ON WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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Abstract. We generalize the results of [11] and [12] for the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$. In particular we show that under the weight condition (B) the weighted H^∞–space on $\mathbb{B}_d$ is isomorphic to ℓ^∞ and thus complemented in the corresponding weighted L^∞–space. We construct concrete, generalized Bergman projections accordingly. We also consider the case where the domain is the entire space $\mathbb{C}^d$.

We also show that for the polydisc $\mathbb{D}^d$, the weighted H^∞–space is never isomorphic to ℓ^∞.

1. Introduction.

We study the structure and projection operators of the spaces $Hv := Hv(\Omega) := H_v^\infty(\Omega)$, consisting of holomorphic functions on Ω ($:= \mathbb{C}^d$ or its open unit ball $\mathbb{B}_d$), the space endowed with a weighted sup-norm

$$\|f\|_v = \sup_{z\in\Omega} |f(z)|v(|z|), \tag{1.1}$$

where $v : [0, R[\to \mathbb{R}_+$, $R = \sup_{z\in\Omega} |z|$, is a suitable bounded continuous weight function (see below). Our aim is to construct projection operators which are bounded with respect to $\|\cdot\|_v$ and which project the corresponding weighted spaces, $Cv(\Omega)$ or $Lv(\Omega) := L_v^\infty(\Omega)$, of continuous or L^∞-functions, respectively, onto $Hv(\Omega)$. The results are nontrivial generalizations of those in [11] and [12]. In [11] the first named author studied the corresponding function spaces on the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ and introduced a large class (B) of radial weight functions v. He proved that the space $Hv(\mathbb{D})$ is isomorphic to the Banach-space ℓ^∞, if and only if v is in the class (B) (otherwise $Hv(\mathbb{D})$ was shown to be isomorphic to H^∞). In [12], concrete bounded projections from $Lv(\mathbb{D})$ or $Cv(\mathbb{D})$ onto $Hv(\mathbb{D})$ were constructed.

Generalizing the weight class (B) to the case of several variables, the following will be our main result for spaces of holomorphic mappings on Ω (for details of the notations and definitions, see below).

Theorem 1.1. *The following are equivalent:*
(i) The weight v satisfies (B).
(ii) $Hv(\Omega)$ is isomorphic to the Banach space ℓ_∞.
(iii) $(Hv)_0(\Omega)$ is isomorphic to c_0.

Since ℓ_∞ is an injective Banach space, the result (ii) implies

Corollary 1.2. *If the weight v satisfies the condition (B), then there exist bounded projections from $Lv(\Omega)$ onto $Hv(\Omega)$.*

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Concrete examples of such projections are constructed in Section 5, see Theorems 5.3 and 5.5.

We also consider the corresponding spaces $Hv(\mathbb{D}^d)$ and $(Hv)_0(\mathbb{D}^d)$ on the polydisc $\mathbb{D}^d = \mathbb{D} \times \ldots \times \mathbb{D}$. Here, we consider weighted norms of the form

$$\|f\|_v = \sup_{z\in\mathbb{D}^d} |f(z)|v(|z|_\infty) \tag{1.2}$$

where $|z|_\infty := \max(|z_1|, \ldots |z_d|)$ for $z = (z_1, \ldots, z_d) \in \mathbb{D}^d$ and v again is a weight on the interval $[0, 1[$.

Proposition 1.3. *If $d \geq 2$ then $Hv(\mathbb{D}^d)$ is never isomorphic to ℓ_∞ and $(Hv)_0(\mathbb{D}^d)$ is never isomorphic to c_0.*

However, it follows from [4] that $Hv(\mathbb{D}^d)$ is almost isometrically isomorphic to a subspace of ℓ_∞. Moreover, on $\mathbb{D}^d$ one can also consider product weights $\omega(z) := \omega_1(z_1)\omega_2(z_2)\ldots\omega_d(z_d)$, where each ω_j, $j = 1, \ldots, d$, is a weight on $\mathbb{D}$. It follows from [2], Corollary 42, and [1], Satz 3.5, that $(H\omega)_0(\mathbb{D}^d)$ is isometrically isomorphic to the ϵ–product of the spaces $(H\omega_j)_0(\mathbb{D})$, and hence, the structural properties of $(H\omega)_0(\mathbb{D}^d)$ and $H\omega(\mathbb{D}^d)$ can be deduced from those of the component spaces. In particular there are examples where $H\omega(\mathbb{D}^d)$ is isomorphic to l_∞ and $(H\omega)_0(\mathbb{D}^d)$ is isomorphic to c_0.

2. Notation.

Let $d \in \mathbb{N} := \{0, 1, 2, \ldots\}$, $d \geq 2$. We use the standard multi–index notation: if $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we denote $|k| := k_1 + \ldots + k_d$ and $z^k := z_1^{k_1} z_2^{k_2} \ldots z_d^{k_d}$. If $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $z = (z_1, \ldots, z_d) = (r_1 e^{i\varphi_1}, \ldots, r_d e^{i\varphi_d}) \in \mathbb{C}^d$, we still denote

$$|k| := |k_1 + \ldots + k_d| \quad \text{and} \quad e_k(z) := \prod_{j=1}^{d} \Big(r_j^{|k_j|} e^{ik_j\varphi_j} \Big). \tag{2.1}$$

By Ω we denote either the whole space $\mathbb{C}^d$ or the Euclidean ball

$$\mathbb{B}_d = \{z \in \mathbb{C}^d : |z| < 1\};$$

here $|z|^2 := \langle z, z\rangle$ with $\langle z, w\rangle := \sum_{k=1}^d z_k \bar{w}_k$ for z, $w \in \mathbb{C}^d$. We also consider the polydisc

$$\mathbb{D}^d = \{z \in \mathbb{C}^d : \max(|z_1|, \ldots, |z_d|) < 1\}.$$

Let $R := +\infty$, if $\Omega = \mathbb{C}^d$, or $R := 1$, if $\Omega = \mathbb{B}_d$, and let us denote $I := [0, R[$. By a weight (on I) we mean a continuous, non-increasing function $v : I \to \mathbb{R}_+$ with $v(r) > 0$ for $r \in [0, R[$ and $\lim_{r\to R} r^m v(r) = 0$ for all $m \geq 0$. We extend v to Ω or to the polydisc as follows. On the set Ω we shall consider weights of the form $v(z) := v(|z|)$, and, on $\mathbb{D}^d$, of the form $v(z) := v(|z|_\infty) := v(\max(|z_1|, \ldots, |z_d|))$ (in the latter case of course $R = 1$). We keep the same notation for all cases; the domain of definition will be indicated or clear from the context.

