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Asteroid orbital inversion using statistical methods

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Academic dissertation

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in Auditorium XV on June 15, 2005, at 12 o'clock noon.

ISBN 952-10-2503-4 (paperback)
ISBN 952-10-2504-2 (PDF)
Helsinki 2005
Yliopistopaino

Abstract

Asteroid orbit computation can be taken to comprise of two problems, the orbital inversion and prediction. For any new celestial object, the inverse problem needs to be solved first and the result, typically a set of orbital parameters, or elements, then serves as a starting point for the prediction problem, which may include topics such as dynamical classification, ephemeris prediction, and planetary impact risk estimation. The present work enlarges on the inverse problem for poorly observed asteroids.

The importance of accurate initial orbit computation has been emphasized due to the realization that asteroids and comets can impact the Earth, and constitute a significant risk for the survival of human species. Unlike many other natural disasters, a cosmic impact might be avoidable given a long enough warning time and adequate knowledge of the dynamical and physical properties of the impacting object.

Reliable predictions, whether for ephemeris or impact risk, rely on solid assessment of orbital uncertainties. For the poorly observed objects, such as new discoveries, orbital uncertainties are known to be large, and conventional techniques giving a single orbit solution and possibly some error estimates can be misleading, and fail to describe the real uncertainties.

In the present thesis, a statistical solution to asteroid orbit computation is described. Following statistical inverse theory, the a posteriori probability density function of the orbital elements constitutes the complete solution to the inverse problem. In addition, adopting Bayesian inference can help to constrain complex inverse problems by introducing a priori information to the statistical model. In particular, two nonlinear numerical techniques for solving the orbital-element probability density are presented, and they are shown to successfully unravel orbital ambiguities, such as nonlinear correlations between orbital elements or multiple solutions.

Acknowledgements

I would like to acknowledge the financial support I have received during the thesis work. Funding has been supplied by the following organizations: the Finnish Graduate School in Astronomy and Space Physics, the Foundation of Jenny and Antti Wihuri, the University of Helsinki Science Foundation, and the Academy of Finland, as well as the BPW-Finland's Helsinki club.

I would like to thank my advisor Karri Muinonen who has been a major driving force behind this thesis. I am greatful to the Planetary group colleagues at the Observatory, both former and current members, for their direct and indirect inputs in this work. Especially Mikael Granvik and Johanna Torppa for their help during the final preparation of the thesis (Johanna also for the music!). I also want to thank the pre-examiners Claes-Ingvar Lagerkvist and Heikki Salo, as well as Ted Bowell, for valuable comments on the summary.

The final thanks belong to Panu (for everything), and to my friends and family for being there.

List of papers

I Virtanen, J., Muinonen, K., and Bowell, E. 2001. Statistical ranging of asteroid orbits. *Icarus* **154**, 2, 412–431.

II Muinonen, K., Virtanen, J., and Bowell, E. 2001. Collision probabilities for Earth-crossing asteroids using orbital ranging. *Celest. Mech. and Dyn. Astron.*, **81**(1), 93-101. (In proc. of the US/European Celestial Mechanics Workshop, Poznan, June 2000).

III Virtanen, J., Tancredi, G., Muinonen, K., and Bowell, E. 2003. Orbit computation for transneptunian objects. *Icarus* **161**, 2, 419–430.

IV Bowell, E., Virtanen, J., Muinonen, K., and Boattini, A. 2002. Asteroid orbit computation. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 27–43. The University of Arizona Press, Tuscon.

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VI Virtanen, J., and Muinonen, K. 2005. Time evolution of orbital uncertainties for the impactor candidate 2004 AS_1. *Icarus*, submitted.

Acronyms used in the text

ESA European Space Agency

JPL Jet Propulsion Laboratory

MBO Main-belt Object

MC Monte Carlo

MPC Minor Planet Center

NEO Near-Earth Object

O–C Observed minus Computed (residuals)

p.d.f. probability density function

PHO Potentially Hazardous Object

TNO Transneptunian Object

Contents

1 Introduction

The study of the motions of celestial bodies is the objective of celestial mechanics. The variety of bodies cover in principle every object observed in the universe, from dust particles in interplanetary medium to stellar systems and galaxy groups, or to the currently so topical planetary systems around other stars, i.e., the numerous exoplanets. However, the present thesis focuses on an object group historically most closely connected to celestial mechanics, the objects of our solar system, which have motivated some of the most fundamental theoretical work.

An elementary problem in celestial mechanics is the orbit determination for the small bodies in the solar system. As solar-system objects are identified due to their movement against the apparently stationary background sky, it is possible to measure their motion as positions on the sky at given times. This information can be used to derive the first quantities characterizing a newly discovered object: the parameters describing its trajectory, i.e., the orbital elements. They are traditionally expressed in terms of the six *Keplerian elements* $\mathbf{P} = (a, e, i, \Omega, \omega, M)$, which are, respectively, the semimajor axis, eccentricity, inclination, longitude of ascending node, argument of perihelion, and mean anomaly. The first two describe the size and shape of the orbit, the three elements (i, Ω, ω) give the orientation with respect to the adopted reference system (the Earth's orbital plane, i.e., the ecliptic at epoch J2000.0), and the last one (M) identifies the position of the object in the orbit at a given epoch. In fact, orbit determination is the *inverse problem* of another basic problem of celestial mechanics: the *direct problem* of computing the path, in particular, the sky positions, of the object in the future when the parameters describing its orbit are known. The inverse problem needs naturally be solved first to make the follow-up observations possible and to enable any detailed physical and dynamical studies.

The (asteroid) orbital inversion is one of the oldest inverse problems in astronomy. Historically, astronomy started as positional astronomy, and still the first information obtainable for any new object is typically its sky positions. In theoretical work, one of the earliest advances was the development of the classical theory of celestial mechanics by Newton. Measuring and predicting of the positions of stars and solar system objects remained in practice as the only objectives of astronomical research until the 19th century when the field of astrophysics saw its first light. Chapter 3 of the thesis describes the historical background in the field of celestial mechanics.

Although the methods developed by, e.g., the famous mathematician Gauss have lived for over 200 years, major advances in the field have been made in the recent years. This ongoing progress is in large part inspired by the impact risk from near-Earth asteroids and comets. The realization of the cosmic threat has led to active monitoring of the impact risk, and to the development of numerous novel techniques to improve the accuracy of asteroid orbit computation. In particular, the emphasis has been on initial orbit computation. The magnitude of the problem became evident when the number of asteroid discoveries began to rise in the 1990's due to several surveys dedicated to discover these potentially hazardous objects. The increasing number of new, fast-moving objects has brought about a situation where not all discoveries can be followed up long enough to secure their orbits with an abundance of observations. In the late-90's, the discovery of the first impactor candidates, i.e., asteroids with non-zero probabilities for an Earth-impact within the next century, boosted the need for efficient techniques to compute initial orbits, which could then be used for predicting follow-up ephemerides, or evaluate the risk of an impact. By then, the complicated nature of initial orbit computation had been realized, and earlier methods had been found not to be applicable to such detailed studies of objects with short observational arcs. The ambiguities of the orbital solution for short-arc objects, such as non-physical parameters or the nonuniqueness, i.e., the existence of multiple solutions, called for more rigorous techniques.

In the present thesis, a new class of rigorous methods for initial orbit computation is presented, and they are shown to successfully unravel the ill-posed inverse problem. The standpoint is that of statistical inverse theory, where the aim is to solve the probability density for the desired, unknown parameters, here, the orbital elements. The orbital-element probability density constitutes the complete solution to the inverse problem; there is no need for additional error analysis, which can be a major obstacle in conventional orbit computation.

While the traditional scheme of orbit computation proceeds from a preliminary orbit using a minimum number of observations to orbit improvement with an abundance of data (standard least-squares fit), today a large variety of techniques exists each with their own application area in terms of observational data. The aim of the ongoing research, as is presented here, is likewise to provide a continuum of optimized statistical techniques that rise to the computational challenge produced by impending all-sky monitoring projects, such as the space observatory Gaia (Chapter 6), that result in immense observational databases for solar-system objects, among others.

A few words are in place about the adopted nomenclature in the present study. First, the term orbit *computation* is adopted instead of orbit determination to emphasize the non-deterministic nature of the problem as it is today understood (taken to include error analysis). Here, orbit computation is further taken to encompass both the inverse and the direct problem, that is, both orbital inversion and prediction.

