The autonomy of mathematical knowledge: Hilbert's program revisited

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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Steve Awodey (Managing Editor), John Burgess, Mark Colyvan, Anuj Dawar, Marcelo Fiore, Noam Greenberg, Hannes Leitgeb, Rahim Moosa, Ernest Schimmerling, Carsten Schürmann, Kai Wehmeier, and Matthias Wille. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA.*

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

STEVO TODORCEVIC. ***Walks on ordinals and their characteristics.*** Progress in Mathematics, vol. 263. Birkhäuser Verlag, Basel, 2007, vi + 324 pp.

The book under review represents a current account of one of the author's celebrated techniques—his *method of minimal walks*. Loosely speaking, this method is a tool for constructing uncountable objects—from set-theoretic trees to non-separable Banach spaces—by utilizing a careful analysis of certain descending sequences of ordinals known as *minimal walks*.

It will without a doubt be a useful reference for those working in a wide variety of areas relating to set theory and its applications. Parts of the book taken by themselves would already be important documents on each of the following rather different areas:

- ZFC constructions associated to ω_1;
- the construction of non separable Banach spaces;
- the combinatorics associated to singular cardinals;
- characterizations of large cardinal properties such as Mahloness and model-theoretic transfer principles in terms of Ramsey-theoretic statements;
- applications of Martin's Axiom and other forcing axioms.

Still, the book is more than a sum of these parts and one only understands the true scope and potential of the methods when viewing the work as a whole.

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The roots of the present text are in Todorcevic's seminal 1987 ***Acta Mathematica*** paper *Partitioning pairs of countable ordinals* where the method of minimal walks was first introduced. This paper is best known for its construction of a "negative square bracket" partition associated to ω_1: an edge coloring of the complete graph on ω_1 vertexes such that every element of ω_1 appears as a color on any uncountable complete subgraph. The Acta paper also contains important insights into the relationship between Mahlo cardinals and the existence of non-special Aronszajn trees, as well what is now considered as the standard construction of a so-called *Countryman line*.

Since then, the methods have matured and grown in their scope considerably. The present work cites 135 references. It reproduces the above constructions along with a wealth of new ones ranging from Banach space geometry to the combinatorics of singular cardinals. It is an expansion of Todorcevic's recent article in the ***Handbook of Set Theory*** (Foreman, Kanamori, eds., Springer, 2010). I will list some of the highlights here:

- If κ is a regular cardinal, then there is a function $f\colon [\kappa^+]^2 \to \kappa^+$ which takes all values on any $[X]^2$ such that $X \subseteq \kappa^+$ has cardinality κ^+.
- If c is Cohen-generic over V, then in $V[c]$ there is a Souslin tree.
- There is Banach space X with a Shauder basis of length ω_1 which contains no infinite unconditional basic sequence.
- If θ is a cardinal, the following are equivalent: (a) θ is Mahlo; (b) if $C \subseteq \theta$ is a club and f is a regressive function on $[C]^3$, then there is an infinite set $H \subseteq C$ such that $f \restriction [H]^3$ depends only on the minimum coordinate.
- If $\square_{\aleph_\omega}$ holds, then there is a linear ordering L on $\aleph_{\omega+1}$ which is not union of countably many well sub-orders but such that every $L_0 \subseteq L$ of cardinality $\aleph_\omega$ is a countable union of well sub-orders.

In some cases the above theorems are original to the author; in others he is providing a different prospective on the work of others. The classic constructions—those from the original Acta paper and a few others which are scattered in the literature—are now given the benefit of hindsight and context. In many cases, they serve as the basic examples of which the more modern can be seen as elaborations. While the role of the text is not primarily to prove independence results, the author does reproduce a number of theorems and their proofs concerning the influence of forcing axioms on coherent sequences.

In summary the book under review will provide an important reference for set theorists and those interested in its applications for generations to come.

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CURTIS FRANKS. ***The autonomy of mathematical knowledge: Hilbert's program revisited.*** Cambridge University Press, Cambridge, 2009, 213 pp.

