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**The Role of Abduction in Self-Similarity:
On the Peircean Concept of the *Map of the Map***

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The third Harvard lecture delivered by Peirce, on April the 9th, 1903, under the title “The Categories Defended”, is considered a major work in Peircean semiotics and epistemology (see Deely 2001:661–663; Sobrinho 2001:29). Relatively less attention is paid to its mathematical contents, as a prefiguration on the methodical study of the self-similarities. Indeed, it is hard to guess why leading mathematicians on this subject—among them Benoît B. Mandelbrot—have largely neglected this fact.

Using a precise language, Peirce formulates three concepts, familiar to self-similarity and fractal theorization. Namely: (1) the *self-sufficiency*, (2) the *economy of the Universe*, and (3) the *map of the map*. Respectively, these notions are directly connected to the modern mathematical concepts of *self-reference*, *power laws*, and *self-similarity*, closely related among them. These concepts can be summarized as follows: self-reference means the “circular causal in feedback mechanisms”, particularly in cybernetics and artificial intelligence (von Foerster 1949). In aesthetics and structural linguistics, self-reference serves as an essential device for sense and meaning—as in René Magritte paintings *La trahison des images* (1929), and *Les deux mystères* (1966), both classical examples of self-referentialism.

Power laws are concerned with constant proportion in generalized phenomena, in which the probability of measuring a specific value of some quantity varies inversely as the power of that value (Newman 2005:323). Many examples of power law can be observed in the proportional relationships between two physical frequencies.

Power laws are valid independently of any scale or moment, as they are universal laws. For instance, Burioni *et alii* (2004:36) consider that “All real physical structures (embedded in three-dimensional space) have been found up to now to exhibit power law behaviour in the low-frequency density of vibrational states”.

Self-similarity is the kind of relationship established when, in different scales, a same object or a same group of behaviours, have the same set of basic symmetries (see Mandelbrot 1967, 1977; Peitgen and Richter 1986). Broadly studied forms of self-similarity are, for example, a *tree*, a *river*, and Brownian motion, whose general development has the same kind of structure in its tiniest parts. Nevertheless, the best example of self-similarity—which bestness is due to its descriptive power, its analytical adequacy, and its aesthetical influence—is the *map of a map* described by Peirce (*CP* 8.122) as it follows:

Imagine that upon the soil of England, there lies somewhere a perfect map of England, showing every detail, however small. Upon this map, then, will be shown that very ground where the map lies, with the map itself in all its minutest details. There will be a part fully representing its whole [...] On that map will be shown the map itself, and the map of the map will again show a map of itself, and so on endlessly.

This are overwhelming words, considering that Mandelbrot, the patriarch of fractal geometry, never devoted any acknowledgement or even the smallest reference to Peirce—this is, however, matter for another debate.

For now, given all these explanations, it is clear that these three categories (*i.e. self-reference, power laws, self-similarity*), are epistemologically and operatively connected to the Peircean categories of *self-sufficiency, economy of the Universe*, and the *map of the map*.

Anyone in acquaintance with Peirce’s theories, will notice that I am not invoking the usual Peircean world of sign trichotomies, like *Qualisign, Sinsign*, and *Legisign*, regarding signs in relation to themselves; or *icon, index*, and *symbol*, about signs in relations to their objects; or *rheme, proposition*, and *argument*, about signs in relation to their interpretants. Neither this is about *Quality, Reaction*, and *Representation*. Rather, this is about other kind of trichotomy, tacitly implied by Peirce, and clockworking in his whole system of trichotomies, as the most important machinery for his postulates. A rationalistic theory of self-similarity is behind this structuralism,

under development during Peirce's investigation and unfinished to his death. From a longer distance, however, one can see how this structuralism is connected to new variations on the same theme.