We introduce the weighted sup-norms (1.1) and (1.2), respectively, where the "sup" is replaced by "ess sup" in the case of measurable functions on Ω. Moreover, for $0 < \rho < R$, put

$$M_\infty(f, \rho) = \sup_{|z|=\rho} |f(z)|.$$

We study the spaces

$$Hv = Hv(\Omega) = \{h : \Omega \to \mathbb{C} \text{ holomorphic } : \|h\|_v < \infty\} \tag{2.2}$$

and

$$(Hv)_0 = (Hv)_0(\Omega) = \{h \in Hv : \lim_{\rho \to R} M_\infty(h, \rho) v(\rho) = 0\}, \tag{2.3}$$

and also the spaces $Hv(\mathbb{D}^d)$ and $(Hv)_0(\mathbb{D}^d)$, which are defined by replacing Ω by the polydisc in (2.2) and (2.3), respectively. In addition, the larger spaces $Cv = Cv(\Omega)$ and $Lv = Lv(\Omega)$ are defined by replacing "holomorphic" by "continuous" or "measurable" in (2.2). All of these spaces are Banach spaces with respect to $\| \cdot \|_v$.

We write $X \sim Y$ to denote that the Banach spaces X and Y are isomorphic, i.e. there exists a bounded linear bijection from X onto Y (the inverse is also bounded by the open mapping theorem). For the bidual we have $(Hv)_0(\Omega)^{**} \sim Hv(\Omega)$, see [3]. Recall that a closed (linear) subspace Z of the Banach space X is called complemented, if there exists a bounded operator, projection, $P : X \to X$ such that $P(X) = Z$ and $P^2 = P$. For Banach space operator theory we use the notation and terminology of [10] and [16].

Let v be a weight on I as above. For all $n > 0$ (not necessarily integers!) we fix a global maximum point ρ_n of the function $\rho \mapsto \rho^n v(\rho)$, $\rho \in [0, R[$. The following concept will only be used for the spaces on Ω.

Definition 2.1. The weight $v : I \to \mathbb{R}_+$ (and the corresponding weight $v : \Omega \to \mathbb{R}_+$) are said to satisfy condition (B) if

$$\begin{aligned}
&\forall b_1 > 1 \ \exists b_2 > 1 \ \exists c > 0 \ \forall m, n > 0 : \\
&\left(\frac{\rho_m}{\rho_n}\right)^m \frac{v(\rho_m)}{v(\rho_n)} \le b_1 \text{ and } m, n, |m - n| \ge c \ \Rightarrow \left(\frac{\rho_n}{\rho_m}\right)^n \frac{v(\rho_n)}{v(\rho_m)} \le b_2
\end{aligned}$$

For the following examples, see [11]. If $R = 1$, $v(\rho) = (1 - \rho)^\alpha$ for some $\alpha > 0$ or $v(\rho) = \exp(-1/(1 - \rho))$ satisfy condition (B). Moreover, all so-called normal weights satisfy (B) (for the definition of normality, see [15] and [5]). An example of a weight not satisfying (B) is $v(\rho) = (1 - \log(1 - |\rho|))^{-1}$ (see [7]).

3. Proof of Theorem 1.1, first part.

The bidual satisfies $(Hv)_0(\Omega)^{**} \sim Hv(\Omega)$, which immediately proves $(iii) \Rightarrow (ii)$.

If v does not satisfy (B), we denote $\Omega_1 := \mathbb{C}$, if $R = \infty$ (respectively, $\mathbb{D}$, if $R = 1$). By [11] (together with [12], Proposition 1), $Hv(\Omega_1) \sim H_\infty$, the space of bounded holomorphic functions $\mathbb{D} \to \mathbb{C}$, endowed with the unweighted sup-norm. We remark that $Hv(\Omega_1)$ is isomorphic to a complemented subspace of $Hv(\Omega)$. Indeed, for $g \in Hv(\Omega_1)$ define $Tg \in Hv(\Omega)$ by $(Tg)(z_1, \ldots, z_d) = g(z_1)$. Then $\|Tg\|_v = \|g\|_v$. Moreover, for $h \in Hv(\Omega)$ define $Sh \in Hv(\Omega_1)$ by $(Sh)(z) = h(z, 0, \ldots, 0)$. Then $\|Sh\|_v \le \|h\|_v$ and TS is a bounded projection from $Hv(\Omega)$ onto $THv(\Omega_1) \sim Hv(\Omega_1)$.

So, if v does not satisfy (B), then H_∞ is complemented in $Hv(\Omega)$. Therefore $Hv(\Omega)$ cannot be isomorphic to ℓ_∞. This proves $(ii) \Rightarrow (i)$.

The proof of $(i) \Rightarrow (iii)$ requires much more work. We complete the proof in Sections 6–7.

4. Proof of Proposition 1.3

Let v be an arbitrary weight on $\mathbb{D}^d$ of the form $v(|z|_\infty)$, as in Section 2. To prove the proposition, it suffices to assume $d = 2$, since $Hv(\mathbb{D}^2)$ and $(Hv)_0(\mathbb{D}^2)$ are complemented in $Hv(\mathbb{D}^d)$ and $(Hv)_0(\mathbb{D}^d)$, respectively, and infinite dimensional complemented subspaces of ℓ_∞ and c_0 are isomorphic to ℓ_∞ and c_0, respectively (see [10], Volume I). In the following we consider the case of the larger spaces; the proof for $(Hv)_0(\mathbb{D}^2)$ is the same.

The idea is to show that for all n, the subspace of n-homogeneous polynomials of $Hv(\mathbb{D}^2)$ is isometric with the space of one variable polynomials of degree at most n, considered as a space on $\partial\mathbb{D}$ with respect to the unweighted sup-norm. The result will follow by combining known facts about projection constants.

