string theory with an antisymmetric B_{ij} background one recovers a field theory in noncommutative space. As it further differentiates the theories with noncommutative time from those with the usual commutative time variable, it is to be considered the main motivation for the latter part of this thesis, which is concentrated on applications with commutative time. String theory as a motivation for noncommutative field theories will be further discussed in section 2.2.

As string theory is based mainly on the lessons learned from particle physics, it is often claimed that it fails to incorporate the insights of general relativity. Since it is based on perturbation theory, a fixed background is required to do those perturbations on. The perturbations then allow the background to change but the general covariance requirement of general relativity still fails to be fulfilled –

string theory is background dependent. To circumvent the problem of background dependence, various non-perturbative approaches are being developed.

Loop quantum gravity. Much of the activity in constructing a non-perturbative theory of quantum gravity has concentrated on loop quantum gravity, or LQG for short [6, 7]. Unlike the particle-physics inspired approach of string theory, it uses the principles of general relativity as a starting point. The name derives from the generally covariant loop states, which were found to be solutions to the Wheeler-DeWitt equation, the "wave-equation of the universe".

In three dimensions LQG has provided a consistent quantization of general relativity with quantized area and volume operators. There is much progress also in the four-dimensional theory, but a consistent treatment, especially of the Lorentzian version, is lacking. A major attraction of LQG models are the big bounce scenarios of loop quantum cosmology, that allow for a workaround for the problems of the big bang singularity.

Three-dimensional loop quantum gravity has been shown to be equivalent to certain types of noncommutative theories [8], again highlighting the connection of different approaches to quantum gravity. Being generally covariant however, it is to be expected that LQG can only be connected to Lorentz invariant formulations of noncommutative space-time. These models, as discussed in chapter 3, have problems with unitarity, causality and energy-momentum conservation and thus for the more mathematically consistent noncommutative models string theory is to be considered as the main motivation.

Other models. The phenomenology of noncommutative quantum field theory is sometimes probed through its connection with doubly special relativity (DSR) [9] (and vice versa), as the structure of DSR has been shown to arise naturally in some noncommutative models [10][4]. By continuity, DSR and LQG have been shown to be related in 2+1 dimensions. In DSR quantum mechanics is extended by simply adding, in addition to the maximum velocity of special relativity, a fundamental observer-independent length scale, usually $l \sim \lambda_{Pl}$. Conceptually, DSR starts with

[4] Both 2+1 -dimensional LQG and DSR are connected to the so-called κ-Poincaré models, and not to the simpler canonical noncommutativity considered in this thesis. The connections therefore point merely to qualitative similarities.

the assumption of unbroken Lorentz invariance familiar from special relativity. Consequently, as with LQG, it is connected to the Lorentz invariant noncommutative models with all of their problems.

Another similar construction is the *very special relativity* (VSR) of Glashow and Cohen [11], again shown to be related to noncommutative quantum field theory [12], giving further possibilities for phenomenological predictions. The connection with VSR also provides insights for the mechanisms by which the Lorentz group could be broken at high energies.

1.3 Problem of time

One of the standing problems in quantum gravity theories that has sparked its fair share of philosophical and conceptual discussion is the problem of time [6,13,14]. The problem typically arises in the canonical approaches to quantum gravity, since a specific "time coordinate" is needed to perform the quantization. As we are then quantizing the generally covariant field equations of general relativity, invariant under the group of diffeomorphisms $\mathrm{Diff}(\mathcal{M})$ of the space-time manifold $\mathcal{M}$, a notion of time needs to be introduced in some consistent manner.

There are various ways to introduce time – theories are commonly grouped into those where time is introduced before quantization, after quantization and those where it is not introduced at all at the fundamental level, the so-called *timeless* approach. In the timeless approach time emerges from the fundamental degrees of freedom in the theory as a phenomenological parameter. The motivation behind the timeless constructions is manifest in the *Wheeler-DeWitt equation* [15]

$$\hat{\mathcal{H}}\,\Psi = 0\,, \tag{1.4}$$

where Ψ is a functional of field configurations on all of space-time, the "wave-function of the universe" and $\hat{\mathcal{H}}$ is the Hamiltonian constraint arising in the canonical quantization of general relativity. In the simplest interpretation this leads to a static universe, far removed from our everyday experience. As one of the proponents of the timeless approach, Julian Barbour, puts it:

> "Unlike the Emperor dressed in nothing, time is nothing dressed in clothes."

We are used to working with Poincaré invariant quantum field theories, commonly taking Lorentz invariance to be a prerequisite for a physically sensible theory. Due to this one often conjectures that also space is emergent, giving rise to emergent space-time. However, there are also approaches where time is taken to be different from the spatial directions as the only emergent coordinate [14], thus renouncing the generally covariant space-time picture at the Planck scale.

As the low-energy limit of string theory, noncommutative quantum field theory lives in a fixed, flat background space-time with a noncommutative algebra for the position operators. Thus there appear no problems related to general covariance – we are not performing a quantization of general relativity. However, as the differences between models with a noncommutative time coordinate and those with only spatial noncommutativity are introduced in chapter 3, it is not hard to convince oneself that also in the models of canonically noncommutative space-time time truly is special. In this sense the role of time on these deformed space-times is reminiscent of the problem of time in quantum gravity in general.

1.4 Experimental searches

Since physics is ultimately about experiments, the test that any theory of quantum gravity should face is to make predictions that are, at least in principle, measurable. Apart from the black hole entropy law and a few other constraints equally far from any possible measurements with current methods, today's theories are not doing too well when it comes to predictions.

As the Planck scale (1.2) is extremely low due to the weakness of the gravitational interaction, any direct experimental tests of quantum gravity are difficult to conceive. However, there are quantum gravitational effects susceptible to measurement even on the scales of current experiments, and especially in some of the projects currently in construction or planning. As this section is only intended to serve as context, it will certainly lack detail and depth. For more information see the recent review [16] and references therein.

The main arenas for quantum gravity experiments are astrophysical observations, collider searches and cosmology. As quantum gravitational effects are expected to be most important in areas of high curvature, black holes and the early universe provide the most interesting testing grounds.

Astrophysical observations are the most important in detecting violations of Lorentz invariance, a particularly important effect in noncommutative quantum field theory. A review of the experimental signatures connected with noncommutative field theory can be found in [17]. Using the bounds on Lorentz invariance violation from experiments an upper limit on the noncommutativity parameter θ can be deduced, usually clearly higher than the square of the Planck length leaving room for speculation. For example, in [18] the bound

$$\theta \sim (\Delta x)^2 \lesssim \left(10^{-24}\text{cm}\right)^2 , \tag{1.5}$$

is presented and is much higher than λ_{Pl}^2. Similar limits have been obtained also in different contexts; for example by consider the Lamb shift in the hydrogen atom [19] a limit slightly weaker than (1.5) was derived.

As the LHC has begun operations at CERN late last year, theorists are eagerly waiting for any clues on physics beyond currently tested energies. However, as the LHC will only access distance scales down to about 10^{-19}m, the focus for quantum gravity is mainly on ideas where the Planck scale is lowered for one reason or another, and on features of string theory required for mathematical consistency, such as supersymmetry. A lowered Plank scale typically appears in theories with extra dimensions. If gravity is allowed to access the hidden dimensions it could on lower energies appear to be weaker, thus leading to the "hierarchy problem" of the fundamental interactions. If these dimensions were probed by the LHC, effects of gravity could be directly observed. The creation of tiny black holes would be a particularly intriguing prospect, as it would allow us to study the theoretically already well-mapped black hole phenomenology.

In cosmological experiments the focus has for long been on the analysis of the cosmic microwave background (CMB). Any preferred direction in the CMB would be direct evidence for the breaking of Lorentz invariance, but no such effect has so far been observed. It seems that the CMB fails to give us any hints to the nature of quantum gravity, but there is hope that it will confirm the existence of primordial gravitational waves that would certainly do just that. There are currently many projects aiming to measure the polarization of the CMB to higher accuracy in the search for tensorial modes, so-called *B-modes*, in the polarization pattern – a direct proof for the existence of these waves. In the upcoming gravity wave experiments (LIGO, VIRGO, LISA and others) it is hoped that the spectrum of these waves could be accessed. As they probe the earliest moments of our universe, the data would certainly shed light on the Planck scale properties of nature.

Chapter 2

Noncommutative space-time

In this chapter the structure and motivation of noncommutative quantum field theory are reviewed. The deformation quantization of space-time is the framework that will be used in the rest of the thesis and thus, in a way, defines everything that is to follow. The two main motivations given for different types of noncommutativity should be understood fully and kept in mind whenever studying theories of noncommutative space-time.

2.1 Quantum space and quantum time

The study of noncommutative quantum field theory originated already in the 1940's by Heisenberg, Snyder and Yang [20–22] with the hope of regulating the ultraviolet divergences that plague quantum field theory. By the success of the renormalization program these works were soon forgotten. However, with the works of Connes, Drinfel'd and Woronowicz [23], the idea re-emerged in the 1980's as a way to model the quantum structure of space-time. For a rigorous treatment of the mathematical structure of noncommutative geometries see [24]. For reviews on noncommutative quantum field theory see [25, 26].

The simplest and best-known way to introduce uncertainty into the Riemannian picture of space-time is to promote coordinates to operators of a suitable Hilbert space-time and impose the commutator

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} , \tag{2.1}$$

where the entries of $\theta^{\mu\nu}$ do not depend on the coordinates. In this thesis the focus is on this simplest choice, i.e. (2.1), where $\theta^{\mu\nu}$ will either be a *constant* or a *tensor*. The study of these two types of theories is motivated below in section 2.2.

Much of this thesis concentrates on the problems connected with noncommutative time and thus implicitly on the problems of all models where $\theta^{\mu\nu}$ is a tensor. This is because Lorentz invariance always allows us to shift the noncommutativity to be in the timelike directions. Thus, for generality, let us consider models where all coordinates are noncommutative. In four dimensions we can always choose a frame where the θ-matrix is in the block-diagonal form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & -\theta' & 0 \end{pmatrix} . \tag{2.2}$$

With this form of $\theta^{\mu\nu}$ it is useful to classify different types of noncommutativity to clarify the causal structure

- Space-space $\theta = 0$.
- Lightlike $\theta^{\mu\nu}\theta_{\mu\nu} = \theta^2 - \theta'^2 = 0$.
- Time-space $\theta \neq 0$.

Lightlike noncommutativity shares most of the properties of space-space noncommutativity, as the former when written in terms of light cone coordinates will only exhibit noncommutativity of the spacelike directions. The major differences in noncommutative models are thus associated with the noncommutativity of the timelike coordinate. As we will see in chapter 3, the theories with noncommutative time exhibit serious problems that are yet to be resolved. Space-space and lightlike models however, although they violate Lorentz invariance, can be defined in a consistent manner.

2.2 Different motivations for different models

In this section the motivations for the different classes of noncommutative models classified in the previous section are briefly reviewed. It is ironic that the two major motivations used in the literature [4, 27] imply different types of noncommutativity. In the Doplicher-Fredenhagen-Roberts (DFR) type models [27, 28] all coordinates are noncommutative, whereas the low-energy limit of string theory considered by Seiberg and Witten in [4] only allows for space-space noncommutativity.

Doplicher-Fredenhagen-Roberts models

Perhaps the most intuitive way of motivating noncommutative models was presented by Doplicher, Fredenhagen and Roberts in [27,28], using only basic principles from quantum mechanics and classical general relativity to obtain a quantum space-time. Imagine performing a measurement near the Planck scale, $\lambda_{Pl} \approx 10^{-33}$cm. According to quantum mechanics, in a measurement confined to a volume of the order of λ_{Pl}^3 there is an intrinsic energy uncertainty of the order of the Planck energy. Then, according to general relativity, the energy density of the space is high enough to create a black hole in the space you perform your measurement. Thus it will be quite impossible to observe anything smaller than λ_{Pl}^3. Ontologically it is tempting to say that nothing smaller can exist.

Now, since space-time can no longer be considered a manifold made of points[1], some form of fuzziness must be introduced to model this uncertainty of space-time. The simplest way is already familiar from the quantization of the phase space operators in quantum mechanics. Hence it is natural to promote coordinates to operators and impose a nonzero commutator for the coordinates as in (2.2). This, in turn, leads to uncertainty relations for the coordinates of the form

$$\begin{aligned} &\Delta x_0 \cdot (\Delta x_1 + \Delta x_2 + \Delta x_3) \geq \lambda_{Pl}^2 \,, \\ &\Delta x_1 \Delta x_2 + \Delta x_1 \Delta x_3 + \Delta x_2 \Delta x_3 \geq \lambda_{Pl}^2 \,. \end{aligned} \tag{2.3}$$

As the approach is based on general relativity it is naturally Lorentz covariant, and $\theta^{\mu\nu}$ is a Lorentz tensor. As such, theories of the DFR type will have to deal with all the peculiarities that stem from noncommutative time further discussed in chapter 3.

String theory motivation

The interest in noncommutative field theories surged after Seiberg and Witten showed [4], that by studying the dynamics of D-branes with a constant Neveu-Schwartz "magnetic" B_{ij} background field a noncommutative field theory is found

[1] Von Neumann coined this "*pointless geometry*".

in the low-energy limit. To see how this comes about, it is instructive to have a look at one of the oldest examples of the appearance of noncommutativity of coordinates: the Landau problem [4, 26, 29, 30].

Consider electrons moving in the plane $\mathbf{x} = (x^2, x^3)$ in a constant, perpendicular magnetic field of magnitude B. The Lagrangian for each electron is given by

$$L = \frac{m}{2}\,\dot{\mathbf{x}}^{\,2} - \dot{\mathbf{x}} \cdot \mathbf{A}\,, \tag{2.4}$$

where $A_i = -\frac{B}{2}\,\epsilon_{ij}\,x^j$ is the corresponding vector potential in the symmetric gauge.

One can map the Hamiltonian of this model onto that of a harmonic oscillator, whose spectrum yields the so-called *Landau levels*. In the limit $m \to 0$ with B fixed, or equivalently $B \to \infty$ with m fixed, the system is projected onto the lowest Landau level, i.e the ground state of the oscillator. The Lagrangian in this limit becomes

$$L_0 = -\frac{B}{2}\,\dot{x}^i\,\epsilon_{ij}\,x^j\ . \tag{2.5}$$

This reduced Lagrangian is of first order in time derivatives. The phase space therefore becomes degenerate and collapses onto the configuration space. Thus canonical quantization gives a *noncommutative space* with the commutator

$$\left[\hat{x}^i\,,\,\hat{x}^j\right] = i\theta^{ij}\,, \quad \text{with} \quad \theta^{ij} = \frac{\hbar\,c}{e\,B}\,. \tag{2.6}$$

This simple example has a direct analog in string theory [4]. Consider bosonic strings moving in flat Euclidean space with metric g_{ij}, in the presence of a constant Neveu-Schwarz two-form B-field and with Dp-branes. The B-field is equivalent to a constant magnetic field on the branes, and it can be gauged away in the directions transverse to the Dp-brane worldvolume. The (Euclidean) worldsheet action is

$$S_\Sigma = \frac{1}{4\pi\alpha'}\int\limits_\Sigma \left(g_{ij}\,\partial_a x^i\,\partial_a x^j - 2\pi i\alpha' B_{ij}\,\epsilon^{ab}\,\partial_a x^i\,\partial_b x^j\right)\,, \tag{2.7}$$

where $\alpha' = \ell_s^2$, Σ is the string worldsheet and x^i is the embedding function of the strings into flat space.

In the low-energy limit $g_{ij} \sim (\alpha')^2 \sim \varepsilon \to 0$, with B_{ij} fixed, the stringy effects decouple and the bulk kinetic terms for the x^i in (2.7) vanish. All that remains are the boundary degrees of freedom of the open strings, which are governed by the action

$$S_{\partial\Sigma} = -\frac{i}{2}\oint\limits_{\partial\Sigma} B_{ij}\,x^i\,\partial_t x^j\,. \tag{2.8}$$

This action coincides with the Landau action describing the motion of electrons in a strong magnetic field (2.5). From this we may infer the noncommutativity

$$[\hat{x}^i, \hat{x}^j] = (i/B)^{ij} \equiv i\theta^{ij}\,, \tag{2.9}$$

of the coordinates of the endpoints of the open strings which live in the Dp-brane worldvolume.

The correlated low-energy limit $\alpha' \to 0$ taken above effectively decouples the closed string dynamics from the open string dynamics. It also decouples the massive open string states, so that the string theory reduces to a field theory describing massless open strings. Only the endpoint degrees of freedom remain and describe a noncommutative geometry.

2.3 Weyl quantization of space-time

The commutation relations of the quantum position operators

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \tag{2.10}$$

are commonly implemented in field theory by the equivalent procedure of deforming the product of functions called *Weyl quantization* [31–33] (for a more pedagogical treatment see [34]).