Second, a diversity of terminology referring to small solar-system bodies exists. In particular, the segregation between asteroids and comets has become difficult with the observation that the activity of an object during its lifetime may vary from completely dormant, asteroid-like stage to active cometary appearance. In the present thesis, the word *asteroid* is in several occasions used in a broad sense in place of the more awkward terms such as "small solar-system body", although the generic term *object* is generally adopted for the purpose.

The thesis consists of the following papers:

Paper I Virtanen J., Muinonen, K., and Bowell, E. 2001. Statistical ranging of asteroid orbits. *Icarus* **154**, 2, 412–431.

Paper II Muinonen, K., Virtanen, J., and Bowell, E. 2001. Collision probabilities for Earth-crossing asteroids using orbital ranging. *Celest. Mech. and Dyn. Astron.*, **81**(1), 93-101. (In proc. of the US/European Celestial Mechanics Workshop, Poznan, June 2000).

Paper III Virtanen, J., Tancredi, G., Muinonen, K., and Bowell, E. 2003. Orbit computation for transneptunian objects. *Icarus* **161**, 2, 419–430.

Paper IV Bowell, E., Virtanen, J., Muinonen, K., and Boattini, A. 2002. Asteroid orbit computation. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 27–43. The University of Arizona Press, Tuscon.

Paper V Muinonen, K., Virtanen, J., Granvik, M., and Laakso, T. 2005. Asteroid orbits using phase-space volumes of variation. *MNRAS*, submitted.

Paper VI Virtanen, J., and Muinonen, K. 2005. Time evolution of orbital uncertainties for the impactor candidate 2004 AS_1. *Icarus*, submitted.

Paper I addresses the problem of initial orbit computation, and describes a new nonlinear inverse technique termed statistical orbital ranging (Ranging) which is based on statistical inverse theory. In Paper II, the Ranging technique is applied in a fully nonlinear assessment of asteroid collision probability. Paper III presents a dynamical study of the orbital distribution of the transneptunian population, again making use of Ranging in orbit computation. The recent advances in the field of orbit computation are reviewed in Paper IV. Whereas Ranging is best suitable for short observational arcs and small numbers of observations (exiguous data), a nonlinear statistical technique applicable to moderate observational data is presented in Paper V. Finally, Paper VI returns to the topic of Paper II, and offers a detailed case study of an impactor candidate, asteroid 2004 AS_1.

The thesis is organized as follows. In Chapter 2, the different populations of small solar-system bodies are described, and a variety of related prediction problems are outlined. As a historical background, the early advances in asteroid orbit determination are reviewed in Chapter 3. In Chapter 4, the inverse problem of asteroid orbit computation is discussed; the fundamental equations of celestial mechanics are written out (Sect. 4.2), and the conventional scheme for their solution is described (Sect. 4.3). Statistical orbit computation is reviewed in Section 4.4, and the new numerical techniques presented in the papers of this thesis are summarized. Section 4.5 reviews other recent advances in inverse techniques and discusses their implication to the current work. Summaries of the papers of the thesis are given in Chapter 5, and implications of future spacebased astrometry are outlined in Chapter 6. Finally, conclusions and future prospects of the presented work are offered in Chapter 7.

2 Small bodies in the solar system

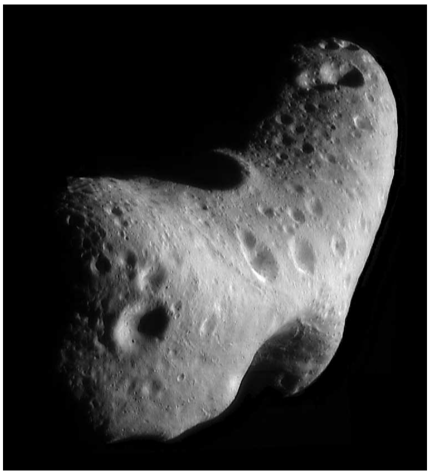

Figure 1: Asteroid Eros as imaged from the NEAR-Shoemaker spacecraft. Eros was the first NEO that was studied intensively at close range, and in February 14, 2001, to complete a scientifically successful mission, the first asteroid (or any atmosphereless planetary body after the Moon) to be landed on (Image courtesy: NASA-JPL).

In our solar system as we know it today, the small planetary bodies have by far outnumbered the nine major planets. Several dynamically different populations of objects are recognized (Fig. 2). Their classifications are mainly based on the orbital parameters, but also on more vague observational differences, such as cometary appearance. Proceeding outwards from the Sun, first, there are the near-Earth objects (NEOs) that have been named after their capability of approaching the Earth on their orbits. The NEO population is further divided into the following three subclasses of asteroids. The Atens have semimajor axes smaller than that of the Earth ($a < a_\oplus = 1\,\mathrm{AU}$), but the orbits do not reside entirely inside Earth's orbit due to their higher eccentricities (aphelion distances $Q = a(1 + e) > q_\oplus = 0.983\,\mathrm{AU}$, where $q_\oplus$ is the perihelion distance of the Earth). As of March 31, 2005, there are two well-observed asteroids, 2003 CP_{20} and 2004 JG_6, that lie permanently closer to the Sun than the Earth, and several other candidates have been discovered.[1] While there are no dynamical reasons why such orbits could not be present in the solar system, their estimated proportion of the entire NEO population is only a few percent (Bottke et al., 2002b). The small number of discoveries is further explained by the difficult groundbased observing geometry. The two other subcategories of NEOs are the Apollos ($a \geq 1.0\,\mathrm{AU}$ and $q \leq Q_\oplus = 1.0167\,\mathrm{AU}$) and the Amors ($Q_\oplus < q \leq 1.3\,\mathrm{AU}$). In addition to asteroids, NEO population is typically considered to contain also those comets that are on Earth-crossing orbits. Finally, a sub-population of NEOs called potentially hazardous objects (PHOs) is distinguished based on their current Earth-approaching orbits. PHOs are generally defined as having absolute magnitude $H < 22\,\mathrm{mag}$ (diameters larger than $\sim 100\,\mathrm{m}$) and orbits that can take them closer than 0.05 AU from the orbit of Earth.[2]

The next population outwards is the asteroid main belt between the orbits of Mars and Jupiter–corresponding to the historical location of the missing planet as predicted

[1] On various occasions such objects have been called Apoheles or Inner-Earth objects (IEOs), although in practice the Aten group could be extended to include these objects, too.

[2] Distance corresponds roughly to the maximum amount of perturbation that could be caused by other solar-system bodies in the object's orbit within the next century.

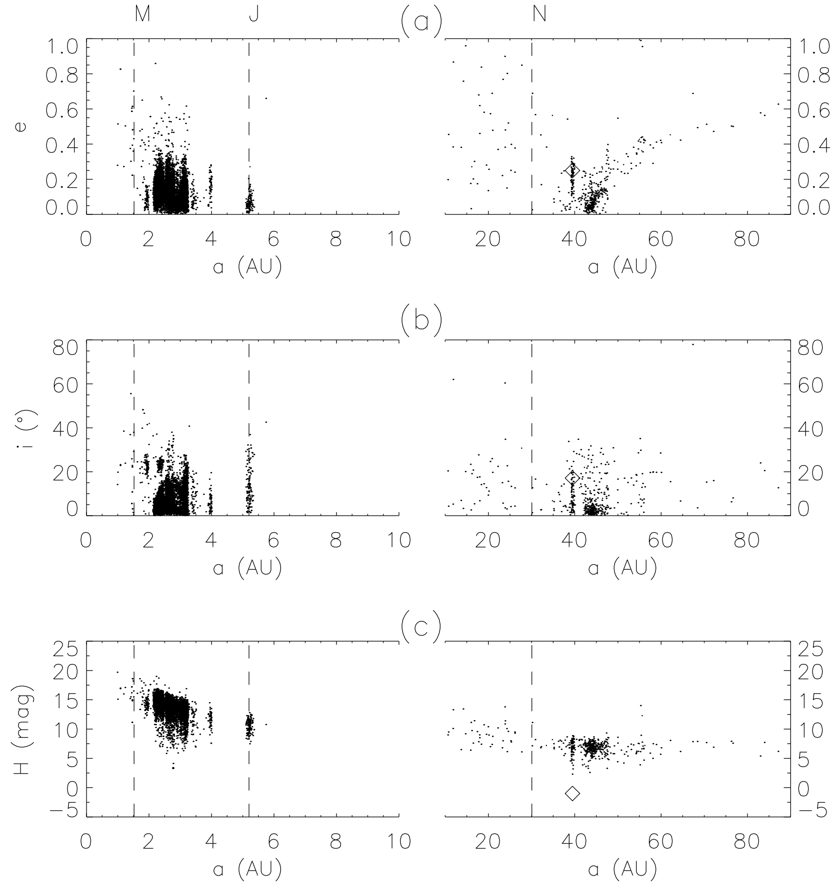

Figure 2: Distribution of the orbital elements (a, e, i) and absolute magnitudes (H) of known asteroids (based on the Lowell database for asteroid orbits, E. Bowell). The location of Mars (M), Jupiter (J), and Neptune (N) are given as a reference (vertical lines), and Pluto is marked with a diamond symbol.

by Bode's law—the semimajor axes of the main-belt objects (MBOs) range from 2.1 AU to 5.3 AU. The region is characterized by numerous resonances caused mostly by the perturbative effect of Jupiter, the most well-known evidence are the Kirkwood gaps at the 2:1 and 3:1 mean-motion resonances. Another feature of the main belt are the asteroid families, dynamically tight groupings of objects which have been traced to be a result from a catastrophic breakup of a large parent body. A separate group of objects is located in Jupiter's vicinity: the Jupiter's Trojans share the same orbit with the planet but they are locked on two specific stable regions around the so-called Lagrangian points, 60 degrees from the position of the planet, the other region leading and the other trailing the planet.