Let P_n be the subspace of $Hv(\mathbb{D}^2)$ consisting of all homogeneous polynomials of degree n. So every $p \in P_n$ has the form

$$p(z_1, z_2) = \sum_{j=0}^{n} \alpha_j z_1^j z_2^{n-j}$$

for some numbers α_j. Let $0 \le r \le s < 1$. Then, applying the maximum principle to the function $f(z_1) = p(z_1, z_2)$ for a fixed z_2, we see that

$$\begin{aligned}
\sup_{\varphi,\psi\in[0,2\pi]} |p(re^{i\varphi}, se^{i\psi})| &\le \sup_{\varphi,\psi} |p(se^{i\varphi}, se^{i\psi})| \\
&= s^n \sup_{\varphi,\psi} |p(e^{i\varphi}, e^{i\psi})| \\
&= s^n \sup_{\varphi} |q(e^{i\varphi})|
\end{aligned}$$

where $q(w) = \sum_{j=0}^{n} \alpha_j w^j$ for $w \in \mathbb{C}$. If $s \le r$, then we get similarly

$$\sup_{\varphi,\psi} |p(re^{i\varphi}, se^{i\psi})| \le r^n \sup_{\psi} |q(e^{i\psi})|$$

This implies $\|p\|_v = \rho_n^n v(\rho_n) \sup_\varphi |q(e^{i\varphi})|$, where ρ_n is a maximum point of the function $s \mapsto s^n v(s)$, $s \in [0, 1[$.

Let A_n be the space of all polynomials of one complex variable and of degree $\le n$. We consider A_n as a space defined on $\partial\mathbb{D}$ and endow it with the sup-norm on $\partial\mathbb{D}$. The preceding shows that the mapping $p \mapsto q\rho_n^n v(\rho_n)$ is an isometry from P_n onto A_n.

There are projections $Q_n : Hv(\mathbb{D}^2) \to P_n$ whose norms are uniformly bounded. Indeed, for $f \in Hv(\mathbb{D}^2)$ put

$$(Q_n f)(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi} z) e^{-in\varphi} d\varphi, \qquad z \in \mathbb{D}^2$$

Hence the relative projection constants

$$\begin{aligned}
&\lambda(P_n, Hv(\mathbb{D}^2)) = \\
&\quad \inf\{\ \|Q\| \ :\ Q : Hv(\mathbb{D}^2) \to P_n \text{ is a bounded surjective projection }\ \}
\end{aligned}$$

are uniformly bounded. If $Hv(\mathbb{D}^2)$ were a $\mathcal{L}_\infty$−space, then, by well known facts which we repeat below, the absolute projection constants

$$\lambda(P_n) = \sup\{\ \lambda(P_n, X) \ :\ X \text{ is a Banach space, } P_n \subset X\ \}$$

would be equivalent to $\lambda(P_n, Hv(\mathbb{D}^2))$, where the constants of equivalence do not depend on n. This would mean that the numbers $\lambda(P_n)$ are uniformly bounded. However, the isometry of P_n and A_n would imply that the numbers $\lambda(A_n) = \lambda(P_n)$ are uniformly bounded. This is a contradiction because it is well-known that $\lambda(A_n) \geq C \log n$ for all n, see [16].

Concerning the statement on the projection constants, if $Hv(\mathbb{D}^2)$ is a $\mathcal{L}_\infty$−space, then it is known by [9] to be injective, which means that there is a constant $\gamma \geq 1$ satisfying the following. For any Banach space $X \supset P_n$ there would be a linear map $T : X \to Hv(\mathbb{D}^2)$ with $T|_{P_n} = \mathrm{id}|_{P_n}$ and $\|T\| \leq \gamma$. This would imply $\lambda(P_n, X) \leq \gamma\lambda(P_n, Hv(\mathbb{D}^2))$. Hence, $\lambda(P_n) \leq \gamma\lambda(P_n, Hv(\mathbb{D}^2))$. □

5. Construction of the bounded projections onto Hv

For the rest of the paper we consider spaces on Ω, and leave the domain out of the notation of the function spaces. We also assume throughout this section that v satisfies (B). We first define the projection $P : Cv \to Hv$.

Take $f \in Cv$, and let $z \in \mathbb{C}^d$, $|z| = 1$, and $0 < \rho < R$. For all $j \in \mathbb{Z}$ we set

$$f_j(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho z e^{i\varphi}) \rho^{-|j|} e^{-ij\varphi} d\varphi, \tag{5.1}$$

and then

$$(Q_1 f)(\rho z) = \sum_n \Big(\sum_{m_{n-1} < j \leq m_{n+1}} t_{n,j} f_j(\rho_{m_n} z) \rho^j \Big), \tag{5.2}$$

where the strictly increasing positive sequence $(m_n)_{n=1}^\infty$ and the numbers $t_{n,j}$, $0 \leq t_{n,j} \leq 1$, will be chosen in Definition 5.6; the series converges at least pointwise for all ρz.

The function $Q_1 f$ is not in general holomorphic (see the following example), so we still define

$$(Q_2 f)(\rho z) = \int_{\partial \mathbb{B}_d} C(z, w) f(\rho w) d\sigma_d(w) \tag{5.3}$$

where σ_d be the normalized rotation invariant measure on $\partial\mathbb{B}_d$ and $C(\cdot, \cdot)$ is the Cauchy kernel

$$C(z, w) = \frac{1}{(1 - \langle z, w \rangle)^d} = \frac{1}{(1 - \sum_{k=1}^d z_k \bar{w}_k)^d} \tag{5.4}$$

Definition 5.1. We define $Pf := Q_2 Q_1 f$ for $f \in Cv$.

Remark 5.2. To see that $Q_1 f$ is not necessarily holomorphic, consider a function of the form

$$f(\rho z) = r(\rho)\gamma(|z_1|, \ldots, |z_d|)\rho^{|k|} e_k(z), \tag{5.5}$$

where $|z| = 1$, $0 \leq \rho < R$, $k \in \mathbb{N}^d$, and r and γ are continuous functions. If $j = k_1 + \ldots + k_d$, then $|j| = |k|$ and

$$f_j(\rho z) = r(\rho)\gamma(|z_1|, \ldots, |z_d|) e_k(z).$$

Hence, $Q_1 f = 0$ for $j < 0$, and

$$Q_1 f(\rho z) = \big(t_{n,j} r(\rho_{m_n}) + t_{n+1,j} r(\rho_{m_{n+1}})\big)\gamma(|z_1|, \ldots, |z_d|)\rho^{|k|} e_k(z),$$

for $m_n < j \leq m_{n+1}$. So even if $j \geq 0$ and r is constant, then $Q_1 f$ need not be holomorphic in case $d > 1$. (If $d = 1$, then $j = k$ and γ must be constant since $|z| = 1$.)

