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Public investment as commitment

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Abstract

Should public assets such as infrastructure, education, and the environment earn the same return as private investments? The long-term nature of public investments provides commitment to current preferences, which justifies lower than private returns for time-inconsistent decision markers. An institutionalized (i.e., exogenous) rule demanding equalized comparable returns removes the bias and implements the standard cost-benefit requirement. We show that such a stand-alone rule has no general welfare content: it implements Pareto efficiency if and only if preferences are time-consistent. Efficiency requires rules not only for the composition of investments but also for overall savings. Without supplementary rules for savings, accepting lower returns for long-term public assets is welfare improving.

JEL Classification: H43; H41; D61; D91; Q54; E21

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1 Introduction

Cost-benefit analysis (CBA) as a way to bring public projects and programs under public scrutiny is a feature of good governance which most economists agree on. CBA is applied not only to individual small projects but also to public programs that influence the economy as a whole as, for example, the Stern Review on climate change (2006) has recently illustrated. Such a cost-benefit analysis with economy-wide costs and benefits requires an integrated assessment of how to best allocate the overall resources between public and private uses. Based on the notion of opportunity cost, the integrated CBA stipulates resource allocations where public investments earn the same, potentially risk-corrected, return as private investments (e.g., Nordhaus 2007). In this paper, we show that there is no basis for such a stand-alone return requirement when we depart from the idealized setting of time-consistent preferences. Even when all present and future agents could commit to equalize comparable returns on public and private uses of savings, the rule would have no welfare content: it implements Pareto efficiency if and only if preferences are time-consistent. Efficiency requires rules not only for the composition of savings between public and private uses but also for the overall savings. Without these, the economy is better off by ignoring the stand-alone cost-benefit requirement for public investments.

The long-term public choices we have in mind are those related to long-term energy-supply, city planning, education, and environmental preservation, which have implications for the course of the economy over time, and where it seems to be a fact of life that the future rankings over the decisions are different from those prevailing at the time of decision making. We consider the allocation of savings for such uses when the future rankings over alternatives are *a priori* known to be different, due to dynamic inconsistencies. Such dynamic inconsistencies can arise from time-variant preferences (Strotz, 1956), intergenerational altruism (Phelps and Pollak, 1968), or self-control problems (Laibson, 1997).[1] In climate change, there is an emerging consensus that the far-distant gains from policies may not be appropriately captured by the cost-benefit calculations based on private capital opportunity costs but, rather, the future benefits should be converted to present values at discount rates declining with the time horizon; however, declining

[1]Non-constant discount rates can also result from aggregation over heterogenous individuals (Gollier and Zeckhauser, 2005, Lengwiler, 2005), or from uncertainty (Weitzman, 2000, Gollier, 2002). These need not to cause time-inconsistencies in the preference structure. We come back to this question in the concluding section.

discount rates introduce inconsistencies into climate policies (Karp 2005).[2]

Casting the analysis in the Phelps-Pollak-Laibson framework, we show that there is more investment in the long-term public asset in the benchmark equilibrium than under the equilibrium where the cost-benefit requirement is institutionalized, i.e., exogenous. Intuitively, the long-term asset provides commitment to a higher welfare for the far-future generations at the cost of the intermediate-future generations, which is an option valued by the current generation. The current decision-maker accepts the costs of this commitment, i.e., a lower than market rate of return for the long-term asset, and would strictly suffer from imposing the cost-benefit rule, even if such a rule is also followed by all later generations. While Pareto efficiency implies the cost-benefit rule — also in our case of time-inconsistent preferences — the inverse does not hold: neither efficiency nor Pareto improvement can be obtained just by insisting CBA as part of the "executive branch". As we will show, a fully efficient policy requires that the cost-benefit rule is supplemented with other policy measures, and without these, CBA unambiguously decreases welfare.[3]

To analyze the cost-benefit rule in a closed economy, we consider a representative-agent Ramsey saving problem where savings are allocated between traditional neoclassical capital and long-term public assets. The former capital can be interpreted as resulting from the aggregation of individual decisions and is thus private by nature, while the latter type of capital is public by assumption. We abstract from the aggregation and political economy aspects of the public decisions in order to pinpoint the allocative distortions not solved by CBA even in the representative agent framework. In this closed economy, the capital stock produces endogenously the rate-of-return requirement, or the opportunity cost, of the public investments; such an approach is needed, for example, in the climate context where the policy has an effect on the growth path of the economy (see, e.g., Weitzman 2007 and Nordhaus 2007).

For the welfare consequences of CBA, we consider welfare Pareto Efficiency (w-PE), where at each point in time welfare depends on both current and future utility levels. In principle, welfare for a given generation can look backwards and forwards in time, i.e., depend also on the utility levels in the past (in a different setting, Caplin and Leahy (2004) consider such a w-PE). In addition to the standard welfare Pareto efficiency criterion,

[2]The Stern Review on climate change (2006) has illustrated the potential scope and complexity of CBA. From the discussion that followed the Stern Review, it is clear that weights attributed to far-future payoffs strongly divide economists (see the September 2007 issue in the Journal of Economic Literature).

[3]Arrow et al. (1996) list eight principles for applying CBA, and, consistent with us, also conclude that cost-benefit rules do not automatically imply good policy.

we employ an auxiliary concept of utility Pareto Efficiency (u-PE), corresponding to the maximization of an intertemporal stream of weighted utilities where weights can be interpreted as discount factors. The hyperbolic discounting models popularized by Laibson (1997), and also O'Donough and Rabin (1999), and Barro (1999) satisfy the u-PE criterion: the equilibrium path of these models maximize a utility stream for some sequence of utility weights. It is clear that u-PE is a weaker efficiency concept, and it is well understood that w-PE implies u-PE but the implication does not hold in the other direction.[4]

This conceptualization helps us to connect to the previous literature and to define the connection between cost-benefit rules, inconsistent preferences, and welfare sharply. First, without the cost-benefit requirement the equilibrium outcome in our model is not even u-PE, i.e., the multiple-asset model shows a distortion not present in popular one capital-good hyperbolic discounting models. Second, adding the cost-benefit requirement as an institutional constraint will restore u-PE. But, if preferences are inconsistent, welfare Pareto efficiency never follows from u-PE and, in addition, the cost-benefit rule is not even Pareto improving. Insisting on such rules thus leads to the satisfaction of a narrow efficiency concept (u-PE), but the wider concept of welfare efficiency (w-PE) is not generally satisfied.

We first present a simple three-period example to illustrate the main results. The framework easily extends to the infinite number of periods, although the equilibrium analysis requires restrictions not present in three periods (see also Krusell et al. 2002, and Karp 2005, and 2007). We seek to formulate the general model such that we can flexibly analyze the relative persistence of the public asset because the persistence is the key determinant of the degree of commitment that the asset provides to the inconsistent decision maker. For a certain degree of persistence, there is no incentive to deviate from the cost-benefit rule, and the resulting equilibrium is observationally equivalent to a consistent preferences equilibrium, even though the underlying preferences are inconsistent, as in Barro (1999). The observational equivalence does not hold generally, however. Moreover, the deviation from the cost-benefit rule (i.e., under- or over-investment) depends both on the persistence parameters and the nature of preference inconsistency.

[4] Our welfare Pareto efficiency is a version of the multi-self Pareto efficiency; see Bernheim and Rangel (2009) for a discussion of the standard definition and suggestion for an alternative concept. Their definition of a weak welfare optimum in Corollary 2 is more restrictive than our w-PE, and requires that the subsequent choices are made by the same identity. Our model allows for an altruistic interpretation as in Phelps and Pollak (1968) with multiple generations.

2 A three-period model

2.1 The setting

We first consider three generations, living in periods $t = 1, 2, 3$. In each period, consumers are represented by an aggregate agent having a utility function and production technology. Consumption programs $(\mathbf{c}, q) = (c_1, c_2, c_3, q) \in A = A_1 \times A_2 \times A_3 \times A_q$ (non-empty intervals) constitute of a consumption level for each generation and the final asset q to the last generation. Generations are assumed to have the following simple welfare representation

$$w_1 = u_1(c_1) + \rho[u_2(c_2) + \theta[u_3(c_3) + v(q)]] \tag{1}$$

$$w_2 = u_2(c_2) + \sigma[u_3(c_3) + v(q)] \tag{2}$$

$$w_3 = u_3(c_3) + v(q), \tag{3}$$

where all utility functions u_t and v are assumed to be continuous and, in addition, strictly concave, differentiable, and satisfying $\lim_{c \to 0} u_t' = \infty$ and $\lim_{q \to 0} v' = \infty$. For interpretation, we assume that parameters $\rho, \theta, \sigma \in [0, 1]$ are discount factors, although this is not necessary in this three period model. Inconsistent preferences are identified by $\theta \neq \sigma$, i.e., the first and second generations disagree on the relative weight given to the last generation's utility. When $\theta > \sigma = \rho$, the near future is discounted more than the far future. Following Phelps and Pollak (1968) or, e.g., Saez-Marti and Weibull (2005) this can be interpreted as pure altruism towards the last generation, or alternatively as lack of (governmental) self control (Laibson, 1997).[5] For completeness, we also allow for the case $\theta < \sigma$. This could represent a situation where the representative agent looks one period ahead with less interest in the future further away.

Generations consider choices in a convex consumption possibility set $A \subseteq R^4_+$. The consumption possibilities are determined by a strictly concave neoclassical production function $f_t(k_t)$, where k_t is the capital stock they receive from the previous generation. The first generation starts with a capital stock k_1, and produces output which can be used to consume c_1, to invest in capital for the immediate next period k_2, or to invest in

[5] We can obtain the common β, δ model as in Phelps-Pollak-Laibson if $\rho = \sigma$, by defining $\beta = \rho/\theta$ and $\delta = \theta$. Then, $w_1 = u_1 + \beta\delta u_2 + \beta\delta^2 u_3$ and $w_2 = u_2 + \beta\delta u_3$. Inconsistencies are indentified by $\beta < 1$, corresponding to $\theta > \sigma$ in our case. For our purposes, it is slightly more straightforward to name the long-run weights as θ and σ. The short-run weight ρ gives degree of freedom in terms of interpretations (as just illustrated) but is inconsequential for the consistency of the preferences.

a durable asset for the third period, q:

$$c_1 + k_2 + q = f_1(k_1). \tag{4}$$

The second agent starts with the capital stock k_2, which produces output $f_2(k_2)$, and can use its income to consume c_2, or to invest in capital for the third period k_3:

$$c_2 + k_3 = f_2(k_2). \tag{5}$$

In this simple example, we abstract from possibilities of the second consumer to invest in the durable asset q. The third consumer derives utility from its consumption,

$$c_3 = f_3(k_3), \tag{6}$$

and from the inherited durable asset $v(q)$.