In Peirce (1903a/1998:162) the *self-sufficiency* of an argument is represented by a schematic trichotomy which structurally can be considered somehow as a precedent for the Chomskyan 'generative trees' (1956), which have been used in many aspects of linguistics and in other analytical fields, such as musicology (e.g. Lerdahl and Jackendoff 1983).



fig.1

This comparison may be seen as trivial, except for the very case that a same kind of structural self-referentialism indicates different things. What is counting here, obviously, is not the schematic representation by itself, but a same kind of mental operation, a systematized analogy that is substantial for the mathematical thought. This operation, by the only intervention of recurrence rules, reproduces the basic relation in the main structure, iterating its own qualities until a system of hierarchies has been plotted:

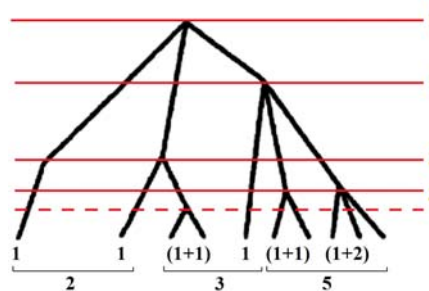


fig.2

Consequently, one can analyze Peirce's tree as a self-sufficient and self-referential construction. The original scheme made by Peirce in 1903 (a) is, very evidently, a

self-referential and (pre)self-similar elaboration. In this scheme Fibonacci distribution is quite perceptible. This is not to say —at all— that Peirce was looking for representing Fibonacci series on it; neither this would be relevant for this case. The true relevance in this elaboration is that one can find here the same intuition for order and hierarchy through self-reference, starting from very few elements and very simple conditions: what Peirce identifies as *Firstness*, *Secondness*, and *Thirdness*, for a mental process. The mind of Peirce works here like the mind of a composer or a mathematician, in search for the most simple —but operational— model for the synthesis and the methodification of self-referentialities. The logic behind Peirce’s trichotomy should be, thus, coherent in all its parts. And it is. If one follows the same logic for extending his tree model, we realize that it is possible to endlessly build Fibonacci trees, typically considered by their self-similarities. Then, the next step in the Peircean tree would be like this:



fig.3

The very point of this exercise is not mere constructivism, but the coherence between this order and the logical effectiveness of self-reference in Peircean mappings. For this, an important issue would be the comprehension of mental mechanisms in such mapping continuity. An elementary question is *what does allow us to continue a basic Peircean trichotomy into a self-similar pattern?* The answer seems to be nested in Peirce’s own theorization over the *map of the map*, particularly the concept of *abduction*.

Based on the Aristotelian criterion referred to as *abductio*, Peirce suggests a method of hypothetical inference, which operates in a different way than the deductive and inductive methods. “Abduction is nothing but guessing”—says Peirce (CP 7.219). This principle is of extreme value for the scrutiny of our understanding of mathematical self-similarity in both of its conceptions: *relative* (or self-affinity) or

absolute (which strictly corresponds to fractal geometry). For the first one, abduction incarnates the quantitative/qualitative relationships of a self-similar object or process; for the second one, abduction makes understandable the statistical treatment of self-similarity, “guessing” —using the Peircean notion of abduction— the continuity of geometric features to the infinity through the use of a systematic stereotype: for instance, the assumption that the general shape of the Sierpiński triangle continues identically into endless segmentation.

Peirce himself explains how abduction works in mathematics, and he uses the following scheme, in which he provides an intuitive proof for demonstrating what he calls “pragmatism as the logic of abduction” (Peirce 1903b:237)

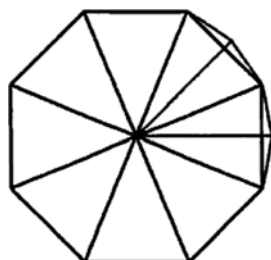
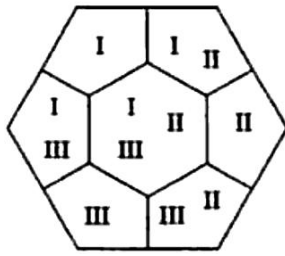


fig.4

In this graph, Peirce suggests that a mental mapping of some known object may be used for a intuitive, logic continuity of the same object. In this example an octagon can turn into a decahexagon. The same operation may lead to the circle, by an infinite progression. The mental tools for doing this are self-reference and abduction, both materialized through analogy (*i.e.* mathematical *αναλογία* or proportion).