On the other hand we obtain $Q_1 f = \rho^{|k|} z^k$, if $k \in \mathbb{N}^d$ and $r = \gamma = 1$. This follows from the definition of the numbers $t_{n,j}$ in (5.11). For f of the form (5.5) we obtain

$$Q_2 Q_1 f(\rho z) = \begin{cases} c_k \rho^{|k|} z^k, & \text{if } k \in \mathbb{N}^d \\ \\ 0, & \text{if } k \notin \mathbb{N}^d, \end{cases}$$

where c_k is a constant. Taking the Stone–Weierstrass theorem into account we see that the linear span of the functions of the form (5.5) in Cv is dense in Cv with respect to the topology of uniform convergence on compact subsets.

Below we complete the missing details of the above definition, show that the definition is rigorous, and prove the following fact:

Theorem 5.3. *P is a bounded projection from Cv onto Hv.*

Before that we also define the projection $P_M : Lv \to Hv$.

Lemma 5.4. *There exists a bounded operator $A : Lv \to Cv$ such that $Af = f$ for $f \in Hv$.*

The following is now an obvious consequence of Theorem 5.3 and Lemma 5.4:

Theorem 5.5. *The operator $P_M := PA$ is a bounded projection from Lv onto Hv.*

Proof of Lemma 5.4. We fix a continuous function $\lambda : [0, R[\to]0, 1[$ such that

$$0 < \rho - \lambda(\rho) \quad , \quad \rho + \lambda(\rho) < R \tag{5.6}$$

and

$$\frac{1}{2} v(\rho - \lambda(\rho)) \leq v(\rho) \leq 2v(\rho + \lambda(\rho)) \tag{5.7}$$

for all $\rho \in [0, R[$. The operator A is defined by

$$Af(z) = \frac{d^{2d}}{\pi^d \lambda(|z|)^{2d}} \int\limits_{\substack{|w_1 - z_1| \\ \leq \lambda(|z|)/d}} \cdots \int\limits_{\substack{|w_d - z_d| \\ \leq \lambda(|z|)/d}} f(w) dw_1 \dots dw_d, \tag{5.8}$$

where dw_j denotes the 2-dimensional real Lebesgue measure. The operator A is bounded with respect to $\| \cdot \|_v$, since the volume of the integration domain is $\pi^d \lambda(|z|)^{2d} d^{-2d}$ and since the weight v is uniformly equivalent to a constant on the integration domain, by (5.7).

Moreover, A keeps the holomorphic functions invariant, since they obey the mean-value principle with respect to each variable separately.

Finally, Af is a continuous function: given $z \in \Omega$, if $\zeta \in \Omega$, the integration domains corresponding to $Af(z)$ and $Af(\zeta)$, respectively, in (5.8), differ from each other by a set of arbitrarily small measure, if $|z - \zeta|$ is small enough. This, and the local boundedness of f imply

$$\lim_{\zeta \to z} Af(\zeta) = Af(z),$$

i.e. Af is continuous at z. $\square$

The rest of this section is devoted to the details of the definition of P and proof of Theorem 5.3. We prove here that Q_1 is bounded with respect to $\|\cdot\|_v$. Assuming that Q_2 is a bounded operator with respect to $\|\cdot\|_v$, we also show that P is idempotent, $P^2 = P$, and that it projects onto the subspace Hv. The only remaining thing, the boundedness of Q_2, is postponed to Section 6.

We define the numbers m_n and $t_{n,j}$ as follows (the definition is the same as in [12]).

Definition 5.6. Fix $b > 2$. Use (B) and induction to find numbers b_1 and $m_0 = 0 < m_1 < m_2 < \dots$ with

$$b \leq \left(\frac{\rho_{m_n}}{\rho_{m_{n+1}}}\right)^{m_n} \frac{v(\rho_{m_n})}{v(\rho_{m_{n+1}})}, \quad \left(\frac{\rho_{m_{n+1}}}{\rho_{m_n}}\right)^{m_{n+1}} \frac{v(\rho_{m_{n+1}})}{v(\rho_{m_n})} \leq b_1 \quad \text{for all } n \tag{5.9}$$

and $\lim_{n\to\infty} m_n = \infty$. (See [11], Lemma 5.1. and Proposition 6.4.) Then [11], Proposition 4.1., implies that there are $\eta > 0$ and $\kappa > 0$ with

$$\eta \leq \frac{m_{n+1} - m_n}{m_n - m_{n-1}} \leq \kappa \quad \text{or} \quad m_{n+1} - m_{n-1} \leq c \quad \text{for all } n, \tag{5.10}$$

where c is the constant of condition (B). (Such indices can be easily computed for special v. For example, if $R = \infty$ and $v(\rho) = \exp(-\rho)$, then $m_{n+1} = m_n + \mathcal{O}(m_n^{3/4})$. If $R = 1$ and $v(\rho) = (1-\rho)^\alpha$ then $m_n = \gamma^n$ for suitable $\gamma > 1$ (see [11]).)

Put

$$t_{n,j} = \begin{cases} \frac{j-[m_{n-1}]}{[m_n]-[m_{n-1}]}, & m_{n-1} < j \leq m_n \\ \\ \frac{[m_{n+1}]-j}{[m_{n+1}]-[m_n]}, & m_n < j \leq m_{n+1} \end{cases} \tag{5.11}$$

Here j is an integer and $[a]$ is the largest integer $\leq a$.

Lemma 5.7. *The series* (5.2) *converges pointwise for all ρz, the operator Q_1 is well-defined and bounded $Cv \to Cv$, and moreover, $Q_1 f = f$ for all $f \in Hv$.*

Proof. Let $f \in Cv$, and let us first assume that f is continuously differentiable in Ω. We fix $z \in \mathbb{C}^d$, $|z| = 1$, and $0 < \rho < R$.