Another large concentration of objects is found in the outer part of the solar system (Fig. 3): the transneptunian objects (TNOs) form a second main belt beyond the orbit of Neptune ($a \sim 30$AU). As in the asteroid main belt, the TNO distribution is far from uniform: the presence of Neptune has a strong influence over the region. In contrast to the asteroid main belt, the mean-motion resonances of Neptune are populated, the best-known example being the planet Pluto in the 3:2 resonance. The TNO region holds

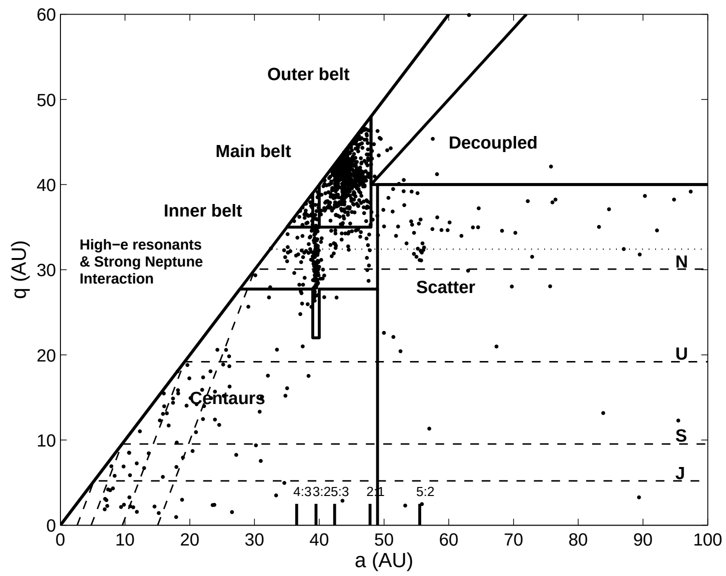

Figure 3: Distribution of known objects in the outer solar system projected to the semimajor axis–perihelion distance $(a - q)$ plane (based on the Lowell database for asteroid orbits, E. Bowell). The location of the outer planets are marked with the horizontal lines. Several Neptune mean-motion resonances are also shown along the abscissa as well as. (Courtesy of G. Tancredi)

several groupings of different dynamical behaviour, the definitions of which are not yet fully established (e.g., Tancredi et al. 2005, in preparation), but the main belt is often described as having a second component, an extension of objects on elongated orbits called scattered disk.

The intermediate space between the main asteroidal populations does not remain unoccupied, although the objects are dynamically more loosely grouped. Centaurs are spread roughly between the orbits of Jupiter and Neptune, and they are typically on outer-planet crossing orbits. These orbits are dynamically unstable, and although the Centaurs are named as asteroids, many of them differ from objects of typical asteroidal material (e.g., the MBOs) by the physical properties (Barucci et al., 2002). In particular, the first Centaur (2060) Chiron has exhibited unusual brightness variations as well as other indications of cometary activity, which has given support to the hypothesis that there is link between the two seemingly different classes of objects, the asteroids and the comets.

Finally, the most dispersed dynamical population in the solar system are the comets. Historically, comets are distinguished from asteroids through their activity observed in terms of coma and/or tails of gas and dust, and they are divided into two groups based on their orbital periods: short-period comets make one revolution in less than 200 years, while long-period (> 200 yr) comets include objects on hyperbolic orbits making only one perihelion passage during their lifetime.

2.1 Observations

The longest known class of small solar-system bodies are the comets, their observations dating back at least to 600 BC in China, possibly even as far as 1500 BC (Sagan and

Druyan 1997), where the oldest records of their appearance have been found. Asteroids made their entrance to our skies millennia later in the beginning of the 19th century; the first asteroid was (1) Ceres discovered by Giuseppe Piazzi in 1801. They then quickly outnumbered the known comets. By the end of the century, there were some four hundred known asteroids, while the number of comets can be estimated as having counted in dozens. In the 20th century, the known asteroid population was in steady growth until, in late-1990's, the numbers practically exploded due to the dedicated NEO surveys; the 10,000th asteroid was numbered in 1999, and two years later the number had doubled after which the increase has again been more steady (Table 1). MBOs were the first to be discovered due to their relatively bright magnitudes compared to the smaller, although closer NEOs.[3] The first recognized NEO[4], and also PHO, was in fact comet Lexell (see Chapter 3), the first asteroid member (433) Eros was discovered in 1898, later classified as the first member of the Amors (Fig. 1). The current number of known NEOs is 3,270 (March 31, 2005).

The distant objects are the newcomers in the known solar system, because only the development of charge-coupled-device (CCD) technology has made it possible to observe these faint targets. Although the existence of the Edgeworth-Kuiper belt was predicted already in the 1940's, the searches did not succeed before 1992. Since then TNO research has become an active field in solar-system studies, and the known population has in a decade grown to nearly 1,000 objects.

The majority of asteroid observations results from few active surveys operated mainly in the United States. The longest-running NEO survey is the Spacewatch program of the University of Arizona's Lunar and Planetary Laboratory while, today, the LINEAR program (Lincoln Laboratory's Near-Earth Asteroid Research) outnumbers the other observing efforts, such as LONEOS (Lowell Observatory NEO Search) or NEAT (NEA Tracking by the Jet Propulsion Laboratory, JPL), by both total and NEO discovery numbers (Stokes et al., 2002). Although all are dedicated to discover NEOs, large numbers of other objects are routinely observed. Only the distant solar-system objects require dedicated search programs due to their slow motions and faint magnitudes; while the first TNOs were discovered in so-called pencil-beam surveys, most of the known objects result from the Deep Ecliptic Survey, which is a broad-areal-coverage survey (Millis et al., 2002).

The increasing archive of astrometric observations is maintained by the Minor Planet Center (MPC)[5]. Although the rate of increase seems already to have reached its maximum (Table 1), the impending deep all-sky surveys, both groundbased and spacebased, will change the trend with their immense databases. Before the end of the decade projects such as the Large Synoptic Survey Telescope (LSST), the Discovery Channel Telescope (DCT, coordinated by Lowell Observatory), Pan-STARRS (coordinated by the University of Hawaii) and the ESA astrometric cornerstone mission Gaia (see Chapter 6) will increase the rate of asteroid detection from 10^2 to 10^4 objects per night.

Large fraction of the discovered objects, currently some 30%, remains as so-called single-apparition objects that have not been observed after their discovery apparition. Due to the relative motion of the observer and the object, solar system objects cannot

[3] Largest MBO is (1) Ceres with a diameter of 930km, while the largest known NEO is $\sim$ 40km across.
[4] Comet Halley is probably the longest *observed* NEO.
[5] http://cfa-www.harvard.edu/iau/mpc.html

Table 1: Discovery statistics of asteroids: numbers of orbits in the MPC database in the beginning of the year. The numbered orbits correspond to well-observed asteroids, while the total number of orbits does not necessarily equal the real number of known objects, since the linkage of short-arc orbits corresponding to the same object (either to another short-arc orbit or to a numbered orbit) may have failed. The growth factor is the ratio of the numbers of orbits for the subsequent years.