Weyl quantization provides a nice way to avoid the use of Hilbert-space operators and allows computations to be done using classical smooth functions. The quantum effects are then encoded in the modified product of functions, the *Moyal $\star$-product.* For two Schwartz functions f, g, i.e. functions on $C^\infty(\mathbb{R}^D)$ that decrease sufficiently fast at infinity, the $\star$-product is given by

$$(f \star g)\,(x) \equiv f(x) e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g(y)\,|_{y=x}\,. \tag{2.11}$$

Using this it is immediately noticed that for the usual coordinate *functions* x^μ and x^ν we get

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}\,, \tag{2.12}$$

since only the first derivatives contribute in the expansion of the exponent. The commutator (2.12) is called the *Moyal bracket.*

Due to the small value of θ, the usual product of functions is only slightly deformed and as a correspondence principle the $\star$-product reduces to usual multiplication in the limit $\theta^{\mu\nu} \to 0$. Hence the method is more generally called *deformation quantization.*

Derivation of the Moyal $\star$-product. As we are working with Schwartz functions, we can consider their Fourier transforms

$$\tilde{f}(k) = \int d^D x \; e^{-ik_\mu x^\mu} \; f(x) \,. \tag{2.13}$$

The corresponding *Weyl operator* is defined as

$$\hat{W}[f] \equiv \int \frac{d^D k}{(2\pi)^D} \; \tilde{f}(k) e^{ik_\mu \hat{x}^\mu} \,. \tag{2.14}$$

This provides us with a mapping between Schwartz functions and the corresponding Hilbert space operators. In this context the function $f(x)$ is called the *Weyl symbol* of the corresponding operator $\hat{W}[f]$. The power series expansion of the exponential automatically gives a symmetric ordering of the operators, i.e. the *Weyl ordering.*

From its definition (2.14), we see that by taking the adjoint we have

$$\hat{W}[f]^\dagger = \hat{W}[f^\dagger] \,, \tag{2.15}$$

and in particular, the Weyl operator is a self-adjoint operator whenever the function $f(x)$ is real-valued.

If we further require that at the level of the symbols f and g of the corresponding Weyl operators $\hat{W}[f]$ and $\hat{W}[g]$ the usual product of operators is reproduced

$$\hat{W}[f]\hat{W}[g] = \hat{W}[f \star g] \,, \tag{2.16}$$

we recover the integral representation of the $\star$-product

$$(f \star g)\,(x) = \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \; \tilde{f}(k) \tilde{g}(k' - k) \; e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{ik'_\sigma x^\sigma} \,. \tag{2.17}$$

Proof: Using (2.14) we have

$$\hat{W}[f]\hat{W}[g] = \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \; \tilde{f}(k) \tilde{g}(k') e^{ik_\mu \hat{x}^\mu} e^{ik'_\nu \hat{x}^\nu} \,. \tag{2.18}$$

With the help of the Baker-Campbell-Hausdorff formula

$$e^{ik_\mu \hat{x}^\mu} e^{ik'_\nu \hat{x}^\nu} = e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{i(k+k')_\sigma \hat{x}^\sigma} \,, \tag{2.19}$$

and by shifting the integration variable $k'_\mu \to k'_\mu - k_\mu$, we get (2.17) with x^σ replaced by $\hat{x}^\sigma$. This is exactly the operator $\hat{W}[f \star g]$ corresponding to the symbol (2.17). $\square$

The integral (2.17) and the differential representations (2.11) of the $\star$-product are identical whenever the entries of θ are constants. In Fourier space, starting from the differential representation, we obtain the expression

$$\begin{aligned}
(f\star g)\,(x) &= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\;\tilde f(k)e^{ik_\sigma x^\sigma}\,e^{\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}\tilde g(k')e^{ik'_\alpha y^\alpha}|_{y=x}\\
&= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\;\tilde f(k)e^{ik_\sigma x^\sigma}\,e^{\frac{i}{2}k_\mu\theta^{\mu\nu}k'_\nu}\tilde g(k')e^{ik'_\alpha y^\alpha}|_{y=x}\\
&= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\;\tilde f(k)\tilde g(k'-k)e^{\frac{i}{2}k_\mu\theta^{\mu\nu}(k'_\nu-k_\nu)}e^{ik_\sigma x^\sigma}\,e^{i(k'_\alpha-k_\alpha)y^\alpha}|_{y=x}\\
&= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\;\tilde f(k)\tilde g(k'-k)\;e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}e^{ik'_\sigma x^\sigma}\,,
\end{aligned} \tag{2.20}$$

where $k'_\nu \to k'_\nu - k_\nu$ has again been used on the third line.

In coordinate space the integral representation of the $\star$-product (2.17) can be written as

$$(f\star g)(x) = \int d^Dy\; d^Dz\; \mathcal{K}(x;y,z)f(y)g(z)\,, \tag{2.21}$$

where the *kernel* is given by

$$\mathcal{K}(x;y,z) = \frac{1}{\pi^D \det\theta}\exp[-2i(x\theta^{-1}y + y\theta^{-1}z + z\theta^{-1}x)]\,. \tag{2.22}$$

Here $\det\theta$ denotes the determinant of the θ-matrix and $x\theta^{-1}y = x^\mu(\theta^{-1})_{\mu\nu}y^\nu$. Equation (2.21) can also be expressed in a form which is insensitive to the singularity of the θ-matrix, as follows:

$$(f\star g)(x) = \frac{1}{(2\pi)^4}\int d^4y\; d^4z\; f\left(x-\frac{1}{2}\theta y\right)g\,(x+z)\,e^{-iyz}\,, \tag{2.23}$$

with the obvious notation $(\theta y)^\mu = \theta^{\mu\nu}y_\nu$. The calculations in this thesis are not sensitive to the choice of representation of the $\star$-product; both (2.11) and (2.21) will be used where they are most practical.

The $\star$-product, although noncommutative, is still associative: $(a\star b)\star c = a\star(b\star c)$, a most useful property in all perturbative calculations as we shall see. Further, by integration by parts one $\star$-product disappears under an integral over the whole space

$$\int d^Dx\; f(x)\star g(x) = \int d^Dx\; f(x)\,g(x)\,. \tag{2.24}$$

Also, due to the correspondence between the operator trace and space-time integration, the integral of $\star$-products,

$$\int d^D x \; f_1(x) \star \cdots \star f_n(x) = \mathrm{Tr}\; \left(\hat{W}[f_1] \cdots \hat{W}[f_n] \right) , \tag{2.25}$$

is invariant under cyclic permutations of the functions f_i.

As most clearly seen from (2.21), the $\star$-product introduces infinite nonlocality in the theory in all the noncommutative directions (the coordinates are integration variables). A nice example of this induced nonlocality was noted in [26], where two Dirac delta functions multiplied with the $\star$-product were shown to give

$$\delta^D(x) \star \delta^D(x) = \frac{1}{\pi^D |\mathrm{det}\theta|} \, . \tag{2.26}$$

Since $\theta^{\mu\nu}$ is constant over the whole space-time, the initially infinitely localized distribution is spread out to cover the entire space-time[2]. This nonlocality can be considered as the source of all the peculiarities of noncommutative quantum field theories that will be discussed in the rest of the thesis.

2.4 Twisted symmetry

For particle physics there is an apparent problem with the particle content of theories with $[\hat{x}^\mu, \hat{x}^\nu] = \theta^{\mu\nu}$, where

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & -\theta' & 0 \end{pmatrix} \tag{2.27}$$

is a constant matrix. Although translational invariance is preserved, by fixing the coordinates in order to have $\theta^{\mu\nu}$ in the form (2.27) we have broken the Lorentz group $SO(1,3)$ down to its subgroup $SO(1,1) \times SO(2)$[3]. Both $SO(1,1)$ and $SO(2)$ are Abelian groups and thus have only one-dimensional irreducible representations. Thus, when assigning particles to representations of $SO(1,1) \times SO(2)$, we would

[2]Naturally, if some coordinates are commutative the spreading only occurs in the noncommutative directions.

[3]Although Lorentz invariance is violated in these models, this does not lead to CPT or spin-statistics violations as in usual quantum field theory [35, 36].

have no higher order representations, including *spinors, vectors or tensors*. We would only be allowed to consider scalar field theory, which of course would make these theories uninteresting.

Luckily however, there is a way around this apparent handicap. In [37, 38] it was found that these theories respect another, *quantum symmetry*[4]. By twisting the Poincaré algebra it was found that the higher-dimensional representations can indeed be included in the theory.

The twisted Poincaré algebra is obtained by a twist element $\mathcal{F}$ in the universal enveloping of the commutative Poincaré algebra $\mathcal{U}(\mathcal{P})$, i.e. $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$. The useful feature of the twist is that it does not affect the multiplication in $\mathcal{U}(\mathcal{P})$ and thus the Lie algebra

$$\begin{aligned} &[P_\mu, P_\nu] = 0\,, \\ &[M_{\mu\nu}, P_\alpha] = -i(\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu)\,, \\ &[M_{\mu\nu}, M_{\alpha\beta}] = -i(\eta_{\mu\alpha} M_{\nu\beta} - \eta_{\mu\beta} M_{\nu\alpha} - \eta_{\nu\alpha} M_{\mu\beta} + \eta_{\nu\beta} M_{\mu\alpha})\,, \end{aligned} \tag{2.28}$$

remains unmodified. The essential implication of this is that *the representation content* of the new theory is identical to that of the usual Poincaré algebra.

Obviously, we are still working in quantum space-time and this needs to be reflected in the calculations. The price we need to pay for the unchanged Lie algebra is a change in the action of the Poincaré generators in the tensor product of representations, the *coproduct*, given in the standard case by

$$\begin{aligned} &\Delta_0 : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})\,, \\ &\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y, \qquad \forall\, Y \in \mathcal{P}\,. \end{aligned} \tag{2.29}$$

When twisting, this coproduct is deformed into the *twisted* coproduct

$$\Delta_0(Y) \longmapsto \Delta_t(Y) = \mathcal{F} \Delta_0(Y) \mathcal{F}^{-1}\,. \tag{2.30}$$

The form of the twist element $\mathcal{F}$ is constrained by the need to satisfy the following *twist equation*:

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes \mathrm{id})\mathcal{F} = (1 \otimes \mathcal{F})(\mathrm{id} \otimes \Delta_0)\mathcal{F}. \tag{2.31}$$

[4]When discussing twisted algebras one needs to be familiar with the language of Hopf algebras and quantum groups [39].

Considering the simplest choice for the twist[5], an *Abelian* twist element written as

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}\,, \tag{2.32}$$

one can check that (2.31) is indeed satisfied.

The twisted coproducts of the generators of Poincaré algebra corresponding to the Abelian twist are given by

$$\Delta_t(P_\mu) = \Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \tag{2.33}$$

$$\begin{aligned}\Delta_t(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\ &- \frac{1}{2}\theta^{\alpha\beta}\left[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right].\end{aligned} \tag{2.34}$$

The unmodified coproduct of the momentum generators signals the preservation of translational invariance in the theory, while the nontriviality of the twisted coproduct of the Lorentz algebra generators, equation (2.34), is a signature of the broken Lorentz symmetry.

The twisted coproduct Δ_t further *requires* a redefinition of the multiplication. When twisting $\mathcal{U}(\mathcal{P})$, in addition to obtaining the twisted coproduct Δ_t, one has to redefine the multiplication, while retaining the usual action of the generators of the Poincaré algebra on coordinates as

$$\begin{aligned}&P_\mu x_\rho = i\partial_\mu x_\rho = i\eta_{\mu\rho}\,, \\ &M_{\mu\nu}x_\rho = i(x_\mu\partial_\nu - x_\nu\partial_\mu)x_\rho = i(x_\mu\eta_{\nu\rho} - x_\nu\eta_{\mu\rho})\,.\end{aligned} \tag{2.35}$$

The required deformation of the commutative multiplication,

$$m_0(f(x) \otimes g(x)) := f(x)g(x)\,, \tag{2.36}$$

is given by the twist (2.32) as

$$\begin{aligned}m_t(f(x) \otimes g(x)) &= m \circ \left(e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu} f(x) \otimes g(x)\right) \\ &= m \circ \left(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \partial_\nu} f(x) \otimes g(x)\right) \\ &= (f \star g)(x)\,.\end{aligned} \tag{2.37}$$

As the deformed multiplication coincides with the Moyal $\star$-product (2.11), the twisted approach is consistent with the Weyl quantization procedure discussed in section 2.3. It should be noted that the condition (2.31) ensures the associativity of the twisted multiplication (2.37).

[5]Different twists satisfying (2.31) have been studied in the literature. In section 4.2 the *quadratic twist* will be considered with symmetry considerations similar to the ones below.

2.5 Dual algebra and noncommutative fields

There are many intriguing proposals that have arisen from the breaking of the Lorentz group $SO(1,3)$ down to its subgroup $O(1,1)\times SO(2)$. One such corollary was proposed in **I** for the definition of fields in noncommutative space-time. The proposal in **I** is to restrict the Lorentz structure of usual field definitions to be applicable in a space-time with broken Lorentz symmetry. To motivate the proposal let us first have a look at finite Poincaré transformations.

Dual algebra. To discuss finite translations (a^μ) and Lorentz transformations (Λ^μ_ν) we need to introduce the dual language of Hopf algebras. The algebra $F(G)$ on the ordinary Poincaré group G, generated by the elements $\mathbf{a}^\mu(g)$ and $\mathbf{\Lambda}^\mu_\nu(g)$, $g\in G$, is dual to the universal enveloping algebra $\mathcal{U}(\mathcal{P})$. The elements $\mathbf{a}^\mu(g)$ and $\mathbf{\Lambda}^\mu_\nu(g)$ are complex valued functions that, when acting on the elements of the Poincaré group, return the familiar real-valued entries of the matrices of finite Lorentz transformations Λ^μ_ν, or the real-valued parameters of finite translations a^μ, as follows (no summation over repeated indices):

$$\begin{aligned}\mathbf{\Lambda}^\mu_\nu\left(e^{i\omega^{\alpha\beta}M_{\alpha\beta}}\right) &= (\Lambda_{\alpha\beta}(\omega))^\mu_{\ \nu}\,, \quad & \mathbf{\Lambda}^\mu_\nu\left(e^{ia^\alpha P_\alpha}\right) &= 0\,,\\ \mathbf{a}^\mu\left(e^{i\omega^{\alpha\beta}M_{\alpha\beta}}\right) &= 0, & \mathbf{a}^\mu\left(e^{ia^\alpha P_\alpha}\right) &= a^\mu\,. \end{aligned} \tag{2.38}$$

The duality is preserved after twisting the Poincaré algebra, but with a deformed multiplication in the dual algebra[6]. The deformed coproduct (2.29) of the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ turns into noncommutativity of translation parameters in the dual $F_\theta(G)$ [40–42]

$$[\mathbf{a}^\mu,\mathbf{a}^\nu] = i\theta^{\mu\nu} - i\mathbf{\Lambda}^\mu_\alpha\mathbf{\Lambda}^\nu_\beta\theta^{\alpha\beta}\,, \tag{2.39}$$

$$[\mathbf{\Lambda}^\mu_\nu,\mathbf{a}^\alpha] = [\mathbf{\Lambda}^\mu_\alpha,\mathbf{\Lambda}^\nu_\beta] = 0, \quad \mathbf{\Lambda}^\mu_\alpha,\mathbf{a}^\mu\in F_\theta(G)\,. \tag{2.40}$$

It was shown in **I** that whenever a Lorentz transformation is considered by which the noncommutative directions are mixed with the commutative directions, *there necessarily appear accompanying translation parameters*, i.e. the commutator in

[6] A basic property of the duality is that the coproduct and multiplication of the deformed Hopf algebra directly influence the multiplication and coproduct, respectively, of the deformed dual Hopf algebra [39].

(2.39) is nonzero. This is to be interpreted as the internal mechanism by which the commutator

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} , \tag{2.41}$$

remains invariant in these transformations, as required from the beginning.

Fields in noncommutative space-time. Since the quantum structure of space-time is introduced via Weyl quantization as discussed above, one would be tempted to say that in the construction of noncommutative quantum field theory this would be equivalent to redefining the multiplication of functions so that it is consistent with the twisted coproduct of the Poincaré generators (2.30). However, the definition of noncommutative fields and the action of the twisted Poincaré transformations on them is not this simple.

In commutative space-time, Minkowski space is realized as the quotient G/L of the Poincaré group G by the Lorentz group L. A classical field is a section of a vector bundle induced by a representation of the Lorentz group, that is an element of $C^\infty(\mathbb{R}^{1,3}) \otimes V$, where $C^\infty(\mathbb{R}^{1,3})$ is the set of smooth functions on Minkowski space and V is a Lorentz module. In noncommutative space-time this construction has no analogue, since Minkowski space cannot be similarly defined as a quotient of groups. This can intuitively be understood by the following:

In noncommutative space-time when acting on the field with a Lorentz generator we need to use the twisted coproduct as discussed in 2.4. This introduces momentum generators that would act on the Lorentz module V. Such action is not defined, however, and it seems we have reached an inconsistency.

The problem has been considered already earlier in [43], where it was proposed that the momentum generators would act trivially on V, i.e to change the properties of the Lorentz module. In **I** a simpler, but more dramatic solution was proposed. The idea is to retain V as a Lorentz module, but to discard the actions of all the generators not in the stability group $O(1,1) \times SO(2)$. This can be implemented by defining the fields to be elements of $C^\infty(\mathbb{R}^{1,1} \times \mathbb{R}^2) \otimes V$, i.e. to replace Minkowski space-time with the subspace $\mathbb{R}^{1,1} \times \mathbb{R}^2$ as the essential background of the fields. For quantization this poses no problems and we recover the same Hilbert space of states as in ordinary QFT (see **I** for details). Minkowski space-time would then only be the low-energy ($\theta^{\mu\nu} \to 0$) manifestation of this deeper structure.