2.2 Welfare and utility Pareto efficiency

Consider an allocation $(\mathbf{c}, q)$ that is Pareto efficient for welfare levels (w_1^*, w_2^*, w_3^*) defined in (1)-(3). If we maximize w_1, subject to the constraints $w_2 \geq w_2^*$, and $w_3 \geq w_3^*$ and feasibility constraints (4)-(6), then we must find the same allocation, and non-negative Lagrange multipliers $(\alpha, \beta) \in R_+^2$ for the welfare constraints. That is, the Pareto efficient allocation is also the solution of a welfare program maximizing

$$\begin{aligned} W(\mathbf{c}, q) &= w_1 + \alpha w_2 + \beta w_3 && (7) \\ &= u_1(c_1) + (\rho + \alpha) u_2(c_2) + (\rho\theta + \alpha\sigma + \beta)[u_3(c_3) + v(q)] && (8) \end{aligned}$$

subject to (4)-(6). The conclusion also holds the other way around: any solution to a welfare maximization program with some $(\alpha, \beta) \in R_+^2$ is Pareto efficient. Strict concavity of the production functions and utility functions ensures the uniqueness of the allocation. Therefore, we can associate any Pareto efficient allocation with a pair of positive welfare weights $(\alpha, \beta) \in R_+^2$.[6] [7]

An alternative, and often easier, approach to describing efficiency is to directly consider the weighted stream of utilities. We say that a feasible allocation (u_1^*, u_2^*, u_3^*) is utility Pareto efficient (u-PE) if no utility level can strictly increase while keeping all

[6]We rule out allocations where the weight on w_1 equals zero. The weight on w_1 approaches zero, in relative terms, when at least one of the other weights becomes sufficiently large.

[7]Note that here the welfare is determined in a forward-looking manner but we could also define backward-looking welfare weights as in Caplin and Leahy (2004).

other utility levels at least constant. To separate the auxiliary concept of u-PE from the true Pareto efficiency we call the latter welfare Pareto efficiency (w-PE). Similar to the approach above, it is easy to see that a feasible allocation is u-PE if and only if there exist utility weights $(\alpha', \beta') \in R^2_+$ such that the allocation maximizes $U(\mathbf{c}, q)$ for these weights, where

$$U(\mathbf{c}, q) = u_1(c_1) + \alpha' u_2(c_2) + \beta'[u_3(c_3) + v(q)]. \tag{9}$$

Note that $W(\mathbf{c}, q)$ and $U(\mathbf{c}, q)$ are not independent welfare functions but rather tools for describing all feasible w-PE and u-PE allocations. Our purpose is to use the concept of utility weights and their connection to welfare weights for verifying whether equilibrium allocations considered below are efficient. The utility weights implied by an equilibrium allocation are easy to infer; see, e.g., Barro (1999) and Saez-Marti and Weibull (2005). If these weights are negative, we can immediately conclude that the allocation cannot be Pareto efficient. Even when the implied utility weights are positive, the welfare weights can be negative and thus the allocation is off the Pareto frontier. Whereas every pair of non-negative welfare weights $(\alpha, \beta) \in R^2_+$ can be converted into a pair of non-negative utility weights $(\alpha', \beta') \in R^2_+$, the inverse conversion is not immediate:

Remark 1 *u-PE implies w-PE if and only if*

$$\alpha' \geq \rho \tag{10}$$

$$\beta' \geq \rho\theta + (\alpha' - \rho)\sigma. \tag{11}$$

It is clear that if the stated inequalities hold, there are positive weights (α, β) corresponding to (α', β'), and respecting the original preference structure (1)-(3). The "only if" part follows from the observation that if one of the inequalities is not met, then one of the implied welfare weights α or β must be negative. Intuitively, efficiency defined in terms of utilities is consistent with welfare efficiency only if the weights on future utilities are sufficiently large so future generations receive a welfare weight *in addition to* the weight they receive indirectly from previous generations, e.g., due to altruism.

If u-PE is established, the gross savings levels $q + k_2$ and k_3 implicitly determine the utility weights α' and β'. The remark thereby implies that if u-PE is complemented with a gross savings policy that secures sufficiently high utility weights for future agents, welfare efficiency w-PE is guaranteed. In the remainder, we study the application of the cost-benefit rule per se, without complementary savings rules, and ask whether the cost-benefit rule suffices to establish welfare efficiency.

2.3 Efficiency and the cost-benefit rule

We describe now how the cost-benefit rule follows from Pareto efficiency. Welfare Pareto efficiency implies utility Pareto efficiency, as we can write $\alpha' = \rho + \alpha > 0$ and $\beta' = \rho\theta + \alpha\sigma + \beta > 0$ for positive welfare weights α and β, and therefore for convenience of notation we use the utility maximization program (9) in this section. The first-order conditions for $\{k_2, k_3, c_1, c_2, c_3\}$ tell us that any Pareto efficient allocation satisfies:

$$1 = [\frac{\alpha' u_2'}{u_1'}] f_2' = [\frac{\beta' u_3'}{u_1'}] f_2' f_3'. \tag{12}$$

Denote by $MRS_{i,j} > 0$ the marginal-rate of substitution of consumptions between periods (i, j) (defined to be positive). Let $R_{i,j}$ denote the (compound) rate of return on capital from period i to j. We can then re-express the first-order conditions as the usual consumption-based asset pricing equation:

$$1 = \frac{R_{1,2}}{MRS_{1,2}} = \frac{R_{1,3}}{MRS_{1,3}}. \tag{13}$$

Thus, the marginal rate of substitution equals the return on savings. For the investment in the public asset q to the last generation, the first-order condition requires $u_1' = \beta' v'$, which we rewrite as

$$1 = MRS_{1,q} \tag{14}$$

where $MRS_{1,q}$ is defined between period 1 consumption and q. To count for the opportunity cost of transferring period 1 output to the asset q, combine $MRS_{q,3} = MRS_{q,1} \cdot MRS_{1,3}$ and $MRS_{1,3} = R_{1,3}$ yielding:

$$1 = \frac{R_{1,3}}{MRS_{q,3}}. \tag{15}$$

This is the consumption-based cost-benefit rule. The benefit of one unit of investment in the long-term asset q is measured in terms of the third-period consumption good. This return to direct long-term investments should equal the opportunity cost determined by the compound return on capital k. Under efficiency, the long-term asset q should yield the same return as the capital asset k. Noticeably, the cost-benefit rule is neutral with respect to, that is, independent of, weights given to each generation's utility. The cost-benefit rule is a positive, and sufficient test for utility efficiency:

Lemma 1 *Utility Pareto efficiency (u-PE) and the cost-benefit rule are equivalent: a feasible allocation with strictly positive consumption, capital and public investment is u-PE if and only if it satisfies the cost-benefit rule.*

Proof. Necessity of the cost-benefit rule has been established above. To prove sufficiency, we notice that given the allocation, we can construct the weights α' and β' from (12). It is straightforward to see that we can construct three non-negative Lagrange multipliers for (4)-(6) to satisfy all first-order conditions for $\{k_2, k_3, c_1, c_2, c_3\}$. The cost-benefit rule then ensures that the first-order condition for q is also satisfied. ■

The equivalence will be instrumental in our equilibrium analysis. First, if the cost-benefit rule is not satisfied, the equilibrium allocation is not u-PE let alone w-PE. We find in the next section that in equilibrium the cost-benefit rule will not hold, so the conclusion for efficiency is immediate. Then, in the following section, we impose the cost-benefit rule as an institutional constraint on the equilibrium. We show that such an equilibrium implies positive utility weights and thus restores u-PE, but the implied welfare weights are not all positive unless preferences are consistent.

2.4 Equilibrium

Consider now the subgame-perfect equilibrium (SPE) of the game where generations choose consumptions and investments in the order of their appearance in the time line, given the preference structure (1)-(3).

The third agent consumes all capital received and enjoys the long-term asset. The second agent decides on the capital k_3 transferred to the third agent, given the long-term asset q chosen by the first agent and the capital inherited k_2. We thus have a policy function $k_3 = g(k_2, q)$, but for the separable utility specification, second-period investments only depends on the stock of capital received, $k_3 = g(k_2)$. The policy function g ensures that the following first-order condition is maintained

$$1 = \frac{\sigma u_3'}{u_2'} f_3'. \tag{16}$$

The strict concavity of utility implies consumption smoothing, and thus if the second agent inherits marginally more capital k_2, the resulting increase in output is not saved fully but rather split between the second and third generation:

Lemma 2 *Policy function g satisfies $0 < g' < R_{1,2}$.*

Proof. Substitute the policy function $k_3 = g(k_2)$ in (16),

$$\sigma u_3'(f_3(g(k_2)))f_3'(g(k_2)) = u_2'(f_2(k_2) - g(k_2))$$

and take the full derivatives with respect to k_2 to obtain

$$\sigma g'(u_3'' f_3' f_3' + u_3' f_3'') = u_2''(f_2' - g')$$

which leads to

$$g' = \frac{f_2' u_2''}{\sigma u_3'' f_3' f_3' + \sigma u_3' f_3'' + u_2''} < f_2' = R_{1,2} \tag{17}$$

as $u_t'', f_t'' < 0$ and $f_t', u_t' > 0$. ■

The first agent decides on consumption and investment in the long-term asset, given the policy function g, to maximize its welfare

$$w_1 = u_1 + \rho[u_2(f_2(k_2) - g(k_2)) + \theta u_3(f_3(g(k_2)) + \theta v_3(q)].$$

The first-order conditions for investments k_2 and q, respectively, are:

$$u_1' = \rho(f_2' - g')u_2' + \rho\theta f_3' g' u_3' \tag{18}$$

$$u_1' = \rho\theta v'. \tag{19}$$

The equations reflect the fact that the marginal cost of investment, i.e., the marginal utility loss, is the same for both types of investments. Rewriting after substitution of (16) gives[8]

$$MRS_{q,3} = [\frac{\sigma}{\theta}(f_2' - g') + g']f_3'. \tag{20}$$

This condition is the equilibrium version of the cost-benefit rule (15). To assess the deviation from the rule (15), consider the difference between the equilibrium market return on capital and the public asset. In view of (20), the gap $R_{1,3} - MRS_{q,3}$ can be written as

$$f_2' f_3' - [\frac{\sigma}{\theta}(f_2' - g') + g']f_3' = (1 - \frac{\sigma}{\theta})(f_2' - g')f_3'.$$

This together with Lemma 2 implies

$$R_{1,3} - MRS_{q,3} > 0 \text{ if and only if } \frac{\sigma}{\theta} < 1. \tag{21}$$

Thus, in equilibrium, the first agent invests in the long-term asset q up to a point where the rate of return falls short of the rate of return of capital over the same period,

[8] Note that the marginal-rate of substitution between q and c_3 is independent of weights on utilities, and therefore there is no need to indicate who's preferences are in question.

if and only if the preferences are near-biased or hyperbolic, $\sigma < \theta$, i.e., the first agent gives a higher weight to the long-term utility than the second agent. The result has a very simple intuition. The first consumer would like to distort investment in favor of the third consumer, compared with the investment pattern of the second consumer. This is possible through the asset q, and thus the long-term asset is more valuable to the first agent, which is reflected in the lower return requirement. The opposite distortion —too little investment— occurs if $\sigma > \theta$.

Proposition 1 *If preferences are inconsistent ($\sigma \neq \theta$), the public investment in the long-term asset does not satisfy the cost benefit rule, i.e., $MRS_{q,3} \neq R_{1,3}$. The equilibrium return falls short of $R_{1,3}$ iff $\sigma < \theta$.*

Proof. *Above.* ■

Considering the welfare properties of the equilibrium, it is clear from Lemma 1 that if $\sigma \neq \theta$, the equilibrium allocation is not welfare Pareto efficient as it is not even utility Pareto efficient: we have shown that an efficient allocation must satisfy the cost-benefit rule (15). Since the equilibrium deviates from this rule, we cannot find positive utility weights that would support the equilibrium outcome as a u-PE outcome. Let us now consider if efficiency can be restored by a cost-benefit requirement.

2.5 Cost-benefit law equilibrium

A simple suggestion for alleviating the efficiency loss due to the deviation from the cost-benefit rule is an intertemporal cost-benefit law requiring that all public investments should earn the same return as the opportunity cost determined by private investments. We impose such a restriction as an institutional constraint on the equilibrium behavior — it can be thought of as a budget office scrutinizing the investment plan at the end of each period. The budget office has no preferences, and it simply enforces the cost-benefit requirement, without restricting the choices of each generation in any other way.