This short but inspiring book questions the received view of Hilbert's contribution to the foundations of mathematics, and offers a newly clarified position within the philosophical attitude known as naturalism, especially of the (fiercely) anti-foundationalist persuasion. And while this reviewer would take issue with the central tenet of the book—that mathematics is "philosophy immune," or in that special sense "autonomous"; and would perhaps take issue with some matters of interpretation, this is without a doubt one of the most thoughtful as well as one of the most beautifully written books on the philosophy of mathematics to have been published in recent memory.

The book's two main claims revolve around the concept of *autonomy*. This is the view that mathematics "should not be subject to criticism from, and does not stand in need of support from, some external, supposedly higher point of view," as the author quotes Penelope Maddy on his page 170.

> Mathematics is a way of knowing that cannot and need not be justified on any *a priori* grounds. As such, it is not properly the the target of skeptical attacks, which in essence demand such grounds. Nevertheless mathematics can be the subject of foundational insight, through a self-evaluation the outcome of which is that questions about how and why mathematical techniques work the way they do can be given purely mathematical answers ... the reliability [of those techniques] is self-witnessing and not founded on any non-mathematical base. (p. 62)

Philosophically loaded skepticism as to the consistency of mathematics, is misplaced: "According to Hilbert," Franks writes

> ... mathematics is justified, though not on any philosophical grounds: Mathematics is justified in application, through a history of successful achievement, through the naturalness with which its methods come to us, through its broad range of applicability, etc. (p. 44)

Moreover, and this is the essence of Franks's rereading of Hilbert:

> This justification earns for mathematics a position of unassailability, but it does not earn for it the position of epistemic bedrock ... the status of mathematics is diminished if its veracity is shown to rest, through however circuitous a route, on non-mathematical grounds. (pp. 44–45)

That is to say there was, for Hilbert, and should be, for us, no question of pursuing the idea of a foundation for mathematics. Indeed, engaging in foundational epistemology of any kind is just to cede to a domain outside of mathematics, i.e., philosophy, what mathematics can, and must, do for itself.

As for how Hilbert is usually read:

> Thus it will not do to interpret "formalism" as the doctrine that mathematics is meaningless or that its subject matter consists just of formal symbols and rules of formula manipulation, as the term is often used in philosophical discussions. Neither is it correct to understand Hilbert's "finitism" as the doctrine that only decidable methods are veracious and that only finitary propositions are contentful ... Both Hilbert's "formalism" and his "finitism," instead of being philosophical perspectives from which he intends to justify mathematical techniques, are methodological constraints *forced* by the type of mathematical self-reliance that he intends to demonstrate. (p. 48)

As for the question of the consistency of mathematics,

> ... Hilbert's epistemological position differs significantly from those of his intellectual adversaries ... the question inspiring him to foundational research is not whether mathematics is consistent, but rather whether or not mathematics can stand on its own—no more in need of philosophically loaded defense than endangered by philosophically loaded skepticism. All the traditional Hilbertian "theses"—formalism, finitism, the essential role of a special proof of consistency—are methodological principles necessitated by this one question. (p. 31)

There is textual support for this reading of Hilbert; and indeed most working mathematicians express anti-foundationalist sympathies in some form from time to time, for important reasons. The burden shifts to Franks, though, of explaining why Hilbert expressed himself—for example in his 1904 address to the International Congress of Mathematicians; in his 1917 "Axiomatisches Denken"; in his "Über das Unendliche"; in his 1922 Hamburg lectures, not to mention in his 1899 correspondence with Frege, in a way which led so many of his contemporaries and subsequent readers to read him as pursuing foundational aims. Herbrand, for example, who made his own naturalistic position very clear, overlooked Hilbert's anti-foundationalism.

How well does Franks make the case? Franks's conceptualization of this material is convincing—if Hilbert wasn't an anti-foundationalist, then perhaps he should have been—and he goes a long way toward explaining what are in his view the various misreadings of Hilbert. But whether the multiplicity of views Hilbert expressed over time coalesce along the lines suggested here, will probably be a matter of debate.