It should be emphasized that the differences between ordinary and noncommutative quantum fields are drastic and there is no way to justify, based on the twisted

Poincaré symmetry, the claim [44] that the noncommutative fields transform under all Lorentz transformations as ordinary relativistic fields.

Chapter 3

Causality, unitarity and noncommutative time

In this chapter the problems that arise when time is a noncommutative coordinate are reviewed. Two fundamental aspects where problems appear are causality and unitarity. It is concluded that in the interaction picture and the Heisenberg picture of canonical quantization, as well as in the path integral formalism, there does not exist a unitary description that would be causal at the same time. The discussion is based mainly on **I** and **II**, concentrating however more on the general picture in the literature.

3.1 Causality

Shortly after the Seiberg-Witten paper [4] the general features of space-space, lightlike and time-space noncommutative theories were taken under study by several groups. The UV/IR mixing problem [45], to be discussed in section 4.1, was the first problem to emerge, followed shortly after by studies on causality [46] and unitarity [47]. Causality of noncommutative space-time in different contexts has been considered for example in [48–52]. In **I** and **II** we considered the space of solutions of the Tomonaga-Schwinger equation to show how in space-space noncommutative theories the so-called *light wedge causality condition* arises naturally, while the introduction of noncommutative time leads to causality violation.

Displaced wave-packets

In [46] a 2-particle scattering was considered in a $2 + 1$ -dimensional noncommutative ϕ^4 theory given by the action

$$S = \int d^4x \left(\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right) . \tag{3.1}$$

In (3.1) the stars in the quadratic terms have been dropped since in the action integral one star vanishes according to (2.24). The nonlocal effects of space-space noncommutativity show up as the displacement of the initial wave-packets in the direction of noncommutativity. The displacement is proportional to the incoming momentum, e.g. for initial momentum P_x the incoming wave splits into two parts displaced by

$$\Delta y = \frac{\theta P_x}{2} \tag{3.2}$$

on each side perpendicular to the original direction of motion. This can be intuitively interpreted [46] as the incoming point-like particles being replaced by rigid rods of length θP, extended perpendicular to their momentum, that only interact when their ends touch. This appears as instantaneous signal propagation in the noncommutative directions, but causality is preserved, as in Galilean causality, since effect never precedes cause.

In the case of time-space noncommutativity effectively the same thing happens, but now the rods are aligned with the momentum. Thus there is an advanced and a delayed wave at a distance $\frac{\theta P}{2}$ from the center of momentum. The advancement of the wave in itself does not violate causality, it is only when used in connection with Lorentz invariance that things start to go wrong. For example, when boosting the particle increasing its velocity, the expectation would be that the “rod” shortens, but now it will expand with the momentum. Thus the nonlocality of time conflicts with the efforts of constructing a Lorentz invariant model of noncommutative space-time. Still, by this argument alone, causality in this picture is not shown to be violated.

Two ways to time-order

For causality, as well as for unitarity (see section 3.2), the time-ordering procedure plays a key role [50, 53]. In [50] this was considered in the time-ordered perturbation theory (TOPT) approach introduced in [54] for the direct application of the Gell-Man-Low formula for Green's functions

$$G_n(x_1,\dots,x_k) = \frac{i^n}{n!}\int d^4z_1\dots d^4z_n \left\langle 0\middle|T\phi(x_1)\dots\phi(x_k)\mathcal{L}_I(z_1)\cdots\mathcal{L}_I(z_n)\middle|0\right\rangle^{con} , \tag{3.3}$$

where $\mathcal{L}_I$ is the interaction Lagrangian and the superscript ^{con} means projection onto the connected part. To match the notation of [50], the $\star$-product (2.23) is written in the form

$$(f\star g)(x) = \int d^4s \int \frac{d^4l}{(2\pi)^4}\, f(x-\tfrac{1}{2}\tilde{l})\, g(x+s)\, \mathrm{e}^{\mathrm{i}ls} , \qquad \tilde{l}^\nu = l_\mu\theta^{\mu\nu} . \tag{3.4}$$

As an example, let us consider the ϕ^4 scalar field theory in four-dimensional Minkowski space given by the action (3.1). In the Green's function

$$G(x,y) = \frac{g}{4!}\int d^4z \left\langle 0\middle|T\big(\phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big)\middle|0\right\rangle , \tag{3.5}$$

one has two natural choices for the time-ordering of the fields. Using the time-ordering with respect to the time coordinates that are *integration variables* as

$$\begin{aligned} G(x,y) = \int d^4z \int \prod_{i=1}^{3}\Big(d^4s_i\frac{d^4l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}l_is_i}\Big)&\Theta(s_1^0{+}s_2^0{+}s_3^0{+}\tfrac{1}{2}\tilde{l}_1^0)\Theta(z^0{-}\tfrac{1}{2}\tilde{l}_1^0{-}x^0)\\ \times\,\Theta(x^0{-}z^0{-}s_1^0{+}\tfrac{1}{2}\tilde{l}_2^0)&\Theta(z^0{+}s_1^0{-}\tfrac{1}{2}\tilde{l}_2^0{-}y^0)\Theta(y^0{-}z^0{-}s_1^0{-}s_2^0{+}\tfrac{1}{2}\tilde{l}_3^0)\\ \times\left\langle 0\middle|\phi(z{+}s_1{+}s_2{+}s_3)\phi(z{-}\tfrac{1}{2}\tilde{l}_1)\phi(x)\phi(z{+}s_1{-}\tfrac{1}{2}\tilde{l}_2)\right.&\left.\phi(y)\phi(z{+}s_1{+}s_2{-}\tfrac{1}{2}\tilde{l}_3)\middle|0\right\rangle , \end{aligned} \tag{3.6}$$

where $\Theta(x)$ is the Heaviside step function, one recovers the results of the "naive Feynman rules" [47], leading to the loss of unitarity as discussed below in section 3.2. If, on the other hand, one uses the time-ordering with respect to the "interaction points"

$$\begin{aligned} G'(x,y) = \int d^4z \int \prod_{i=1}^{3}\Big(d^4s_i\frac{d^4l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}l_is_i}\Big)&\Theta(x^0{-}z^0)\Theta(z^0{-}y^0)\\ \times\left\langle 0\middle|\phi(x)\phi(z{-}\tfrac{1}{2}\tilde{l}_1)\phi(z{+}s_1{-}\tfrac{1}{2}\tilde{l}_2)\right.&\left.\phi(z{+}s_1{+}s_2{-}\tfrac{1}{2}\tilde{l}_3)\phi(z{+}s_1{+}s_2{+}s_3)\phi(y)\middle|0\right\rangle , \end{aligned} \tag{3.7}$$

one is faced with the loss of causality, already clear from the acausal time-ordering under the integral. This latter time-ordering leads to a unitary theory [55, 56], but results in the the loss of the positive energy condition as well as the violation of causality [50, 53].

Time-ordering in path integral formalism. The same results for the different time-orderings were arrived at in the path integral formalism in [53].

Considering scalar field theory as an example, it is assumed, as usual, that the vacuum-to-vacuum transition amplitude from infinitely distant past to infinitely distant future is given by the path integral

$$\langle 0, +\infty | 0, -\infty \rangle_J = \int \mathcal{D}\phi \; exp \; \left[i \int d^4x \mathcal{L}_J \right] , \tag{3.8}$$

where $|0, \pm\infty\rangle_J$ are the asymptotic vacuum states at the times $t = \pm\infty$ and the Lagrangian with the local source function $J(x)$ is given by

$$\mathcal{L}_J(x) = \frac{1}{2}\partial^\mu \phi(x) \star \partial_\mu \phi(x) + \frac{1}{2} m^2 \phi(x) \star \phi(x) - \frac{\lambda}{3!} \phi(x) \star \phi(x) \star \phi(x) + \phi(x) \star J(x) \, . \tag{3.9}$$

Green's functions and hence all physics are given through Schwinger's action principle by functional derivatives of the path integral as, for example,

$$\langle 0, +\infty | T^\star \hat{\phi}(x) \hat{\phi}(y) | 0, -\infty \rangle = \frac{\delta}{i\delta J(x)} \frac{\delta}{i\delta J(y)} \langle 0, +\infty | 0, -\infty \rangle_J \Big|_{J=0} \, . \tag{3.10}$$

In the construction of Green's functions space-time is sliced with Heisenberg picture states and this choosing of the "path" automatically selects the time-ordering as the time-ordering with respect to the "times of the fields", denoted here by $T^\star$. Using these Green's functions one obtains the "naive Feynman rules" used in [47], and consequently the violation of unitarity.

In the path integral the time-ordering is thus inherently taken with respect to the times of the fields and not the times of the Hamiltonians (the "interaction points" in [50]), but one can formally check whether the different time-ordering could lead to a consistent theory. The main conclusion in [53], similarly as in [50], is that while the time-ordering with respect to the times of the interaction Hamiltonians indeed leads to a unitary theory, the positive energy condition is lost, i.e. negative energy particles are allowed to propagate in the forward time direction. This also leads to the well-known fact that the Wick rotation from Minkowski space to an Euclidean theory does not work as in commutative theories [55].

A further, and more detailed analysis of causality, with direct applicability to unitarity and energy-momentum conservation was presented in **I** and further elaborated on in **II** by analyzing the integrability condition of the Tomonaga-Schwinger equation.

3.1.1 Solutions of Tomonaga-Schwinger equation

The Tomonaga-Schwinger equation [57, 58] (see also [59, 60]) is the Lorentz covariant generalization of the single-time Schrödinger equation to include arbitrary Cauchy surfaces. It turns out that, as in commutative space-time, also in noncommutative space-time the integrability condition equals the microcausality condition. Further, it is a necessary requirement for energy-momentum conservation.

Commutative space-time. We consider the interaction picture, where operators evolve with the free Hamiltonian, while the states evolve with the Hamiltonian of interaction. In commutative theory the Tomonaga-Schwinger equation in the interaction picture reads

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma] = \mathcal{H}_{int}(x)\Psi[\sigma]\,. \tag{3.11}$$

When the hypersurface σ is a surface of constant time, the Tomonaga-Schwinger equation reduces to the single-time Schrödinger equation.

The existence of a *unique* solution to the Tomonaga-Schwinger equation is ensured if the integrability condition

$$\frac{\delta^2\Psi[\sigma]}{\delta\sigma(x)\delta\sigma(x')} - \frac{\delta^2\Psi[\sigma]}{\delta\sigma(x')\delta\sigma(x)} = 0, \tag{3.12}$$

with x and x' on the hypersurface σ, is satisfied. This integrability condition (3.12), plugged into (3.11), implies

$$[\mathcal{H}_{int}(x), \mathcal{H}_{int}(x')] = 0\,. \tag{3.13}$$

Since in the interaction picture the field operators satisfy free-field equations, they automatically satisfy Lorentz invariant commutation rules. The Lorentz invariant commutation relations are such that (3.13) is fulfilled only when x and x' are space-like separated,

$$(x-y)^2 < 0\,, \tag{3.14}$$

i.e. when σ is a space-like surface. As a result, the integrability condition (3.13) is equivalent to the microcausality condition for local relativistic QFT.

Noncommutative space-time. The use of the interaction picture has the advantage that the free-field equations for the noncommutative fields are identical to the corresponding free-field equations of the commutative case. Here our interest lies in what replaces the space-like surface σ in Weyl-Moyal space-time, i.e. for which type of separation of the coordinates x and y is the integrability condition satisfied. The Tomonaga-Schwinger equation in the noncommutative case reads

$$i\frac{\delta}{\delta\sigma'}\Psi[\sigma'] = \mathcal{H}_{int}(x)_\star \Psi[\sigma'] \,, \tag{3.15}$$

where σ' is to be determined and we make a simple choice for $\mathcal{H}_{int}(x)_\star$ as

$$\mathcal{H}_{int}(x)_\star = \lambda[\phi(x)]^n_\star = \lambda\phi(x) \star \phi(x) \star \ldots \star \phi(x) \,. \tag{3.16}$$

The corresponding integrability condition for (3.15) is

$$[\mathcal{H}_{int}(x)_\star, \mathcal{H}_{int}(y)_\star] = 0 \,, \quad \text{for } x, y \in \sigma' \,. \tag{3.17}$$

Using the integral representation of the $\star$-product (2.17), we can write (3.17) as

$$\lambda^2\big[(\phi \star \ldots \star \phi)(x), (\phi \star \ldots \star \phi)(y)\big] = \lambda^2 \int \prod_{i=1}^{n} da_i \, \mathcal{K}(x; a_1, \cdots, a_n)$$
$$\times \int \prod_{j=1}^{n} db_j \, \mathcal{K}(y; b_1, \cdots, b_n) \big[\phi(a_1) \ldots \phi(a_n), \phi(b_1) \ldots \phi(b_n)\big] \,. \tag{3.18}$$

Using the fact that in the interaction picture the field ϕ satisfies the same free-field equations and commutation relations as in the commutative case, it was shown in **I** and **II** that the necessary condition for the expression (3.18) to vanish is given by

$$\big[\phi(a_i), \phi(b_j)\big] = \Delta(a_i - b_j) = 0 \,. \tag{3.19}$$

Here, $\Delta(a_i - b_j)$ is the causal Δ-function of ordinary QFT. The condition (3.19) is satisfied outside of the mutual light cone

$$(a_i^0 - b_j^0)^2 - (a_i^1 - b_j^1)^2 - (a_i^2 - b_j^2)^2 - (a_i^3 - b_j^3)^2 < 0 \,. \tag{3.20}$$

In order to satisfy (3.18), it is necessary that (3.20) holds for all values of ${a_i}^k$ and ${b_j}^k$. However, since the coordinates are integration variables in the range

$$0 \le (a_i^k - b_j^k)^2 < \infty \,, \tag{3.21}$$

the requirement (3.20) is clearly not satisfied for the whole space of a_i^k and b_j^k. This in turn means that the integrability condition (3.18) is not satisfied for any x and y. Thus, the Tomonaga-Schwinger equation does not have a uniquely determined solution in a time-space noncommutative quantum field theory.

The fact that the condition (3.17) is not satisfied in general is a special case of the fact that the commutator of "local" observables composed with the $\star$-product, $[\mathcal{O}_\star(x), \mathcal{O}_\star(y)]$, does not vanish for any x and y. This clearly shows the violation of microcausality, which complements nicely the macroscopic analysis of [46].

When considering space-space noncommutativity, we would clearly end up integrating only over the spatial directions in (3.20). As an example, when only $\theta^{23} = -\theta^{32} \neq 0$ we would integrate over the second and third coordinate and end up with

$$(x^0 - y^0)^2 - (x^1 - y^1)^2 < 0\,, \tag{3.22}$$

which is the *light wedge causality condition.* The light wedge was first introduced in [48] purely on symmetry grounds as it is the natural modification of the light cone[1] symmetric under $O(1,1) \times SO(2)$, the stability group of $\theta^{\mu\nu}$.

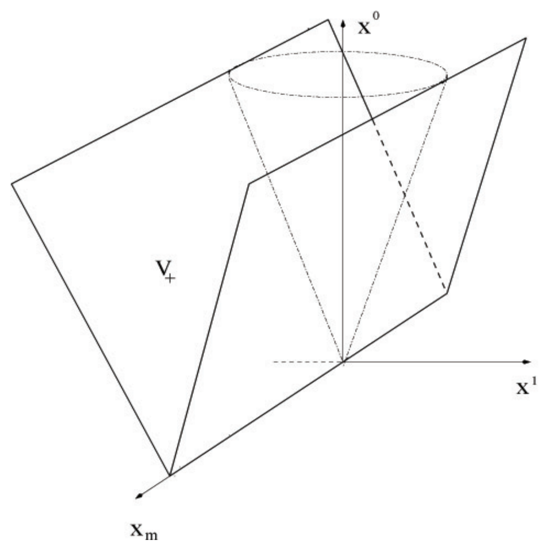


Figure 3.1: The light wedge symmetric under the group $O(1,1) \times SO(2)$. Figure adapted from [49].

[1]The light cone is symmetric under the full Lorentz group $SO(1,3)$.

3.2 Unitarity

The problem of unitarity violation in time-space noncommutative theories [47] appeared soon after the work of Seiberg and Witten [4]. As [47] used the covariant formalism in the interaction picture, it was natural to check whether these problems could be cured by using the Hamiltonian formalism or by using the Heisenberg picture. It turns out that in the Hamiltonian approach unitarity can indeed be kept intact [55, 56], but only at the expense of introducing other problems [48, 50, 53, 61, 62]. In **II** we used old results from nonlocal field theory to show that also in the Heisenberg picture unitarity violation appears as expected. For space-space noncommutativity no problems appear, as can be seen also from the string theory analysis.

3.2.1 String theory and unitarity

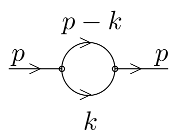
The unitarity of both space-space and lightlike noncommutative theories is clear from the string perspective. As space-space noncommutative theories are effectively unitary string theories in the limit where massive open strings and closed strings decouple rendering these theories to describe the dynamics of massless open strings, it is natural that they too are unitary.