In three periods, the law will constrain only the first generation's choices for consumption and investments in the two purposes. Given the policy function g of the second generation, the first generation maximizes

$$w_1 = u_1 + \rho[u_2(f_2(k_2) - g(k_2)) + \theta u_3(f_3(g(k_2)) + \theta v_3(q_3)]$$

subject to the budget equation and the cost-benefit requirement

$$MRS_{q,3} = R_{1,3}.$$

While the consumption-based cost-benefit rule (CBR) implies a complicated-looking constraint on the current actions, there is a simple way to model it. Note that the CBR is reducing the first generation's control of the equilibrium allocation: it can only decide on the total savings as the cost-benefit rule determines the allocation of the savings among the two assets. Let I denote the total savings by generation 1. Now, when facing savings I the budget office needs the imputed equilibrium returns on the two assets in order to allocate the savings among the two assets such that the CBR is satisfied. The imputed returns depend on generation 2 policy function, so the budget office needs to solve the generation 2 problem to fulfill its task of allocating savings for the two purposes. But as the second generation has no time-inconsistency problem, it therefore cannot gain by deviating from the cost-benefit rule. The budget office's task and the second generation's preferences thus run parallel, and we can interpret the equilibrium as one where the budget office at the end of period 1 and the second generation are joined.

Given that the budget office is known to behave this way, we may then solve the equilibrium behavior under the following budget sets:

$$c_1 + I = f_1(k_1) \tag{22}$$

$$c_2 + k_3 = f_2(I - q) \tag{23}$$

$$c_3 = f_3(k_3), \tag{24}$$

where I indicates the overall saving of generation 1, q is the public investment that the second generation sets apart for the third generation, and k_3 is the capital stock transferred to generation 3. Note that this change in the timing of the decision on public investment q leaves the production possibility set of the economy unaltered.

The second generation finds the optimal investments portfolio in the two stocks k_2 and q under budget constraints (23)-(24) and given wealth from the previous generation I by solving

$$\max_{k_3,q} u_2(c_2) + \sigma[u_3(c_3) + v(q)], \tag{25}$$

leading to equilibrium conditions

$$u_2' = \sigma u_3' f_3' \tag{26}$$

$$u_2' f_2' = \sigma v', \tag{27}$$

and thus

$$\frac{v'}{u_3'} = MRS_{q,3} = R_{1,3} = f_2' f_3'.$$

We see therefore immediately that the cost-benefit rule will be satisfied, irrespective of the wealth transfer I from generation 1. This is no surprise since, as pointed out above, generation 2 has no time-inconsistency problem.

While the CBR restores the "productive efficiency" in the public investment, the first generation can still decide on transfer I following its own preferences. It is therefore not clear whether the CBR restores efficiency in terms of welfare. To explore this, consider conditions (26)-(27) defining generation 2 policy functions $g(I)$ and $h(I)$ for capital k_3 and public investment q, respectively.[9] Using the policies, we can write the continuation value for generation 1 as

$$V_2(I) = u_2(f_2(I - h(I)) - g(I)) + \theta u_3(f_3(g(I))) + \theta v(h(I))$$

to obtain the return for investment I as

$$\begin{aligned} V_2'(I) &= [(1 - h') - g']f_2'u_2' + \theta f_3'g'u_3' + \theta h'v' \\ &= [1 + (\frac{\theta}{\sigma} - 1)(h' + g')]f_2'u_2', \end{aligned}$$

where the latter line follows from using (26)-(27). Note that $h' > 0$ and $g' > 0$. The first generation balances costs and benefits of the transfer by choosing I to satisfy

$$u_1'(f_1(k_1) - I) = \rho V_2'(I),$$

implying

$$\alpha' = \frac{u_1'}{u_2'f_2'} = \rho[1 + (\frac{\theta}{\sigma} - 1)(h' + g')] \geq 0. \tag{28}$$

The equilibrium thus puts this implicit value for the utility weight α' in the program that maximizes the utility-weighted value $U(\mathbf{c}, q) = u_1(c_1) + \alpha' u_2(c_2) + \beta'[u_3(c_3) + v(q)]$. Similarly, we have

$$\frac{\beta'}{\alpha'} = \frac{u_2'}{u_3'f_3'} = \sigma \tag{29}$$

so that the implied β' is

$$\beta' = \sigma\rho[1 + (\frac{\theta}{\sigma} - 1)(h' + g')] \geq 0. \tag{30}$$

We can now state the welfare consequences of the cost-benefit requirement.

Proposition 2 *The welfare implications of the CBR:*

[9]By the assumptions made on the primitives of the model, the policy function are continuous, increasing, and differentiable.

1. *The cost-benefit rule implements utility Pareto efficiency (u-PE) for $\theta \neq \sigma$ and $\theta = \sigma$.*

2. *The cost-benefit rule implements welfare Pareto efficiency (w-PE) iff $\theta = \sigma$.*

Proof. We have seen in Lemma 1 that the CBR and the concept of u-PE are equivalent. Above we constructed the allocation satisfying the cost-benefit rule, and derived the implied non-negative weights (α', β'), without any restrictions on the discount factors. This proves the first item. For the second item, we show that inequalities in Remark 1 can hold if and only if $\theta = \sigma$. Thus, only for consistent preferences are the implied welfare weights non-negative. For inequality (10), note that

$$\alpha' = \rho[1 + (\frac{\theta}{\sigma} - 1)(h' + g')] \geq \rho \Leftrightarrow \theta \geq \sigma. \tag{31}$$

For inequality (11), substitute (29) and write

$$\beta' = \sigma\alpha' \geq \rho\theta + (\alpha' - \rho)\sigma,$$

which simplifies to

$$\sigma \geq \theta \tag{32}$$

We see that (10) and (11) are in contradiction unless $\theta = \sigma$, a case in which equalities hold in (31) and (32). If $\theta > \sigma$, then (31) and thus (10) is satisfied but (32) violated. If $\theta < \sigma$, then by (31) condition (10) is violated. ■

It is worth emphasizing why the CBR equilibrium violates Pareto efficiency. When $\theta > \sigma$, the CBR equilibrium implies that the welfare weight on the last generation is negative, $\beta < 0$. This is intuitive as the first generation would like to transfer more wealth to the last generation but cannot do so due to the CBR. The fact that the first generation is prevented from implementing its altruistic plan for the future distorts the overall savings below the minimum level that supports Pareto efficiency. On the other hand, if $\theta < \sigma$, the implied weight on the middle generation is negative, $\alpha < 0$.

Corollary 1 *The CBR does not imply a welfare Pareto improvement vis-a-vis the equilibrium without the cost-benefit law.*

The reason for this result is simple: the cost-benefit law is only a constraint on the first generation, as it could have implemented such a law without consulting the later generations. Therefore, enforcing the CBR must decrease welfare of the first generation if $\theta \neq \sigma$. If preferences are time-consistent, the CBR does not change equilibrium. In three periods, generation 1 cannot benefit from the later generations adherence to the CBR.

Though the results above clearly establish inefficiency of the cost-benefit rule, we stress that it is not possible to observe ex-post violation of Pareto efficiency, since the equilibrium is observationally equivalent to an equilibrium following from time-consistent preferences:

Corollary 2 *The inefficient CBR equilibrium is observationally equivalent to a w-PE equilibrium associated with time consistent preferences*

The corolloary directly follows from the construction of such an alternative time preference: consider $\widetilde{\rho} = \alpha'$, $\widetilde{\theta} = \widetilde{\sigma} = \sigma$, and (28) and (29) then show the equivalence.

The assumption in the three-period model that investments in q can be used to directly transfer welfare from the first to the last generation is a strong one. To assess the potential welfare gains from the CBR in a context with more flexible intertemporal substitution, we consider next if such benefits can exist in an infinite-horizon setting with continual subsequent investments in the public assets.

2.6 Discussion

The main lessons will carry over to the more general model, so we may discuss some policy implications after this preliminary analysis. It should first be emphasized what is not implied by the analysis: we do not want to implicate that fully efficient policies should not satisfy the cost-benefit rule. The cost-benefit requirement is a simple policy rule to advocate and something that could potential arise as an "intergenerational social contract" regarding the good public governance. We have demonstrated only that the cost benefit rule, when it dictates the allocation of current resources among alternative uses, cannot internalize all inefficiencies, if the overall amount of resources left for the future is open to choice. The core of the welfare inefficiency is that the first generation cannot directly transfer income to any but the immediately next future generation. The cost benefit rule prevents the use of public assets for altruistic purposes, which then reduces the overall savings, thereby adding to the existing intergenerational welfare-transfer distortion. This key problem of the cost-benefit rule has already been discussed by Lind (1995), but qualitatively. In order to benefit all parties the cost benefit requirement should be accompanied by policy rules steering the savings rate. While we do have various "golden rules" for the public sector finances, the macro-economic savings decisions are inherently private, and the structure of time preferences cannot be derived from savings observations as noted by Barro (1999). Our analysis, stated in Corollary 2, extends this result and shows that if all investments have to satisfy the cost-benefit requirement, then underlying preferences cannot be derived from the savings decisions. It is thus less clear if anything as easy to interpret as a rule as the cost-benefit check can be devised for savings.

We do not provide an explicit political economy justification for the time-inconsistencies, but it is useful to contrast our findings with the central questions in political economy where we often see that various restrictions on the set of policies that democratically elected governments can implement are viewed as welfare-improving. An example is the European Union public deficit restrictions as stated in the Maastricht Treaty. The political economy literature provides various arguments for restrictions on policies that would otherwise be used, through some persistent fundamentals of the economy, to influence future outcomes. For example, Persson and Svensson (1989) show that without institutional constraints, time-inconsistent preferences will press the current government to exert control over its successors behavior by running deficits.[10] Tabellini and Alessina (1990) argue that the lack of current majority's control over future voters most-preferred composition of spending tends to create current deficits, as a solution to the commitment problem. More directly related to our setting, Glazer (1987) finds that uncertainty of future voting outcomes biases current public investment towards durable long-term physical capital, and, more normatively, Bassetti and Sargent (2006) argue in favor of the golden rule where physical long-term public investments should be exempted from deficit restrictions.

Our results share the positive tone of this literature, as the current investments –in the absence of cost-benefit rules– are used to tie the hands of the future agents. However, on the normative side, we argue against simple behavioral rules eliminating discretion by the current decision maker as not welfare enhancing. The normative conclusion we reach is that such rules must be part of a larger package that not only corrects for distortions in the composition of temporal spending but also in the intertemporal choices. Thus, while reasons differ, we concur with Tabellini and Alessina (1990): "There is a role for institutions that enable society to separate its intertemporal choices from decisions concerning the allocation of resources within any given period".

[10]Interestingly, Fiva and Natvik (2010) find evidence using data from Norwegian municipalities that public investments are stimulated by higher continuation probabilities for the current majority.

3 Infinite horizon model

3.1 The setting

The more general framework, introduced in this section, allows us to flexibly analyze the relative persistence of the commitment provided by the public asset, and also the welfare implications of the cost-benefit law when it influences not only the current public investments but also the future ones. This section on the infinite-horizon model proceeds, after introducing the setting, by first developing the conceptual tools for welfare analysis, i.e., the infinite-horizon versions of the utility (u-PE) and welfare Pareto efficiency (w-PE). The main plot has already been outlined: we infer the utility weights from the equilibrium outcome, and their implications for the welfare weights. We obtain the same results as in three periods but the infinite horizon setting allows addressing a richer set of questions. For example, the relative persistence of the public asset determines the over- or under-investments together with the preference inconsistency, and the model provides a tool for gauging the welfare losses from pursuing the cost-benefit rules. Some proofs for this section are directed to the Appendix, as will be indicated.