Year	Numbered orbits (growth factor)	All orbits (growth factor)
1997	7,000 (1.0)	33,000 (1.0)
1999	10,000 (1.4)	47,000 (1.4)
2001	21,000 (2.1)	110,000 (2.3)
2003	52,000 (2.5)	210,000 (1.9)
2005	96,000 (1.8)	264,000 (1.3)

be observed continuously. The typical time interval between successive apparitions varies for the different populations: for MBOs, it is $\sim$ 15 months, and, for TNOs, only $\sim$ 6 months[6] because the Earth's orbital motion dominates the observing geometry. For NEOs, the observing geometries evolve in a more complicated way, and the time between two apparitions fluctuates from a few months to several years. In addition to the need of efficient ephemeris predictions and optimized observing strategies for NEOs, efficient tools for linking of the discrete sets of observations are highly called for.

2.2 Prediction problems

For any newly discovered object, the orbital elements serve as a starting point for further analysis. Asteroid orbital prediction problems include topics such as ephemeris prediction and uncertainty estimation, identification, collision probability assessment, and optimization of observing strategies.

Ephemeris prediction

Asteroid ephemeris computation is the classical example of the direct problem. Since asteroid observations consist of discontinuous sets, reliable ephemeris predictions are due in order to keep track of the objects discovered.

Asteroids observed over short time arcs after their discovery stand the highest chance of becoming lost. Efficient ephemeris predictions are particularly important for TNOs and hazardous NEOs, which require observing time at large telescopes due to their typically faint magnitudes and, thus, are expensive to recover. In the space era of asteroid research, an extreme case of ephemeris prediction is faced with space missions (also with artificial satellites), where the accuracy of the predictions for both the target's and the spacecraft's position is of uttermost importance.

[6] In the long-term, large eccentricities complicate TNO observations, since the most favorable observing circumstances correspond to perihelion passages, where most of the discoveries are also made.

While predicting an object's position at a given time is mathematically straightforward when the orbital parameters are known, the planning of follow-up observations is by no means a simple task. This stems from several factors. First, the orbital uncertainties have to be considered. In the linear approximation (cf. Sect. 4.3.2), the projection of orbital uncertainties to the sky-plane results in a relatively small and linear distribution, and can be given in terms of a confidence ellipsoid, or even more simply, as a one-dimensional line of variation.[7] But for objects observed over short time arcs, the orbital uncertainties are often large, and their projection to ephemeris uncertainties, the observation function, is highly nonlinear. NEOs and TNOs represent two extreme cases of this behaviour. While newly discovered TNOs have widely spread orbital-element p.d.f.'s and compact regularly behaving sky-plane p.d.f.'s, NEO discoveries are characterized by compact orbital-element p.d.f.'s and widely spread sky-plane p.d.f.'s (cf. Papers I and III). Second, uncertainties in the position evolve with time, typically growing roughly linearly as a function of time elapsed from the last observation; the longer an object remains unobserved, the more difficult its recovery becomes.

Several online services have been established that offer assistance in planning asteroid observations. These include the Minor Planet Center Web interface to the observation archives and Lowell Observatory's asteroid services that allow, e.g., the building of asteroid observability plots and finding charts. Ephemeris computation systems are also provided by the research groups on asteroid dynamics at the JPL and University of Pisa. However, a defect of some of the existing services is that they make use of linear approximation when giving uncertainty estimates also for single-apparition objects. TNOEPH service at the Lowell Observatory site (Granvik et al., 2003; see also Paper III) relies on the nonlinear technique presented in Sect. 4.4 and provides thus nonlinear ephemeris uncertainties. A similar service for NEOs is ready for implementation by the same authors.

Collision probability

Since the realization that NEOs constitute a significant risk for the prosper and survival of human species, a major effort has been coordinated world-wide to discover and follow up these potentially hazardous objects as well as to predict the timing and likelihood of impacts.

The collision probability estimation relies on the assessment of orbital uncertainty of the given object. Since the uncertainty evolves rapidly after an object has been discovered, also the asteroid collision probability evolves when the observational arc and the number of observations grow. In fact, for most of the discovered impactor candidates the probabilities for an Earth impact have been shown to vanish after days or weeks of observations from the discovery. The desire to be able to deduce the risk of an impact as early as possible after discovery calls for techniques for initial orbit computation.

Automatic monitoring systems with Web interfaces have been put out, such as the Near Earth Object Dynamic Site (NEODys) at the University of Pisa and the Sentry system at the JPL. They probe for possible impact intervals for typically the next 100 years for all new objects in a routine-like manner. While the current online services are

[7]Line of variation is aligned with the principle axis of the ellipsoid, corresponding to the principal eigenvector of the projected orbital-element covariance matrix.

restricted by the linear approximation, the rigorous assessment of collision probability for short-arc objects is put forward in Papers II and VI.

3 Fundamentals of orbits: From Kepler to Gauss

Nature and Nature's laws
lay hid in night
God said, Let Newton be!
And all was light
– Alexander Pope (1688-1744)

The field of orbit computation saw its first light during the Copernican revolution which begun with Nicholaus Copernicus and his heliocentric model for the solar system. The new model called for a new explanation for the motion of the planets to replace the Ptolemaic epicycle theory. Johann Kepler (1571-1630) came up with the laws describing the planetary motion in 1601-1619 trying to find a theory that would fit to the accurate observations of planet Mars made by Tycho Brahe. Kepler's laws are:

I The orbit of each planet is an ellipse, with the Sun at one of its foci.

II The line joining the planet to the Sun sweeps out equal areas in equal intervals of time.

III The squares of the periods of any two planets are in the same proportion as the cubes of their mean distances from the Sun.

Tycho Brahe was probably the first in the history of astronomy to attempt to find an orbit for a celestial body other than a planet, namely for the comet of the year 1577. While Kepler's explanation of the planetary motion was in a way the first solution to the inverse problem of orbit determination, the theoretical work matured with cometary orbits until the discovery of the first asteroids 200 years after the times of Kepler and Tycho Brahe. However, Tycho Brahe did not succeed in his task since he had not totally accepted the Copernican model of the solar system, and could not detach himself from the theory of epicycles. Although Kepler described the motion of the planets around the Sun, neither he nor anyone at the time could make the obvious connection to the orbits of comets. Kepler himself assumed that the trajectories of comets were straight lines which is in agreement with his thought that they did not return periodically. Only shortly before Newton was it supposed that the focus of cometary orbits could be the Sun.

Some 50 years after Kepler, sir Isaac Newton (1642-1727) gave the explanation and mathematical form for the *Keplerian* motion in his famous *Principia*. However, history was once again not made overnight. Edmond Halley and Robert Hooke have the credit for bringing Newton's achievements known to the world. Among others, Hooke and Halley had been puzzled by the explanation of the motion of the planets around the Sun, and had considered a law similar to magnetism, where the force is proportional to the inverse of the square of distance. Hooke was offered 40 shillings if he could produce the proof but, within the two weeks that were given to him, nothing more was heard. In 1685, Halley found out that Newton had already solved the problem as a young student twenty years ago. Halley was visiting Newton at Cambridge when he casually posed the question : 'If the Sun pulled the planets with a force inversely proportional to the square of their distances, in what paths ought they to go?' 'Why, in ellipses, of course. I have already calculated it and have the proof among my papers somewhere. Give me a few days and

I shall find it for you.' was Newton's reply to Halley's complete astonishment. The visit from Halley together with the arguments between Newton and Hooke about the discovery of the form of the force finally got Newton to publish his results in 1687. (Bate et al. 1971, Karttunen 1996).

In *Principia*, Newton introduced his three laws of motion:

I Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

II The rate of change of momentum is proportional to the force impressed and is in the same direction as that force.

III To every action there is always opposed an equal reaction.

Newton's explanation for the Keplerian motion was given in terms of the law of universal gravitation, which is still the fundamental law of celestial mechanics. It expresses how particles act when impressed by forces:

$$F = k^2 \frac{m_1 m_2}{r^2}. \tag{1}$$

Newton did not really give proof to the original question of Halley–he had instead solved the inverse problem and proved that force impressed on a body moving on a conic section is of this form. Nevertheless, his work had importance beyond the application to the motion of celestial bodies; in *Principia*, Newton laid the foundations for the whole field of modern mechanics.

Halley (1656-1742) was the first to successfully apply Newton's theories to orbit determination in 1705. He computed parabolic orbits for a bunch of comets and discovered that they belonged to one single object (now known as comet Halley) that was on a periodic orbit around the Sun. He also predicted that the comet would reappear in the year 1758 which it did although the accurate calculations were performed by a French mathematician and astronomer Alexis Clairaut. He computed the perihelion time for comet Halley which was missed by only one month. Newton's orbit computation method from three observations was a graphical one and applicable only to parabolic orbits. The first purely analytical method was proposed by Euler in 1744. Lambert generalized Euler's method for the other conic sections, i.e., for elliptic and hyperbolic orbits in his works of 1761 to 1771. The contribution of the mathematician Lagrange to this development was the introduction of 'mathematical elegance' (Bate et al. 1971).