The definition (5.1) implies for all $j \in \mathbb{Z}$

$$f_j(\rho z e^{i\varphi}) = f_j(\rho z) e^{ij\varphi}. \tag{5.12}$$

(Notice that if f is holomorphic then, for $j \geq 0$, $\rho^j f_j$ is a homogeneous polynomial of degree j while, for $j < 0$, $f_j = 0$.) Since we assume $f \in C^1(\Omega)$, the Fourier-series of the C^1-function $\psi_f : \varphi \mapsto f(\rho z e^{i\varphi})$ converges uniformly on the set $[0, 2\pi]$. The numbers $f_j(\rho z)$ are the Fourier-coefficients of ψ_f, hence, taking $\varphi = 0$, we see that

$$f(\rho z) = \sum_{j\in\mathbb{Z}} f_j(\rho z) \rho^{|j|}, \tag{5.13}$$

where the sum converges pointwise for all z and ρ.

Let us still keep z fixed, but let ρ vary, and define the function $g^{(z)}$ of one complex variable on Ω_1 (see Section 3),

$$g_f^{(z)}(\rho e^{i\varphi}) = f(\rho z e^{i\varphi}). \tag{5.14}$$

We immediately get the z-independent bound

$$\|g_f^{(z)}\|_v := \sup_{\zeta\in\Omega_1} |g_f^{(z)}(\zeta)| v(\zeta) \leq \|f\|_v. \tag{5.15}$$

Recall that in [12] the authors defined a projection P_C, which is a bounded projection from $Cv(\Omega_1)$ onto $Hv(\Omega_1)$. In view of (5.15), $P_C g_f^{(z)} \in Hv(\Omega_1)$ with a bound

$$\|P_C g_f^{(z)}\|_v \le C\|f\|_v. \tag{5.16}$$

(5.2) and (5.12) imply

$$(Q_1 f)(\rho z e^{i\varphi}) = \sum_n \Big(\sum_{m_{n-1}<j\le m_{n+1}} t_{n,j} f_j(\rho_{m_n} z)\rho^j e^{ij\varphi}\Big).$$

But this is exactly the definition of $P_C g_f^{(z)}$ according to (16), [12]. So we have

$$(P_C g_f^{(z)})(\rho e^{i\varphi}) = (Q_1 f)(\rho z e^{i\varphi}). \tag{5.17}$$

Letting now also z vary, (5.17) and (5.16) imply

$$\|Q_1 f\|_v = \sup_{z\in\partial\mathbb{B}_d} \|P_C g_f^{(z)}\|_v \le C\|f\|_v, \tag{5.18}$$

i.e. Q_1 is bounded with respect to $\|\cdot\|_v$.

Using the density of C^1-functions in Cv we extend the definition of P to Cv as a bounded linear operator.

If $f \in Hv$, then, according to (5.14), for a fixed z, $|z| = 1$, $g_f^{(z)} \in Hv(\Omega_1)$. We obtain $P_C g_f^{(z)} = g_f^{(z)}$ and hence $Q_1 f = f$. $\square$

Lemma 5.8. *If the operator Q_2 is bounded with respect to $\|\cdot\|_v$, then $P = Q_2Q_1 : Cv \to Hv$ is a bounded projection.*

Proof. Obviously, the assumption and Lemma 5.7 imply that P is bounded. Looking at (5.3) and (5.4) it is also clear that Q_2 leaves polynomials invariant, hence, it does the same for all holomorphic functions: $Q_2 f = f$ for all $f \in Hv$.

We need to show that $Q_2 f$ is holomorphic for all $f \in Q_1Cv$. Assume first that f is, for some $k \in \mathbb{Z}^d$, of the form

$$f(\rho z) = \rho^{|k|}\gamma(|z_1|,\dots,|z_d|)e_k(z),$$

where $z \in \mathbb{C}^d$, $|z| = 1$ and $0 \le \rho < R$. We claim that $Q_2 f = 0$, if $k \notin \mathbb{N}^d$, and $Q_2 f(\rho z) = c_k \rho^{|k|} z^k$ for some constant c_k, if $k \in \mathbb{N}^d$. In the first case we may assume without loss of generality that $k_d < 0$. Let $w = (w', w_d) \in \mathbb{C}^d$, where $w' \in \mathbb{C}^{d-1}$ and w_d is the last coordinate, and recall that $w \in \partial\mathbb{B}_d$ if and only if $(w', e^{i\theta}w_d) \in \partial\mathbb{B}_d$ for all $\theta \in [0, 2\pi]$. By (6) in the proof of Proposition 1.4.7 of [13],

$$\int_{\partial\mathbb{B}^d} \frac{f(\rho w)}{1-\langle z,w\rangle} d\sigma_d(w) = \int_{\partial\mathbb{B}^d} \frac{1}{2\pi}\int_0^{2\pi} \frac{f((\rho w', \rho e^{i\theta}w_d))}{1-\langle z,(w', e^{i\theta}w_d)\rangle} d\theta d\sigma_d(w)$$

$$= \int_{\partial\mathbb{B}^d} \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^{|k|}\gamma(|w_1|,\dots,|w_d|)e_k(w',0)e^{ik_d\theta}}{1-\langle z,(w', e^{i\theta}w_d)\rangle} d\theta d\sigma_d(w). \tag{5.19}$$

Now the kernel

$$\frac{1}{1-\langle z,(w', e^{i\theta}w_d)\rangle}$$

is a sum of terms of the form $f_n(z,w')e^{-in\theta}\bar{w}_d^n$ with $n \in \mathbb{N}$ and some continuous functions f_n; taking into account the sign of k_d, the integration with respect to θ renders (5.19) equal to zero.

If $k \in \mathbb{N}^d$, then a similar argument shows that all the integrals

$$\int_{\partial \mathbb{B}^d} f(\rho w) z^l \bar{w}^l d\sigma_d(w) \tag{5.20}$$

vanish, if $l \neq k$. We obtain

$$\int_{\partial \mathbb{B}^d} \frac{f(\rho w)}{1-\langle z,w\rangle} d\sigma_d(w) = \int_{\partial \mathbb{B}^d} f(\rho w) z^k \bar{w}^k d\sigma_d(w) = c_k \rho^{|k|} z^k, \tag{5.21}$$

i.e. a holomorphic function.