Consider a sequence of periods $t \in \{1, 2, ...\}$ where gradual public investments, denoted by $q_t \geq 0$, are made to build up a public-asset, denoted by $s_t \geq 0$. The public asset accumulates as a function of the existing stock s_t and current investment $q_{t+1} \geq 0$ in the next-period asset:

$$s_{t+1} = \varphi(s_t, q_{t+1}),$$

where we assume that $\varphi(.)$ is increasing, bounded, and twice continuously differentiable in its arguments. This formulation is general enough to allow for multiple interpretations. In climate change, s_t can measure the reduction of the greenhouse-gas stock from a predetermined level, and q_{t+1} is the current abatement effort. Variable s_t could also be an index for biodiversity which is maintained by continual effort. The model could also be interepreted as a stylized model of education where the future human capital depends not only on past investments but also on past levels of human capital, or we can think of s_t as public infrastructure where the final service depends on the quality of current infrastructure determined by past investments.

We make the same assumptions on utility and production functions as in the three-period model, except that we impose stationarity by assuming that the neoclassical production technology does not change over time. The budget accounting equations between the periods are then

$$c_t + k_{t+1} + q_{t+1} = f(k_t) \quad (33)$$

$$s_{t+1} = \varphi(s_t, q_{t+1}). \quad (34)$$

In each period, the representative consumer makes the consumption and investment decisions, and derives utility from its own consumption and the public good. The consumer's welfare is

$$w_t = u(c_t) + v(s_t) + \rho \sum_{\tau=t+1}^{\infty} \theta^{\tau-t-1}[u_\tau(c_\tau) + v_\tau(s_\tau)], \quad (35)$$

where we identify dynamically consistent preferences by $\rho = \theta$, so that each future period $\tau > t$ is discounted with the same discount factor $\theta^{\tau-t}$. The dynamically inconsistent preferences are identified by $\rho \neq \theta$, and this model lends itself to the interpretations suggested by Phelps and Pollak (1968), Saez-Marti and Weibull (2005), and Laibson (1997).[11] In particular, $\rho < \theta$ is consistent with pure altruism towards later decision makers, or with near-term self-control problems.[12] We also allow for $\rho > \theta$.[13]

Analogously to the three periods model, we can consider the utility and welfare weights implied by an equilibrium allocation $(\mathbf{c}, \mathbf{q}, \mathbf{k}) = \{c_t, q_t, k_t\}_{t=1}^{\infty}$. We consider the welfare aggregator

$$W(\mathbf{c}, \mathbf{q}, \mathbf{k}) = \sum_{t=1}^{\infty} \alpha_t w_t$$

When the mass of weights is bounded, $\sum_{t=1}^{\infty} \alpha_t < \infty$, and welfare $W(\mathbf{c}, \mathbf{q}, \mathbf{k})$ is maximized subject to the resource constraints of the economy, the allocation is Pareto efficient. As in the three periods, $W(\mathbf{c}, \mathbf{q}, \mathbf{k})$ is not an independent welfare objective but rather an

[11]If we wish to reformulate the preferences consistent with the β, δ model of Phelps-Pollak (1968) and Laibson (1997): define $\beta = \rho/\theta$, $\delta = \theta$ and indentify inconsistent prefrences by $\beta < 1$ to obtain their framework. We want to indentify inconsistencies by $\rho \neq \theta$ to maintain an easy comparison with the three period model. Our framework also allows for the inverse of quasi-hyperbolic, e.g. 'linear' time preferences, with $\rho > 0$, $\theta = 0$, where the current generation cares about the immediate future, but not about those in the future further away.

[12]Gollier and Zeckhauser (2005) and also Weitzman (2000) show that the aggregation of unequal time preferences over consumers creates non-constant discount factors on the aggregate level. However, as such, this is not a source of inconsistency for preferences as long as the underlying individual preferences are consistent. In our case, the representative agent is the only agent capable of making the public-good decision, and the preferences under which these decisions are made are inconsistent by assumption. We do not explicitly model the source of the inconsistency in the public decision making.

[13]The models of self-control typically focus on hyperbolic preferences due to the empirical and experimental support for the case (see Della Vigna 2009). We do not want to rule out the opposite of the hyperbolic case in the context of government decision making.

auxiliary function to connect efficient equilibrium with weights implied by the welfare program. Similarly, we define the utility aggregator

$$U(\mathbf{c},\mathbf{q},\mathbf{k})=\sum\nolimits_{t=1}^{\infty}\alpha_t'[u(c_t)+v(s_t)].$$

When the mass of weights is bounded, $\sum_{t=1}^{\infty}\alpha_t'<\infty$, and welfare $U(\mathbf{c},\mathbf{q},\mathbf{k})$ is maximized subject to the resource constraints of the economy, the allocation is u-PE. Again, any allocation that maximizes $W(\mathbf{c},\mathbf{q},\mathbf{k})$ will also be u-PE, as we can choose $\alpha_1'=\alpha_1$ and $\alpha_\tau'=\sum_{t=1}^{\tau}\alpha_t\rho\theta^{\tau-t-1}$ for $\tau>1$,and obtain the equivalence of objectives $W(\mathbf{c},\mathbf{q},\mathbf{k})=U(\mathbf{c},\mathbf{q},\mathbf{k})$. But the converse is not true: the equilibrium utility weights need not imply positive welfare weights. Below, we solve the equilibrium with and without the cost-benefit rule, and characterize the Pareto efficiency by investigating whether the implied utility and welfare weights are positive.

Before describing the equilibrium, consider first the efficient benchmark, i.e., an allocation $(\mathbf{c},\mathbf{q},\mathbf{k})$ that maximizes $W(\mathbf{c},\mathbf{q},\mathbf{k})$ and also $U(\mathbf{c},\mathbf{q},\mathbf{k})$ for some positive respective weights. It is easier to work with utility weights, so let us use the utility weights α_t' associated with the Pareto efficient allocation in the analysis. Capital investment $k_{t+1}>0$ satisfies

$$\alpha_t'u_t'=\alpha_{t+1}'u_{t+1}'f_{t+1}'. \tag{36}$$

This conditions holds between any two periods with positive investment, implying that for any $\tau\geq t+1$

$$\alpha_t'u_t'=\alpha_\tau'u_\tau'R_{t,\tau}, \tag{37}$$

where $R_{t,\tau}=f_{t+1}'\cdot f_{t+2}'\cdot ...\cdot f_\tau'$ is the compound rate of return for k. On the other hand, investment $q_{t+1}>0$ in the public asset satisfies

$$\alpha_t'u_t'=\left[\alpha_{t+1}'J_{t+1,t+1}v_{t+1}'+\alpha_{t+2}'J_{t+1,t+2}v_{t+2}'+...\right]\varphi_{q,t+1}, \tag{38}$$

where we use the short-hand notation $\varphi_{q,t+1}=\varphi_q(s_t,q_{t+1})$ and $\varphi_{s,t+1}=\varphi_s(s_t,q_{t+1})$, and also $J_{t+1,\tau}=\varphi_{s,t+1}\cdot\varphi_{s,t+2}\cdot ...\cdot\varphi_{s,\tau}$ for the compound rate of return for the public asset ($J_{t+1,t+1}\equiv 1$). Rearrange (36) to obtain

$$1=\varphi_{q,t+1}[\frac{\alpha_{t+1}'J_{t+1,t+1}v_{t+1}'}{\alpha_t'u_t'}+\frac{\alpha_{t+2}'J_{t+1,t+2}v_{t+2}'}{\alpha_t'u_t'}+...].$$

The expression on the right gives the benefit-cost ratio for a marginal increase in the public asset. The benefit from the investment q is the increase in the next period public asset stock $\varphi_{q,t+1}$ times the sum of the utility-weighted compound returns $J_{t+1,\tau}v_\tau'$ in periods $\tau\geq t+1$. The cost of the investment is the current utility loss which equals the return on capital investment k. To obtain an expression that does not depend on utility weights, we can replace $\alpha_t' u_t'$ with (37) to obtain

$$1 = \left[\varphi_{q,t+1} \sum\nolimits_{\tau=t+1}^{\infty} \frac{J_{t+1,\tau}}{R_{t,\tau}} \frac{v_\tau'}{u_\tau'} \right]^{-1}. \quad (39)$$

As in three periods (cf. equation (15)), the cost-benefit rule depends only on marginal-rates of substitutions and returns but not directly on utility weights (the inverse is taken to maintain the expression as cost-benefit rather than benefit-cost ratio).

The final remark on the efficiency concepts is the one that connects above cost-benefit rule and utility Pareto efficiency. The following extension of Lemma 1 to the infinite horizon is helpful as it implies that if an equilibrium allocation deviates from the cost-benefit rule, efficiency of neither type can be reached whatever the inconsistencies in the underlying preference structure:

Lemma 3 *Assume that for given allocation* $(\mathbf{c}, \mathbf{q}, \mathbf{k})$*, the utility weight sequence constructed from (36) has a bounded mass,* $\sum_{t=1}^{\infty} \alpha_t' < \infty$*. Then,* $(\mathbf{c}, \mathbf{q}, \mathbf{k})$ *is u-PE if and only if the cost-benefit ratio (39) holds.*

The proof in the Appendix exploits the bounded mass condition to show that the cost-benefit condition, quite intuitively, rules out utility improving perturbations in the allocation. Further on, in the equilibrium analysis we will see that all equilibrium allocations will satisfy the bounded mass condition. We see therefore that any efficient equilibrium allocation must have the cost-benefit ratio equal to one. We will see also when the equilibrium will deviate from the cost-benefit ratio, and how the cost-benefit requirement can put the economy on the u-PE frontier but not on the true Pareto frontier (i.e., w-PE), without additional policy measures.

3.2 Constant investment-share policies

As is well known, the equilibrium outcome of the infinite-horizon Ramsey problem under dynamically inconsistent preferences depends on the restrictions made on the strategies available (see Krusell et al. 2002, and Karp 2007). To obtain a comparison with the consistent preferences case ($\rho = \theta$), we impose differentiability and symmetry restriction on the equilibrium strategies, i.e., each generation is assumed to use the same pair of differentiable policy functions $k_{t+1} = g(k_t, s_t)$, and $q_{t+1} = h(k_t, s_t)$.[14] Moreover, we

[14] See Krusell et al. 2002 for implications of relaxing these assumptions.

confine attention to Cobb-Douglas production functions $f(k_t) = Ak_t^{\alpha}$ and $\varphi(s_t, q_{t+1}) = Bs_t^{\delta} q_{t+1}^{1-\delta}$, where $0 < \alpha < 1$ and $0 < \delta < 1$,[15] and assume logarithmic utilities $u(c_t) = \ln(c_t)$, and $v(s_t) = \omega \ln(s_t)$.Under these assumptions, we can find equilibrium strategies where investments shares $1 > g > 0$ and $1 > h > 0$ are constant fractions of the current output:

$$k_{t+1} = gf(k_t) \tag{40}$$

$$q_{t+1} = hf(k_t). \tag{41}$$

The stationarity investment shares is well-known for consistent preferences ($\rho = \theta$) under this specification, and we will derive such policies explicitly for inconsistent preferences ($\rho \neq \theta$).[16] Since all policies in this paper take the form (40)-(41) irrespective of the degree of the dynamic inconsistency in the preferences, we can state some general properties of the policies in this class before the equilibrium analysis in Section 3.3.