A new chapter in the history of orbit determination began with two mathematicians, Laplace and Gauss. Before, beginning from Newton, the aim of the methods had been to find a set of elements that would satisfy as precisely as possible the three observations chosen to the analysis. Pierre de Laplace, a French mathematician (1749-1827), was the first in 1780 to introduce an orbit determination method, where the number of observations included was (in principle) not limited. This was accomplished by introducing truncated power series expansions of position vectors into the computation and solving the differential equations from their numerical values. Laplace's method (see Danby 1992 for a modern representation) results in orbital elements that fit accurately to only one chosen observation (in the case of three observations, usually the middle one), the rest of the data entering the computation through series of approximations. Although, analytically, the method is very complete, it never proved itself very useful in practice. In his main work *Traité de mécanique céleste* (ca. 1800), he constantly turns to the expression '*Il est aisé à voir*', '*it is easily seen*', which can still be found in mathematical textbooks in place of complicated proofs.

German mathematician Johann Carl Friedrich Gauss (1777-1855) developed the first practical method for computing an orbit from three observations in 1801 (published in 1809). What led Gauss to derive his theory, was the discovery and loss of the first minor planet, asteroid (1) Ceres, observed by the Italian astronomer Giuseppe Piazzi earlier that year. Gauss completed the work of Lagrange who had proposed the use of a constraining equation based on the fact that three heliocentric position vectors are necessarily coplanar if they represent the motion of the object on a Keplerian orbit. The method is based upon an analytical solution of the differential equations and solves the numerical values of the constants of integration. This method overcomes the one of Laplace by its accuracy and applicability; Gauss's method continues to be the standard method for initial orbit determination even today. It also led to the recovery of the asteroid Ceres in 1802 by Olbers who used the ephemeris computed by Gauss. Gauss also introduced the so-called least-squares method to deal with large numbers of observations which is a standard tool for anyone working with inverse problems or data analysis in general (see Sect. 4.3.2).

So far, all the orbital motion has taken place in a system consisting of only two bodies, the central body (usually the Sun) and the orbiting body (planet or asteroid/comet). However, already Newton realized that this was only a useful approximation for the preliminary orbit and, for improvement, the perturbations of other major bodies in the solar system had to be considered. He was able to explain most of the variations in the orbit of the Moon caused by the gravitational effects of the Sun: the twisting of the orbital plane and the motion of perigee. Also Clairaut, in his calculations for the orbit of comet Halley in 1758, had included Jupiter and Saturn as perturbing bodies. He was also one of the numerous mathematicians who tried to find an analytical solution for the problem of three bodies.

It is worth mentioning the work of an mathematician and astronomer of Finnish origin, Anders Lexell (1740-1784), a co-worker of Euler at the St. Petersburg Academy of Sciences. He demonstrated the importance of the perturbing effect of Jupiter on bound cometary orbits in pursuance of calculating the orbit for an object that was in fact the first near-Earth object.[8] First, he deduced that the unusual elliptic orbit was a result of perturbations by Jupiter before the discovery. Second, he predicted that the comet's second close approach to Jupiter in 1779 would flung the comet out of the inner solar system, which received proof from the fact that it made no reappearance in 1782, nor later, as it otherwise should have.

The next success with perturbation theory came with the prediction of the presence of a new planet, Neptune, based on the perturbed motion of planet Uranus. Lexell was also the first to compute the orbit for Uranus showing it was not a comet as had been first thought. Although he did not predict the position of Neptune, his calculations showed that Uranus was being perturbed, and he deduced that this was due to another more distant

[8]The comet was discovered by Messier in 1770 during its close-approach to the Earth but later named after Lexell. The encounter to only six lunar distances still remains as the closest cometary approach.

planet. Later, the independent analytical derivations of Adams and LeVerrier in 1845-46 resulted in the discovery of the planet immediately after the search was started. LeVerrier further studied the orbit of Mercury and was able to explain most of the variation in the motion of perihelion with Newtonian mechanics; the rest was left unresolved until the theory of general relativity by Albert Einstein in 1915.

4 Orbital inverse problem

In this Chapter, the inverse problem of asteroid orbit computation from astronomical observations is described. The current work is based on the statistical inverse theory. Thus first, the basic concepts of inverse theory in general and the philosophy of statistical inversion are outlined. Next, the fundamental equations for orbit computation, the equations of motions, are written out for two dynamical models, the two-body approximation and the full many-body model. After presenting the general solution for the equations, the conventional scheme of orbit determination is described. Then, statistical orbit computation is formulated, and the solution is given in terms of two new nonlinear numerical techniques which are described in detail in Papers I and V. Finally, a variety of other recently developed inverse techniques are reviewed.

4.1 About inverse theory

The inverse theory tries to obtain useful information about physical phenomena by solving for parameters that describe some desired properties which are not necessarily directly measurable themselves but can be related to some measurable quantities (e.g., Menke 1989). What is needed is a physical (or mathematical) model that relates the model parameters to the outcome of the measurement, the observations. The model can usually be expressed in terms of some function f, the value of which gives the theoretical, computed result of the measurement $\mathbf{m}$ when the parameters $\mathbf{x}$ of the function are assumed known: $\mathbf{m} = f(\mathbf{x})$. This corresponds to the direct problem of computing the observable quantity when the parameters of the model are known, the solution of the inverse problem is then obtained by inverting this equation for $\mathbf{x}$. However, the physical world does not follow this simple equation since real measurements of nature are corrupted by noise, both random and systematic measurement errors. The general equations for the problem are thus the following *observation equations*:

$$\mathbf{m} = f(\mathbf{x}) + \epsilon, \tag{2}$$

where ϵ is a vector containing the measurement errors.

The goal of statistical inversion is to solve the probability density function (p.d.f.) of the desired, non-measurable parameters given the (indirect) observational data. The fundamental idea is that our inference of natural phenomena is taken to be probabilistic in nature. Since all measurements contain errors, our data points are just one possible outcome of the measurement, and they can mathematically be considered as realizations of random variables. This in turn requires that the physical quantities that are solved are also modeled as random variables and assumed to follow certain distributions.

Now, in Eq. (2) the vector containing the measurements, $\mathbf{m}$, should be interpreted as realizations of the corresponding random variable $\tilde{m}$, and $\mathbf{x}$ and ϵ as random variables with distributions $p(\mathbf{x})$ and $p(\epsilon)$. The solution for the inversion can be written in terms of the *a posteriori* probability density of the parameters, $p_{\mathrm{p}}(\mathbf{x})$, which is the conditional probability $p(\mathbf{x} \mid \mathbf{m})$. In Bayesian formulation (e.g., Lehtinen 1988, Vallinkoski 1988, Vallinkoski and Lehtinen 1990ab), conditional probabilities can be written as follows:

$$p_{\mathrm{P}}(\mathbf{x}) = p(\mathbf{x} \mid \mathbf{m}) = \frac{p(\mathbf{x}, \mathbf{m})}{\int p(\mathbf{x}, \mathbf{m})\, d\mathbf{x}} = C\, p_{\mathrm{pr}}(\mathbf{x})\, p(\mathbf{m} \mid \mathbf{x}), \tag{3}$$

where $p(\mathbf{x}, \mathbf{m})$ is the joint probability density of $\mathbf{x}$ and $\mathbf{m}$, p_{pr} is the *a priori* p.d.f. for the parameters, $p(\mathbf{m} \mid \mathbf{x})$ is the *likelihood function* of the measurements, i.e., the probability to obtain the measured values given the particular parameters, and $C = \int p_{\mathrm{pr}}(\boldsymbol{x})\, p(\boldsymbol{m} \mid \boldsymbol{x})\, d\boldsymbol{x}$. In practice, $p(\mathbf{m} \mid \mathbf{x})$ is given in terms of the error p.d.f., $p(\epsilon) = p(\mathbf{m} - f(\mathbf{x}))$.

The statistical model should thus contain the following ingredients: a physical model interrelating the observations and the parameters, information of the noise statistics connected to observational errors, modeling errors etc., and possibly some prior information of the parameters to be determined (from previous measurements, physical constraints, common-sense reasoning etc.).