The heart of the book addresses the question, in connection with the Hilbert program, whether so-called "intensionally correct" versions of Gödel's Second Incompleteness Theorem can be obtained for weak arithmetic theories. We saw that Franks reads Hilbert as having a strong pre-theoretical belief in the consistency of mathematics, while at the same time having "high standards as to what counts as a proof of it." The question then becomes urgent, whether Gödel's second theorem genuinely establishes its claim; that is, whether "every set of propositions sufficient to make a formula of T a fit expression of T's consistency is also sufficient to make that formula unprovable in T (if T is consistent)," as the author quotes M. Detlefsen.

As Feferman has pointed out, there is a distinction between the two incompleteness theorems: Gödel's First Incompleteness Theorem exhibits a sentence $\mathcal{G}$ in the language of the relevant theory, which is undecided. Nothing about the correctness of the claim that e.g., Peano Arithmetic is incomplete, turns on the "meaning" of $\mathcal{G}$. This is not the case with the second theorem, where the general claim must depend on the meaning of the consistency statement *as read by the theory*. That is, we should grant the meta-theoretical claim that a theory T cannot prove its own consistency only when there is a sentence both which T "recognizes" as a consistency statement, and which T cannot prove. This is not to question the legitimacy of Gödel's second theorem of course, but rather to point to possible limitations on its range of application—an observation Gödel himself made, famously, in his 1931 paper.

Franks suggests that in order for a proof predicate to be counted as intensionally correct, it must be formulated in terms of Herbrand provability. Herbrand proofs are propositional proofs, and therefore, Franks suggests, computationally simpler; they are sensitive to the computational resources of the theory, in that a game semantics can be given which relativizes in a natural way to the computational strength of the relevant theory. But the concepts of Herbrand provability and provability separate in weak theories. (For example $I\Delta_0 + \Omega_1$ does not prove the equivalence of Herbrand consistency with the usual formalized consistency statement.) The crux of the matter then, for Franks, is extending what one might call the Feferman/Detlefsen project to Robinson's weak arithmetic theory Q—a theory whose resources are non-controversial.

It is just here that the non-proof-theorist might lose track of Franks's compelling but rather delicate argument. Also, the suggestion that natural deduction without the cut rule captures accepted mathematical methods of proof is one most mathematicians would find puzzling—in spite of the virtues of Herbrand proofs otherwise. One would think that intensional adequacy has more to do with the standard notion of proof.

Franks's reading of Hilbert aside, his book raises the question: is mathematics autonomous? Franks's suggestion is that mathematics is completely free-standing; answerable to *aesthetic* constraints, if anything, and certified, if such is needed, by applications. Franks is asking us to shift our attention away from the foundational project; not to naturalize mathematics necessarily, but to historicize it.

But is abandoning foundations the right thing to do? Taking autonomy in the absolute sense of the word, what about the old problem of applicability? One of Franks's strengths is his consequentiality, meaning he'd probably dismiss the applicability problem as just another in a long line of pseudo-problems, which philosophy has thrown at mathematics.

There is also the problem, or problems, of set theory. Franks remarks that "the trail of philosophy never crosses into mathematical terrain." (Page 62.) Set theory is not really mentioned by Franks, but it must be said that in that field, philosophy has managed to build not a trail, but a four lane highway. "Second order reflection," as Franks calls it, on the nature of the infinite, on the notion of maximality, on the bivalent—or not—nature of mathematical truth, has been going on in the set theory community for some time. It is one of the main sources of the discovery and further development of reflection principles and of large cardinal concepts. It simply cuts to the core of the subject—and always has.

This puts Franks in the position of having to become involved in the taxonomy of subjects—a rather pedestrian topic to come across in a book of this quality; in the position of classifying as mathematics or philosophy Woodin's argument, for example, against the generic multiverse view—an argument seemingly rife with second order considerations. On the other hand Franks could simply suggest, along with Quine, that (this area of) set theory should not be considered real mathematics. But this goes against the grain, presumably. The whole point of this book is to put an end to such prohibitions.