Given (35), we can express the equilibrium welfare as

$$w_t = u(c_t) + v(s_t) + \rho V(k_{t+1}, s_{t+1}), \tag{42}$$

where the (auxiliary) value function satisfies

$$V(k_t, s_t) = u((1 - g - h)f(k_t)) + v(s_t) + \theta V(gf(k_t), \varphi(s_t, hf(k_t))).$$

We derive in Appendix the parametric form for the value function, applying to all equilibria considered in this paper:

Lemma 4 *The value function implied by policies (40)-(41) has the following parametric form*

$$V(k_{t+1}, s_{t+1}) = \xi \ln(k_{t+1}) + \frac{\zeta}{1-\delta} \ln(s_{t+1}) + \theta\mu[\xi \ln(g) + \zeta \ln(h)] + \mu \ln(1 - g - h)$$

where $\xi, \zeta, \mu > 0$.

The lemma is very useful as it immediately establishes some important features of any equilibrium with constant investment shares. Notice that g and h in the value

[15] We follow the custom use of α for the capital-output elasticity. When using time subscripts, the α_t refer to welfare weights while α_t' refer to utility weights.

[16] The focus on linear strategies is motivated by the ease of comparison with the consistent preferences case. We do not consider non-linear symmetric stationary strategies; on that, see Karp (2007). Moreover, there could be equilibria in symmetric but non-stationary strategies.

function refer to the *future* investment shares, from period $t+1$ onwards. The variables k_{t+1} and s_{t+1} are the current investments. The lemma establishes the fact that there is no interaction between g, h, and k_{t+1}, s_{t+1}, so that the current optimal investments in k_{t+1} and s_{t+1} are independent of future investment shares g and h. As $\ln(s_{t+1}) = \delta\ln(s_t)+(1-\delta)\ln(q_{t+1})$, and one unit of investment in k_{t+1} should yield the same marginal value as one unit investment in q_{t+1}, the lemma shows the current investment ratio is a constant given by $k_{t+1}/q_{t+1} = \xi/\zeta$ (see Appendix for the expressions of the parameters ξ and ζ). This ratio is independent of the immediate time-preference parameter ρ . Furthermore, the lemma shows that the investment shares g^* and h^* maximizing the value function also satisfy $g^*/h^* = \xi/\zeta$. Thus, the time-inconsistent preferences thereby maintain the optimal investment shares, but will affect the aggregate investment level as will be seen shortly.

Using the Cobb-Douglas form, the state equations, and the stationarity of investment shares, we can write the compound productivity variables as

$$\begin{aligned} J_{t+1,\tau} &= \delta^{\tau-t-1}\frac{s_\tau}{s_{t+1}} \\ R_{t,\tau} &= g^{t-\tau}\alpha^{\tau-t}\frac{k_{\tau+1}}{k_{t+1}}. \end{aligned}$$

Substituting in (39), and using constant investment shares identities, we can write the cost-benefit ratio explicitly for any (g,h) –policy that implies bounded payoffs as (we derive this expression in the Appendix):

$$1 = \left[\varphi_{q,t+1}\sum\nolimits_{\tau=t+1}^{\infty}\frac{J_{t+1,\tau}}{R_{t,\tau}}\frac{v'_\tau}{u'_\tau}\right]^{-1} = \frac{h(\alpha-\delta g)}{g\omega(1-\delta)(1-g-h)}. \quad (43)$$

We know from Section 3.1 that any u-PE allocation must satisfy this rule, and thus any (g,h)–policy that is u-PE must satisfy the cost-benefit rule (43). Moreover, u-PE allocation with constant investment shares must also satisfy (36) and thus the policy pins down the ratio of the subsequent utility weights as follows:

$$\gamma \equiv \frac{\alpha'_{t+1}}{\alpha'_t} = \frac{u'_t}{u'_{t+1}R_{t,t+1}} = \frac{c_{t+1}}{c_t R_{t,t+1}} = \frac{c_{t+1}}{c_t}\frac{g}{\alpha}\frac{k_{t+1}}{k_{t+2}} = \frac{g}{\alpha} \quad (44)$$

where the last step uses stationarity of $c_t/k_{t+1} = (1-g-h)/g$.

For interpretation it is useful to note that when $\gamma < 1$ we can view the equilibrium as if the choices were made by a representative dynastic agent facing (consistent) discount factor γ, provided the cost-benefit rule (43) holds.[17] The discount factors γ^{t-1} are the

[17] We notice that if $\gamma > 1$, we have a dynamically inefficient allocation, and the capital stock converges

utility weights for periods $t \geq 1$, and then it follows from Lemma 3 that such an allocation is utility Pareto efficient:

Lemma 5 *For an equilibrium with constant investment shares g and h, and γ in (44) strictly less than unity, the equilibrium is u-PE if and only if the cost-benefit rule (43) holds.*

As we can take $\gamma < 1$ as the discount factor, it is clear that upon observing the equilibrium path *ex post* we cannot distinguish consistent decision makers from inconsistent ones, i.e., the decisions could have been made by a representative agent with time-consistent preferences.

Remark 2 *An equilibrium with constant investment shares g and h that satisfies the cost-benefit rule (43) and with $\gamma < 1$ in (44) is observationally equivalent with an allocation that follows from geometrical utility discounting.*

The remark resembles the observational equivalence noticed by Barro (1999), but deviates in an important point: in our case, there are two assets and we need additional conditions for the observational equivalence to hold, i.e., the relative investments shares in the two assets must satisfy the cost-benefit rule, and there is nothing as of yet that implies the satisfaction of this rule. Below, in the equilibrium analysis, we find values for production-side parameters α and δ such that the equivalence holds irrespective of the time preference parameters ρ and θ. In general, as we will see and in contrast with Barro, the time-inconsistent preferences will not result in equilibria that are observationally equivalent to those resulting from exponentially decreasing welfare weights when there is more than one capital good.

When observing a constant investment share equilibrium satisfying the cost-benefit rule, the previous results imply that we have a utility efficient allocation at hand, but how to verify true welfare efficiency? The answer turns out to be simple:

Lemma 6 *The u-PE equilibrium with $\gamma = g/\alpha < 1$ is w-PE if and only if $\gamma \geq \max\{\rho, \theta\}$.*

In the Appendix, we use the lower bound on the "equilibrium discount factor" γ, i.e. the condition $\gamma \geq \max\{\rho, \theta\}$, to show that the welfare weights remain positive, and also

to a level at which $f' < 1$. We can then construct a strict utility Pareto improvement by lowering the capital stock to k^* with $f'(k^*) = 1$. For $\gamma = 1$, dynamic efficiency requires $f'(k_0) > 1$. But as we will see that all equilibria satisfy $\gamma < 1$, we do not go into the details for $\gamma \geq 1$.

that only in this case such weights can be found. Intuitively, γ can be seen as the discount factor that makes the first generation look like a fictional consistent-preferences planner; when this "planner" puts a per-period weight larger than ρ and θ on each generation's utility, then the implied equilibrium utility weights are large enough to leave room for positive welfare weights. It is not obvious whether this can hold in equilibrium — in particular so when the cost-benefit requirement is imposed as a rule of the game.

3.3 Equilibrium

Given the background from the previous section, it is now straightforward to assess the efficiency properties of the equilibrium. Considering the symmetric equilibrium where each period representative consumer chooses the same pair (g, h), we can readily see the continuation value for each investment level from Lemma 4, and determine the equilibrium investment shares g and h from the first-order conditions for k_{t+1} and q_{t+1},

$$u'(c_t) = \rho V_k(k_{t+1}, s_{t+1}), \tag{45}$$

$$u'(c_t) = \rho \varphi_{q,t+1} V_s(k_{t+1}, s_{t+1}). \tag{46}$$

Given functional forms from Lemma 4, the equilibrium best-responses (45) and (46) can be written as

$$k_{t+1} = \rho \xi c_t \tag{47}$$

$$q_{t+1} = \rho \zeta c_t. \tag{48}$$

Using $k_{t+1}/q_{t+1} = g/h$ and $c_t/k_{t+1} = (1-g-h)/g$ together with (47)-(48), we can express the equilibrium policies as follows (using the expressions for ξ and ζ in Appendix):

$$\begin{aligned} g &= \frac{\rho\xi}{1+\rho\zeta+\rho\xi} \\ &= \rho\alpha \frac{1-\delta\theta+\theta\omega(1-\delta)}{1-\delta\theta+\rho\omega(1-\delta)+\alpha(1-\delta\theta)(\rho-\theta)}, \end{aligned} \tag{49}$$

$$\begin{aligned} h &= \frac{\rho\zeta}{1+\rho\zeta+\rho\xi} \\ &= \rho\omega \frac{(1-\alpha\theta)(1-\delta)}{1-\delta\theta+\rho\omega(1-\delta)+\alpha(1-\delta\theta)(\rho-\theta)}. \end{aligned} \tag{50}$$

We see that when preferences are time-consistent ($\theta = \rho$), the equilibrium investment in k has the familiar form $g = \alpha\rho$, and the equilibrium discount factor is, as it should, $\gamma = \frac{g}{\alpha} = \rho < 1$. When preferences are time-inconsistent ($\theta \neq \rho$), we can obtain the intuitive result that the equilibrium discount factor is between the two conceivable extremes:

Lemma 7 *For all $\rho \neq \theta$, the equilibrium policy g satisfies*

$$\min\{\rho, \theta\} < \gamma = \frac{g}{\alpha} < \max\{\rho, \theta\} < 1.$$

The reasoning for this result (formally proved in the Appendix) is straightforward. Suppose the long-term discount factor θ is larger than the short-term factor ρ, which can be thought of as altruism towards later generations. Had the equilibrium discount factor γ been larger than θ, the equilibrium savings would exceed those in the case where the most altruistic discount factor θ is applied in each period. Savings this high cannot occur in equilibrium, as the true discount factor is determined by the short- and long-term discount factors jointly. Similarly, for γ lower than ρ, the savings would be lower than those under the short-term discounting. Clearly, the equilibrium savings must be somewhere between the extremes.

We can now describe the equilibrium outcome as depending on the inconsistency of the preferences and the relative persistency of the public asset. For ease of exposition, we use CBR as a shorthand for the cost-benefit ratio, expressed on the right-hand side of the cost-benefit rule (43). We plug in the equilibrium policies (49) and (50) to (43) to obtain:

$$CBR = 1 + \frac{1 - (\delta - \omega + \omega\delta)\rho - \alpha(\theta - \rho)}{(\delta - \alpha - \omega + \omega\delta)(\theta - \rho)}.$$

This is a closed form expression for the equilibrium cost-benefit ratio, implying:

Proposition 3 *Returns on public investments fall short of returns on capital ($CBR > 1$) in equilibrium if and only if $(\theta - \rho)(\delta - \frac{\alpha+\omega}{1+\omega}) > 0$. The equilibrium is u-PE if and only if either (i) $\theta = \rho$, or (ii) $\delta = \frac{\alpha+\omega}{1+\omega}$.*

The proof (in the Appendix) is a matter of straightforward verification. The latter part follows by the equivalence of the utility Pareto efficiency and the cost-benefit rule ($CBR = 1$) that we explicated in lemma 3. While the equilibrium deviation from the cost-benefit rule is not surprising given our arguments from three periods, the result gives more structure to the determinants of the deviation. In particular, since commitment provided by the public asset depends on its persistence relative to the traditional capital, the degree of over- or under-investment depends not only on preferences but also on persistence. A large long-term discount factor ($\theta > \rho$) was previously shown to be a reason for over-investment (i.e., costs exceeding benefits, $CBR > 1$), but now the public asset should also be persistent enough to satisfy $\delta > \frac{\alpha+\omega}{1+\omega}$. Otherwise, the agent will under-invest in the public asset.

