Dark Solitons in Holographic Superfluids

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(Dated: November 10, 2009)

We construct dark soliton solutions in a holographic model of a relativistic superfluid. We study the length scales associated with the condensate and the charge density depletion, and find that the two scales differ by a non-trivial function of the chemical potential. By adjusting the chemical potential, we study the variation of the depletion of charge density at the interface.

a. Introduction. One of the exciting theoretical developments in recent years has been the evolution of holographic gauge/gravity duality. The holographic principle, a dimensional reduction in the reorganization of all information, was proposed to be a general feature of any gravitational theory [1]. So far the most concrete realization is the correspondence between $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills gauge theory in 3+1 dimensions and type IIB supersymmetric string theory in 9+1 dimensions [2], where the spacetime manifold consists of a 4+1 dimensional anti-de Sitter spacetime and a 5-dimensional sphere, with the structure $AdS_5 \times S^5$. The mapping of gauge theory into string theory becomes tractable and useful in the strong t'Hooft coupling $\lambda = g_{YM}^2 N$ limit of the Yang-Mills theory, which corresponds to the low energy supergravity limit of string theory in weakly curved spacetime. The gravitational theory can then be Kaluza-Klein reduced to the AdS_5 spacetime, yielding the holographic gauge-gravity duality between strongly coupled gauge theory in 3+1 Minkowski space and gravitational theory in 4+1 dimensions. There are many ways to deform and extend the theory to obtain other cases of dual theories, some of which are reviewed in [3].

The duality is not only a profound theoretical statement about a non-gravitational theory secretly being a theory of gravity in higher dimensions (or vice versa). It is also a useful calculational tool, in particular for computing correlation functions in a strongly coupled gauge theory, when perturbative methods are no longer tractable [4]. Most recently this connection has been applied to various strongly interacting condensed matter systems (reviewed *e.g.* in [5, 6, 7]). There are various constructions for holographic gravity duals, allowing new insights and methods for the analysis of correlation functions. In this document we investigate a recently formu-

lated gravitational dual theory [8, 9], see also [10, 11, 12], in the context of (relativistic) superfluids with a spontaneously broken global U(1) symmetry. We will work in the so called probe limit where we can neglect gravitational backreaction to the metric. This backreaction would be important as one approaches zero temperature.

A characteristic structure supported by superfluids is a vortex. However, in this work, we will investigate other solitonic configurations which interpolate between two phases. Vortex solutions were recently reported in the holographic model [8] by different groups [13, 14, 15], in the context of interpreting [8, 11] as a dual model of a Type II superconductor.

As one source of inspiration, we note the fascinating crossover from BCS superconductors to BEC superfluids which can be experimentally realized in ultracold gases of fermionic atoms. Secondly we note that the condensate superfluid can support interesting localized but extended defects, called bright or dark solitons – interfaces of increased or reduced charge density between two superfluid phases (for a review, see *e.g.* [16, 17]). They can be created in experiments, and have many interesting and incompletely understood properties. In particular one can study what happens to a dark soliton during the BCS-BEC crossover. In a theoretical study it was found that during the transition to the BCS regime, the charge depletion diminishes and becomes visibly modulated by Friedel oscillations [18]. Finally, we note that at strong coupling dark solitons, in particular multidimensional ones, are difficult to study with standard theoretical tools and *e.g.* the spectrum of excitations is ill-understood.

In this document we construct dark soliton solutions at finite temperature in (relativistic) holographic superfluids, with a dual gravitational description that describes strongly coupled dynamics in field theory. We find that the depletion of charge density varies as we dial the only available parameter of the model, the chemical potential. The existence of such solutions might have been anticipated due to generic topological arguments, but their detailed properties could very well be different than those of the solution to the GP equation. Indeed, we find that there are qualitative differences. The depletion fraction is far from 100% unlike the GP example. Also there are two characteristic length scales governing the condensate

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density and charge density as opposed to a single length scale in the GP equation.