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JACQUES SAKAROVITCH. ***Elements of automata theory.*** Cambridge University Press, Cambridge, 2009, xxiv + 758 pp.

Automata theory is one of the longest established areas in Computer Science and its applications include pattern matching, syntax analysis, software verification and linguistics.

Jacques Sakarovitch's ***Elements of automata theory***, published by Cambridge University Press in 2009, was originally published in French as ***Éléments de théorie des automates*** by Vuibert, Paris in 2003. It covers in great detail the classical theory of finite automata and the part of its mathematical foundations that rely on noncommutative algebra: semigroups, semirings and formal power series. It only treats finite automata on finite words, but includes automata with outputs, also called transducers, and automata with coefficients, also known as weighted automata or formal power series. It does not cover connections with logic, except for decidability results.

Reuben Thomas' remarkable translation succeeds in conveying the inimitable style of the author, including puns and citations. The reader should prepare himself for a deluge of footnotes and marginal notes. The original French version already contained a lot of them, but the English edition adds a number of *Notes added in translation*. Marginal notes are used to give references to related matter like exercises or propositions elsewhere in the book, which turns out to be very convenient. Footnotes are reserved for comments on terminology and notation, further references and entertaining personal thoughts of the author. I recommend the reading of the footnotes on pages xxiv, 24, 103 and 158 to catch a glimpse of Sakarovitch's humoristic style.

The author provides a great variety of exercises, ranging from straightforward results to more challenging ones taken from research papers. Following a practice that is common nowadays, part of these exercises form an integral part of the text and are sometimes used later in the book. But contrary to most authors, who simply throw the responsibility of providing intricate proofs onto the reader, Sakarovitch takes the pain to give detailed solutions to most of his exercises. Even for this reason alone, the book should be highly recommended as a reference textbook for students and researchers interested in automata theory.

Chapter 0 covers reminders on relations, words and languages, algebraic structures such as monoids, semirings and matrices, terminology from graph theory and basic results on complexity and decidability. The exercises also introduce a number of nontrivial notions and results: Fine and Wilf's theorem on periods of a word, conjugate words, equidivisible monoids. The native English speaker will also enjoy the fact (proved at the bottom of page 41) that the Robert & Collins Dictionary does not follow any lexicographic order: the problem arises when the hyphenation symbol is used.

According to the author, Chapter 1 presents the traditional theory of finite automata on finite words. It contains indeed the material expected to be found in any basic course on finite automata: recognisable languages (i.e., recognised by a finite automaton), rational (also called regular) expressions and rational languages, determinisation, minimisation, derivatives and Kleene's theorem. However, it actually contains much more. I recommend in particular Section 4, which gives an exhaustive survey on rational expressions and on derivatives and which compares, for the first time, the algorithms of Brzozowski and McCluskey (BMC), McNaughton and Yamada (MNY), Thompson, and Antimirov. Further, already in this first chapter, the author takes the opportunity to bring up some famous problems of automata theory. Rational identities are introduced on p. 128 and used to analyse the BMC and MNY algorithms. Conway's problem of finding a complete set of identities for rational expressions is clearly out of the scope of this book, but references to Krob's solution to this problem are given in the references of the chapter. The star height problems are illustrated by two early results on the (restricted) star height problem: Eggan's theorem relating star height and loop complexity and Dejean and Schützenberger's proof that the star height hierarchy is infinite. The author also presents a deep improvement over the usual pumping lemma, due to Ehrenfeucht, Parikh and Rozenberg, which permits him to *characterize* recognisable languages. Further, the chapter includes the Knuth–Morris–Pratt string matching algorithm (p. 156), Schützenberger's proof of Shepherdson's result on two-way automata (p. 173) and a number of classical results on rational languages (pp. 176–178). A very short section (p. 175) is devoted to Moore and Mealy machines, probably for historical reasons. These machines are indeed special cases of the sequential transducers studied in Chapter 4.