In practice, solving an arbitrary a posteriori probability density can turn out difficult. Localized p.d.f.'s corresponding to small parameter uncertainties can often be described using mathematically well-defined distributions such as the Gaussian distribution. For complicated p.d.f.'s, analytical solutions are rarely found–in fact, the whole problem can be ill-posed–but Monte Carlo (MC) simulations can be used to map the discretized p.d.f. as described in Sect. 4.4.

4.2 Equations of motion in celestial mechanics

The basic problem of celestial mechanics is to compute the trajectory of a body with mass m_i as a function of time, $\mathbf{r}_i(t)$. In a system of $N+1$ bodies, each body is subject to reciprocal gravitational attractions according to Newtonian mechanics (Eq. 1), which leads to the following equations of motion for the system in Cartesian coordinates,

$$\frac{d^2\mathbf{r}_i}{dt^2} = \gamma \sum_{j=0, j\neq i}^{N} m_j \frac{\mathbf{r}_{ij}}{r_{ij}^3} \quad (i = 0, N), \tag{4}$$

where γ is the universal constant of gravity, $\mathbf{r}_i$ are the Cartesian position vectors of the $N+1$ bodies and $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ their relative position vectors. Some assumptions have already been made: (1) the rigid bodies are spherically symmetric, with their masses concentrated at the center, (2) no external forces are impressed upon them, and (3) there exists an inertial reference frame which is absolutely at rest and relative to which all motion takes place. In the case of the solar system, the Sun (m_0) can be adopted as the origin of the coordinate system whereupon the relative positions now refer to $\mathbf{r}_i \leftarrow \mathbf{r}_i - \mathbf{r}_0$, and the equations (4) become

$$\frac{d^2\mathbf{r}_i}{dt^2} = -\gamma(m_0 + m_i)\frac{\mathbf{r}_i}{r_i^3} + \gamma \sum_{j=1, j\neq i}^{N} m_j \left(\frac{\mathbf{r}_{ij}}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_j^3}\right) \quad (i = 1, N). \tag{5}$$

The equations of motion now consist of two terms: the two-body motion of the central body and m_i, and the perturbative term which represents the effect of the gravity of the other bodies on the Sun (an apparent force that causes perturbations from Keplerian motion). In several applications, we can neglect the second term but, e.g., in the case of close encounters between small bodies and planets (with asteroids usually the Earth should be considered, with comets Jupiter), the gravitational effects of the planets become significant.

Perturbations arise also from sources other than gravitational interactions. If the restrictions we had to make in order to come up with the above equations of motion are

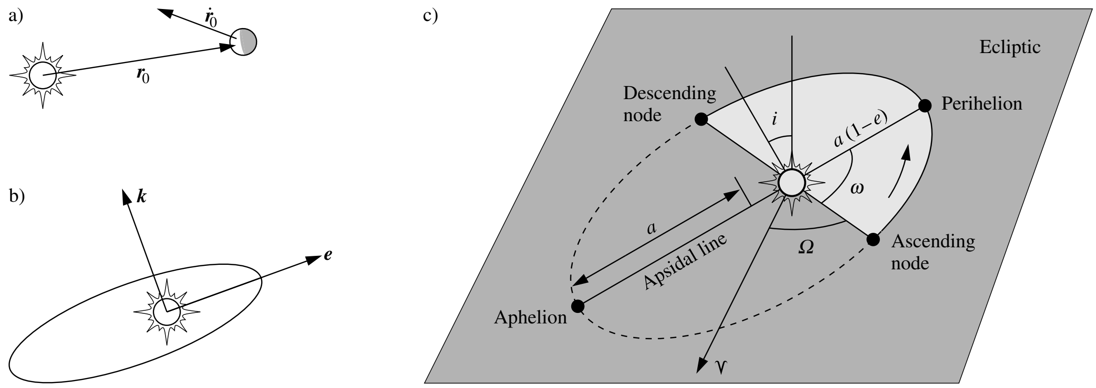

Figure 4: Different choices of parameters describing the orbit: a) Cartesian position and velocity at some epoch t_0, b) angular momentum $\boldsymbol{k}$ and direction of perihelion $\boldsymbol{e}$, c) Keplerian orbital elements (see text). From Fundamental astronomy (Karttunen et al., 2003).

not fulfilled, also the equations in (4) are not valid. For example, the nonspherical shapes of the Earth and the Moon have to be considered when computing the orbit of the Moon or artificial Earth satellites (failure of assumption (1)). Also, nongravitational forces may be present (failure of assumption (2)): for comets, the external forces result from the momentum on the nucleus by dust and gas escaping due to sublimation; for asteroids, a radiation force caused by the diurnal and seasonal heating of a rotating object–termed Yarkovsky effect–can lead to large secular effects in the orbit (e.g., Bottke et al. 2002a).

The relativistic effects have been shown to be important in explaining the motion of small solar-system bodies. Sitarski (1983) suggested a simple modification to the Newtonian equations of motion in Eq. (5) by including a relativistic term due to the Sun (one-body Schwarzschild problem). The relativistic equations of motion have also been implemented for the work presented here (from Paper V onwards).

4.2.1 Two-body problem

In the particular case of $N = 1$, Eqs. (5) describe the Keplerian two-body system and take the form

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu \frac{\mathbf{r}}{r^3}, \tag{6}$$

where $\mu = \gamma(m_0 + m_1)$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$. This is equal to three second-order differential equations that require six constants of integration for their complete solution. Finding and interpreting these constants constitutes the kernel of the two-body orbit computation problem.

Choice of coordinates. There are several ways to choose the six constants but some parameterizations have attained a more permanent status. For numerical purposes, the solution of the equations can be given in terms of the Cartesian position and velocity, $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, of m_1 at some epoch t_0 (Fig. 4a). Another set of parameters is ($\mathbf{k}$, $\mathbf{e}$, τ) (Fig. 4b), where $\mathbf{k}$ is the angular momentum vector, $\mathbf{e}$ is a vector along the major axis of the orbit pointing toward the closest approach between the two bodies (perihelion if m_0 is the Sun; $|\boldsymbol{e}| = e$, the orbital eccentricity), and τ is the moment of time for the approach. For illustrative purposes, it is common to use the six Keplerian orbital elements (a, e, i, Ω, ω, M_0) (Fig. 4c) describing a bound elliptic orbit and its orientation in space and with M playing the role of time (with small adjustments applicable also to other conic sections, i.e., cometary-type unbound orbits). The problem with the definition of Keplerian elements is faced in practical computations; some of the angular elements are not well defined when the orbit is nearly circular or the orbital plane is close to the ecliptic, i.e., when $e, i \to 0$. This can be avoided by introducing non-singular parameters with the following replacements: $\omega \to \widetilde{\omega} = \Omega + \omega$, $M \to L = M + \widetilde{\omega}$. The equinoctial elements provide still another parameter set that avoids the singularities by defining parameters which are functions of the Keplerian angles.

It is entirely application-dependent which set of parameters one should choose to use. The Cartesian elements are attractive for physical approaches, such as numerical integration, and we have adopted them as basic parameters for computational purposes and use Keplerian elements for illustration. Also, for orbital inversion, the question of which are the optimum parameters to be utilized has turned out to be case-sensitive. Milani et al. (2005a) discuss the significance of the choice of coordinates when evaluating orbital uncertainties.

Classes of inverse problems: constraints from observational data. The six boundary conditions, or initial values, needed for solving the six integration constants are obtained from the observations. The common astrometric observations consist of two topocentric angular coordinates, right ascension and declination (R.A. and Dec.; (α, δ)), at given observing dates, suggesting the minimum requirement of three such observations. This corresponds to an evenly determined inverse problem where the number of parameters, N_{par}, is equal to the number of observations, N_{obs}. The problem has only one solution that fulfils the exact observation equations in Eq. (2) where the vector containing the errors is omitted. However, Bayesian a priori information can be used to impose additional constraints to the problem when the original observational data is inadequate (see Sect. 4.4.2). Such an underdetermined case is the orbit computation from two astrometric observations, a novel solution to which is given in the present thesis. Other techniques for solving both even- and underdetermined inverse problem are discussed in Sects. 4.3.1 and 4.5.

While the inversion has historically been based on the angular astrometric data, other types of observations may also be available and equally well used in inversion. Such data as radar measurements or independent measurement of the angular coordinates and their change rates (see Chapter 6) contain more information than the traditional astrometry, and thus the number of data points needed for the inversion is reduced. In fact, a single radar observation obtained with a tracking radar would be adequate since it gives us all six boundary values: three from the position vector and three from the velocity vector (e.g., Bate et al., 1971). Unfortunately, radar observations (usually only time-delay or Doppler shift measurements) are rarely available for short-arc asteroids; due to their high accuracy they would efficiently improve the accuracy of initial orbit computation (Yeomans et al. 1987; 1992).