When preferences are time-consistent ($\theta = \rho$), the cost-benefit rule will hold and the equilibrium is, of course, u-PE. But this outcome also arises when the persistence of the public asset exactly matches the persistence of welfare transferred to future generations through capital ($\delta = \frac{\alpha+\omega}{1+\omega}$), i.e., the equilibrium will be u-PE irrespective of the structure of time preferences ($\rho = \theta$, and $\rho \neq \theta$). This result sheds light on the generality of the observational equivalence between the equilibrium outcome and that obtained under consistent preferences, pointed out by Barro (1999) and discussed in Remark 2. With more than one capital good, the observational equivalence follows only in the knife-edge case identified here.

The observational equivalence does not imply welfare efficiency, however. Lemma 7 implies that the exponential decrease in utility weights γ associated with the equilibrium is too large.

Proposition 4 *Suppose preferences are inconsistent, $\theta \neq \rho$, but $\delta = \frac{\alpha+\omega}{1+\omega}$ so that the equilibrium is u-PE. The equilibrium is not w-PE.*

Efficiency requires $\gamma \geq \max\{\rho, \theta\}$ but this contradicts Lemma 7 above. The result thus implies that the equilibrium can never reach w-PE when preferences are dynamically inconsistent. This result is not surprising; while the equilibrium satisfies temporal efficiency in the sense that the composition of savings is optimal, the overall savings still deviate from the efficient savings for the reasons known from the one capital-good Ramsey saving problems with hyperbolic preferences.

3.4 Cost-benefit law equilibrium

We explore now whether the cost-benefit law, similar to that studied in three-periods, can improve welfare. We assume that the cost-benefit requirement is an institutional constraint dictating that all public investments must earn the same return as capital investments. As in three periods, we may think that the requirement is implemented administratively, e.g., through a budget office scrutinizing the investment plan at the end of each period. Other than this per-period check on the composition of spending, each generation is free to choose, within the resource constraints, the overall level of investment and consumption. With infinite horizon, the welfare implications of the cost-benefit law are less obvious than in three periods, as the current generation can potentially benefit from the future generation's adherence to the law — in three periods we could not address the full dynamic potential of the cost-benefit law, as it was only binding for the first generation by construction.

Formally, we consider a game where each generation chooses investments k_{t+1} and q_{t+1} subject to the constraint that the cost-benefit ratio must equal unity ($CBR = 1$), and the restriction on strategies that each future generation applies a constant investment share policy. We can think of each period involving two steps. In the first, the agent decides only on the overall investment $I_{t+1} = k_{t+1} + q_{t+1}$ and, in the second, the amount I_{t+1} is divided between the two purposes such that $CBR = 1$ is satisfied, understanding that each future generation will follow the same procedure.

Since we are focusing on the constant investment share policies, and the cost-benefit rule (43) was derived for any such policy, we can solve for the investment shares from the cost-benefit rule (43):[18]

$$\frac{q_{t+1}}{k_{t+1}} = \frac{\omega(1-\delta)(1-g-h)}{\alpha - \delta g} \equiv \frac{\eta}{1-\eta}.$$

The left-hand side refers to current investment decisions, and the right hand side refers to future investment decisions that are considered as given by the future agents' strategies in the subgame-perfect equilibrium. By definition, η is the share of the public asset investment in total investments I_{t+1}. Given the future policies, we only need to consider the best-response today for total savings I_{t+1}, as the shares follow by $k_{t+1} = (1-\eta)I_{t+1}$, and $q_{t+1} = \eta I_{t+1}$. We must thus have

$$\frac{dw_t}{dI_{t+1}} = -\frac{du_t}{dc_t} + \rho((1-\eta)\frac{dV_{t+1}}{dk_{t+1}} + \eta\frac{dV_{t+1}}{dq_{t+1}}) = 0 \tag{51}$$

We have derived the form for the value function for any pair of (g, h)-policies, so we can readily assess the implications of the cost-benefit rule on total savings:

Remark 3 *The cost-benefit law does not change total investments, but only the shares of capital and the public good. Investment in the public good decreases if and only if* $(\theta - \rho)(\delta - \frac{\alpha+\omega}{1+\omega}) > 0$

Formally, we can see the first part of the result from the first-order condition (51) which, given Lemma 4, can be restated as

$$I_{t+1} = \rho(\xi + \zeta)c_t \tag{52}$$

By $I_{t+1} = \frac{g+h}{1-g-h}c_t$,

$$g + h = \frac{\rho\xi + \rho\zeta}{1 + \rho\zeta + \rho\xi}.$$

[18] This follows by rearranging equation (58) in Appendix where we derive the closed form for the cost-benefit rule.

which equals the equilibrium total savings implied by (49) and (50). This result is already indicative of the fact that the cost-benefit requirement alone cannot deliver a Pareto efficient outcome (w-PE), as it does not correct for the distortions in overall savings. For the second part, note that in this equilibrium we must have $CBR = 1$ so that, if $CBR > 1$, the public-asset investment share declines when compared to the equilibrium without the cost-benefit rule.[19] Thus, the cost-benefit law either pulls resources away from public investment or towards it, depending on the relative persistence of the public asset and preference inconsistencies as indicated by the condition in the Proposition (that we discussed in the previous section).

The cost-benefit law restores productive efficiency in the sense that all assets earn seemingly appropriate returns, so that by observing such an outcome we might conclude that efficiency has been achieved. However, in levels the outcome is inefficient due to the fact that there are distortions in savings, when preferences are inconsistent:

Proposition 5 *If $\rho \neq \theta$, the cost-benefit law equilibrium is not w-PE.*

We can verify the result by noting that the law implements an equilibrium that is observationally equivalent to a consistent-preferences equilibrium with discount factor $\gamma < 1$; see our Remark 2. Such a fictional consistent-preferences economy grows by investing fraction $g = \alpha\gamma$ of the output in capital k. In the true equilibrium, the first-order condition for capital investment implies a constant investment-consumption ratio:

$$\frac{k_{t+1}}{c_t} = \rho\alpha[\frac{1}{(1-\alpha\theta)} + \frac{(1-\delta)\omega\theta}{(1-\alpha\theta)(1-\delta\theta)}].$$

Since observationally equivalent consistent-preferences equilibrium must satisfy the same ratio, we have

$$\rho\alpha[\frac{1}{(1-\alpha\theta)} + \frac{(1-\delta)\omega\theta}{(1-\alpha\theta)(1-\delta\theta)}] = \gamma\alpha[\frac{1}{(1-\alpha\gamma)} + \frac{(1-\delta)\omega\gamma}{(1-\alpha\gamma)(1-\delta\gamma)}],$$

where the right-hand side is the consistent-preferences version of the ratio. However, if $\gamma \geq \max\{\rho, \theta\}$, the equation cannot hold (the right-hand side is larger). Thus, we must have $\gamma < \max\{\rho, \theta\}$, and by Lemma 6, full efficiency (w-PE) is not achieved.The construction of γ immediately provides observational equivalence between various economies, as in Barro (1999):

[19]The cost-benefit ratio in (43) strictly decreases in h when $g + h$ remains constant, as is the case in this comparison.

Corollary 3 *The inefficient cost-benefit law equilibrium is observationally equivalent to an efficient equilibrium resulting from time-consistent preferences with discount factors* $\widetilde{\rho} = \widetilde{\theta} = \gamma$, *where* $\min\{\rho, \theta\} < \gamma < \max\{\rho, \theta\}$.

In analogy to the three-period model, the corollary shows a difficulty if one wishes to check efficiency ex-post. Recall that from Lemmas 5 and 6 it immediately follows that the cost-benefit rule in combination with a requirement for sufficiently large overall savings, to guarantee $g \geq \alpha \max\{\rho, \theta\}$, establishes welfare efficiency. However, if the cost-benefit rule is imposed, the preference structure as defined by ρ and θ cannot be deduced from equilibrium, and the condition $g \geq \alpha \max\{\rho, \theta\}$ cannot be checked.

To illustrate, suppose $\rho > \theta$ so that the decision maker is hyperbolic, and thus the equilibrium total savings fall short of the efficient savings; the agent would like to save more but cannot do so due to self-control problems, under this interpretation of the dynamic consistency. Now, with the cost-benefit requirement, the agent is still hyperbolic and the distortions in savings remain, as shown by the result that the savings are not changed but only their composition.

While the cost-benefit law does not restore full efficiency, it might be argued that the productive inefficiency removed produces at least a Pareto improvement. However, not even this can be achieved:

Proposition 6 *The implementation of the cost-benefit law from period* t *onwards implies a welfare loss for generation* t, *compared to the equilibrium without the law.*

The result shows that the three period conclusion extends to infinite horizon: the first generation under the law cannot sufficiently benefit from the later generations' adherence to the law. In this sense, the cost-benefit rule does noy create overall economic surplus that could be used to justify more complicated behavioral strategies supporting the rule as an equilibrium outcome without imposing it as an institutional constraint.

For the proof of the result, recall that the cost-benefit law does not change the total savings, but only the composition. We will first establish that for given total savings in the benchmark SPE without the cost-benefit requirement the composition of savings maximizes the continuation welfare given by value function V_t, so that the cost-benefit law must strictly decrease the value of future welfare to the current generation. From Lemma 4, it is clear that the pair (g, h) maximizing V given $g + h = I$ for some exogenous I must satisfy $g/h = \xi/\zeta$. As this ratio is preserved in the benchmark SPE, labeled with BAU, we thus have

$$V(k_{t+1}, s_{t+1}; g^{CBR}, h^{CBR}) < V(k_{t+1}, s_{t+1}; g^{BAU}, h^{BAU})$$

if $g^{CBR} \neq g^{BAU}$, where CBR and BAU refer to policies in the two cases. We can then conclude that

$$\begin{aligned} w_t^{CBR} &= u_t^{CBR} + v_t^{CBR} + \rho V(k_{t+1}^{CBR}, s_{t+1}^{CBR}; g^{CBR}, h^{CBR}) \\ &< u_t^{CBR} + v_t^{CBR} + \rho V(k_{t+1}^{CBR}, s_{t+1}^{CBR}; g^{BAU}, h^{BAU}) \\ &< u_t^{BAU} + v_t^{BAU} + \rho V(k_{t+1}^{BAU}, s_{t+1}^{BAU}; g^{BAU}, h^{BAU}) \\ &= w_t^{BAU} \end{aligned}$$

The second inequality follows from the fact that welfare without constraints on investments, as in the benchmark SPE, must exceed welfare with additional constraints.