For lightlike noncommutativity the decoupling limit was shown to exist for several *D-brane* configurations in [63]. Intuitively this is easy to understand when working in light cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}}\,(x^0 \pm x^1)$. Performing a light cone quantization with x^+ as the time coordinate one has $\theta^{0+} = 0$; the theory is local in the (light cone) time direction and thus acts like space-space noncommutative theories in this respect.

Using S-duality one can go from the string theory with a constant magnetic B_{ij} background to a dual theory with an electric E_{ij} background. Thus one might expect that by taking the decoupling limit of this perfectly unitary string theory a unitary noncommutative quantum field theory could be reached. In this case the noncommutativity would be of time-space type due to the electric field background. However, as shown in [64] (see also [65]), when taking the limit it is impossible to decouple the massive open strings while keeping θ^{0i} finite. In the decoupling limit a noncommutative quantum field theory is not reached and the limit describes a noncommutative open string theory. In [48] the nonunitarity of time-space noncommutative theories was interpreted in the string context as the production of extra propagating particles that are necessarily *tachyonic*. As propagating degrees of freedom these particles cannot be considered as formal devices and must be included in any attempt to construct a theory with noncommutative time. As we will see below, the string perspective agrees nicely with the general analysis of unitarity violation.

3.2.2 Unitarity in covariant formalism

The first results on unitarity in [47] are based on the modified (sometimes called "naive") Feynman rules first formulated in [66] and use the generalized unitarity rules, or *cutting rules*, of noncommutative scalar field theories. The noncommutative effects appear from the *nonplanar diagrams* that are further discussed in section 4.1 (see also figure 4.1). The modified Feynman rules introduce oscillatory factors at the vertices that depend on the external as well as loop momenta. As an example, the one loop diagram in ϕ^3 theory was shown to be proportional to



$$\propto \int d^4k \frac{1 + \cos\left(p_\mu \theta^{\mu\nu} k_\nu\right)}{((p-k)^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)}\,, \tag{3.23}$$

where the term with $\cos\left(p_\mu \theta^{\mu\nu} k_\nu\right)$ is the nonplanar contribution.

The unitarity analysis of [47] boils down to the sign of the term ($\theta^{\mu\nu}$ is given by (2.2))

$$p \circ q \equiv -p^\mu \theta_{\mu\alpha} \theta_{\alpha\nu} q^\nu = \theta(p_0^2 - p_1^2) - \theta'(p_2^2 + p_3^2)\,. \tag{3.24}$$

It was shown, assuming energy-momentum conservation, that whenever $p \circ q$ is negative definite perturbative unitarity is recovered. With the Minkowskian signature of the metric this only holds true for space-space noncommutative theories. Thus in theories with noncommutative time based on the modified, or "naive", Feynman rules perturbative unitarity in lost. The above treatment was extended to *lightlike* noncommutativity in [63], where the unitarity relation was shown to hold using string theory considerations.

3.2.3 Unitarity in Hamiltonian formalism

An alternative way for the quantization of field theories in noncommutative space-time was proposed in [27], further considered in [55] and reviewed in [61]. It is based on the introduction of the interaction Hamiltonian

$$H_{int}(t) = \int_{x_0=t} d^3x(\phi \star \phi \star \ldots \star \phi)(x) \tag{3.25}$$

and, assuming that the S-matrix exists, the time-ordering is taken with respect to the overall times of the Hamiltonians of interactions (and not with respect to the "times of the fields"), as

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \ldots \int_{-\infty}^{+\infty} dt_n \Theta(t_1 - t_2) \ldots \Theta(t_{n-1} - t_n) H_{int}(t_1) \ldots H_{int}(t_n) \,. \tag{3.26}$$

Since $H_{int}(t)$ is formally self-adjoint, the resulting S-matrix is formally unitary.

This seems to lead to a contradiction with the results from the modified Feynman rules discussed above. If, as in commutative theories, the Lagrangian and Hamiltonian formalisms described the same physics, unitarity would be violated also in the Hamiltonian approach. This is not the case, and the discrepancy can be traced back to the different time-ordering used, as already discussed in section 3.1 in connection with causality. When one uses the time-ordering with respect to the times of the fields and not the times of the Hamiltonians one is faced with unitarity violation, whereas in the latter case one finds violations of causality along with the failure of the positive-energy condition [50, 53].

In addition to the above, in [61] it was noted that in the Hamiltonian formalism with noncommutative time the interacting fields do not satisfy the usual equations of motion, raising doubt on the physical nature of these fields. In noncommutative QED it has further been shown [62] to lead to the violation of Ward identities. Thus it seems that by introducing a different time-ordering one merely shifts the problem from unitarity to other features of the theory. The problems introduced are severe and we are still missing a consistent approach to include the noncommutativity of time.

In **II** we explained the time-ordering ambiguity as a consequence of the failure of Matthew's theorem [67] discussed below. This in turn is a consequence of the non-existence of a unique solution to the Tomonaga-Schwinger equation discussed in section 3.1.1. Before examining Matthew's theorem, let us have a look at one more problem in the Hamiltonian formalism, the non-conservation of energy.

Non-conservation of energy. When time is noncommutative, the Lagrangian and Hamiltonian formulations of quantum field theory do not lead to the same predictions. This is hinted at already by the fact that the conjugate momentum $\pi(x) = \frac{\partial \mathcal{L}_\star}{\partial(\partial_0\phi(x))}$ cannot be uniquely defined as the Lagrangian contains an infinite amount of time derivatives. Taking $\pi(x)$ to be formally defined and using formally the Hamiltonian as in [55, 56]

$$\mathcal{H}_\star(x) = \frac{1}{2}\pi^2(x) + \frac{1}{2}(\partial_i\phi(x))^2 + \frac{1}{2}m^2\phi^2(x) + \frac{\lambda}{3!}\phi_\star^3(x)\,, \quad \pi(x) = \partial_t\phi(x)\,, \tag{3.27}$$

we can make this difference more transparent.

The time evolution of an operator in the interaction picture is given by

$$i\frac{dA^I(t)}{dt} = [A^I(t), H_0] + iU_0^\dagger\frac{\partial A^S}{\partial t}U_0, \qquad U_0 = e^{-iH_0t}, \tag{3.28}$$

where $A^I = U_0^\dagger A^S U_0$ is the operator in the interaction picture, A^S is the same operator in the Schrödinger picture and H_0 is the free Hamiltonian. In **II** we calculated the time evolution of (3.27) in a time-space noncommutative theory. The final result can be written with the help of the causal Δ-function of commutative quantum field theory

$$\big[\phi(a_i), \phi(b_j)\big] = \Delta(a_i - b_j)\,, \tag{3.29}$$

already used in section 3.1.1, as

$$\begin{aligned}
i\frac{\partial}{\partial t}H^\star_{int}(t) =& [H^\star_{int}(t), H_0] \\
=& \frac{\lambda}{3!}\int d^3x\, d^3y \int d^4a_1\, d^4a_2\, d^4a_3 \frac{1}{\pi^4 \det\theta}\mathcal{K}(y; a_1, a_2, a_3)\times \\
& \Big[\big(\partial_0\Delta(a_1 - x)\partial_0\phi(x) - \Delta(a_1 - x)\partial_0^2\phi(x)\big)\,\phi(a_2)\phi(a_3) \\
& + \phi(a_1)\,\big(\partial_0\Delta(a_2 - x)\partial_0\phi(x) - \Delta(a_2 - x)\partial_0^2\phi(x)\big)\,\phi(a_3) \\
& + \phi(a_1)\phi(a_2)\,\big(\partial_0\Delta(a_3 - x)\partial_0\phi(x) - \Delta(a_3 - x)\partial_0^2\phi(x)\big)\Big]\,.
\end{aligned} \tag{3.30}$$

The terms proportional to $\Delta(a_i - x)$ vanish only when a_i^0 coincides with t and we get contributions from all other times, including the distant future. Thus the evolution of the interaction Hamiltonian at the time t is influenced by field configurations in its future, signalling the lack of causality. The non-trivial time-dependence can be interpreted as the non-conservation of the energy associated with the interacting system.

In section 3.1.1 we saw that in these theories the integrability condition of the Tomonaga-Schwinger equation cannot be fulfilled. The integrability condition has been shown to be a requirement for energy-momentum conservation in the interaction picture in [68] and thus these two results agree nicely. It is further connected to the fact that the equation of motion for the interacting scalar field differs from the one in commutative theory [55, 61].

Matthew's theorem. The source of all the inconsistencies can be understood as the failure of Matthew's theorem [67] in time-space noncommutative theories. The theorem in commutative space-time states that the S-matrix given in the Hamiltonian formalism agrees with the one in the Lagrangian formalism

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} d^4x_1 \dots d^4x_n \, T[\mathcal{H}_{int}(x_1) \dots \mathcal{H}_{int}(x_n)] \tag{3.31}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} d^4x_1 \dots d^4x_n \, T^\star[\mathcal{L}_{int}(x_1) \dots \mathcal{L}_{int}(x_n)] \, , \tag{3.32}$$

where $T^\star$ is the covariant modification of the usual time-ordering

$$T^\star \left[\frac{\partial\phi(x)}{\partial x^\mu} \frac{\partial\phi(y)}{\partial y^\nu} \right] = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} T[\phi(x)\phi(y)] \neq T \left[\frac{\partial\phi(x)}{\partial x^\mu} \frac{\partial\phi(y)}{\partial y^\nu} \right] . \tag{3.33}$$

It should be noted that for theories with higher-derivative interactions we have

$$\mathcal{H}_{int}(x) \neq -\mathcal{L}_{int}(x) \tag{3.34}$$

and thus care must be taken when comparing the dynamics of the two formulations.

If Matthew's theorem would hold also in time-space noncommutative theories, there would be no ambiguity between the two formalisms; the predictions of both the Hamiltonian and the Lagrangian approaches would coincide. However, a crucial requirement for the proof of the theorem is the uniqueness of the solutions of the Tomonaga-Schwinger equation. This can always be achieved in a theory with a finite number of time derivatives in the interaction term, but not when the number of such derivatives is infinite as in the $\star$-product in the case of a noncommutative time, as explicitly shown in section 3.1.1.

Thus the two "S-matrices" (3.31) and (3.32) are inequivalent, leading to different predictions. In the covariant formalism unitarity is violated, while in the Hamiltonian approach with the time-ordering T there appear violations of causality and energy-momentum conservation.

3.2.4 Unitarity in Heisenberg picture

As the interaction picture shows inconsistencies in the Hamiltonian as well as the covariant approach when time is noncommutative, the Heisenberg picture has sometimes been used instead [61, 69–71]. Historically, for the study of nonlocal field theories the approach of Yang, Feldman and Källén [72, 73] has customarily been used [74–78]. The advantage of this approach is that the elements of the S-matrix are calculated directly in terms of Heisenberg picture n-point correlation functions (Green's functions), avoiding the use of the interaction picture entirely.

As noncommutative field theories are infinitely nonlocal field theories with the nonlocal kernel given by (2.22), the results of nonlocal field theories from the 1950's to the 1970's can be used in studying their properties. In **II** we reviewed the careful analysis of Marnelius [77, 78] on the nonlocal Kristensen-Møller model [75] to show the expected failure of unitarity of time-space noncommutative theories also in the Heisenberg picture [79].

The possible violation of unitarity was recognized already in [61] in connection with the non-trivial asymptotics of the fields. As a possible resolution to the unitarity problem, and to help with renormalizability, the *quasiplanar Wick products* were introduced (see also [80]) leading to changes in the dispersion relations. This affects the asymptotics and, it is claimed, could lead to a unitary description. This modification of the dispersion relations as a manifestation of Lorentz invariance violation was noted in [70] to imply that the "conceptual basis of the present approach is rather shaky". In [71] it is further noted that the modified dispersion relations give rise to acausal effects that, however small, raise doubt on the consistency of the Heisenberg picture.

One technical issue to be noted in the discussion of [61] is the use of usual test-functions. As argued in [81], in noncommutative space-time the proper test-function space is a Gel'fand-Shilov space that takes into account the nonlocality of space-time. Thus, whenever using test-functions in noncommutative space-time, proper care should be taken in order not to lose information of the inherent nonlocality.

Unitarity of nonlocal field theories

The work of Marnelius on the nonlocal Kristensen-Møller model is a generalization of [82] to include q-number variations in addition to the previously used

c-number variations. This was done with the hope of restoring asymptotic completeness in the theory in order to define a unitary S-matrix. As it turns out, including q-number variations does not help, and a unitary S-matrix cannot be derived.

The Kristensen-Møller model is described by the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu\phi(x)\partial^\mu\phi(x) - \mu^2\phi^2(x)) + i\frac{1}{2}\bar{\psi}(x)\partial\!\!\!/\psi(x) - m\bar{\psi}(x)\psi(x) + \mathcal{L}_{int}(x)\,,$$
$$\mathcal{L}_{int}(x) = -g\int d^4a_1d^4a_2d^4a_3K(x;a_1,a_2,a_3)i\bar{\psi}(a_1)\gamma_5\phi(a_2)\psi(a_3)\,, \tag{3.35}$$

from which the following equations of motion follow:

$$(\partial_\mu\partial^\mu + \mu^2)\phi(x) = -\,g\int d^4\xi d^4\eta\ K(\xi,\eta)\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi) = -g\rho(x)\,, \tag{3.36}$$
$$i(\partial\!\!\!/ - m)\psi(x) = g\int d^4\xi d^4\eta\ K(\xi,\eta)i\gamma_5\phi(x-\eta)\psi(x-\eta+\xi) = gf(x)\,, \tag{3.37}$$

where $\xi = a_3 - a_2$, $\eta = a_1 - a_2$ and $x = a_2$.

In the Yang-Feldman-Källén procedure the solutions of (3.36) and (3.37) are written in terms of either the *in* or *out* fields, as

$$\phi(x) = \phi_{in}(x) - g\int d^4y\Delta_R(x-y)\rho(x) \quad ; \quad \phi(x) = \phi_{out}(x) - g\int d^4y\Delta_A(x-y)\rho(x)\,,$$
$$\psi(x) = \psi_{in}(x) + g\int d^4yS_R(x-y)f(x) \quad ; \quad \psi(x) = \psi_{out}(x) + g\int d^4yS_A(x-y)f(x)\,. \tag{3.38}$$

Here Δ_R, Δ_A, S_R and S_A are the usual retarded and advanced Green's functions for bosonic and fermionic fields. The asymptotic *in* or *out* fields defined at $t \to \mp\infty$ satisfy free-field equations. The lack of causality is manifest in (3.38), since the behaviour of an interacting quantum field at a given space-time point is determined by its *entire past and future history.* There seems to be a conflict between the equations of motion and the boundary conditions, but the above formal solutions are assumed.

Marnelius looked for solutions that can be expressed iteratively as a series expansion in the *in* fields or alternatively in the *out* fields as

$$\phi(x) = \phi_{in/out}(x) + \sum_{n=1}^{\infty} g^n\phi^{(n)}(x;in/out)\,,$$
$$\psi(x) = \psi_{in/out}(x) + \sum_{n=1}^{\infty} g^n\psi^{(n)}(x;in/out)\,, \tag{3.39}$$

where the $\phi^{(n)}(x; in/out)$ and $\psi^{(n)}(x; in/out)$ are functionals of the *in* or *out* fields, respectively. For the theory to be consistent the same interacting fields $\phi(x)$ and $\psi(x)$ should be obtained by using either expansion. This turns out not to be the case.

By looking at the *in* and *out* representations of generators, given by the limits

$$\begin{aligned} F_0(t; in) &= \lim_{t_0 \to -\infty} F_{t_0}(t)\,, \\ F_0(t; out) &= \lim_{t_0 \to +\infty} F_{t_0}(t)\,. \end{aligned} \tag{3.40}$$

we get the difference of the *in* and *out* representations as

$$F(t; out) - F(t; in) = \frac{g}{2} \int_{-\infty}^{+\infty} d^4x\ \delta_0 A(x)\,, \tag{3.41}$$

where $\delta_0 A(x)$ in the Kristensen-Møller model is given by

$$\begin{aligned} \delta_0 A(x) = \int d^4\eta\, d^4\xi F(\xi, \eta)\big([\delta_0\phi(x), \bar\psi(x+\eta)]i\gamma_5\psi(x+\xi) \\ - \bar\psi(x+\eta)]i\gamma_5[\delta_0\phi(x), \psi(x+\xi)]\big)\,. \end{aligned} \tag{3.42}$$

The main result[2] is that when considering the momentum generators there is a difference in the *in* and *out* representations in fourth order of the coupling constant g, i.e.

$$P^\nu(t; out) - P^\nu(t, in) = g^4\ F[\psi_{in}; \bar\psi_{in}] + \mathcal{O}(g^5)\,, \tag{3.43}$$

where $F[\psi_{in}; \bar\psi_{in}]$ is a functional of ψ_{in} and $\bar\psi_{in}$ given in [78] and in **II**. The nonvanishing of (3.43) will result in the expansions (3.39) in terms of *in* and *out* fields giving different expressions for the interacting quantum fields $\phi(x)$ and $\psi(x)$. This follows from the requirement that the momentum generators perform the transformations that they are assumed to perform

$$\begin{aligned} [P^\nu, \phi(x)] &= -i\partial^\nu\phi(x)\,, \\ [P^\nu, \psi(x)] &= -i\partial^\nu\psi(x)\,, \end{aligned} \tag{3.44}$$

irrespective of the representation. If the fields in the two representations coincided, we would have for example

$$[P^\nu(t; out) - P^\nu(t; in), \psi(x)] = 0\,. \tag{3.45}$$

[2]See [78] for details.