This document is organized as follows. We begin with a brief discussion of the dark soliton and its properties using the Gross-Pitaevskii equation as a guideline. This will enable us to determine quantities of interest which will be computed using the holographic description. The configurations of interest are obtained by solving a system of partial differential equations in AdS space. We present the equations and the boundary conditions which sustain the soliton. We then discuss the numerical methods and present the results. We conclude with a brief discussion of the results and future directions.

b. Dark Solitons in Superfluids. Dark solitons are spatial interfaces of reduced charge density between two superfluid phases. As a starting point, we can model them by using the time independent Gross-Pitaevskii (GP) equation for the superfluid order parameter Φ

$$-\frac{1}{2m_B}\partial_x^2\Phi + (2V - \mu)\Phi + g\Phi|\Phi|^2 = 0 \qquad (1)$$

where the coefficients depend on temperature and chemical potential. The dark soliton is a spatially varying solution which interpolates between the potential minima (phases)

$$\Phi(\infty) = \Delta, \ \Phi(-\infty) = -\Delta; \qquad \Delta = \sqrt{\frac{(\mu - 2V)}{2g}}. \quad (2)$$

For fermionic systems, this equation may be obtained from the microscopic Bogoliubov-de Gennes equation [19] (in a strong coupling limit) as a mean field description - which nevertheless may be expected to give a reasonably accurate description of the essential physics. However, we will not assume any particular dependence for the coefficients.

The GP equation (1) has a well known exact soliton solution

$$\Phi = \sqrt{\frac{\mu - 2V}{2g}} \tanh(x/\xi), \quad \rho(x) = q|\Phi|^2 \qquad (3)$$

which interpolates between the two vacua $\Phi = \pm\Delta$. Here ρ is the charge density and q is the unit of U(1) charge. The coherence length ξ, can then be written in a useful form in terms of the parameters of the equation

$$\xi = \frac{1}{\sqrt{4gm_B\Delta}} \quad = \frac{1}{\sqrt{2(\mu - 2V)m_B}}. \qquad (4)$$

This will act as a guide in determining the quantities of interest for the holographic bulk description without assuming any particular dependence for the coefficients (or even assuming the GP equation).

c. Holographic Description. The holographic modelling of the superfluid system was constructed in [8] following the ideas in [9]. We consider a Maxwell-Scalar system in a 4-D planar AdS black hole background with a metric

$$ds^2 = L^2(-\frac{fdt^2}{z^2} + \frac{dz^2}{fz^2} + \frac{d\vec{x}^2}{z^2}); \ f = 1 - \left(\frac{z}{z_T}\right)^3 \quad (5)$$

where L is the AdS radius and which will be set to unity in the following discussion. The bulk Lagrangian is taken to be

$$\mathcal{L} = \sqrt{-g}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\Psi D^\mu\bar{\Psi} + 2\Psi\bar{\Psi}). \qquad (6)$$

In the gauge $A_z = 0$, the nontrivial equations of motion for the bulk fields $\Psi = \frac{1}{\sqrt{2}}z\tilde{R}$ and $A_0 = A$, after suitable rescaling, are

$$f\tilde{R}'' + f'\tilde{R}' - z\tilde{R} + \partial_x^2\tilde{R} + \tilde{R}(\frac{A^2}{f}) = 0 \qquad (7)$$

$$fA'' + \partial_x^2 A - \tilde{R}^2 A = 0.$$

Using the equations, it can be easily seen that, close to the boundary at $z = 0$, the fields Ψ, A behave as

$$\tilde{R} = \tilde{R}^{(1)} + z\tilde{R}^{(2)} + ... \quad A = A^{(0)} + zA^{(1)} + ... \qquad (8)$$

in an expansion along the z-direction.

The dual field theory (in this case, the superfluid system) is said to be at a temperature $T = T_H = \frac{3}{4\pi z_T}$ which is the Hawking temperature of the black hole in AdS space. According to the recipe provided by the AdS/CFT correspondence, the boundary values of the bulk fields are related to the (dual) superfluid system as follows

$$\tilde{R}^{(1)} = \sqrt{2}z_T J(x) \ , \qquad \tilde{R}^{(2)} = \sqrt{2}z_T^2 \langle O_2\rangle(x) \quad (9)$$

$$A^{(0)} = z_T \mu(x) \ , \qquad A^{(1)} = z_T^2 \rho(x) \qquad (10)$$

where μ is the chemical potential, ρ is the (total) charge density, O_2 is a charged operator of mass dimension 2 and $J(x)$ is to be regarded as an external source in the field theory for this operator. We will set such sources to zero in this work. The expectation values of this operator (in the ground state) may be taken to be the order parameter of the superfluid system (since superfluidity is the spontaneous breaking of a global U(1) symmetry). We will work in the probe limit of this background where one may ignore gravitational backreaction. Such backreaction would become important near zero temperature.