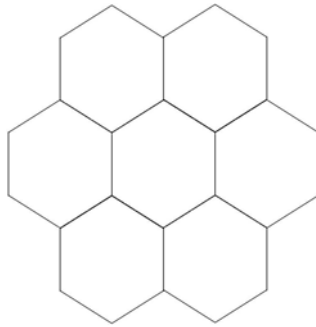
The analogy coined by Peirce, of an *exact map* containing itself the same exact map, is not only the most important precedent of Mandelbrot’s problem of measuring the boundaries of a continuous irregular surface with a logarithmic ruler, but also still being a useful abstraction for the conceptualization of relative and absolute self-similarities, and its mechanisms of implementation. It is useful for explaining some of the most basic geometric ontologies as mental constructions: in the notion of infinite convergence of points in the corners of a triangle, or the intuition for defining two parallel straight lines as two lines in a plane that “never” intersect; but also in the contradiction of this notions, by a non-Euclidian principle. Abduction, ultimately, is the means for investigating what happens in infinite relations, by a principle of finiteness.

a)

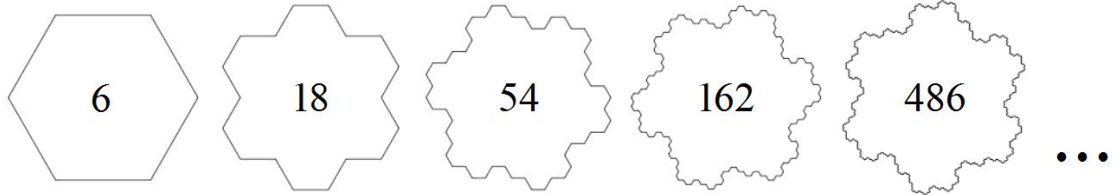


(Peirce 1903a/1998:164)

b)



c)



d)

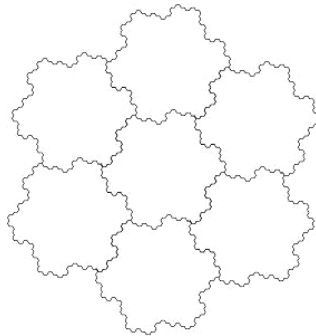


Fig. 5. Abductive steps for the construction of the Gosper tessellation.

In figure 5 it is shown the evolution of a mental model, starting by (a) the possible classes of what Peirce calls “all possible systems of metaphysics”. A remark should be made in the fact that this hexagon inscribed into a bigger hexagon is completely designed by Peirce, for the purpose of portrait the relationship between actual and potential, being the finite object the first, and its infinite expansion, the second, in which “all possible systems” have place. This proposition and its representation are exactly of the same kind the previous example with expansive trees is. Actual and potential, finite and infinite, are associated identically in the same mode. However, the intuition for continuity in this example has more difficult aspects, provided that in the first example we were dealing with a one-dimensional model, and in the latter we deal with surfaces in two dimensions. In an outstanding way, Peirce reflects his intuition for self-similarity and its coherent continuity. But he could not make it better in the absence of the modern informatic tools which allow the infinite progression of the corresponding model into an infinite, fractal curve.

In figure 5–b we see a hexagon surrounded by six identical hexagons. But the whole does not reflect a holistic coherence, as the bigger figure is not similar to the smaller: this tessellation of hexagons does not make bigger hexagons (see Schroeder 1991:14). This is why, tentatively, Peirce rejects this kind of figure: because its sole repetition does not purport any coherence between actual and potential, or between identity and multiplicity. By contrast, in figure 5–c gives a solution for constructing coherence after considering the hexagon as a first step for an infinite tessellation, by the consistent segmentation of its sides. In the ultimate result, portrayed by figure 5–c with the so called Gosper tessellation, we finally see how structural derivations from the original model by Peirce, can also get the form of a fractal continuity.

As a matter of fact, this view is also a possible solution to what C. W. Spinks (1991) justifiably criticizes as Peircean “triadomania”. Peirce’s obsession for looking three levels or stages of inference in any logical structure, can be nuanced in the light of multiplicity, as “boundaries cannot be trichotomized” (Spinks 1991:45-46). Furthermore, this expansion of the Peircean dialectics, from three to infinite faces, fits as well into Peirce’s definition of science and mathematics as a self-correcting device (see Peirce *CP* 6.40–6.42). Rationalistic thought, and not only geometry and the geometrical representation of thought, is what counts here for the analysis of knowledge as a self-constructive process.

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