Since the difference of the momentum generators (3.43) is proportional to the fields ψ_{in} and $\bar{\psi}_{in}$ at different times, the commutator (3.45) is nonzero in fourth order in g. It follows that the interacting fields derived from the *in* fields can not be the same as those derived from the *out* fields.

The nonuniqueness of solutions, in turn, leads to the nonstationarity of the action for both sets of solutions. This is because the variation of the total action turns out to be equal to the difference of the generators in the *in* and *out* representations

$$\int \delta \left(d^4x \mathcal{L}(x) \right) = F(t; out) - F(t; in) = \frac{g}{2} \int_{-\infty}^{+\infty} \delta_0 A(x).$$

From the lack of asymptotic completeness it is concluded that there does not exist a unitary S-operator that would relate the *in* and *out* fields by a similarity transformation

$$\begin{aligned} \phi_{out}(x) &= S^{-1} \phi_{in}(x) S \,, \\ \psi_{out}(x) &= S^{-1} \psi_{in}(x) S \,. \end{aligned} \tag{3.46}$$

Rather, as was shown in [78, 82], from

$$\psi_{out}(x) = \psi_{in}(x) - g \int d^4y S(x-y) f(y) \,, \tag{3.47}$$

a direct calculation in fourth order g gives

$$S^\dagger \psi_{in}(x) S = \psi_{out}(x) + g^4(\cdots) \neq \psi_{out}(x) \,, \tag{3.48}$$

i.e. there is no unitary S-operator satisfying (3.46) in this picture.

In conclusion, if we consider that quantum fields satisfy the equations of motion in the Yang-Feldman-Källén approach, the infinitesimal generators will be modified and the field expressions in the *in* or *out* representations will not coincide. This discrepancy further leads to the nonexistence of a unitary S-matrix in the Heisenberg picture. Thus in the Heisenberg picture, as in the interaction picture, there are unresolved problems in constructing theories with noncommutative time that would be both unitary and causal.

Chapter 4

Effects of quantization of space

In the previous chapter the problems of theories with noncommutative time were examined, showing violations of unitarity and causality in various approaches. The remainder of the thesis is concentrated on applications with non-trivial commutators in spatial directions only, i.e. $\theta^{0i} = 0$. The infinite nonlocality introduced by the $\star$-product leads to the mixing of ultraviolet and infrared divergences as well as technical problems in the use of different coordinate systems.

4.1 UV/IR mixing

The original hope in introducing the uncertainty of space-time to quantum field theories was to rid them of short scale (UV) singularities and renormalization problems as a consequence [20, 21]. This hope was not to be realized. Although much of this thesis is concentrared on highlighting the problems associated with noncommutative time, there is a standing problem in all noncommutative quantum field theories, the mixing of low-energy (IR) and high-energy (UV) divergences [45] (see also [83]).

As in [45], let us consider the simplest example, the ϕ^4 scalar field theory in four-dimensional Euclidean space given by the action

$$S = \int d^4x \left(\frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) . \tag{4.1}$$

The free theory is identical to the commutative free theory and thus the 1-particle irreducible two point function in lowest order is, as usual, the inverse propagator

$$\Gamma_0^{(2)} = p^2 + m^2 . \tag{4.2}$$

The noncommutative corrections arise in 1-loop order and are given by the following diagrams:

Figure 4.1: Planar and nonplanar one loop corrections to $\Gamma^{(2)}$ in ϕ^4 theory. Figure adapted from [45].

The one loop corrections from these *planar* and *nonplanar* graphs are given respectively by

$$\begin{aligned}\Gamma^{(2)}_{1\,planar} &= \frac{\lambda}{3(2\pi)^4}\int \frac{d^4k}{k^2+m^2}\,,\\ \Gamma^{(2)}_{1\,nonplanar} &= \frac{\lambda}{6(2\pi)^4}\int \frac{d^4k}{k^2+m^2}e^{ik_\mu\theta^{\mu\nu}p_\nu}\,.\end{aligned} \tag{4.3}$$

Regularizing the momentum integrals at the energy scale Λ, these expressions are expanded as

$$\begin{aligned}\Gamma^{(2)}_{1\,planar} &= \frac{\lambda}{48\pi^2}\left[\Lambda^2 - m^2\ln\left(\frac{\Lambda^2}{m^2}\right)+\mathcal{O}(1)\right]\,,\\ \Gamma^{(2)}_{1\,nonplanar} &= \frac{\lambda}{96\pi^2}\left[\Lambda^2_{eff} - m^2\ln\left(\frac{\Lambda^2_{eff}}{m^2}\right)+\mathcal{O}(1)\right]\,,\end{aligned} \tag{4.4}$$

where

$$\Lambda^2_{eff} = \frac{1}{1/\Lambda^2 + p\circ p}, \qquad p\circ q \equiv -p^\mu\theta_{\mu\alpha}\theta^\alpha_\nu q^\nu\,. \tag{4.5}$$

The planar contribution is proportional to the expression in the commutative theory, which diverges quadratically in the limit $\Lambda\to\infty$. Thus the novelty due to noncommutativity is given by the nonplanar graphs that introduce the term $p\circ q$.

When $\Lambda\to\infty$, $\Lambda_{eff}\to\frac{1}{p\circ q}$, which is finite as long as $\theta^{\mu\nu}p_\nu$ is nonzero. Thus $\theta^{\mu\nu}$ introduces a regularization in the UV limit. In particular, for the full expression of the nonplanar part we have

$$\Gamma^{(2)}_{1\,nonplanar} \xrightarrow{\Lambda\to\infty} \frac{\lambda}{96\pi^2}\left[\left(\frac{1}{p\circ q}\right)^2 - m^2\ln\left(\frac{1}{m^2\,(p\circ q)^2}\right)+\mathcal{O}(1)\right]\,. \tag{4.6}$$

This expression diverges in the low-energy limit $p \to 0$ of the external momentum, or in general when $\theta^{\mu\nu} p_\nu \to 0$. Taking the limits in the other order the divergence structure is different – the high-energy and low-energy limits do not commute. This mixing of the high-energy and low-energy scales is called *UV/IR mixing*. Considered here only in the simplest case of a scalar field theory, the effect has also been confirmed in gauge field theories [83, 84]. As an example, in [83] it was shown that noncommutativity induces linear and quadratic divergences absent in commutative gauge theories. Including supersymmetry removes these extra poles at one loop level, but logarithmic divergences persist at small values of $\theta^{\mu\nu} p_\nu$.

The mixing of divergences can be considered a consequence of the inherent nonlocality of the theory discussed in section 2.3. In [45] the effect was qualitatively explained as follows: Let ϕ be a free field spread in the x, y-plane with the (approximate) widths Δ_x and Δ_y in the x and y directions respectively. Then $\phi \star \phi$ has the corresponding widths $\delta_x \approx \max\left(\Delta_x, \frac{\theta}{\Delta_x}\right)$ and $\delta_y \approx \max\left(\Delta_y, \frac{\theta}{\Delta_y}\right)$. Thus when ϕ is very well localized, i.e. when Δ is very small (UV), the corresponding spread $\frac{\theta}{\Delta}$ in $\phi \star \phi$ becomes large (IR).

In interacting quantum theory the effect will be more pronounced due to high-energy virtual particles that contribute even to low-energy processes. In the non-planar graphs, a virtual particle of energy ω will, upon interacting, spread and thus contribute to events at energies $\sim \frac{1}{\theta\omega}$. Thus placing a UV cutoff Λ effectively induces an IR cutoff $\frac{1}{\Lambda}$ and again, the high-energy and low-energy limits do not commute. This effectively spoils the hope for improved renormalization behaviour [85] for which the noncommutativity was originally introduced.

Naturally, there are attempts to construct a noncommutative theory without UV/IR mixing in order to restore renormalizability. Most notably, there have been significant advances in the Grosse-Wulkenhaar model [86, 87] defined by the action

$$S_{GW}[\phi] = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{1}{2}\mu_0^2\, \phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x)\,, \tag{4.7}$$

where $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu$ and the Euclidean metric is used. The inclusion of the harmonic potential term has been shown to make the theory renormalizable to all orders along with other nice features such as a vanishing β-function.

However, the inclusion of the explicitly translational invariance breaking term leads to non-conservation of energy-momentum. To obtain similar renormalizability, but still retain translational invariance, modified models have since been put forward. As an example, the model of Gurau *et al* [88] is given by the action

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi - \frac{1}{2}\phi \frac{a^2}{\tilde{\partial}^2}\phi + \frac{1}{2}\mu_0^2\, \phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x)\,, \tag{4.8}$$

where $\tilde{\partial}^\mu = \theta^{\mu\nu}\partial_\nu$. The effect of the extra term is to modify the propagator as

$$\frac{1}{p^2 + m^2 + \frac{a^2}{\theta^2 p^2}}\,, \quad a > 0\,. \tag{4.9}$$

This modification counter-acts the IR divergent contributions and indeed makes the ϕ^4 theory renormalizable.

There has been much effort to extend these results to gauge theories, with no success to date. As the UV/IR mixing is in the end due to the infinite nonlocality induced by the commutator (2.1), apparent in the $\star$-product (2.21), one would (at least naively) expect that for a consistent removal of all divergences in the theory the structure (2.1) should be modified to tame the nonlocality.

4.2 Spherical coordinates in quantum space

The direct generalization of the commutation relations of quantum mechanics leads to the commutator (2.1), explicitly written in terms of cartesian coordinates. When dealing with systems where the commutative theory is spherically symmetric, such as black holes or magnetic monopoles[1], one is faced with the question whether to stay in the cartesian basis to use (2.1) straightforwardly, or to switch to spherical coordinates in order to exploit the spherical symmetry of the corresponding commutative theory? A further question is whether it would be justified, in order to simplify the calculations, to replace (2.1) by a similar, but clearly inequivalent, commutator among the spherical coordinates such as

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu} \quad x^\mu = t, r, \phi, \theta\,. \tag{4.10}$$

Just such questions have been considered explicitly in [89] and implicitly for example in [90–92]. However, as there has been some confusion on the equivalence

[1] As the next chapter focuses on noncommutative magnetic monopoles, this question is especially relevant.

of the different choices of commutator, it is appropriate to consider the matter more fully.

The commutator (4.10) is different from (2.1) already by dimensional count and thus clearly describes different physics. The main idea in [89], further used in [90], is to look at the difference of the physics one gets when using the commutator

$$\left[\frac{\hat{r}^2}{2}, \hat{\phi}\right] = i\theta\,, \tag{4.11}$$

instead of (2.1) in 2+1 dimensions. The program is to calculate noncommutative quantities directly in polar coordinates using (4.11) and then compare the results to the expression one gets by the *correct* method

1. Performing the change of variables from polar coordinates to rectangular coordinates.
2. Calculating the deformation expansion with the cartesian noncommutativity (2.1).
3. Changing variables back to polar coordinates.

In [89] it was found that the results of the two calculations are different already in first order in θ. This was not expected, as it was argued, directly in the operator formalism, that the two commutators are equivalent in first order in the deformation parameter.

In order to investigate the difference in the deformed structures given by the two commutators (4.11) and (2.1), we will next do an explicit comparison up to third order in θ using the twist approach. It is further shown that this discussion is only sensible in $2+1$ dimensions, since in $3+1$ dimensions it is clear already in first order that the two structures cannot be claimed to be equal. Finally, in section 4.2.1 the use of the more complicated quadratic twist element is considered with the hope of simplifying the results.

Difference in polar coordinates

To compare the symplectic structures described by the two commutators (2.1) and (4.11), let us calculate the commutators

$$[x, y]_{\star/\star'}\,; \qquad \left[\frac{r^2}{2}, \phi\right]_{\star/\star'}\,, \tag{4.12}$$

using both the usual $\star$-product induced by (2.1) and the $\star'$-product given by the twist element corresponding to (4.11)

$$\mathcal{F}'_* = \exp\left[-i\theta\left(\frac{\partial}{\partial r^2}\otimes\frac{\partial}{\partial\phi}-\frac{\partial}{\partial\phi}\otimes\frac{\partial}{\partial r^2}\right)\right]. \tag{4.13}$$

The reason why this turns out to be interesting is that for $[\frac{r^2}{2},\phi]_{\star/\star'}$ both $\star$-products give the same result.

To begin with, it is good to note that in *any $\star$-commutator all even order terms vanish* due to the antisymmetry of $\theta^{\mu\nu}$. Further, as $r^2 = x^2 + y^2$, it turns out that all the terms in $[\frac{r^2}{2},\phi]_\star$ that are higher than second order in θ vanish. This follows, since in each term there is a total of more than two derivatives with respect to x and y, acting on r^2 and

$$\partial_x^n\partial_y^m(x^2+y^2) = 0\,, \tag{4.14}$$

whenever $n, m > 2$ or when both n and m are nonzero. Thus the only contribution comes in first order and is given by

$$\begin{aligned}\left[\frac{r^2}{2},\phi\right]_\star = i\theta\Big[&\partial_x\left(\frac{x^2+y^2}{2}\right)\partial_y\arctan\left(\frac{y}{x}\right)\\ &-\partial_y\left(\frac{x^2+y^2}{2}\right)\partial_x\arctan\left(\frac{y}{x}\right)\Big] = i\theta\,.\end{aligned} \tag{4.15}$$

Since using the $\star'$-product corresponding to (4.13) obviously gives the same result, one might hastily conclude that the two commutators describe the same deformation of space-time. This would be premature however, as can be seen by considering the other non-trivial combination, $[x,y]_{\star'}$, up to third order in θ (with the notation $R \equiv \frac{r^2}{2}$)

$$\begin{aligned}[x,y]_{\star'} =&[x,y] + i\theta\left(\partial_R x\partial_\phi y\right)\\ &+\left(\frac{i}{2}\right)^3\frac{1}{3!}\theta^3\left(2\partial_R^3x\partial_\phi^3y - 6\partial_R^2\partial_\phi x\partial_\phi^2\partial_R y + 6\partial_R\partial_\phi^2x\partial_\phi\partial_R^2y - 2\partial_\phi^3x\partial_R\phi^3y\right) + \mathcal{O}(\theta^5)\\ =&2i\theta - i\frac{\theta^3}{r^4} + \mathcal{O}(\theta^5)\,.\end{aligned} \tag{4.16}$$

Thus the two structures given by (2.1) and (4.11) are different and describe different deformations. In first order there is a numerical factor of 2 difference, and in higher orders one can see that also the functional form is different.

To give a flavour of how the different structures affect calculations we can, as an example, note the difference in a simple commutator calculated with $\star$ and $\star'$

$$\begin{aligned} [r,\phi]_{\star'} &= i\frac{\theta}{r}\,, \\ [r,\phi]_{\star} &= i\frac{\theta}{r} - \frac{1}{3!}\left(\frac{i}{2}\right)^3 \theta^3 \frac{12}{r^5} + \mathcal{O}(\theta^5)\,, \end{aligned} \tag{4.17}$$

where the result on the first line is exact. Thus it is not justified to use (4.11) in calculations that aim to describe physics in the Moyal-deformed space-time corresponding to (2.1). As shown in the following, in $3+1$ dimensions this is evident already in first order of perturbation.

Noncommutative spherical coordinates. To look at the more relevant 3+1 -dimensional space-times, let us consider the commutator (2.1), where the θ-matrix in the most general form is given by

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_3 & -\theta_2 \\ 0 & -\theta_3 & 0 & \theta_1 \\ 0 & \theta_2 & -\theta_1 & 0 \end{pmatrix}\,, \tag{4.18}$$

in four-dimensional spherical coordinates (t, r, ϕ, θ) parametrized as usual

$$x = r\cos\phi\sin\theta; \qquad y = r\sin\phi\sin\theta; \qquad z = r\cos\theta\,. \tag{4.19}$$

The commutators of coordinates in first order are readily calculated to give

$$\begin{aligned} [r,\phi]_\star &= i[\theta_1(\partial_y r\partial_z\phi - \partial_z r\partial_y\phi) \\ &\quad - \theta_2(\partial_x r\partial_z\phi - \partial_z r\partial_x\phi) \\ &\quad + \theta_3(\partial_x r\partial_y\phi - \partial_y r\partial_x\phi)] + \mathcal{O}(\theta^2) \\ &= \frac{i}{r}\left[-\theta_1\frac{\cos\phi\cos\theta}{\sin\theta} - \theta_2\frac{\sin\phi\cos\theta}{\sin\theta} + \theta_3\right] + \mathcal{O}(\theta^2)\,, \\ [r,\theta]_\star &= \frac{i}{r}\left[-\theta_1\sin\phi + \theta_2\cos\phi\right] + \mathcal{O}(\theta^2)\,, \\ [\phi,\theta]_\star &= -\frac{i}{r^2}\left[\theta_1\cos\phi + \theta_2\sin\phi + \theta_3\frac{\cos\theta}{\sin\theta}\right] + \mathcal{O}(\theta^2)\,. \end{aligned} \tag{4.20}$$

Thus in 3+1 dimensions the noncommutativity of spherical coordinates has a more complicated functional form than (2.1) already in first order of perturbation. From this one can directly conclude that these deformations describe different physics and no confusion, as in $2+1$ dimensions, should arise.