The first two sections of Chapter 2, entitled *The power of algebra*, cover automata over a monoid and their matrix representations, [unambiguous] rational and recognisable sets over arbitrary monoids, syntactic congruences and syntactic monoids. This material can also be found in Eilenberg's two volumes ***Automata, languages and machines*** or in Berstel's book ***Transductions and context-free languages***. But the topics presented in the remaining sections are rarely found in books. Section 3 on coverings and Section 4 on universal automata (p. 273) are some of the highlights of the book. They present, for the first time, a comprehensive survey on a notion frequently rediscovered since its introduction by J. H. Conway in 1971. I just have a slight regret: the author misses the opportunity to mention residual finite state automata, a notion introduced by Denis, Lemay and Terlutte [***STACS 2001***, LNCS 2010, p. 144–157], that is also strongly related to residuals. Section 5 is devoted to two results: Higman's theorem that the subword order is a well quasi-order and an extension of this result due to Ehrenfeucht, Haussler and Rozenberg. The last two sections describe the rational subsets of two important monoids: the free group and the free commutative monoids.

Chapter 3 is devoted to formal power series (in noncommutative variables) over a semiring. This topic is also covered in other books, notably in the book of Berstel and Reutenauer, ***Rational series and their languages***, Springer 1988. The chapter deals with series over a semiring, weighted automata and their matrix representations. Derivatives, Hankel matrices and Kleene–Schützenberger's theorem on the equivalence between rational and recognisable series are carefully explained. Algorithmic and decidability issues are also considered, notably the equality problems for two recognisable series (decidable) and for the supports of two $\mathbb{Z}$-rational series (undecidable). Another interesting question concerns the rationality of the support of a series. In particular, in the colourful terminology of the author, the *skimming theorem* states that if s is an $\mathbb{N}$-rational series, then $s - \text{support}(s)$ is also $\mathbb{N}$-rational. Sections 5 and 6 treat some more advanced matters: series on an arbitrary monoid with coefficients in a complete semiring and rational subsets in free products. The chapter ends with a primer on noncommutative linear algebra (modules over a ring and vector spaces over a skew field). As everywhere in the book, the numerous exercises and their solutions are an invaluable source of examples and research topics.

Chapter 4 on transducers deals with relations realised by finite automata. The first section covers rational and recognisable relations, their representations by real-time transducers and the Rabin–Scott model. It follows roughly the presentation given by Berstel [***Transduction and context-free languages***, Teubner 1979] or Eilenberg [***Automata, languages and machines***, vol. A, Academic Press 1974]. Section 2 introduces K-relations in a very general setting and Section 3 gives an exhaustive survey on rational K-relations and their representations. Section 4 and Section 7 form one of the most substantial and valuable parts of the book. It contains a proof of several important decidability and undecidability results, dealing with intersection, recognisability and equality of rational relations over various semirings. Section 5 is devoted to deterministic transducers and deterministic relations, their matrix representations and the related decidability questions. The highly valuable Section 6 presents for the first time a unified treatment of various variants of deterministic relations: letter-to-letter, bounded-length discrepancy and synchronising relations.

Chapter 5 deals with functions realised by finite automata. The chapter starts with a proof that deciding whether a finite transducer computes a function is a decidable property. It follows that the inclusion and the equivalence of rational functions is decidable. Then the author introduces a new terminology which, in my opinion, should be strongly supported since it is much better than the traditional one. He calls *sequential* a function that can be realised by a deterministic automaton with a possible initial output and possible final outputs. This definition, originally introduced by Schützenberger under the name of a subsequential function, turns out to be the really important notion. For instance, it allows a one-state representation for the simple functions $x \mapsto xu$ and $x \mapsto ux$, where u is a fixed word, which would be impossible with the old definition. The captivating Section 2 investigates the uniformisation problem from descriptive theory in the context of rational relations. It is proved that every rational relation from A^* to B^* can be uniformised by an unambiguous rational function. As a consequence, every rational function can be obtained as the composition of a sequential function with a co-sequential function. An extended version of the skimming theorem and various decidability results complete this nice section. Section 3 turns to a related question, namely cross-sections of rational functions. Section 4 covers two important topics: Choffrut's topological characterization of sequential functions and the minimisation of sequential transducers.