To prepare the ground for the illustration, we conclude this section by studying the effect of the cost-benefit requirement on the steady-state welfare. Let us denote log-variables by tildes and write the steady state stocks as

$$\begin{aligned} \widetilde{k}^* &= \frac{\widetilde{g}}{1-\alpha} \\ \widetilde{s}^* &= \widetilde{h} + \frac{\alpha}{1-\alpha}\widetilde{g}. \end{aligned}$$

Substituting, we can write the steady-state utility level as

$$u^* + v^* = \frac{\alpha(1+\omega)}{1-\alpha}\widetilde{g} + \omega\widetilde{h} + \ln(1-g-h),$$

and consider the investment shares that maximize steady state utility and welfare:

$$\frac{g^*}{h^*} = \frac{\alpha}{1-\alpha}\frac{1+\omega}{\omega}.$$

Proposition 7 *The cost-benefit law decreases the steady state welfare if preferences are hyperbolic ($\theta > \rho$)*

Let us use $\delta^* = \frac{\alpha+\omega}{1+\omega}$ for the critical persistence of the public asset. Comparing the ratio in investments between the benchmark SPE without the cost-benefit rule, denoted by BAU, and the steady state optimum, we get

$$\begin{aligned} \frac{g^{BAU}}{h^{BAU}}\frac{h^*}{g^*} &= \frac{\xi}{\zeta}\frac{1-\alpha}{\alpha}\frac{\omega}{1+\omega} \\ &= \frac{1-\alpha\theta-(\delta-\delta^*)\theta(1+\omega)}{1-\alpha\theta}\frac{1-\alpha}{1-\alpha-(\delta-\delta^*)(1+\omega)}, \end{aligned}$$

where ratio ξ/ζ is obtained from the Appendix for the value function. When $\delta = \delta^*$, the ratio equals one. This is the case where imposing the cost-benefit rule has no bite since rule is satisfied anyways; the equilibrium is u-PE as the persistence of the public asset happens to match the persistence of the other asset. When $\delta > \delta^*$, the first ratio decreases relatively less as $1 - \alpha\theta > 1 - \alpha$ and $\theta < 1$, so that the overall ratio exceeds one. In the benchmark SPE there is thus too much investment in the neoclassical capital. If the public good is persistent ($\delta > \frac{\alpha+\omega}{1+\omega}$) and preferences are hyperbolic, the cost-benefit law will further increase investments in the capital stock, at the cost of the public good, and steady state utility must decrease. If the public good is fluid ($\delta < \frac{\alpha+\omega}{1+\omega}$) and preferences are hyperbolic, the cost-benefit law will decrease investments in the capital stock, and increase the public good, but the above ratio is below one and steady state utility still decreases.

3.5 Illustration

As seen from the above steady-state analysis, the cost-benefit law can pull investments from the public good into capital, and this may go at the cost of long-term utility. To see whether ballpark numbers can make this effect visible, we carry out a simple exercise. Consider the case where one must choose between investments in capital or in a very durable public asset such as the global environmental quality. Say time steps are 20 years so that we may treat the neoclassical capital as a broad man-made stock that is fully depreciated in one period, and $a = 0.5$. For the public asset, we consider the global environment, such as the climate, involving extremely slow global processes; e.g., the uptake of antropogenic emissions can imply that atmospheric CO2 particles depreciate annually 0.5 per cent. This implies $\delta = 0.9$. Let us assume that agents are relatively impatient in the short term discounting at annual rate of 2.5 per cent, implying $\rho = 0.5$, but do not differentiate much after the first 20 years; we set $\theta = 0.95$. These parameters would lead to a gross investment rate in capital of $g = 0.329$. By choosing the weight given to the public good as $\omega = 0.1$, we determine the optimal investment in the public good as $h = 0.022$. An interpretation could be that in the benchmark equilibrium 2.2 per cent of income is used to preserve the environment, e.g. to reduce greenhouse gas emissions. However, the net present value of benefits of public investments is less then half of the immediate costs, suggesting a potential welfare improvement by implementing the cost benefit rule.

If we implement the cost-benefit rule, part of the resources invested in the public asset are diverted to the capital stock. The optimal investment in capital increases while investments in the public good would about halve, $h = 0.011$. The long-term capital stock increases by almost 7 per cent, consumption increases by about 3 per cent, but weighted utility from the public good decreases by an amount equivalent to a decrease in consumption of the private good of about 6 per cent. Enforcing the cost benefit rule decreases substantially overall long-term welfare by as much as would be caused by a drop in capital of 7 per cent.

parameter		variable	Benchmark Equilibrium	Cost-benefit law
α	0.5	g	0.329	0.340
ω	0.1	h	0.022	0.011
δ	0.9	k^*	1	1.068
ρ	0.5	s^*	1	0.525
θ	0.95	u^*	0	0.033
		v^*	0	-0.064
		CBR	2.136	1

4 Concluding remarks

Public investments are often extremely long term by nature. Due to the long time horizon and difficulties in evaluating the future benefits, they present a challenge to the traditional cost-benefit analysis. We introduced a different complication: if preferences are known to change in the future such that the future ranking of current public decisions will be different from that today, how should the principles of the CBA be altered? We found that the persistence of the effects of current decisions lead to incentives to deviate from the standard cost-benefit requirements. Almost by definition the public investments provide commitment to current preferences, and it makes sense to use this commitment to overcome the inconsistencies in public decision making over time.

We found no normative reason to insist on the use cost-benefit rules when preferences are inconsistent: the overall welfare is not maximized under such rules. The cost-benefit analysis is based on a narrow concept of efficiency, and imposing the cost-benefit rule does not even imply a Pareto improvement, let alone achievement of welfare Pareto efficiency.

One extension of the current infinite-horizon model is a more detailed application to climate change, e.g., by using a numerical integrated assessment model (IAM) linking the economy and the climate development. Based on our results on observational equivalence between consistent and inconsistent preferences equilibria, we can conjecture that a standard IAM solution can also be interpreted as an equilibrium resulting from inconsistent preferences with an enforced cost-benefit rule. One can then explore with little effort the welfare loss from pursuing the cost-benefit rules (typically justified by a consistent preference framework) if the true underlying preferences are in fact inconsistent.

On a theory level, a natural alternative formulation is one where the current government understands that the future preferences are likely to be different but is unsure in which way. Alternatively, one may want to consider more deeply the source of inconsistency in public decision making. For example, it is well known that aggregation over individual heterogenous discount factors leads to average discount rates that decline with the time horizon (Weitzman 2000, Gollier and Zeckhauser 2005). As such this is not a source inconsistency in decentralized economy with heterogenous but consistent agents (Lengwiler 2005). However, in public decision making one may be forced to aggregate over individuals such that inconsistencies arise. We leave these interesting questions open for future research.

Appendix: Lemma 3

The if-part of the lemma is straightforward. Once the utility weights are constructed, if first order conditions are satisfied, the allocation must be u-PE. Consider then the only-if -part, and optimal utility sequence $\{u_t^* + v_t^*\}_{t\geq 1}$ that maximizes $U(\mathbf{c}, \mathbf{q}, \mathbf{k})$. Strict concavity of utility and production functions means that for any non-zero $\{\Delta_t\}_{t=1}^{\infty}$ with $\sum_{t=1}^{\infty} \alpha_t' \Delta_t \geq 0$, $\{u_t^* + v_t^* + \Delta_t\}_{t\geq 1}$ is infeasible as utility sequence. For $\sum_{t=1}^{\infty} \alpha_t' \Delta_t < 0$, there is a $\varepsilon > 0$ such that $u^* + v^* + \varepsilon\Delta$ is feasible as utility vector. We notice that the first order condition for k_{t+1} defines the (direction of) perturbations dc_t, dc_{t+1}, dk_{t+1} that are consistent with perturbations in utility pairs (du_t, du_{t+1}) such that $\alpha_t' du_t + \alpha_{t+1}' du_{t+1} = 0$. That is, if we have a Δ_t with $\sum_{t=1}^{\infty} \alpha_t' \Delta_t < 0$, then we can construct a sequence of perturbations dc_t, dk_{t+1} such that the associated change in utility satisfies $du_t \geq \varepsilon\Delta_t$.

If the first-order condition for q_t is not met, then there is a feasible perturbation $dq_t, (ds_\tau)_{\tau=t+1}^{\infty}$ such that the resulting $du_t, (dv_\tau)_{\tau=t+1}^{\infty}$ satisfies $\alpha_t' du_t + \sum_{\tau=t+1}^{\infty} \alpha_\tau' dv_\tau > 0$. Now take $\Delta_t = -du_t$, and $\Delta_\tau = -dv_\tau$, and we thus construct a perturbation dc_t, dk_{t+1} such that the associated change in utility satisfies $du_\tau \geq \varepsilon\Delta_\tau$ for $\tau = t, ..., \infty$. If we now add ε times the perturbation in $q_t, (s_\tau)_{\tau=t+1}^{\infty}$, we have a feasible perturbation that substitutes capital for public investment, or other way around, and that strictly increases the utility path.

Appendix: expression (43)

We obtain by expanding,

$$\varphi_{q,t+1}\sum\nolimits_{\tau=t+1}^{\infty}\frac{J_{t+1,\tau}}{R_{t,\tau}}\frac{v'_\tau}{u'_\tau} = (1-\delta)\frac{s_{t+1}}{q_{t+1}}\sum\nolimits_{\tau=t+1}^{\infty}\frac{\delta^{\tau-t-1}\frac{s_\tau}{s_{t+1}}}{g^{t-\tau}\alpha^{\tau-t}\frac{k_{\tau+1}}{k_{t+1}}}\frac{\omega c_\tau}{s_\tau} \tag{53}$$

$$= (1-\delta)\frac{s_{t+1}}{q_{t+1}}\sum\nolimits_{\tau=t+1}^{\infty}\frac{\delta^{\tau-t-1}\frac{s_\tau}{s_{t+1}}}{g^{t-\tau}\alpha^{\tau-t}\frac{k_{\tau+1}}{k_{t+1}}}\frac{(1-g-h)\omega k_{\tau+1}}{gs_\tau} \tag{54}$$

$$= (1-\delta)\frac{s_{t+1}}{q_{t+1}}\sum\nolimits_{\tau=t+1}^{\infty}\frac{\delta^{\tau-t-1}}{g^{t-\tau}\alpha^{\tau-t}}\frac{(1-g-h)\omega k_{t+1}}{gs_{t+1}} \tag{55}$$

$$= \frac{g_t}{h_t}\omega(1-\delta)(1-g-h)\sum\nolimits_{\tau=t+1}^{\infty}\frac{\delta^{\tau-t-1}g^{\tau-t}}{\alpha^{\tau-t}} \tag{56}$$

$$= \frac{g_t}{h_t}\frac{\omega(1-\delta)(1-g-h)}{\alpha}\sum\nolimits_{\tau=0}^{\infty}(\frac{\delta g}{\alpha})^\tau \tag{57}$$

$$= \frac{g_t}{h_t}\frac{\omega(1-\delta)(1-g-h)}{(\alpha-\delta g)}. \tag{58}$$

Line (53) follows from the definition of $\varphi_{q,t+1}$ and the state equation for s_{t+1} together with expressions for compound returns from the main text. Line (54) uses $gc_\tau = (1-g-h)k_{\tau+1}$. Line (55) follows by simplification. Line (56) uses $k_{t+1}/s_{t+1} = g/h$. Line (58) uses the boundedness assumption. We used the subscript t for the investment shares when they refer to current investment decisions as opposed to future investment shares that are given from the present point of view.

Appendix: Lemma 6

The equilibrium implies geometric utility weights $\alpha'_t = \gamma^{t-1}$. If $\gamma < \rho$ or $\gamma < \theta$ one cannot construct a sequence of non-negative welfare weights α_t consistent with the sequence of utility weights α'_t. Suppose the contrary, that welfare weights $\alpha_t \geq 0$ consistent with α'_t exist. Then, using the definition of w-PE, we see that for some $\tau > t$, the relationship between the two is $\alpha'_1 = \alpha_1$ and $\alpha'_\tau = \sum_{t=1}^{\tau}\alpha_t\rho\theta^{\tau-t-1}$ for $\tau > 1$. Expanding the latter gives

$$\alpha'_\tau = \alpha_1\rho\theta^{\tau-2} + \alpha_2\rho\theta^{\tau-3} + ... + \alpha_{\tau-1}\rho + \alpha_\tau. \tag{59}$$

If $\gamma < \theta$ and $\alpha_1 > 0$, we see that the equation cannot hold with $\alpha_\tau \geq 0$ for sufficiently large τ. If $\gamma < \rho$, we can write from (59)

$$\alpha'_{\tau+1} \geq \rho\alpha_\tau + \alpha_{\tau+1},$$

or

$$\gamma\alpha'_\tau - \rho\alpha_\tau \geq \alpha_{\tau+1}.$$

Again, since $\gamma < \rho$, this cannot hold with $\alpha_{\tau+1} \geq 0$ for sufficiently large τ.