4.2.1 Using a quadratic twist element

Using the Abelian deformation (2.32) helps us retain the translational invariance of the corresponding commutative theory, as can be seen from the undeformed coproduct (2.33). Thus it is worthwhile to check whether using a different twist could help us retain at least a part of the rotational symmetry of the corresponding commutative theory. To investigate this, we consider *quadratic deformations*, considered with rectangular coordinates in [93].

The quadratic deformation is given by the twist element

$$\mathcal{F}_{(2)} = e^{\frac{i}{2}\theta_{(2)}^{\alpha\beta\gamma\delta} M_{\alpha\beta}\otimes M_{\gamma\delta}}\,, \tag{4.21}$$

where from the antisymmetry of $M_{\alpha\beta}$ directly follows: $\theta_{(2)}^{\alpha\beta\gamma\delta} = -\theta_{(2)}^{\beta\alpha\gamma\delta} = -\theta_{(2)}^{\alpha\beta\delta\gamma} = -\theta_{(2)}^{\gamma\delta\alpha\beta}$. In order to satisfy the twist condition (2.31) we further require that all the indices of $\theta_{(2)}$ are different and fixed, i.e. we fix the frame of reference so that there is only one nonzero parameter in the object $\theta_{(2)}$.

Symmetries. The preservation or breaking of any symmetry under twisting can be seen from the deformed coproduct (2.29) of the corresponding generator. For example, using the Abelian twist results in preserved translation invariance since the coproduct of translation generators (2.33) remains trivial. On the other hand, the breaking of Lorentz symmetry is apparent from the deformed coproduct of Lorentz generators (2.34).

To follow this line of thought let us have a look at the deformations of coproducts for Poincaré generators using the quadratic twist (4.21). Using the Campbell-Baker-Hausdorff formula (2.19) we get for the Lorentz generators

$$\begin{aligned}
\Delta_{(2)}(M_{\mu\nu}) =&\, \Delta_0(M_{\mu\nu}) + \frac{i}{2}\theta_{(2)}^{\beta\alpha\gamma\delta}\left[M_{\alpha\beta}\otimes M_{\gamma\delta}, M_{\mu\nu}\right] \\
&+ \frac{1}{2!}\left(\frac{i}{2}\right)^2 \theta_{(2)}^{ijkl}\theta_{(2)}^{\beta\alpha\gamma\delta}\left[M_{ij}\otimes M_{kl},\left[M_{\alpha\beta}\otimes M_{\gamma\delta}, M_{\mu\nu}\right]\right] + \mathcal{O}(\theta^3) \\
=&\, \Delta_0(M_{\mu\nu}) + \frac{1}{2}\theta_{(2)}^{\beta\alpha\gamma\delta}\big[(\eta_{\alpha\mu}M_{\beta\nu} - \eta_{\alpha\nu}M_{\beta\mu} - \eta_{\beta\mu}M_{\alpha\nu} + \eta_{\beta\nu}M_{\alpha\mu})\otimes M_{\gamma\delta} \\
&+ M_{\alpha\beta}\otimes(\eta_{\gamma\mu}M_{\delta\nu} - \eta_{\gamma\nu}M_{\delta\mu} - \eta_{\delta\mu}M_{\gamma\nu} + \eta_{\delta\nu}M_{\gamma\mu})\big] + \mathcal{O}(\theta^2)\,,
\end{aligned} \tag{4.22}$$

with nonvanishing contributions in all orders of θ, and different from Δ_0 for all $M_{\mu\nu}$.

Similarly, for the translation generators we have

$$\begin{aligned}\Delta_{(2)}(P_\mu) =& \Delta_0(P_\mu) + \frac{i}{2}\theta_{(2)}^{\beta\alpha\gamma\delta}\left[M_{\alpha\beta}\otimes M_{\gamma\delta}, P_\mu\right] \\ &+ \frac{1}{2!}\left(\frac{i}{2}\right)^2 \theta_{(2)}^{ijkl}\theta_{(2)}^{\beta\alpha\gamma\delta}\left[M_{ij}\otimes M_{kl},\left[M_{\alpha\beta}\otimes M_{\gamma\delta}, P_\mu\right]\right] + \mathcal{O}(\theta^3) \qquad (4.23)\\ =& \ \Delta_0(P_\mu) + \frac{1}{2}\theta_{(2)}^{\beta\alpha\gamma\delta}\left[(\eta_{\alpha\mu}P_\beta - \eta_{\beta\mu}P_\alpha)\otimes M_{\gamma\delta} + M_{\alpha\beta}\otimes(\eta_{\gamma\mu}P_\delta - \eta_{\delta\mu}P_\gamma)\right] + \mathcal{O}(\theta^2)\,,\end{aligned}$$

which is nonzero in all orders of θ and for all components of P_μ.

Unlike the Abelian case, the infinite series expansions (4.22) and (4.23) make the coproducts intractable for anything but perturbative calculations. Nevertheless we can deduce symmetries of the theory from the form of these coproducts: as the coproducts of all Poincaré generators are non-trivial, we are led to conclude that the original symmetry is broken completely. The residual symmetries that were present when using the Abelian twist do not appear when considering the quadratic twist.

Example calculation using the quadratic twist. To compare the expressions one gets when using the quadratic twist with those derived with the Abelian twist, let us calculate $[r,\phi]_\star$ to third order in θ in a quadratically deformed Minkowski space and compare with (4.17). We choose the frame of reference such that only $\theta_{(2)}^{0123} \neq 0$ and for the Lorentz generators we use the usual realization $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$. In first order we have

$$\begin{aligned}[r,\phi]_\star =& \, r e^{\frac{i}{2}\theta_{(2)}^{\alpha\beta\gamma\delta}M_{\alpha\beta}\otimes M_{\gamma\delta}}\phi - \phi e^{\frac{i}{2}\theta_{(2)}^{\alpha\beta\gamma\delta}M_{\alpha\beta}\otimes M_{\gamma\delta}} r \\ =& \frac{i}{2}\theta_{(2)}^{\alpha\beta\gamma\delta}\left(M_{\alpha\beta}rM_{\gamma\delta}\phi - M_{\gamma\delta}rM_{\alpha\beta}\phi\right) + \mathcal{O}(\theta^2) \\ =& \, i\theta_{(2)}^{\alpha\beta\gamma\delta}M_{\alpha\beta}rM_{\gamma\delta}\phi + \mathcal{O}(\theta^2) \\ =& \, 4i\theta_{(2)}^{0123}M_{01}rM_{23}\phi + \mathcal{O}(\theta^2) \\ =& \, -4it\theta_{(2)}^{0123}\cos\theta\cos^2\phi + \mathcal{O}(\theta^2)\,. \qquad (4.24)\end{aligned}$$

In second order we once again have a vanishing contribution and by including the third order correction we get

$$\begin{aligned}[r,\phi]_\star =& -4it\theta_{(2)}^{0123}\cos\theta\cos^2\phi + 2\left(\frac{i}{2}\right)^3\frac{1}{3!}\theta_{(2)}^{\alpha\beta\gamma\delta}\theta_{(2)}^{abcd}\theta_{(2)}^{ijkl}M_{\alpha\beta}M_{ab}M_{ij}rM_{\gamma\delta}M_{cd}M_{kl}\phi + \mathcal{O}(\theta^5) \\ =& -4it\theta_{(2)}^{0123}\cos\theta\cos^2\phi + 8\left(\frac{i}{2}\right)^3\frac{1}{3!}\theta_{(2)}^{\alpha\beta\gamma\delta}\theta_{(2)}^{abcd}\theta_{(2)}^{0124}M_{\alpha\beta}M_{ab}M_{01}rM_{\gamma\delta}M_{cd}M_{23}\phi + \mathcal{O}(\theta^5) \\ =& -4it\theta_{(2)}^{0123}\cos\theta\cos^2\phi - i\frac{8}{3}\left(\theta_{(2)}^{0124}\right)^3 M_{01}^3 rM_{23}^3\phi + \mathcal{O}(\theta^5)\,. \qquad (4.25)\end{aligned}$$

This expression is significantly more complicated than (4.17), agreeing nicely with the expectations from the symmetry considerations above.

Thus it is concluded that using the more complicated quadratic twist (4.21) does not help us to obtain simpler results and only leads to the breaking of the entire Poincaré group. In describing the usual Weyl-Moyal space-time given by (2.11), the use of the commutator (4.11) is unfounded and it is clear that the two describe different structures, both in 2+1 and in 3+1 dimensions. Nothing prevents one from using different deformations of space-time, but as most studies are concentrated on using the Moyal deformation (2.11), motivated in section 2.2, care must be taken in order not to cause undue confusion.

Chapter 5

Noncommutative gauge theory and magnetic monopoles

The infinite nonlocality induced by the noncommutativity makes gauge theories particularly difficult subjects to handle. In addition to the already discussed UV/IR mixing effect that persists, one implication of the $\star$-product structure is that there are no Abelian noncommutative gauge theories. Further, as discussed in the following, there appears a natural charge quantization as well as the subsequent problem of charging fields under more than two groups. We used this deformed structure in **III** and **IV** to study magnetic monopoles in noncommutative space with results clearly different from the commutative case.

5.1 Gauge theory in quantum space

5.1.1 Charge quantization and no-go theorems

To highlight the novelties introduced by the noncommutativity into gauge theory let us look at the $U_\star(1)$ Yang-Mills theory [84] (see also [94]), given by the action

$$S_{YM} = \int d^dx \left(-\frac{1}{4g^2}\right) F_{\mu\nu} \star F^{\mu\nu} \, . \tag{5.1}$$

The main difference compared with ordinary $U(1)$ is that the $U_\star(1)$ group is non-Abelian, leading to a self-coupling of the gauge field, clearly seen from the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu,\, A_\nu]_\star \, . \tag{5.2}$$

The corresponding gauge transformations of the field A_μ in the adjoint representation are given by the usual formula for non-Abelian theories, only now with $\star$-products inserted between the factors

$$A_\mu(x) \to U(x) \star A_\mu(x) \star U^{-1}(x) - iU(x) \star \partial_\mu U^{-1}(x)\,, \tag{5.3}$$

where the gauge group elements are given by

$$U^{-1}(x) = e_\star^{i\lambda(x)} = 1 + i\lambda(x) + \frac{i^2}{2!}\lambda(x) \star \lambda(x) + \ldots\,. \tag{5.4}$$

Equation (5.3) is used below in (5.32)-(5.34) as an expansion up to second order in θ.

When fermions are added we can consider noncommutative QED, with the action

$$S_{QED} = \int d^d x \left(-\frac{1}{4g^2} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star \gamma^\mu i D_\mu \psi - m\bar{\psi} \star \psi \right)\,, \tag{5.5}$$

where gauge fields are coupled to matter fields through the covariant derivative

$$D_\mu \psi = \partial_\mu \psi - iA_\mu \star \psi\,. \tag{5.6}$$

This behaves covariantly, as can be seen by using (5.3) and the transformation of the matter fields charged under the fundamental representation of the gauge group

$$\psi(x) \to \psi'(x) = U(x) \star \psi(x)\,. \tag{5.7}$$

Plugging (5.7) into (5.6) we get, as required,

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = U(x) \star D_\mu \psi(x)\,. \tag{5.8}$$

Similarly, for $\bar{\psi}(x)$ in the antifundamental representation, we have

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) \star U^{-1}(x)\,, \tag{5.9}$$

and the covariant derivative

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + i\bar{\psi} \star A_\mu\,. \tag{5.10}$$

Considering the commutative limit $\theta^{\mu\nu} \to 0$ indicates that the fields are charged under the $U_\star(1)$ group with opposite charges, ± 1 in the appropriate units. The extension of the covariant derivative (5.6) to higher charges is given by

$$D_\mu \psi^{(n)} = \partial_\mu \psi^{(n)} - inA_\mu \star \psi^{(n)}\,, \tag{5.11}$$

with integral multiple n of the unit charge. However, equation (5.11) fails to transform covariantly meaning that higher charges cannot be consistently introduced, thus reducing the space of possible charges to $0, \pm 1$. This is to be expected from the analogy with commutative non-Abelian gauge theory, where similar charge quantization occurs.

A corollary of charge quantization was noticed in [95], where it was pointed out that in noncommutative gauge theories it is impossible to charge a field under more than two groups. This follows from the fact that the $\star$-products in the group algebra prevent one from charging a field under more than one group in the fundamental representation as in (5.7), let us call this group $U_\star(n)$. As it is further possible to charge the same field under a different group, say $V_\star(n)$, in the antifundamental representation, as in (5.9), one has

$$\begin{aligned} &\psi \to \psi' = U \star \psi \star V^{-1} \,, \\ &U \in U_\star(x) \,, \quad V \in V_\star(x) \,. \end{aligned} \tag{5.12}$$

No further charges can be added, as can be seen from an analysis similar to that in equation (5.11). This obviously poses problems in the efforts of constructing a noncommutative standard model, as one would like to charge the quarks under three groups, which is not allowed by the above argument.

The $\star$-product further spoils the closure condition of $SU_\star(n)$ groups, thus limiting the group structure of noncommutative gauge theories even more. Consider two traceless hermitian $n \times n$ matrices g_1 and g_2, i.e. elements of the usual $su(n)$ algebra. The commutator $[g_1, g_2]_\star$ fails to give a traceless matrix and consequently the group does not close. For this reason, when constructing the noncommutative standard model for example, one is lead to the use of $U_\star(n)$ groups (that close), along with their extensions.

Attempts have been made to by-pass all of these restrictions by the use of enveloping algebra methods [96]. Whether this can be done consistently has been debated [97] and a detailed analysis is in preparation.

5.1.2 Seiberg-Witten map

The parallel between commutative and noncommutative gauge theories was considered at length in [4], where noncommutative Yang-Mills theory was shown to arise as a low-energy limit of string theory, as discussed above in section 2.2. There is a corollary to this, as using a different regularization method[1] leads to a commutative Yang-Mills theory instead of a noncommutative one. It turns out that one can then introduce a mapping from the noncommutative fields and gauge parameters to the corresponding quantities in the commutative theory by requiring the gauge equivalence of fields derived by the two different regularization methods. One cannot require a simple mapping of the form

$$\begin{cases} \hat{A} = \hat{A}(A, \partial A, \partial^2 A, \dots)\,, \\ \hat{\lambda} = \hat{\lambda}(\lambda, \partial\lambda, \partial^2\lambda, \dots)\,, \end{cases} \tag{5.13}$$

as from this would follow that the two gauge groups, $U(1)$ and $U_\star(1)$, are isomorphic – an untrue statement.

Instead, a weaker condition is required, that of gauge equivalence of the two fields A and $\hat{A}$, given by the expression

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}(A) = \hat{A}(A + \delta_\lambda A)\,. \tag{5.14}$$

The different variations in (5.14) are given by

$$\delta_\lambda A_i = \partial_i\lambda + i[\lambda, A_i]\,, \tag{5.15}$$

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_i = \partial_i\hat{\lambda} + i\hat{\lambda}\star\hat{A}_i - i\hat{A}_i\star\hat{\lambda}\,. \tag{5.16}$$

Expanding (5.14) up to first order in θ, one gets the following mapping between the fields and gauge parameters

$$\begin{cases} \hat{A}_i = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + \mathcal{O}(\theta^2)\,, \\ \hat{\lambda} = \lambda + \frac{1}{4}\theta^{ij}\{\partial_i\lambda, A_j\} + \mathcal{O}(\theta^2)\,. \end{cases} \tag{5.17}$$

Equation (5.17) is referred to as the *Seiberg-Witten map* and is often used in perturbative studies of noncommutative gauge theories. However, the gauge equivalence principle is an extra structure coming from the regularization procedure and it is not always clear *a priori* whether the mapping holds. This is especially true for topologically non-trivial space-times, as is discussed below in connection with noncommutative magnetic monopoles.

[1] Pauli-Villars instead of point splitting regularization.

5.1.3 Gauge invariant observables

In usual gauge field theory one is often interested in the correlation functions of gauge invariant local observables such as $\mathrm{Tr}[F^2(x)]$. In noncommutative space these operators are not gauge invariant but rather transform non-trivially. This raises the question of how to define gauge invariant observables, for example the electric and magnetic fields. This can be achieved by using the IIKK construction [98][2]. The starting point is that if we can find gauge invariant combinations of the noncommutative potential that reduce to the electric and magnetic fields in the $\theta \to 0$ limit, it is justified to call these combinations the noncommutative electric and magnetic field, respectively.