In the overdetermined inversion, the number of data points exceeds the number of parameters to be estimated ($N_{obs} > N_{par}$) (Sect. 4.3.2). The problem no longer has an exact solution (unless both the model is exact and the data flawless), a typical approach is to go for the next best solution: to find the orbital parameters that best describe the data. The standard least-squares technique finds a unique solution by minimizing the O–C (observed – computed) residuals. The nonlinear model between the data and the orbital parameters requires a nonlinear technique, in orbit computation the nonlinear Eqs. (2) are linearized with an iterative procedure (for a general scheme, see Menke, 1989).

4.2.2 Many-body problem

While in the two-body approximation only the mean anomaly changes with time, in the real solar system, all the elements must be treated as functions of time. At a certain epoch, the *osculating elements* describe the two-body orbit on which the body would move if all perturbations were to cease instantaneously. The two-body orbit serves as a preliminary solution but in practice it is often impossible to exclude the effects of planetary, or even asteroidal, perturbations. In particular, in the improvement of the preliminary orbit towards a definitive orbit, the full many-body approach has to be adopted. Also in the initial orbit computation for near-Earth objects, perturbations should optionally be included since the observations are often obtained at the time of a close-encounter with the Earth.

There are two alternative approaches which can be used to solve the many-body equations of motion in Eqs. (5); the so-called special perturbations which are purely numerical integration methods, and analytical methods under the theory of (absolute) perturbations. The absolute perturbations are interesting because they give better understanding of the source of the perturbations and give an insight into what is happening to the orbit in a geometrical sense. However, they lead to the rather difficult and lengthy computation of series expansions. Thus, the absolute perturbations are not useful in numerical applications, and special perturbations are adopted. Two classes of special perturbation techniques are usually considered; one is concerned with the variation of orbital elements and results in the osculating elements, i.e., the elements as a function of time, the other works in terms of the coordinates and determines the perturbed coordinates themselves. The latter technique is useful in many applications, where the actual orbital elements are not needed (explicitly).

In applications involving large-scale integrations, e.g., over long time scales, the choice of the numerical integrator should be considered in terms of its accuracy and efficiency. One of the numerical integrators used in the current work is the Bulirsch-Stoer extrapolation method (Press et al., 1994), which is well-suited for orbit computation due to its robustness and accuracy.

4.3 Conventional orbit determination

4.3.1 Preliminary orbits

In initial orbit computation, one aims at finding a preliminary orbit from a minimum number of observations. The inverse problem is then typically an even-determined one ($N_{\rm obs} = N_{\rm par}$) or an underdetermined one ($N_{\rm obs} < N_{\rm par}$).

Gauss's method from 1801 falls to the first class and is still attractive, in fact, there exist today a family of methods originating from the ideas of Gauss. Marsden (1985, 1991) outlines two three-observation methods, GEM (Gauss-Encke-Merton) and MVC (Moulton-Väisälä-Cunningham), names referring to the various contributors (see also Danby, 1992). For cases where three observations are not available or the observational arc is very short, two-observation methods have been developed. As described in Sect. 4.2, the problem is now underdetermined, thus orbit computation from two observations must be carried out by introducing constraining assumptions about the motion of the body, often not verifiable until additional observations are obtained. In some cases, it is possible to assume a circular orbit (e.g., Dubyago 1961), which reduces the number of elements to be determined to four ($e = 0$ and ω is not defined). The methods of Väisälä (1939) and Orlov (1939) assume that the object tends to be near perihelion at discovery and fix one parameter a priori (perihelion distance or eccentricity, respectively). While a valid assumption for earlier asteroid observations (Marsden, 1991), the current deep surveys certainly result in a large fraction of their discoveries made far from perihelion. Nevertheless, Väisälä orbits are widely used for discriminating between newly discovered NEOs and MBOs with observations from one or two nights as well as for their ephemeris predictions, for example at the MPC.

Bowell et al. (1990) have studied orbit determination from asteroid motion vectors. They made use of the correlation of opposition motion and semimajor axis and inclination to restrict the position uncertainty for follow up. Kristensen (1995) has described an orbit determination method that uses four observations from two oppositions. The method can be used to link two groups of short-arc observations for orbital elements, even though orbits cannot be derived from the separate sets.

4.3.2 Linear approximation

When an abundance of data is available, we can search for the best estimate orbit through a weighted nonlinear least-squares analysis (see, e.g., Press et al. 1994). Assuming that N_{obs} pairs of right ascensions and declinations $\psi_i = (\alpha_i, \delta_i)$ have been observed at certain times t_i, let the theoretical, computed sky-plane positions be described by the vector $\Psi(\mathbf{P})$ for the osculating elements $\mathbf{P}$ at a given epoch t_0. The observation equations in Eq. (2) are rewritten for orbit computation,

$$\psi = \Psi(\mathbf{P}) + \varepsilon. \tag{7}$$

We minimize the square of O-C (observed–computed) residuals, i.e. the difference between the prediction of the model and observations $\varepsilon = \psi - \Psi(\mathbf{P})$,

$$\phi = \varepsilon^T \, \mathbf{W} \, \varepsilon, \tag{8}$$

where ϕ is the function to be minimized, and $\mathbf{W} = \Lambda^{-1}$ is the weight matrix for the observations which is equal to the inverse of the covariance matrix of the noise. Thus, the observational errors are assumed to follow a Gaussian distribution. If the data is uncorrelated, the diagonal elements of $\mathbf{W}$ are the weights of individual observations, σ_i^{-2}. The condition for the existence of an extremum point, which in this case should be the minimum, is $\frac{\partial \phi}{\partial \mathbf{P}} = 0$.

Since the observation equations of orbit determination (Eq. (7)) describe a nonlinear relationship between the observations and parameters, we have to linearize them and use successive approximations. The method is now composed of the following steps (e.g., Muinonen and Bowell, 1993):

1. obtaining a good initial estimate for the parameters $\boldsymbol{P}_{\mathrm{ls}} = \boldsymbol{P}_0$
2. linearizing the observation equations in the vicinity of the $\boldsymbol{P}_{\mathrm{ls}}$ with a Taylor expansion

$$\begin{aligned} \alpha(\mathbf{P}, t) &= \alpha(\mathbf{P}_{\mathrm{ls}}, t) + \sum_{j=1}^{6} \Delta p_j \frac{\partial \alpha}{\partial P_j}(\mathbf{P}_{\mathrm{ls}}, t) \\ \delta(\mathbf{P}, t) &= \delta(\mathbf{P}_{\mathrm{ls}}, t) + \sum_{j=1}^{6} \Delta p_j \frac{\partial \delta}{\partial P_j}(\mathbf{P}_{\mathrm{ls}}, t), \end{aligned} \tag{9}$$

   where $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{ls}} + \Delta \boldsymbol{p}$. The partial derivatives of right ascension and declination with respect to the six orbital elements in two-body approximation are derived with the help of partial derivatives of heliocentric coordinates.

3. solving for a new estimate for the parameters from

$$\left\{ \begin{array}{c} \mathbf{P}_{\mathrm{ls}} \leftarrow \mathbf{P}_{\mathrm{ls}} + \Sigma\, \Phi^T\, \Lambda^{-1}\, (\psi - \Psi(\mathbf{P}_{\mathrm{ls}})) \\ \Sigma^{-1} = \Phi^T\, \Lambda^{-1}\, \Phi \end{array} \right. \tag{10}$$

$$\left\{ \begin{array}{r} \Phi_{2k-1,j} = \cos\delta_k \frac{\partial \alpha}{\partial P_j}(\mathbf{P}_{\mathrm{ls}}, t_k), \\ \Phi_{2k,j} = \frac{\partial \delta}{\partial P_j}(\mathbf{P}_{\mathrm{ls}}, t_k), \end{array} \right. \quad j = 1, \ldots, 6;\ k = 1, \ldots, N$$

   where Σ is the covariance matrix of the orbital elements.

4. iterating the procedure until convergence is reached, i.e., until the estimated correction $\Delta \boldsymbol{p}$ becomes sufficiently small.

The widely used pseudo-Newtonian iteration is called differential correction. Together with the covariance matrix, the least-squares orbit gives the solution for the linearized inverse problem. Least-squares is the standard procedure for orbit improvement, where one starts with an initial estimate obtained from, e.g., the Gauss' method and stepwise corrects it: First, approximate correction is carried out in two-body approximation and with equal observational weights. Second, many-body approach in weighted least-squares is adopted. However, since the solution is based on an approximation, the linearization of observation equations, the convergence of the iteration and the existence of a unique solution is by no means guaranteed (e.g., Menke, 1989). In particular, for poorly observed objects convergence is uncertain, and the validity of the approximation itself is a subject to a critical study (Papers II and V).