The solution is then uniquely determined by the boundary conditions

$$\tilde{R}^{(1)} = 0, \quad A^{(0)} = z_T\mu = \text{constant} \qquad (11)$$

and regularity conditions at the horizon of the black hole (one each for R and A_0).

In this particular system, it is also possible to identify $\tilde{R}^{(1)}$ with the vev of a charged operator of mass dimension 1 in which case $\tilde{R}^{(2)}$ has the source interpretation

in the dual theory. This case will be discussed in a later publication.

In the work [8], it was shown that for sufficiently small T one can find solutions where the scalar field is nonzero. That is to say, for low enough temperature we can have nonzero charged condensates, and hence superfluidity. Since there is only one independent parameter determining the solution of (7) we can equivalently vary the chemical potential μ and find the condensate phase at large enough μ. The critical value of μ depends on whether we consider $\tilde{R}^{(1)}$ or $\tilde{R}^{(2)}$ as the order parameter.

d. Numerical method. We place the system in a large box, $(z,x) \in [0,1] \times [-L_x/2, L_x/2]$ and use numerical methods to solve the field equations (7). Searching for dark soliton solutions, the basic strategy is to assume an initial configuration with an interface at $x=0$ which satisfies the boundary conditions and then use the Gauss-Seidel relaxation method to obtain the solution. The topological information is included with the initial seed configuration and is preserved during relaxation. We have used Neumann boundary conditions at $x = \pm L$ and confirmed that the results are insensitive to the size L. For derivatives in Gauss-Seidel, we use a second order representation in the bulk and a one sided, third order representation at $x = \pm L$.

In the iteration method there is a danger that what looks like a solution ceases to be one after suitably many iterations. However, consider the topological stability criterion in two dimensions, the space dimension of ∂AdS_4 which is the space in which the dual theory resides. The spacetime independent solutions to (1) define a space of vacua $\mathcal{M}$. In our case clearly $\mathcal{M} = U(1)$. Topologically stable one-dimensional objects (kinks) are not possible in this system because $\mathcal{M}$ is arc-connected. But topologically stable zero-dimensional objects (vortices) are possible since $\pi_1(\mathcal{M}) = \mathbb{Z}$.

However, the starting point for dark soliton solutions is (7) which involves the *real* valued field $\tilde{R}$ and have a $\tilde{R} \mapsto -\tilde{R}$ symmetry. So the $U(1)$ symmetry has been reduced to $\mathbb{Z}_2$. Moreover, we require translational invariance in the y direction which means that the spatial boundary is also $\mathbb{Z}_2$. Thus, the solutions we discuss are topologically *nontrivial* maps from $x = \pm\infty$ to $\mathbb{Z}_2$.

This provides the underlying reason that we expect our numerical solutions to be stable approximations to the actual solutions. Further, we find that the solutions we obtain asymptote to those of [10] far away from the interface. We also find that even if we perturb these configurations (preserving the boundary conditions), the perturbations decay rapidly (in iteration time).

We have identified two main sources of error: discretization error from the finiteness of the lattice and algebraic error from solving the discretized equations at each site. By substituting the numerically determined solutions into the differential equations we have checked that the error decreases monotonically as one increases the lattice size.