The gauge invariant operators are given in momentum space, but they can be transformed back to coordinate space by a usual, *commutative* inverse Fourier transformation. Using this, one may define a gauge invariant object constructed from the $U_\star(1)$ field strength tensor $F^{\mu\nu}$ as

$$G^{\mu\nu} = \int d^4k e^{-ikx} \Big[\int d^4x F^{\mu\nu} \star W(x,C) \star e^{ikx} \Big] , \tag{5.18}$$

where $W(x,C)$ is the noncommutative $U_\star(1)$ generalization of the Wilson line

$$W(x,C) = P_\star \exp\left(ig \int_0^1 d\sigma \frac{d\zeta^\mu}{d\sigma} A_\mu(x+\zeta(\sigma)) \right) , \tag{5.19}$$

and where C is the curve which is parameterized by $\zeta^\mu(\sigma)$ with $0 \leq \sigma \leq 1$, $\zeta(0) = 0$, $\zeta(1) = l$ and satisfies the condition $l^\nu = k_\mu \theta^{\mu\nu}$, l being the length of the curve. $P_\star$ denotes path ordering with respect to the star product

$$\begin{aligned} W(x,C) = \sum_{n=0}^{\infty} (ig)^n \int_0^1 d\sigma_1 \int_{\sigma_1}^1 d\sigma_2 ... \int_{\sigma_{n-1}}^1 d\sigma_n \\ \times \zeta'_{\mu_1}(\sigma_1) ... \zeta'_{\mu_n}(\sigma_n) A_{\mu_1}(x+\zeta(\sigma_1)) \star ... \star A_{\mu_n}(x+\zeta(\sigma_n)). \end{aligned} \tag{5.20}$$

Equation (5.18) is a gauge invariant combination of the noncommutative potential, that reduces to the commutative field strength in the limit $\theta \to 0$. Therefore the F^{0i} and F^{ij} parts of the noncommutative field strength may be attributed to the noncommutative electric and magnetic fields, such that G^{0i} is the noncommutative electric field and $\epsilon_{ijk} G^{jk}$ is the noncommutative magnetic field.

[2] The IIKK construction was further analyzed and extended in [99].

Different choices for the shape of the curve C give rise to different gauge invariant objects, and therefore the definition of the magnetic and electric fields in (5.18) is ambiguous. It may be that straight Wilson lines are the best choices as then the point of attachment of $F^{\mu\nu}$ to the Wilson line does not matter as argued in [99]. However, the definitions of the gauge invariant fields are only given here for a better understanding of the noncommutative Maxwell's equations of section 5.2.3 and thus a qualitative understanding is sufficient.

5.2 Magnetic monopoles

Magnetic monopoles have been under continuous study for decades even though not a single one has been observed to date[3]. In 1931 Dirac showed that the existence of a magnetic monopole would imply the quantization of electric charge [101]. This and the duality-like symmetry of Maxwell's equations are the two major motivations for the study of monopoles. The *Dirac quantization condition* (DQC)

$$\frac{2ge}{\hbar c} = \text{integer} = N, \tag{5.21}$$

is a topological property of space, i.e. independent of the local structure of the theory. In 1975 the singular potentials that Dirac's derivation results in were better understood when Wu and Yang rederived the DQC by a new method based on singularity-free gauge transformations [102].

In **III** and **IV** we applied the method of Wu and Yang to study the DQC in Moyal space using the deformed Maxwell's equations and gauge structure that the noncommutativity of space-time brings with it. The final result is that, at least within the perturbative framework we used, it is not possible to have a consistent noncommutative gauge theory while retaining the DQC.

5.2.1 Wu-Yang approach

When describing a magnetic monopole in the Dirac approach [101], one is led to a singularity in the gauge potential A_μ for the magnetic field – the Dirac string. The string is rotatable by a gauge transformation and thus cannot be observed, but the gauge transformations used for the rotation are also singular. In the approach of

[3] For a review on the experimental searches, see [100].

Wu and Yang [102], the singularity problem is circumvented by dividing the whole space into two overlapping hemispheres and by defining a singularity-free potential in each hemisphere. In the original paper the space R is divided as

$$R^N : 0 \leq \theta < \pi/2 + \delta,\ \ r > 0,\ \ 0 \leq \phi < 2\pi,\ \ t \in (-\infty, \infty)\,,$$
$$R^S :\ \pi/2 - \delta < \theta \leq \pi,\ \ r > 0,\ \ 0 \leq \phi < 2\pi,\ \ t \in (-\infty, \infty) \tag{5.22}$$

and the two gauge fields A^N_μ and A^S_μ are taken to be

$$A^N_t = A^N_r = A^N_\theta = 0,\ \ A^N_\phi = \tfrac{g}{r\sin\theta}(1-\cos\theta),$$
$$A^S_t = A^S_r = A^S_\theta = 0,\ A^S_\phi = -\tfrac{g}{r\sin\theta}(1+\cos\theta). \tag{5.23}$$

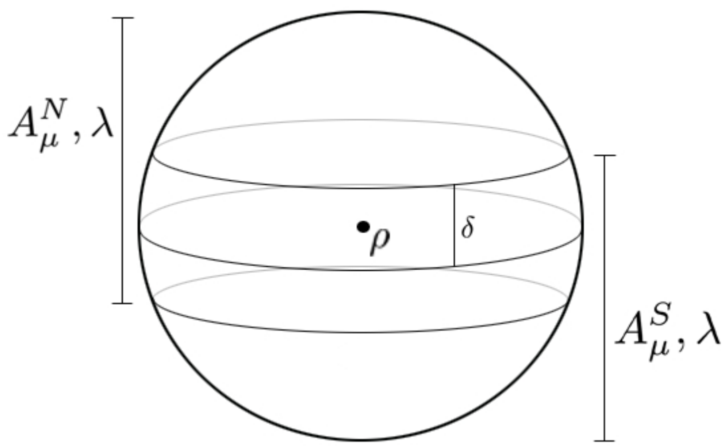


Figure 5.1: Schematic illustration of the potentials A^N_μ and A^S_μ defined in the two hemispheres enclosing the monopole ρ.

The conditions the potentials need to satisfy are the following:

1. In the overlapping region they are gauge transformable to each other.
2. Their curls give the magnetic field.
3. Both potentials are singularity-free in their respective regions of validity.

For the potentials (5.23), the gauge transformation from one hemisphere to the other is given by

$$S = S_{ab} = e^{-i\alpha} = e^{\frac{2ige}{\hbar c}\phi}. \tag{5.24}$$

The main result in [102] was that this gauge transformation remains single-valued only if the condition

$$\frac{2ige}{\hbar c} = \text{integer} = N\,, \tag{5.25}$$

is satisfied. Equation (5.25) is exactly (5.21), the quantization condition due to Dirac.

5.2.2 Wu-Yang procedure in noncommutative space

To check the validity of the DQC in noncommutative space, we will use a slightly modified version of the original Wu-Yang procedure. In commutative space-time, Wu and Yang looked for a gauge transformation from one hemisphere to the other and required that the potentials in each hemisphere give the magnetic field. In noncommutative space-time the situation is modified since the $U_\star(1)$ group is non-Abelian and the gauge invariant magnetic field needs to be constructed via the IIKK construction as discussed in section 5.1.3.

The procedure we used in noncommutative space is the following: we look for a potential in each hemisphere, $A_\mu^N(x)$ and $A_\mu^S(x)$, such that

1. The potentials are gauge transformable to each other in the overlapping region of the potentials. For the non-Abelian group $U_\star(1)$ this means that we require
$$A_\mu^{N/S}(x) \to U(x)\star A_\mu^{N/S}(x)\star U^{-1}(x) - iU(x)\star\partial_\mu U^{-1}(x) = A_\mu^{S/N}(x)\,. \tag{5.26}$$
2. Both potentials satisfy Maxwell's equations with an appropriate source for the magnetic charge.
3. The potentials remain singularity free in their respective regions of validity. That is, Maxwell's equations are solved in such a way that noncommutativity does not produce new singularities into the potentials.

In **III** and **IV** we treated the problem as a perturbation series up to second order in θ. In the notation used, the noncommutative gauge field A_μ is expanded as $A_\mu = A_\mu^0 + A_\mu^1 + A_\mu^2 + O(\theta^3)$. Here the upper index corresponds to the order in θ for each correction. In this notation the gauge transformation parameter is (symbolically) expanded as $\lambda = \lambda^0 + \lambda^1 + \lambda^2 + O(\theta^3)$. To preserve the DQC we required that the θ-corrections to λ can be put to zero (or a constant), i.e. $\lambda = \lambda^0 + C$, while satisfying the three above requirements. This is because the higher order corrections $\lambda^1, \lambda^2 \dots$ necessarily bring about a dependence on space-time points to the DQC, as is clear already from dimensional arguments. This will lead to observer-dependence and thus make the DQC uninteresting. The main result of **IV** is that it is not possible to find solutions when $\lambda^1 = 0$, and thus the DQC will need to be modified and its topological nature will be lost.

5.2.3 Deformed gauge transformations and Maxwell's equations

Gauge transformations. As already mentioned above, the noncommutative gauge transformations for the gauge field of the $U_\star(1)$ theory are given by

$$A_\mu(x) \rightarrow U(x) \star A_\mu(x) \star U^{-1}(x) - iU(x) \star \partial_\mu U^{-1}(x) \,. \tag{5.27}$$

The gauge group element $U^{-1}(x) = e_\star^{i\lambda(x)}$ up to second order in θ was calculated in **IV** and is given by

$$e_\star^{i\lambda} = e^{i\lambda} + \frac{\theta^{ij}\theta^{kl}}{8} e^{i\lambda}\partial_j\partial_l\lambda \left(\frac{1}{2}\partial_i\partial_k\lambda + \frac{i}{3}\partial_i\lambda\partial_k\lambda\right) + \mathcal{O}(\theta^3) \,. \tag{5.28}$$

Using (5.28), we further derived the full expression for the gauge transformation (5.27) up to second order. Order by order, these are

$$A_i^0(x) \rightarrow A_i^0(x) + \partial_i\lambda \,, \tag{5.29}$$

$$A_i^1(x) \rightarrow A_i^1(x) + \theta^{kl}\partial_k\lambda\partial_l A_i^0(x) + \frac{\theta^{kl}}{2}\partial_k\lambda\partial_l\partial_i\lambda \,, \tag{5.30}$$

$$\begin{aligned} A_i^2(x) \rightarrow A_i^2(x) &+ \theta^{kl}\partial_k\lambda\partial_l A_i^1 - \frac{1}{2}\theta^{kl}\theta^{pq}\Big(\partial_k A_i^0\partial_p\lambda\partial_q\partial_l\lambda - \\ &\partial_k\partial_p A_i^0\partial_q\lambda\partial_l\lambda + \frac{1}{3}\left(\partial_k\partial_p\lambda\partial_l\lambda\partial_q\partial_i\lambda - \partial_k\lambda\partial_p\lambda\partial_l\partial_q\partial_i\lambda\right)\Big) \,. \end{aligned} \tag{5.31}$$

As we want the two gauge potentials to be gauge transformable to each other, we require that the following equations hold in order to satisfy the first requirement of section 5.2.2:

$$A_i^{N_0}(x) = A_i^{S_0}(x) + \partial_i\lambda \,, \tag{5.32}$$

$$A_i^{N_1}(x) = A_i^{S_1}(x) + \theta^{kl}\partial_k\lambda\partial_l A_i^{S_0}(x) + \frac{\theta^{kl}}{2}\partial_k\lambda\partial_l\partial_i\lambda \,, \tag{5.33}$$

$$\begin{aligned} A_i^{N_2}(x) = A_i^{S_2}(x) &+ \theta^{kl}\partial_k\lambda\partial_l A_i^{S_1} - \frac{1}{2}\theta^{kl}\theta^{pq}\Big(\partial_k A_i^{S_0}\partial_p\lambda\partial_q\partial_l\lambda - \\ &\partial_k\partial_p A_i^{S_0}\partial_q\lambda\partial_l\lambda + \frac{1}{3}\left(\partial_k\partial_p\lambda\partial_l\lambda\partial_q\partial_i\lambda - \partial_k\lambda\partial_p\lambda\partial_l\partial_q\partial_i\lambda\right)\Big) \,. \end{aligned} \tag{5.34}$$

Maxwell's equations. In noncommutative space Maxwell's equations for a static monopole are

$$\epsilon^{\mu\nu\gamma\delta} D_\nu \star \mathcal{F}_{\gamma\delta} = 0\,, \tag{5.35}$$

$$D_\mu \star \mathcal{F}^{\mu\nu} = J^\nu\,, \tag{5.36}$$

where $\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\gamma\delta}F_{\gamma\delta}$ is the dual field strength tensor, while the noncommutative $U_\star(1)$ field strength tensor and the covariant derivative are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_\star\,, \tag{5.37}$$

$$D_\nu = \partial_\nu - ie[A_\nu, \cdot]_\star\,. \tag{5.38}$$

In **III** and **IV** we expanded equations (5.35) and (5.36) perturbatively, using the $\star$-product (2.11), up to second order in θ. The resulting equations need to be solved order by order. For "Ampère's law" (5.35), the deformed equations are given by

$$\begin{cases} (\nabla\times B^0)^i = 0\,, \\ (\nabla\times B^1)^i = -\theta^{\gamma\delta}\left[\partial_j(\partial_\gamma A_0^i \partial_\delta A_0^j) + \partial_\gamma A_j^0 \epsilon^{ijk}\partial_\delta B_k^0\right]\,, \\ (\nabla\times B_2)^i = -\theta^{pq}\{\partial_j(\partial_p A_1^i\partial_q A_0^j + \partial_p A_0^i \partial_q A_1^j) + \partial_p A_j^0\partial_q(\partial^i A_1^j - \partial^j A_1^i) \\ \qquad\qquad +\partial_p A_j^1\partial_q(\partial^i A_0^j - \partial^j A_0^i) + \theta^{kl}\partial_p A_j^0 \partial_q(\partial_k A_0^i \partial_l A_0^j)\}\,, \end{cases} \tag{5.39}$$

where we have denoted $(\partial^i A_2^k - \partial^k A_2^i)$ by $\epsilon^{ikp}B_p^2$. The corresponding equations for "Gauss's law" (5.36) have the simpler form

$$\begin{cases} \nabla\cdot\mathbf{B}_0 = -4\pi\delta^3(r)\,, \\ \nabla\cdot\mathbf{B}_1 = -\rho^1(x)\,, \\ \nabla\cdot\mathbf{B}_2 = -\rho^2(x)\,, \end{cases} \tag{5.40}$$

where $\rho = -4\pi\delta^3(r) - \rho^1(x) - \rho^2(x) + \mathcal{O}(\theta^3)$ is the perturbative source.

Using the well-known identity from vector calculus, $\nabla^2\vec{B} = \nabla(\nabla\cdot\vec{B}) + \nabla\times(\nabla\times\vec{B})$, the first order equations combine to give

$$(\nabla^2 B_1(A_1))^i = -\partial^i\rho^1 - \theta^{pq}\{\epsilon^{ijk}\partial^l(\partial^p A_0^k \partial^j\partial^q A_0^l) \\ -2\partial^m(\partial^p A_0^m \partial^q B_0^i) - \partial^m(\partial^p B_0^m \partial^q A_0^i)\}\,, \tag{5.41}$$

while in second order we get the expansion

$$\begin{aligned}(\nabla^2 B_2)^m =& \partial^m(\nabla\cdot B_2) + (\nabla\times(\nabla\times B_2))^m \\ =& -\partial^m\rho^2 - \epsilon^{mni}\partial^n\theta^{pq}\Bigg(\partial_j(\partial_p A_1^i \partial_q A_0^j + \partial_p A_0^i\partial_q A_1^j) \\ &+ \partial_p A_j^0 \partial_q \epsilon^{ijl} B_1^l + \partial_p A_j^1\partial_q\epsilon^{ijl}B_0^l + \theta^{kl}\partial_p A_j^0\partial_q(\partial_k A_0^i \partial_l A_0^j)\Bigg)\,. \end{aligned} \tag{5.42}$$

5.2.4 The Dirac quantization condition in quantum space

The aim is to compare the deformed gauge transformations and Maxwell's equations given above and see whether we can consistently retain the DQC. In order for the DQC to remain unmodified we require that $\lambda = \lambda^0 + \lambda^1 + \cdots = \lambda^0 = \frac{2ge}{\hbar c}\phi$, where $\phi = \arctan\left(\frac{x}{y}\right)$.

In the following we shall choose the noncommutative plane as $\theta = \theta^{12} = -\theta^{21}$, while other components are set to zero. With this choice the solutions to the first order Maxwell's equations (5.41) for the potential differences $A^{N_1} - A^{S_1}$ were derived in **III** and are given by

$$A_1^{N_1} - A_1^{S_1} = \frac{2\theta yz(2(x^2+y^2)+z^2)}{(x^2+y^2)^2 r^3}\,, \tag{5.43}$$

$$A_2^{N_1} - A_2^{S_1} = -\frac{2\theta xz(2(x^2+y^2)+z^2)}{(x^2+y^2)^2 r^3}\,, \tag{5.44}$$

$$A_3^{N_1} - A_3^{S_1} = 0\,. \tag{5.45}$$

In first order in θ, the two sets of equations (5.41) and the corresponding ones derived from (5.43)-(5.45) agree perfectly, component by component, as shown in **III**. That is, by taking the curl and then operating with the Laplace operator on the separation of the potentials (5.43)-(5.45) one gets exactly (5.41). In second order this turn out not to be the case.