On the whole, the book is well written and pleasant to read, but given its length (758 p.), I would recommend reading it chapter by chapter. The book starts with a preface and a prologue on Pascal's division machine. Every chapter begins with a discussion of the motivations and ends with detailed notes and references. Many of the more elaborate proofs are preceeded by a discussion of the ideas of the proof, which greatly aids the understanding of the constructions. As mentioned earlier, many interesting results are presented as exercises, for most of which complete solutions are provided. ***Elements of automata theory*** will be a very valuable resource for students and researchers working in automata theory.

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Kurt Gödel. Essays for his centennial, edited by Solomon Feferman, Charles Parsons, and Stephen G. Simpson, Cambridge University Press, Cambridge and others, 2010, x+373 pp.

If the *Association for Symbolic Logic* is preparing an anniversary for a certain logician, it is no wonder that the resulting volume is written by the Who is who in these fields of interest. This holds in particular in the case of Kurt Gödel and the centennial of his birthday in 2006. The reader of the volume under review will be rewarded by 17 articles, written by a very distinguished team consisting of 19 Gödel scholars and covering a remarkable range of Gödel's fields of interest. Because of the limited space, the reviewer cannot adequately take every contribution into account. Therefore we give only some general hints on the content of each contribution. The volume is organized into the four parts "General" (pp. 1–42), "Proof Theory" (pp. 43–141), "Set Theory" (pp. 143–225), and "Philosophy of Mathematics" (pp. 227–373), whereas the emphasis on proof theory, set theory, and the philosophy of mathematics was chosen with respect to the represented fields of the *Association for Symbolic Logic*. Within this threefold partition of Gödelian subjects, the editors take care for contributions dealing primarily with recent research interests in mathematical logic, the foundations and philosophy of mathematics as well as their history. The volume was not prepared as a collection of introductory essays but as a contribution to Gödel scholarship. Therefore, it is less appropriate for novices. Since 11 articles had already been published elsewhere (most of them in ***The Bulletin of Symbolic Logic*** 11(2), 2005), we indicate these contributions by an asterisk *.

The first part is explicitly dedicated to the recent context and the possible further aims of the Gödel scholarship. Solomon Feferman's personal report "The Gödel editorial project: A synopsis"* (3–20) offers interesting background information about the decades long project of Gödel's ***Collected Works***. The presented synopsis documents a lot of systematic and ordinary problems arising from such an enormous editorial project. In the complementing contribution "Future Tasks for Gödel Scholars"* (pp. 21-42) John W. Dawson, Jr. and Cheryl A. Dawson prepare a list of further challenges related to still unpublished items and archival sources. In particular they discuss interesting items that were not published in the ***Collected Works***. Additionally, the paper contains some appendices (pp. 27–42) listing entries from Gödel's ***Arbeitshefte*** and his Notebooks.

Jeremy Avigad's paper "Gödel and the metamathematical tradition" (pp. 45–60) opens the second part and is dedicated to the tensions between the tradition of Hilbert's program and Gödel's methodological conception of logic. He outlines, among other things, that Gödel's opposition to Hilbert's outlook was constant throughout his career. On the basis of the discussion of the incompleteness theorems and their implications, Wilfried Sieg's contribution "Only two letters: The correspondence between Herbrand and Gödel"* (pp. 61–73) analyzes the intellectual similarities between Herbrand and Gödel. In "Gödel's reformulation of Gentzen's first consistency proof for arithmetic: The no-counterexample interpretation"* (pp. 74–87) W. W. Tait offers a detailed reconstruction of Gödel's reformulation of Gentzen's first consistency proof for Peano Arithmetic. The same author analyzes in "Gödel on Intuition and on Hilbert's finitism" (pp. 88–108) Gödel's conception(s) of intuition with respect to the Kantian notion of pure intuition and in a convincing manner argues for the thesis that Kant's philosophy of mathematics leads precisely to primitive recursive arithmetic—a remarkable result also for the ongoing discussion about Hilbert's concept of finitism and the Kant scholarship. The aim of Stephen G. Simpson's paper "The Gödel hierarchy and reverse mathematics" (pp. 109–127) is to raise attention to several recent research questions arising from Gödel's incompleteness theorems. He does this on the topics "Gödel Hierarchy", "reverse mathematics", "Foundational consequences of reverse mathematics", and "partial realization of Hilbert's program." With respect to the Lucas–Penrose fallacy, John P. Burgess in his contribution "On the outside looking in: A caution about conservativeness" (pp. 128–141) advises caution towards some neo-Hilbertian interpretations or applications of conservativeness metatheorems.