The dark soliton is known to be an unstable object in two or more dimensions via the snake instability. This happens as follows. If we break the translational invariance in y direction (or reinstate the phase d.o.f.) by small perturbations, the superfluid dark soliton will decay into topologically stable vortices by snake instability (see for example [20] and references therein), which has been conjectured as due to tachyonic states in the Bogoliubov spectrum around the dark soliton.

e. Results. Generically, the solutions for the scalar field $\tilde{R}$ and the vector field A in the bulk look as in Figure 1. Note that all graphs are plotted using dimensionless variables (denoted with a tilde) - dimensions maybe restored by using eqn(10). In order to high-
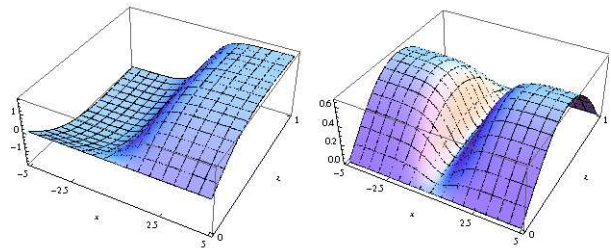

FIG. 1: Profiles for the scalar field $\tilde{R}$ (left) and the normalized gauge field $A - z_T \mu(1-z)$ (right) in the bulk for $\mu = 5.18$.

light the structure of the solution for the vector field, we have subtracted a "background" contribution and plot $A - z_T \mu(1-z)$ instead of A. From the numerical solutions we obtain, we can plot the boundary profiles of the charge density $A^{(1)}$ and the condensate $\tilde{R}^{(2)} = (\partial_z \tilde{R})|_{z=0}$. The numerical data and expectations from the GP equations suggest that the condensate can be fitted by a $\tanh(\frac{x}{\xi})$ profile and the charge density by a $\text{sech}^2(\frac{x}{\xi_q})$ profile (with *a priori* unrelated coherence lengths to be determined from the fit). The data along with the fitted curves are shown in the Figure 2. Using a least square fit, we then
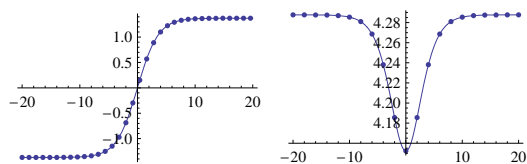

FIG. 2: Condensate and charge density vs. x.

extract the coherence length ξ from the condensate profile and ξ_q from the charge density.

Motivated by the earlier discussion of the GP equations, we plot the dependence of the coherence lengths on the chemical potential μ in Figure 3. Using the degree to which our solutions satisfy the differential equations as an estimate of the total error, we find that the error bars on the curves are too small to be visible. One might anticipate a divergence as we reach the critical value for
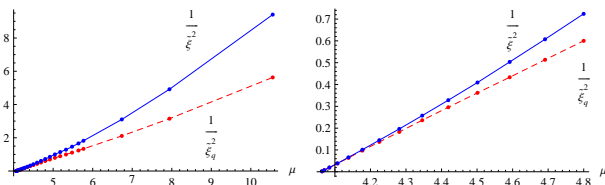
FIG. 3: [Coherence length]$^{-2}$ vs. chemical potential.

the chemical potential. We find that for small values of
the chemical potential

$$\tilde{\xi}(\mu) \approx 0.99(\tilde{\mu} - \tilde{\mu}_0)^{-\frac{1}{2}}, \qquad (12)$$

where $\tilde{\mu} = A(z=0) = \mu z_T$ is the dimensionless chemical
potential and $\tilde{\mu}_0 \approx 4.07$ is the critical value below which
there is no condensate for $\langle \mathcal{O}_2 \rangle$ (cf. eqn. (4)). As can
be seen from equation (3) the two scales ξ and ξ_q are the
same in GP model. However, in the holographic model,
it is seen that the two scales are different and their dif-
ference is a non-trivial function of the temperature (or
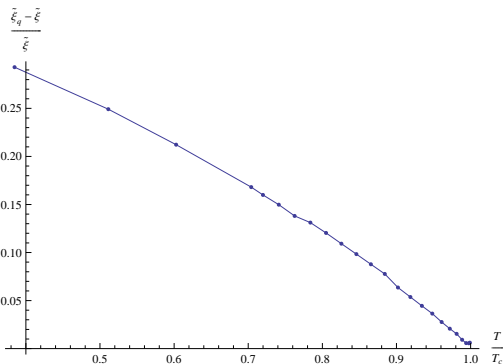inverse chemical potential) as displayed in figure 4. As

FIG. 4: Difference in Length scale vs. Temperature.

might be expected, the difference between the two scales
tends to zero as we approach the critical chemical po-
tential (or temperature). However, for low temperatures
(or large chemical potentials), the difference appears to
asymptote to a constant value. This is a very exciting
feature because the presence of two length scales would
be detectable in any physical realization of the dual field
theory.