To make a comparison between Maxwell's equations (5.42) and the gauge transformation (5.34) in second order we need to solve for the potentials A^{N_1} and A^{S_1} (not just their difference) in first order, because this quantity appears in (5.34). These were derived in **III** and are given by

$$A_1^{N_1} = \theta\Big(\frac{-2x\arctan(\frac{x}{y})}{(x^2+y^2)^2} + \frac{y}{4}\Big[\frac{7}{r^4} - \frac{2}{(x^2+y^2)r^2} + \frac{4z(x^2+y^2+r^2)}{(x^2+y^2)^2 r^3}\Big]\Big)\,, \tag{5.46}$$

$$A_2^{N_1} = -\theta\Big(\frac{2y\arctan(\frac{x}{y})}{(x^2+y^2)^2} + \frac{x}{4}\Big[\frac{7}{r^4} - \frac{2}{(x^2+y^2)r^2} + \frac{4z(x^2+y^2+r^2)}{(x^2+y^2)^2 r^3}\Big]\Big)\,, \tag{5.47}$$

$$A_3^{N_1} = 0. \tag{5.48}$$

This is of course just one choice of A^{N_1}, but it is a singularity-free choice, thus satisfying the third requirement of section 5.2.2. From these potentials it is straightforward to obtain the expression for $A_i^{S_1}$, using (5.43), (5.44) and (5.45).

In second order we can write down Maxwell's equations for the difference $B^{N_2} - B^{S_2}$ from (5.42) as

$$\nabla^2(B^{N_2} - B^{S_2})_1 = \frac{4\theta^2 xz}{(x^2+y^2)^3 r^{10}}\Big[-375(x^2+y^2)^3 + 131z^2(x^2+y^2)^2 - 2z^4(x^2+y^2) - 4z^6\Big] - \partial_x\left(\rho^{N_2} + \rho^{S_2}\right), \tag{5.49}$$

$$\nabla^2(B^{N_2} - B^{S_2})_2 = \frac{4\theta^2 yz}{(x^2+y^2)^3 r^{10}}\Big[-375(x^2+y^2)^3 + 131z^2(x^2+y^2)^2 - 2z^4(x^2+y^2) - 4z^6\Big] - \partial_y\left(\rho^{N_2} + \rho^{S_2}\right), \tag{5.50}$$

$$\nabla^2(B^{N_2} - B^{S_2})_3 = \frac{4\theta^2}{(x^2+y^2)^4 r^{10}}\Big[120(x^2+y^2)^5 - 900(x^2+y^2)^4 z^2 - 1285(x^2+y^2)^3 z^4 - 1289(x^2+y^2)^2 z^6 - 652(x^2+y^2)z^8 - 132z^{10}\Big] - \partial_z\left(\rho^{N_2} + \rho^{S_2}\right). \tag{5.51}$$

These equations are difficult to solve analytically. Fortunately this is not needed however, since we only want to compare these equations to the ones coming from the gauge transformation in the overlap of the potentials (5.27). Taking the curl and then the Laplace operator of the transformations (5.34), component by component, we get the following equations:

$$\nabla^2(B^{N_2} - B^{S_2})_1^{GT} = \frac{4\theta^2 xz}{(x^2+y^2)^3 r^{10}}\Big(-321(x^2+y^2)^3 + 205(x^2+y^2)^2 z^2 + 26(x^2+y^2)z^4 + 4z^6\Big), \tag{5.52}$$

$$\nabla^2(B^{N_2} - B^{S_2})_2^{GT} = \frac{4\theta^2 yz}{(x^2+y^2)^3 r^{10}}\Big(-321(x^2+y^2)^3 + 205(x^2+y^2)^2 z^2 + 26(x^2+y^2)z^4 + 4z^6\Big), \tag{5.53}$$

$$\nabla^2(B^{N_2} - B^{S_2})_3^{GT} = \frac{4\theta^2}{(x^2+y^2)^4 r^{10}}\Big(144(x^2+y^2)^5 - 564(x^2+y^2)^4 z^2 - 455(x^2+y^2)^3 z^4 - 403(x^2+y^2)^2 z^6 - 188(x^2+y^2)z^8 - 36z^{10}\Big). \tag{5.54}$$

In order to satisfy criteria 1 and 2 of section 5.2.2 for the DQC, the equations (5.49)-(5.51) and (5.52)-(5.54) need to be satisfied simultaneously. We may simplify this system of equations by subtracting one set from the other to get

$$-\partial_x(\rho^{N_2} - \rho^{S_2}) = \frac{8\theta^2 xz}{(x^2+y^2)^3 r^8}\Big(27(x^2+y^2)^2 + 10(x^2+y^2)z^2 + 4z^4\Big)\,, \tag{5.55}$$

$$-\partial_y(\rho^{N_2} - \rho^{S_2}) = \frac{8\theta^2 yz}{(x^2+y^2)^3 r^8}\Big(27(x^2+y^2)^2 + 10(x^2+y^2)z^2 + 4z^4\Big)\,, \tag{5.56}$$

$$\begin{aligned}-\partial_z(\rho^{N_2} - \rho^{S_2}) = \frac{2\theta^2}{(x^2+y^2)^3 r^8}\Big(&48(x^2+y^2)^4 + 624(x^2+y^2)^3 z^2 + 1036(x^2+y^2)^2 z^4 \\ &+ 736(x^2+y^2)z^6 + 192z^8\Big)\,. \end{aligned}\tag{5.57}$$

We can then differentiate equation (5.55) with respect to y and equation (5.56) with respect to x and perform a subtraction between the two. The derivatives commute[4] and the resulting function needs to vanish. This is true for this first combination

$$(\partial_x\partial_y - \partial_y\partial_x)(\rho^{N_2} - \rho^{S_2}) = 0\,. \tag{5.58}$$

We get the following two additional equations in a similar manner, this time with non-vanishing functions

$$\begin{aligned}0 = (\partial_x\partial_z - \partial_z\partial_x)(\rho^{N_2} - \rho^{S_2}) = \frac{24\theta^2 x}{(x^2+y^2)^5 r^8}\Big(&41(x^2+y^2)^4 + 426(x^2+y^2)^3 z^2 \\ &+ 704(x^2+y^2)^2 z^4 + 496(x^2+y^2)z^6 + 128z^8\Big)\,, \end{aligned}\tag{5.59}$$

$$\begin{aligned}0 = (\partial_y\partial_z - \partial_z\partial_y)(\rho^{N_2} - \rho^{S_2}) = \frac{24\theta^2 y}{(x^2+y^2)^5 r^8}\Big(&41(x^2+y^2)^4 + 426(x^2+y^2)^3 z^2 \\ &+ 704(x^2+y^2)^2 z^4 + 496(x^2+y^2)z^6 + 128z^8\Big)\,. \end{aligned}\tag{5.60}$$

The equations (5.59) and (5.60) have no solution (except when $x = y = 0$). Thus we can conclude that there do not exist potentials A_μ^N and A_μ^S that would simultaneously satisfy Maxwell's equations and be gauge transformable to each other by (5.27). In our case, the inclusion of the source does not change the contradiction in equations (5.59) and (5.60). This is more fully explained in **III**, here in short: The only contribution that the source (5.63) or (5.64) has on the function B_i^2 is at the origin $r = 0$, where the noncommutativity of space makes the theory more singular. As this point is not included in the zeroth order potentials (5.23), it follows that it is not included in the full expressions A_i^N and A_i^S. Thus we do not need to consider the second order source contribution in the calculation of B_i^2.

[4]The derivatives of $(\rho^{N_2} - \rho^{S_2})$ are given by (5.55)-(5.56) and are continous functions outside the origin, thus $(\rho^{N_2} - \rho^{S_2})$ itself must also be continuous in this region.

Therefore we have our final result: The DQC cannot hold topologically in this model, even by considering the possibility of having an arbitrary perturbative noncommutative source. This does not mean that we could not introduce a perturbative source for the monopole, it only states that one could maximally retain the DQC to first order in θ. In higher orders it receives corrections that depend on space-time points and therefore the DQC would no longer be a topological property of noncommutative space-time.

5.2.5 Discussion and comparison to earlier results

In hindsight, it is not surprising that the failure of the DQC is only apparent in the second order of perturbation, as it is there that the first order correction λ_1 influences the gauge transformations (5.29)-(5.31). In fact, the breaking is a first order effect that only shows up in the second order calculation.

The most intuitive explanation for our result would be the breaking of rotational invariance. Since rotational invariance is directly related to the fibre bundle construction of the Wu-Yang potentials in commutative space-time, it may indeed be that the breaking of it leads to a nontopological DQC in noncommutative space-time.

In **IV** we also proposed the following related possible explanation: NCQED is CP-violating [103] and it is known that in flat commutative space-time [104] and even in curved space-time [105], a CP-violating theory necessarily leads to the monopole aquiring an electric charge and the failure of the DQC. One could be led to believe that this phenomenon, also called charge dequantization, is what we have observed. It should be pointed out however, that the noncommutative Maxwell's equations are manifestly CP-violating whereas the CP-violation observed in the charge dequantization phenomenon of commutative electrodynamics only occurs if one adds extra terms to the free Lagrangian of electrodynamics. This does indicate some difference in the two approaches.

Comparison with the Seiberg-Witten map (5.17) shows that our potentials are not the same as those derived from the map. We checked explicitly that the first order potentials derived from (5.17) do not satisfy the equations of motion (5.41). This could be due to the physical singularity at the origin, but irrespective of the reason for this failure it should be noted that any similar construction based on the Seiberg-Witten map would immediately conclude that the DQC *cannot be satisfied*, since it necessarily produces a $\hat{\lambda}$ that depends on the potential A^0_μ. If it did not, the gauge groups $U(1)$ and $U_\star(1)$ in the ordinary and noncommutative theories respectively, would be identical, a point already mentioned in section 5.1.2.

Although a direct generalization of the Wu-Yang formulation of the Dirac monopole, such as here, has not been considered previously, there have been many studies of noncommutative BPS-monopoles. The works include perturbative studies of the $U_\star(2)$ [106] and $U_\star(1)$ [107] BPS-monopoles, as well as non-perturbative studies of the $U_\star(1)$ [108] BPS-monopoles, generalized to other groups in [109, 110]. These constructions share the assumption that the definition of magnetic charge in the BPS-limit may be taken over, without change, to the noncommutative case. Our result shows that this assumption fails in the perturbative case. Therefore it is to be expected that the definition of noncommutative magnetic charge in the non-perturbative treatment is also subject to problems.

Finally, the failure of the DQC could be due to the perturbative approach used, as the infinite non-locality induced by the $\star$-product is only apparent in the non-perturbative approach[5]. It seems clear that the DQC cannot be recovered with the inclusion of any finite number of θ-corrections, but a final verdict for the DQC cannot be given before a non-perturbative treatment of this problem has been accomplished. A full non-perturbative calculation with the method considered here would thus be an interesting continuation of our work.

Assuming that a non-perturbative treatment leads to the same conclusion, it is interesting to speculate over the implications this result might have. First of all, since the charge of the matter fields is quantized in noncommutative theories [84], we are not in need of another explanation for the quantization of charge. Furthermore, since the Aharonov-Bohm effect can be formulated in a gauge invariant manner in noncommutative quantum mechanics [112], the noncommutative theories seem to make a difference of outcome between the experimentally observed Aharonov-Bohm effect and the DQC. Thus these two results, closely related in commutative space-time, seem to be unrelated in the noncommutative theory. As a consequence, one might argue that the lack of observations of magnetic monopoles is related to the deformed structure of space-time at very small scales.

[5] A similar perturbative modification of quantization has been noticed for the magnetic flux of vortex solutions in 2+1 -dimensional Chern-Simons theory coupled to a scalar field in the BPS setting [111].

5.3 A noncommutative particle source

The above calculation shows that, at least in perturbation theory, we cannot force the DQC to hold topologically by choosing a source term with the zeroth order contribution given by the Dirac delta function. However, we might still ask what a possible noncommutative particle source might look like, whether it is a monopole, an electrically charged particle or something else. Naturally, even if we cannot find a DQC supportive source for the monopole, it should be possible to find a source term e.g. for an electrically charged particle, as they must be described somehow in this theory if it is to have any connection with commutative Maxwellian electrodynamics.

To determine a possible noncommutative particle source, we need to discuss the symmetry of the equations. Firstly, equation (5.36) transforms as $U(x) \star D_\mu \star \mathcal{F}^{\mu 0} \star U^{-1}(x)$ under gauge transformations on the left-hand side. Namely, it is gauge covariant. Therefore, the source must also transform this way. Secondly, the left-hand side is $O(1,1) \times SO(2)$ symmetric and consequently, the source must also be that. Thirdly, as a correspondence principle when $\theta \to 0$, we want to recover the Dirac delta function for the source.

To begin with, any realistic generalization of a particle-source must in the noncommutative Maxwell's equations transform covariantly under gauge transformations. Therefore extensions of the delta function, such as

$$\delta^3_{NC}(r) = \frac{1}{\sqrt{(4\pi\theta)^3}} \exp\big(\frac{-r^2}{4\theta}\big), \tag{5.61}$$

must be discarded. They do not contain the gauge potential and therefore do not transform under gauge transformations.

Two covariant sources. For the consistency of the deformed Maxwell's equations (5.36) we need to find a source that is covariant up to second order of perturbation. We have indeed found two such expansions, which surprisingly have all of their coefficients uniquely fixed. The form of the possible sources is thus strongly constrained by gauge covariance.

If we suppose we have a source that transforms gauge covariantly in the second order in θ and we call it ρ_2, where $\rho_{NC} = \rho_0 + \rho_1 + \rho_2 + ...$, one can calculate the gauge transformation it must satisfy using the gauge group element (5.28). It is given by

$$\rho_2 \to \rho_2 + \theta^{ij}\partial_i\lambda\partial_j\rho_1 + \frac{\theta^{ij}\theta^{kl}}{2}\Big(\partial_k\lambda\partial_i\lambda\partial_j\partial_l\rho_0 - \partial_j\lambda\partial_l\rho_0\partial_i\partial_k\lambda\Big), \tag{5.62}$$

where λ is the local gauge group parameter. Equation (5.62) is the gauge covariance requirement for the source in second order in θ.

If we start with a first order source of the form $\rho_1 = \theta^{kl}\partial_k\left(A_l\delta^3(r)\right)$ we find[6] a gauge covariant source up to second order in θ, satisfying all our symmetry requirements, as

$$\begin{aligned}\rho =&\rho^0 + \rho^1 + \rho^2 + \mathcal{O}(\theta^3) = \ 4\pi g\Bigg(\delta^3(r) - \theta^{kl}\partial_k\left(A_l\delta^3(r)\right) \qquad (5.63)\\ &+\theta^{ij}A_j^1\partial_i\delta^3(r) + \theta^{ij}\theta^{kl}\left[A_j^0\partial_k\left(\partial_iA_l^0\delta^3(r) + A_l^0\partial_i\delta^3(r)\right) + \frac{1}{2}A_i^0A_k^0\partial_j\partial_l\delta^3(r)\right]\Bigg) + \mathcal{O}(\theta^3)\,.\end{aligned}$$

Due to the requirement of gauge covariance, the first order contribution to the source is unique up to the position of the partial derivative and the numerical coefficient in front. The second order contribution was found by using the most general ansatz possible, performing the transformation according to (5.29) and (5.30) and finally comparing with the gauge covariance condition (5.62). An interesting point is that the second order coefficients as well as the coefficient for the first order term are all uniquely determined merely by specifying the form of the first order contribution $\theta^{kl}\partial_k\left(A_l\delta^3(r)\right)$.

The other first order source term leading to a gauge covariant expansion in second order in θ is $\rho_1' = \theta^{kl}A_l\partial_k\delta^3(r)$. The corresponding expansion is given by

$$\rho' = \ 4\pi g\left(\delta^3(r) - \theta^{ij}A_j^0\partial_i\delta^3(r) - \theta^{ij}A_j^1\partial_i\delta^3(r) + \frac{1}{2}\theta^{ij}\theta^{kl}A_i^0A_k^0\partial_j\partial_l\delta^3(r) + \mathcal{O}(\theta^3)\right). \tag{5.64}$$

The two second order sources (5.63) and (5.64) are the only gauge covariant expansions consistent with the noncommutative Maxwell's equations (5.36).

Of course, there remains the possibility to construct a non-perturbative source similar to (5.61) that is also gauge covariant. This would allow for a full non-perturbative study and is currently under investigation.

[6] The first order contribution, up to a sign change, was found in [113] and was also considered in **III**.

Chapter 6

Conclusions

In this thesis the structure of noncommutative field theory has been examined placing special focus on the role of time as a noncommutative coordinate. There remain unsolved problems in all the approaches where noncommutative time has been introduced. We still lack the means to construct a consistent, infinitely non-local, Lorentz invariant quantum field theory.

There is interesting physics in theories where only spatial coordinates are non-commutative, such as the UV/IR mixing problem and the deformed gauge theory structure. The problems connected with charge quantization and restrictions on group structure need to be solved in order to build a working noncommutative standard model. For the magnetic monopoles, it seems that at least in the perturbative setting there is not much hope to accommodate the Dirac quantization condition. In a full non-perturbative treatment the situation might be different as it would allow us to probe the global effects of noncommutativity in full.

We live in quantum space-time. We are still missing a consistent theory to describe it, but significant progress has been made and there is no reason to think that such a theory would be completely beyond our reach, or even hopelessly far in the future. Noncommutative quantum field theory is sure to give us some hints on the structure of quantum space-time, especially on its possible nonlocal nature.

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