If $f = Q_1 g$ for some $g \in Cv$, then (5.2) and the Stone-Weierstraß theorem ([14], Section 5.7.) imply that $f = f(\rho z)$ can be approximated at least uniformly on the compact subsets of $\mathbb{B}_d$ by functions of the form

$$\sum_{l=0}^{N} \rho^l \sum_{\substack{k\in\mathbb{Z}^d \\ |k|=l}} \gamma_k(|z_1|,\ldots,|z_d|) e_k(z) \tag{5.22}$$

with $\gamma_k \in C(\partial\Omega)$. For fixed ρ and z we infer from (5.3) and the previous case that $Q_2 Q_1 g$ can be approximated by holomorphic functions uniformly in a neighbourhood of ρz, hence $Q_2 Q_1 g$ is holomorphic. $\square$

As the last topic of this section we consider operators connected with the coefficients $t_{n,j}$.

Consider $h \in Hv(\Omega)$. Then for every multi–index k there exists an $a_k \in \mathbb{C}$ such that

$$h = \sum_{j=0}^{\infty} h_j, \quad \text{where} \quad h_j(z) = h_j(z_1,\ldots,z_d) = \sum_{\substack{k\in\mathbb{N}^d \\ |k|=j}} a_k z^k \tag{5.23}$$

Define

$$T_n h = \sum_{m_{n-1}<j\leq m_n} t_{n,j} h_j + \sum_{m_n<j\leq m_{n+1}} t_{n,j} h_j. \tag{5.24}$$

Fix the number $z \in \mathbb{C}^d$, $|z| = 1$, and consider the function of one complex variable, $f(\rho e^{i\varphi}) = h(\rho e^{i\varphi} z)$,$\rho \in [0, R[$. An application of [12], Theorem 1, to f yields

Theorem 5.9. *There are numbers $c_1 > 0$, $c_2 > 0$ such that for any $h \in Hv(\Omega)$ we have*

$$c_1 \sup_n M_\infty(T_n h, \rho_{m_n}) v(\rho_{m_n}) \leq \|h\|_v \leq c_2 \sup_n M_\infty(T_n h, \rho_{m_n}) v(\rho_{m_n})$$

and

$$c_1 M_\infty(T_n h, \rho_{m_n}) v(\rho_{m_n}) \leq \|T_n h\|_v \leq c_2 M_\infty(T_n h, \rho_{m_n}) v(\rho_{m_n}) \quad \textit{for all } n.$$

Theorem 5.9 also implies that $h \in Hv$ provided that h is holomorphic and $\sup_n M_\infty(T_n h, \rho_{m_n}) v(\rho_{m_n}) < \infty$.

Since $\lim_{n\to\infty} m_n = \infty$ we obtain $\lim_{n\to\infty} \rho_{m_n} = R$. Hence, if $h \in (Hv)_0$, we even have

$$\lim_{n\to\infty} M_\infty(T_n h, \rho_{m_n}) v(\rho_{m_n}) = 0 \tag{5.25}$$

6. Proof of Theorem 5.3, continued

As remarked in the previous section, the remaining thing is to prove that Q_2 is bounded with respect to the weighted sup-norm.

Fix $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ and, for $z = (z_1, \dots, z_d) \in \partial\mathbb{B}_d$ and $z_j = r_j e^{i\varphi_j}$, let $e_k(z)$ be as in (2.1). It is well-known that the functions e_k are orthogonal with respect to the scalar product in $L_2(\partial\Omega, \sigma_d)$ ([13]). Let m_n be the indices of (5.9) and let X_n be the $\|\cdot\|_v$–closed linear span of the functions f of the form

$$f(\rho z) = \rho^l \gamma(|z_1|, \dots, |z_d|) e_k(z) \tag{6.1}$$

where $z \in \partial\mathbb{B}_d$, $0 \le \rho < R$, $\gamma \in C(\partial\mathbb{B}_d)$, $k \in \mathbb{Z}^d$ with $|k| = l$ and $m_{n-1} < l \le m_{n+1}$.

We have $X_n \subset Q_1 Cv$ for all n, since Q_1 keeps functions of the form (6.1) invariant.

Now we extend the operators T_n of Theorem 5.9.

Proposition 6.1. *There are linear operators $\tilde{T}_n : Q_1 Cv \to X_n$ and $c_1 > 0$, $c_2 > 0$ such that, for any $f \in Q_1 Cv$,*

$$\sup_n c_1 M_\infty(\tilde{T}_n f, \rho_{m_n}) v(\rho_{m_n}) \le \|f\|_v \le \sup_n c_2 M_\infty(\tilde{T}_n f, \rho_{m_n}) v(\rho_{m_n})$$

and

$$c_1 M_\infty(\tilde{T}_n f, \rho_{m_n}) v(\rho_{m_n}) \le \|\tilde{T}_n f\|_v \le c_2 M_\infty(\tilde{T}_n f, \rho_{m_n}) v(\rho_{m_n})$$

Moreover, $\tilde{T}_n|_{Hv} = T_n$. Finally, $\tilde{T}_n \tilde{T}_m = 0$ if $|n - m| > 1$.

Proof. Fix $f \in Q_1 Cv$ and $z \in \partial\mathbb{B}_d$. Then, according to (5.2) and (5.14), $g_f^{(z)} \in Hv(\Omega_1)$. Let $T_{n,1}$ be the operators of Theorem 5.9 in the case $d = 1$. Put $(\tilde{T}_n f)(\rho z) = (T_{n,1} g_f^{(z)})(\rho)$. Then, clearly $\tilde{T}_n h = T_n h$ if $h \in Hv(\Omega)$. Now the proposition follows from (5.18), Theorem 5.9 and (5.24). □

Next we bring the Cauchy kernel into play.

Lemma 6.2. *We have $Q_2 X_n \subset X_n \cap Hv$. Moreover, if $f \in Q_1 Cv$ then*

$$\tilde{T}_n Q_2 f = \tilde{T}_n Q_2 (\tilde{T}_{n-1} + \tilde{T}_n + \tilde{T}_{n+1}) f$$

Proof. In the proof of Lemma 5.8 we already verified that $Q_2 f$ is holomorphic for all $f \in Q_1(Cv)$. This implies $Q_2 X_n \subset X_n \cap Hv$; see also the arguments leading to (5.21).