Consider now $\gamma \geq \max\{\rho, \theta\}$. We show that now one can construct the non-negative welfare weights. We construct an algorithm for finding the weights. Let $\widetilde{\alpha}_1 = \{\alpha_\tau^1\}_{\tau \geq 1}$, $\widetilde{\alpha}_2 = \{\alpha_\tau^2\}_{\tau \geq 2}$, and so on. Define

$$\begin{aligned}
\alpha_\tau^1 &= \gamma^{\tau-1}, \tau \geq 1 \\
\alpha_\tau^2 &= \alpha_\tau^1 - \alpha_1^1 \theta^{\tau-t-1}, \tau \geq 2 \\
&\dots \\
\alpha_\tau^{t+1} &= \alpha_\tau^t - \alpha_t^t \theta^{\tau-t-1}, \tau \geq t.
\end{aligned}$$

The value of α_τ^t measures the weight remaining for generation τ after all altruistic weights of generations 1 to $t-1$ have been subtracted. Note that the equilibrium implies utility weights $\widetilde{\alpha}_1$, and $\{\alpha_t^t\}_{t \geq 1}$ is the sequence of welfare weights consistent with $\widetilde{\alpha}_1$. The main intermediate result that we need, in order to prove that the sequence of welfare weights $\{\alpha_t^t\}_{t \geq 1}$ is non-negative, is that for all $\tau \geq t$:

$$\frac{\alpha_{\tau+1}^t}{\alpha_\tau^t} > \max\{\rho, \theta\}. \tag{60}$$

By construction, this condition is satisfied for $t = 1$. It implies that next sequence $\widetilde{\alpha}_2$, induced by the algorithm, is non-negative, as

$$\alpha_\tau^2 = \gamma^{\tau-1} - \theta^{\tau-t-1} > \alpha_\tau^1 \{(\max\{\rho, \theta\})^{\tau-1} - \rho\theta^{\tau-t-1}\} > 0, \tau \geq 2.$$

By induction, if the condition holds for $\widetilde{\alpha}_t$, the sequence $\widetilde{\alpha}_{t+1}$ is non-negative:

$$\alpha_\tau^{t+1} > \alpha_\tau^t \{(\max\{\rho, \theta\})^{\tau-t} - \rho\theta^{\tau-t-1}\} > 0, \tau \geq t.$$

Thus, we are done if we can show that condition (60) holds. Notice that

$$\alpha_{\tau+1}^{t+1} = \alpha_{\tau+1}^t - \alpha_t^t \rho\theta^{\tau-t} > \max\{\rho, \theta\}\alpha_\tau^t - \alpha_t^t \rho\theta^{\tau-t} \geq \theta\{\alpha_\tau^t - \alpha_t^t \rho\theta^{\tau-t-1}\} = \theta\{\alpha_\tau^{t+1}\}.$$

If $\theta > \rho$, this proves that $\alpha_{\tau+1}^{t+1} > \theta\{\alpha_\tau^{t+1}\} > \rho\{\alpha_\tau^{t+1}\}$. On the other hand, if $\theta < \rho$, we have

$$\alpha_{\tau+1}^{t+1} = \alpha_{\tau+1}^t - \alpha_t^t \rho\theta^{\tau-t} > \max\{\rho, \theta\}\alpha_\tau^t - \alpha_t^t \rho\theta^{\tau-t} \geq \rho\{\alpha_\tau^t - \alpha_t^t \rho\theta^{\tau-t-1}\} = \rho\{\alpha_\tau^{t+1}\},$$

which completes the proof.

Appendix: deriving the value function

Proof of Lemma 4. We proceed in the following steps. First, we show that there are parameters $\xi, \zeta, a_g, a_h, \mu$ such that the value function can be written as

$$V_t = \xi \ln(k_t) + \frac{\zeta}{1-\delta}\ln(s_t) + a_g \ln(g) + a_h \ln(h) + \mu \ln(1-g-h).$$

Then we analyze how a_g and a_h relate to the other parameters.

Given stationary investment shares, we can fully calculate all forward capital and public good levels. We use tildes to denote log-variables. The stock dynamics can then be written recursively as

$$\widetilde{k}_{t+1} = \widetilde{g} + \alpha\widetilde{k}_t \tag{61}$$

$$\widetilde{s}_{t+1} = \delta\widetilde{s}_t + (1-\delta)(\widetilde{h} + \alpha\widetilde{k}_t) \tag{62}$$

Substitution allows us to find the complete future path of stocks k_t and s_t as a function of initial stocks k_1 and s_1, and the policy functions:

$$\widetilde{k}_{t+1} = \alpha^t\widetilde{k}_1 + \frac{1-\alpha^t}{1-\alpha}\widetilde{g} \tag{63}$$

$$\widetilde{s}_{t+1} = \delta^t\widetilde{s}_1 + (1-\delta^t)\widetilde{h} + (1-\delta)\alpha\widetilde{k}_1\sum_{\tau=0}^{t-1}\alpha^\tau\delta^{t-\tau-1} + (1-\delta)\alpha\widetilde{g}\sum_{\tau=1}^{t-1}\frac{1-\alpha^\tau}{1-\alpha}\delta^{t-\tau-1} \tag{64}$$

$$= \delta^t\widetilde{s}_1 + (1-\delta^t)\widetilde{h} + (1-\delta)\alpha\frac{\delta^t-\alpha^t}{\delta-\alpha}\widetilde{k}_1 + \frac{\alpha(1-\delta)}{1-\alpha}\left(\frac{1}{1-\delta} - \frac{\delta^t-\alpha^t}{\delta-\alpha}\right)\widetilde{g} \tag{65}$$

First, we observe that using logarithms denoted by a tilde, we have $u_t = \widetilde{c}_t = \ln(1-g-h) + \alpha\widetilde{k}_t$, $v_t = \omega\widetilde{s}_t$, and by definition

$$\begin{aligned} V_t &= \sum_{\tau=0}^{\infty}\theta^\tau(u_{t+\tau} + v_{t+\tau}) \\ &= \frac{1}{1-\theta}\ln(1-g-h) + \alpha\sum_{\tau=0}^{\infty}\theta^\tau\widetilde{k}_{t+\tau} + \omega\sum_{\tau=0}^{\infty}\theta^\tau\widetilde{s}_{t+\tau} \end{aligned}$$

If we have a look at (63) and (64), the general parametric form is obvious. For the parameters, we find

where

$$\begin{aligned} \xi &= \rho\alpha\frac{1+\omega\theta-\delta\theta-\omega\delta\theta}{(1-\alpha\theta)(1-\delta\theta)}, \\ \zeta &= \rho\frac{(1-\delta)\omega}{(1-\delta\theta)}. \end{aligned}$$

$$
\begin{aligned}
\xi &= \alpha \sum_{\tau=0}^{\infty}(\theta\alpha)^{\tau} + \alpha(1-\delta)\omega \sum_{\tau=0}^{\infty}(\alpha\theta/\delta)^{\tau} \\
&= \frac{\alpha}{1-\alpha\theta} + \frac{(1-\delta)\theta\omega}{(1-\alpha\theta)(1-\delta\theta)} \\
&= \alpha\frac{1+\omega\theta-\delta\theta-\omega\delta\theta}{(1-\alpha\theta)(1-\delta\theta)} \\
\zeta &= (1-\delta)\omega \sum_{\tau=0}^{\infty}(\delta\theta)^{\tau} \\
&= \frac{(1-\delta)\omega}{(1-\delta\theta)} \\
\mu &= \frac{1}{1-\theta}
\end{aligned}
$$

We now want to determine a_g and a_h. As above, we could directly calculate the coefficients by summing all terms over time, but we can also derive the coefficients by a more subtle reasoning. For time consistent preferences, $\rho = \theta$, we can calculate the investment shares $g^* = k_{t+1}/y_t$ and $h^* = q_{t+1}/y_t$ that maximize

$$w_t = u_t(y_t - k_{t+1} - q_{t+1}) + \theta V(k_{t+1}, s_{t+1}; g, h)$$

which gives

$$
\begin{aligned}
g^* &= \frac{\theta\xi}{1+\theta\xi+\theta\zeta} \\
h^* &= \frac{\theta\zeta}{1+\theta\xi+\theta\zeta}
\end{aligned}
$$

These values must be the same as those we can calculate directly from maximizing V :

$$
\begin{aligned}
g^* &= \frac{a_g}{\mu+a_g+a_h} \\
h^* &= \frac{a_h}{\mu+a_g+a_h}
\end{aligned}
$$

It follows directly that $a_g = \mu\theta\xi$ and $a_h = \mu\theta\zeta$. Q.E.D.

Appendix: Lemma 7

We will show that $\rho < \theta$ gives $\rho < g/\alpha < \theta$, while $\theta < \rho$ results in $\theta < g/\alpha < \rho$. First compare g/α in (49) with ρ:

$$
\begin{aligned}
g/\alpha &< \rho \Leftrightarrow \\
0 &< [\alpha(1-\delta\theta) + \omega(1-\delta)](\rho-\theta) \Leftrightarrow \\
\theta &< \rho
\end{aligned}
$$

The second equivalence follows because all terms between the square brackets are positive. It follows that $\rho < \theta$ gives $\rho < g/\alpha$, while $\theta < \rho$ results in $g/\alpha < \rho$. Now compare g/α and θ:

$$\begin{aligned} g/\alpha &< \theta \Leftrightarrow \\ \rho[1-\delta\theta+\theta\omega(1-\delta)] &< \theta[1-\delta\theta+\rho\omega(1-\delta)+\alpha(1-\delta\theta)(\rho-\theta)] \Leftrightarrow \\ 0 &< (\alpha\theta-1)(1-\delta\theta)(\rho-\theta) \Leftrightarrow \\ \rho &< \theta \end{aligned}$$

The last equivalence follows because the first term between brackets is negative while the second is positive. This shows the second half of the lemma: $\rho < \theta$ gives $\upsilon < \theta$, while $\theta < \rho$ results in $\theta < \upsilon$.

Appendix: Proposition 3

We can now plug in the equilibrium policies to the cost-benefit ratio in (43):

$$CBR = 1 + \frac{1-(\delta-\omega+\omega\delta)\rho-\alpha(\theta-\rho)}{(\delta-\alpha-\omega+\omega\delta)(\theta-\rho)}.$$

The numerator of the fraction is positive,

$$\begin{aligned} 1-(\delta-(1-\delta)\omega)\rho-\alpha(\theta-\rho) &> 1-\rho-\alpha(\theta-\rho) \\ &= 1-(1-\alpha)\rho-\alpha\theta > 0. \end{aligned}$$

The first part of the proposition follows immediately from the comparison of the denominator with $(\theta-\rho)(\delta-\frac{\alpha+\omega}{1+\omega})$. For the second statement, Lemma 7 implies that the utility weight γ associated with the equilibrium is below unity. From Lemma 5 we can see that $\gamma < 1$ together with $CBR = 1$ implies utility Pareto efficiency.

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