For the solutions discussed in [10], we know the depen-
dence of the condensate $\langle \mathcal{O}_2 \rangle$ on the chemical potential
(for values close to the critical value $\tilde{\mu}_0$)

$$z_T^2 \langle \mathcal{O}_2 \rangle \approx 3.46 \sqrt{\tilde{\mu} - \tilde{\mu}_0}. \qquad (13)$$

Using equations (12) and (13) we get the connection be-
tween the coherence length and the condensate for small
chemical potentials

$$z_T^2 \tilde{\xi} \approx 0.29 / \langle \mathcal{O}_2 \rangle . \qquad (14)$$

Another quantity that is of great interest is the amount
of depletion of charge density at the interface and its vari-
ation with the chemical potential for fixed temperature.
This is shown in the Figure 5 where we plot the spatial
variation of the charge density for a few values of the
chemical potential. A striking feature is that the charge
density does not appear to vanish at the interface even
at zero temperature in sharp contrast to the dark soli-
ton solution of the GP equation. The (total) density
depletion fraction increases monotonically with increas-
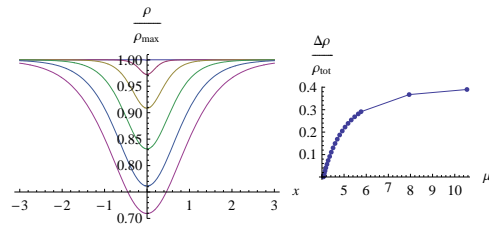ing chemical potential. In the second graph in Figure

FIG. 5: Density depletion profile vs. chemical potential.

5, we see that the percentage of depletion seems to tend
to a constant as we tend to zero temperature (or large
chemical potential). However, our numerical results are
less accurate for the larger values of the chemical poten-
tial since it is computationally much more expensive to
obtain solutions when the field values become large.

Note that our system is at finite temperature and
therefore we might expect that only a part of the total
charge density comes from the condensate, while some of
it is due to quasiparticles which are not in the conden-
sate. This is supported by the fact that the condensate
vanishes at a nonzero value of the charge density.

f. Discussion. In this work we have shown that the
holographic model of a superfluid constructed in [8] sup-
ports dark solitons, that are stationary structures in the
condensate, stabilized by non-linear interactions. At the
simplest level such solutions can be obtained from the
Gross-Pitaevskii equation (1), which includes the neces-
sary interaction terms to stabilize the dark soliton. Still
the GP equation has its limitations since it is applicable
only in the limit of small temperatures and densities.

Our work shows that the functional form of the con-
densate profile of the holographic dark soliton is similar
to the one obtained from the GP equation, but there
are striking differences. First of all the charge density
does not go to zero at the core of the dark soliton, which
means that not all of the particles are in the condensate.
The density depletion is seen to vary as a function of the
chemical potential.

Furthermore we find that the length scale associated
with the charge density depletion is different from the
coherence length determined from the condensate. This
difference is observed to be a non-trivial function of the
chemical potential. It would be interesting to see if such

a difference of scales of the dark soliton could be seen in currently available experimental systems.

It will be important to study dark solitons in non-relativistic duals that are closer to BEC-BCS systems, when they become available. However, we find it interesting that even in a relativistic dual, dark solitons have similar features as those observed in BEC-BCS systems.

In a longer forthcoming manuscript, we will study dark solitons in the $\langle \mathcal{O}_1 \rangle$ theory, which turn out to have interesting differences to the ones studied here. In the immediate future, one might also construct vortices and compare them with the dark solitons [21]. In a closely related work, the hydrodynamic properties of this system was studied [22]. It would be important to examine the relation between that study and ours.

It will of course be interesting to study the effects of magnetic fields since one can use an external magnetic field to control the density of states of this system. An-other exciting prospect is the Abrikosov vortex lattice. An important shortcoming of our work is that we work in the probe limit. Taking the gravitational backreaction into account would be doubly interesting - both from the field theory point of view as well as from the gravitational point of view. Some of these projects are currently under study.

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