To prove the second part of Lemma 6.2 fix $f \in Q_1 Cv$ which is of the form (5.22). We use $\tilde{T}_n \gamma_k e_k = 0$, if $k \in \mathbb{N}^d$ and if $|k| \le m_{n-1}$ or $|k| > m_{n+1}$, which is a consequence of Proposition 6.1. Hence $\tilde{T}_n Q_2 f = \tilde{T}_n Q_2 (\tilde{T}_{n-1} + \tilde{T}_n + \tilde{T}_{n+1}) f$. □

We need a technical lemma. Recall that, if $f(re^{i\varphi}) = \sum_{j\in\mathbb{Z}} f_j(r) e^{ij\varphi}$ and

$$(W_n f)(re^{i\varphi}) = \sum_{|j|\le n} \frac{n - |j|}{n} f_j(r) e^{ij\varphi} \tag{6.2}$$

then

$$\int_0^{2\pi} |(W_n f)(re^{i\varphi})| d\varphi \le \int_0^{2\pi} |f(re^{i\varphi})| d\varphi$$

for any r (convolution with a Fejer kernel; see [6] for this inequality).

Lemma 6.3. *There is a function $c(t) > 0$ satisfying the following:*
For all integers $0 < p < q$ there exists a function

$$K(z) = \sum_{l=0}^{\infty} g_l(|z|)z^l, \qquad z \in \mathbb{B}_1$$

such that

$$\int_0^1 \int_0^{2\pi} (1-r^2)^{\max(d-2,0)} |K(re^{i\varphi})| r d\varphi dr \leq c(\frac{q}{p})$$

and

$$g_l(r) = \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \qquad \textit{if} \quad p \leq l \leq q$$

Proof of Theorem 5.3, continued. Combining Proposition 6.1, Lemma 6.2 and the remark after Theorem 5.9 we see that, for the proof of Theorem 5.3, it suffices to show that the maps $Q_2|X_n$ are uniformly bounded with respect to the norm $\|\cdot\|_v$. To this end we shall construct uniformly bounded operators $Q_{2,n}$ with $Q_{2,n}|_{X_n} = Q_2|_{X_n}$.

Fix n and take K of Lemma 6.3 with $p = [m_{n-1}]$ and $q = [m_{n+1}]$. For $f \in Cv$, $z \in \partial\mathbb{B}_d$, $0 \leq \rho < R$, define

$$(Q_{2n}f)(\rho z) = \int_{\partial\mathbb{B}_d} K(\langle z, w\rangle) f(\rho w) d\sigma_d(w)$$

By [13], Lemma 1.4.2., we have

$$(Q_{2n}f)(\rho z) = \frac{1}{2\pi} \int_{\partial\mathbb{B}_d} \int_0^{2\pi} K(\langle z, we^{i\varphi}\rangle) f(\rho we^{i\varphi}) d\varphi d\sigma_d(w)$$

In view of Lemma 6.3 this implies, for $f(\rho z) = \rho^l \gamma(|z_1|, \ldots, |z_d|) e_k(z)$, where $|k| = l$, $m_{n-1} < l \leq m_{n+1}$,

$$\begin{aligned} (Q_{2n}f)(\rho z) &= \int_{\partial\mathbb{B}_d} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \langle z, w\rangle^l f(\rho w) d\sigma_d(w) \\ &= (Q_2 f)(\rho z) \end{aligned}$$

We use (see [13], 1.4.5.(2),)

$$\int_{\partial\mathbb{B}_d} |K(\langle z, w\rangle)| d\sigma_d(w) = \frac{d-1}{\pi} \int_0^{2\pi} \int_0^1 (1-r^2)^{d-2} |K(re^{i\varphi})| r dr d\varphi,$$

if $d > 1$, to show that $\|Q_{2n}\| \leq c(m_{n+1}/m_{n-1})$, where c is the function of Lemma 6.3. The bound (5.10) yields $m_{n+1}/m_{n-1} \leq 1 + \kappa^2$ which proves that the Q_{2n} are uniformly bounded. $\square$

The only remaining missing part is the following

Proof of Lemma 6.3. Put $\tilde{p} = (q-p)/2$ and $\tilde{q} = (q+p)/2$. We can assume that $\tilde{p}$ and $\tilde{q}$ are integers (otherwise take $q+1$ instead of q). For a function $f(re^{i\varphi}) = \sum_{j\in\mathbb{Z}} f_j(r) e^{ij\varphi}$ put

$$Wf = \frac{\tilde{q}W_{\tilde{q}} - \tilde{p}W_{\tilde{p}}}{\tilde{q} - \tilde{p}} f.$$

In view of (6.2) we have

$$(Wf)(re^{i\varphi}) = \sum_{|j|\le\tilde p} f_j(r)e^{ij\varphi} + \sum_{\tilde p<|j|\le\tilde q} \frac{\tilde q-|j|}{\tilde q-\tilde p} f_j(r)e^{ij\varphi}.$$

Put $(Vf)(re^{i\varphi}) = e^{i\tilde q}W(e^{-i\tilde q\varphi}f(re^{i\varphi}))$. Since $(\tilde q+\tilde p)/(\tilde q-\tilde p) = q/p$ we have

$$\int_0^{2\pi} |(Vf)(re^{i\varphi})|d\varphi \le \frac{q}{p}\int_0^{2\pi} |f(re^{i\varphi})|d\varphi \quad \text{for all} \quad r.$$

Moreover, $(Vf)(re^{i\varphi}) = \sum_{l\ge 0} \tilde f_l(r)e^{il\varphi}$ where $\tilde f_l = f_l$ if $p \le l \le q$.

Let $u(z) = (1-|z|^2)/|1-z|^2$ be the Poisson kernel. If $d = 1$ then put $K(z) = (Vu)(z)$.