The third part of the volume starts with Akihiro Kanamori's paper "Gödel and set theory"* (pp. 145–180). He chronicles Gödel's work in set theory by an integrated view of the historical and mathematical development. Sy-David Friedman's article "Generalisations of Gödel's universe of constructible sets" (pp. 181–188) deals with the questions whether we can combine the mathematical power of $V = L$ with the consistency power of large cardinals, and if large cardinals are relevant solely for the calibration of consistency strength? While using results of contemporary set theory, Peter Koellner in his paper "On the question of absolute undecidability"* (pp. 189–225) brings the question of absolute undecidability into sharper relief.

Martin Davis in his opening contribution (to the last part) "What did Gödel believe and when did he believe it?"* (pp. 229–241) exhibits some evidence for changes in Gödel's philosophical point of view, in particular with respect to his evaluation of Hilbert's program and of his attitude concerning a realist treatment of sets. In his "On Gödel's way in: The influence of Rudolf Carnap"* (pp. 242–251), Warren Goldfarb argues for the importance of Carnap for the intellectual development of Gödel. He shows that it was Carnap who introduced Gödel to logic and that this influence changed into the other direction after the incompleteness theorems. Steve Awodey and A. W. Carus consolidate this in their contribution on the topic "Gödel and Carnap" (pp. 252–274). While considering several biographical stages of Gödel and Carnap, they argue in detail for the thesis that kinship with (the philosophy of) Leibniz was something both had in common and that especially that motive spurred each of them to refine and sharpen their own views in the face of the respective other's successive improvements. Awodey and Carus portray a fascinating picture of the interaction of two critical, open minded, and brilliant logicians. The extensive contribution "On the philosophical development of Kurt Gödel"* (pp. 275–325), by Mark van Atten and Juliette Kennedy, deals with the change of Gödel's favored philosophy from Leibniz to Husserl. In particular they discuss the question of how to interpret this turn, why Gödel specifically did turn to Husserl's later transcendental philosophy, and whether there are provable influences from Husserl on Gödel's writings. Since this investigation belongs to Gödel as well as to Husserl scholarship, the paper makes also a valuable contribution to the latter. (Small remark: In the presentation of the varieties of idealism, p. 289, "dogmatic idealism" is unfortunately attributed to George Berkeley. This ascription—of course also done by Kant—is misleading, since Berkeley does not claim that ideas exist only one's mind or that external objects are illusions.) Charles Parsons in his "Platonism and mathematical intuition in Kurt Gödel's thought"* (pp. 326–355) explicates Gödel's understanding of Platonism and argues for the thesis that this epistemology is not quite so intrinsically connected to his conception of intuition as other commentators have thought. Parsons suggests interpreting Gödel's conception of intuition rather in terms of a Kantian inspired theory of reason. The final paper "Gödel's conceptual realism"* (pp. 356–373), by Donald A. Martin, deals once again with Gödel's Platonism. Martin focuses his attention on Gödel's brand of Platonism "conceptual realism" and analyzes Gödel's understanding of the term "concept" while construing the concepts in question (stets, numbers, etc.) as concepts of set structures, number structures, etc.

Despite the fact that several contributions have already been published elsewhere, the volume under review represents an excellent composition of highly interesting articles on several fields of Gödel's interest. The contributions document exemplarily the widespread influence and actuality of several Gödelian achievements as well as the potential of Gödelian thoughts for further investigations in logic, mathematics, and philosophy. This volume should be found in the library of every Gödel scholar.

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