Now let $d > 1$. Fix $j \in \mathbb{Z}_+$ with $1 \le j \le d-1$ and define

$$k_j(z) = z^p\left(\frac{1-z^{q-p+1}}{1-z}\right)^2 + (p+j-1)V(z^pu(z)).$$

Recall

$$z^p\left(\frac{1-z^{q-p+1}}{1-z}\right)^2 = \sum_{l=p}^{q}(l+1-p)z^l + \sum_{l>q} a_lz^l \qquad \text{for some} \quad a_l.$$

Moreover

$$\begin{aligned} V(z^pu(z)) &= V\left(\sum_{l=-\infty}^{-1} z^p\bar z^l + \sum_{l=0}^{\infty} z^{l+p}\right) \\ &= \sum_{l=0}^{p-1} b_l(r)r^le^{il\varphi} + \sum_{l=p}^{q} r^le^{il\varphi} + \sum_{l=q+1}^{q+p} b_l(r)r^le^{il\varphi} \end{aligned}$$

for some b_l. Hence

$$k_j(re^{i\varphi}) = \sum_{l\ge 0}\gamma_l(r)r^le^{il\varphi} \tag{6.3}$$

with $\gamma_l(r) = l+j$ for $p \le l \le q$. Fix r and put

$$\tilde k_j(e^{i\varphi}) = k_j\left(r^{1/(d-1)}e^{i\varphi}\right).$$

Then define

$$K(re^{i\varphi}) = \frac{1}{(d-1)!}(\tilde k_1 * \cdots * \tilde k_{d-1})(e^{i\varphi}), \tag{6.4}$$

where $f*g(e^{i\varphi}) := (2\pi)^{-1}\int_0^{2\pi} f(e^{i(\varphi-\theta)})g(e^{i\theta})d\theta$. In view of (6.3) we have $K(re^{i\varphi}) = \sum_{l\ge 0}\delta_l(r)r^le^{il\varphi}$ with

$$\delta_l(r) = \frac{(l+1)\cdots(l+d-1)}{(d-1)!} = \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \quad \text{if} \quad p \le l \le q.$$

Finally,

$$\begin{aligned} &\int_0^{2\pi} |k_j\left(r^{1/(d-1)}e^{i\varphi}\right)|d\varphi \\ &\le r^{p/(d-1)}\int_0^{2\pi}|\sum_{l=0}^{q-p} r^{l/(d-1)}e^{il\varphi}|^2d\varphi + 2\pi(p+j)\frac{q}{p}r^{p/(d-1)} \end{aligned}$$

$$\leq 2\pi \sum_{l=0}^{q-p} r^{(2l+p)/(d-1)} + 2\pi \frac{p+d-1}{p} q r^{p/(d-1)}.$$

Here we used $\int_0^{2\pi} |u(re^{i\varphi})| d\varphi = 2\pi$ for any r. In view of (6.4) we obtain

$$\begin{aligned} &\int_0^{2\pi} |K(re^{i\varphi})| d\varphi \\ &\leq \frac{1}{(d-1)!} \left(2\pi \sum_{l=0}^{q-p} r^{(2l+p)/(d-1)} + 2\pi \frac{q+d-1}{p} q r^{p/(d-1)} \right)^{d-1} \end{aligned} \tag{6.5}$$

To bound this we remark that, for some constant c_2,

$$r^{p/(d-1)}(1-r^2) \leq \frac{c_2}{p+2d-2} \quad \text{and} \quad \sum_{l=0}^{\infty} r^{(2l+p)/(d-1)}(1-r^2) \leq c_2$$

for all $0 \leq r \leq 1$. (To see the first one, use elementary calculus to find the maximum of the given function of r. For the second one, use the sum of a geometric series.) Hence (6.5) implies

$$\begin{aligned} &\int_0^1 \int_0^{2\pi} (1-r^2)^{d-2} |K(re^{i\varphi})| r d\varphi dr \\ \leq\ & \frac{(c_2)^{d-2}(2\pi)^{d-1}}{(d-1)!} \left(1 + \frac{q+d-1}{p+2d-2} \cdot \frac{q}{p} \right)^{d-2} \\ &\cdot \int_0^1 \left(\sum_{l=0}^{q-p} r^{(2l+p)/(d-1)} + \frac{q+d-1}{p} q r^{p/(d-1)} \right) dr \\ \leq\ & \frac{(c_2)^{d-2}(2\pi)^{d-1}}{(d-1)!} \left(1 + \frac{q+d-1}{p+2d-2} \cdot \frac{q}{p} \right)^{d-2} \\ &\cdot \left(\frac{q-p+1}{p+d-1}(d-1) + (d-1)\frac{q}{p} \right) \\ \leq\ & c\left(\frac{q}{p}\right) \end{aligned}$$

for a suitable function c. $\square$

7. Proof of Theorem 1.1, continued.

We only have to show $(i) \Rightarrow (iii)$. Go back to Theorem 5.9 and to the operators T_n. The definition (5.24) implies that T_n has finite rank. Let C_n be the space of all continuous functions on $K_n := \rho_{m_n} \partial \mathbb{B}_d$ endowed with the norm $\|f\| = M_\infty(f, \rho_{m_n}) v(\rho_{m_n})$. The space C_n is, of course, isometrically isomorphic to $C(K_n)$. Hence we find finite dimensional subspaces $E_n \subset C_n$ such that

$$\begin{aligned} &\sup_n d(E_n, \ell_\infty^{\dim E_n}) := \\ &\sup_n \inf\{\|T\| \cdot \|T^{-1}\| : T : E_n \to \ell_\infty^{\dim E_n} \text{ an isomorphism } \} < \infty \end{aligned}$$

and $E_n \supset T_n Hv|_{K_n}$ for all n. Theorem 5.9 together with (5.25) imply that the map $h \mapsto (T_n h|_{K_n})$ is an isomorphism from $(Hv)_0$ into $(\sum_n \oplus E_n)_{(c_0)} \sim c_0$. On the other hand Cv is a $C(K)-$space where K is the Stone-Czech compactification of Ω.

According to Theorem 5.3, $(Hv)_0^{**} \sim Hv$ is complemented in Cv. This means that Hv and hence also $(Hv)_0$ are $\mathcal{L}_\infty-$spaces. Thus $(Hv)_0$ is isomorphic to a subspace of c_0 and a $\mathcal{L}_\infty-$space. Then, by [8], $(Hv)_0 \sim c_0$. $\square$

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