The weighted least-squares analysis actually accommodates a probabilistic treatment of the inverse problem, since, to solve Eqs. (10), some *a priori* information on the noise variances is needed. In fact, if the observational noise can be assumed to be Gaussian as above, the solution can be expressed in terms of a Gaussian probability density for the orbital elements. The covariance matrix defines a six-dimensional uncertainty region, a hyperellipsoid, centered on the least-squares orbit: $\Delta\chi^2 = (\mathbf{P} - \mathbf{P}_{\mathrm{ls}})^T\, \Sigma^{-1}\, (\mathbf{P} - \mathbf{P}_{\mathrm{ls}})$.

The orbital uncertainty analysis in the linear approximation has been a subject of extensive research. Some of the pioneering work was carried out by Cappellari et al. (1976) in their least-squares analysis of spacecraft trajectories, while Brower and Clemence (1961) already gave an introduction to orbital error analysis. Herget (1965) described a method that was a compromise between methods that make use of few observations, which are satisfied precisely, and fitting procedures that aim to find the best estimate orbit that minimizes the observational residuals of a set of observations. He varied the topocentric ranges of two observations to find the values that minimize the residuals in the least-squares sense.

4.4 Statistical inverse problem

The statistical interpretation of orbit computation was already put forward by Gauss when he presented the least-squares method in the beginning of the 19th century. The space-exploration era starting from 1960's strengthened the probabilistic treatment of orbits in astrodynamic applications such as in the analysis of spacecraft trajectories. But although most of the work presented in previous sections can be taken to be probabilistic in nature, only in Muinonen and Bowell (1993) was the asteroid orbit computation, i.e., orbits computed from ground-based observations, first given a fully statistical treatment using the concepts of statistical inversion.

The statistical inverse theory for asteroid orbit computation has been described in detail in Paper I of this thesis. An important addition to this formulation was put forward in Paper II and expanded in Paper VI, where the statistical treatment was completed by introducing a regularization that maintains the invariance of the statistical analysis in parameter transformation. The main equations are summarized first, and then, two new numerical inverse techniques based on MC simulations are described.

4.4.1 Orbital-element probability density

In Bayesian inference, the orbital-element probability density function (p.d.f.) $p_{\rm p}$ is proportional to the a priori ($p_{\rm pr}$) and observational error (p_ϵ) p.d.f.'s, the latter being evaluated for the sky-plane (O-C) residuals $\Delta\boldsymbol{\psi}(\boldsymbol{P})$,

$$p_{\rm p}(\boldsymbol{P}) \quad = \quad C\, p_{\rm pr}(\boldsymbol{P})\, p_\epsilon(\Delta\boldsymbol{\psi}(\boldsymbol{P})), \tag{11}$$

where p_ϵ can usually be assumed to be Gaussian. The normalization constant is $C = (\int p(\mathbf{P},\psi)\, d\mathbf{P})^{-1}$ where the joint p.d.f. is $p(\mathbf{P},\psi) = p_{\rm pr}(\boldsymbol{P})\, p_\epsilon(\Delta\boldsymbol{\psi}(\boldsymbol{P}))$ (compare to Eq. (3)). For the mathematical form of $p_{\rm p}$ to be invariant in transformations from one orbital element set to another (e.g., from Keplerian to equinoctial or Cartesian), we regularize the statistical analysis by Jeffreys' noninformative a priori p.d.f. (Jeffreys 1946, Box and Tiao 1973),

$$\begin{aligned} p_{\rm pr}(\boldsymbol{P}) &\propto \sqrt{\det \Sigma^{-1}(\boldsymbol{P})}, \\ \Sigma^{-1}(\boldsymbol{P}) &= \Phi(\boldsymbol{P})^T \Lambda^{-1} \Phi(\boldsymbol{P}), \end{aligned} \tag{12}$$

where Σ^{-1} is the information matrix (or the inverse covariance matrix) evaluated for the local (i.e., not in the global least-squares sense) orbital elements $\boldsymbol{P}$, Φ contains the partial derivatives of R.A. and Dec. with respect to the orbital elements, and Λ is the covariance matrix for the observational errors.

The final a posteriori orbital-element p.d.f. is

$$\begin{aligned} p_{\rm p}(\boldsymbol{P}) &\propto \sqrt{\det \Sigma^{-1}(\boldsymbol{P})}\ \exp\left[-\frac{1}{2}\chi^2(\boldsymbol{P})\right], \\ \chi^2(\boldsymbol{P}) &= \Delta\boldsymbol{\psi}^T(\boldsymbol{P})\Lambda^{-1}\Delta\boldsymbol{\psi}(\boldsymbol{P}). \end{aligned} \tag{13}$$

As a consequence of securing the invariance in orbital-element transformations, e.g., ephemeris uncertainties and collision probabilities based on the orbital-element p.d.f. are independent of the choice of the orbital-element set (Paper VI). Although for localized p.d.f.'s it is acceptable to assume constant $p_{\rm pr}$, the invariance principle should be considered in all inverse problems in which the parameter uncertainties can be substantial.

Propagation of probabilities: the prediction problem

The utilization of the orbital-element p.d.f.'s constitutes a prediction problem, where additional p.d.f.'s are derived for parameters that are functions of the orbital elements. Following Muinonen and Bowell (1993), the joint p.d.f. for a given set $\boldsymbol{F} = (F_1, \ldots, F_K)^T$ of functions of orbital elements can be derived according to

$$p(\boldsymbol{F}) = \int d\boldsymbol{P}\, p_{\rm p}(\boldsymbol{P})\, \delta_{\rm D}(F_1 - F_1(\boldsymbol{P})) \cdot \ldots \cdot \delta_{\rm D}(F_K - F_K(\boldsymbol{P})), \tag{14}$$

where $\delta_{\rm D}$ is Dirac's delta function. In particular, the p.d.f.'s for other orbital element sets, including sets propagated to other epochs, as well as ephemerides can be established using the relationship in Eq. (14).

4.4.2 Statistical vs. deterministic inversion: discussion

In statistical inverse theory, the a posteriori p.d.f. is the complete solution to the inverse problem. At the same time the p.d.f. provides a full error analysis for the problem in question. In the deterministic framework one typically computes single (point) estimators for the parameters, e.g., maximum likelihood estimates, together with some error estimates. In statistical inversion this corresponds to summarizing the solved p.d.f. by finding values for some parameters that describe the distribution, such as moments and confidence intervals. For many practical purposes, in particular in the case of well-constrained p.d.f.'s, such single estimates are often the objective of statistical inversion, too (compare to Sect. 4.3.2 about linear approximation), and the classification between statistical and deterministic methods in somewhat arbitrary, the difference between the two approaches being more one of emphasis.

Problems arise when the parameter uncertainties are expected to be significant, i.e., the a posteriori p.d.f. is complicated, which is the case with initial orbit computation. The deterministic inverse problem may be *ill-posed* which can be manifested in different ways. For one thing, the solved single estimates for the parameters may not be meaningful, e.g. not physical, or in the worst case no solution is found (e.g., the differential correction scheme in Sect. 4.3.2 does not converge). Or, the solution may not be unambiguous; the function measuring the goodness of the fit (often called the merit function) may have multiple minima of which the global one might be difficult to find, or the parameter space may just be very complex, e.g., multidimensional, both of which lead to ambiguities in the inversion. Finally, the inversion may be unstable, i.e., sensitive to errors in the measurement, which means that a slightly different data set may result in grossly different parameter estimates.

From a deterministic point of view, all the information available for the inversion is included in the observational data, and only some simple constraints can be posed, e.g., in terms of regularization schemes to make the solution stable. But it is not unremarkable that the observational data can be problematic, e.g., it may be insufficient in number or temporal coverage, or it may contain a lot of noise. As already discussed, the inverse problem can be purely underdetermined so that the data does not suffice for finding a unique solution. Even the model can sometimes be the cause of trouble: the model may contain a large number of parameters, i.e., it is complex, or the adopted model may be inexact, i.e., the model is noisy. In orbit computation, our model is completely physical, and exact as far as Newtonian mechanics are considered.

Adopting the statistical approach to inversion does not make a complex inversion simple. But it can help to discern the complexity of the problem, if not known a priori, and, in ambiguous cases, give more realistic estimates for the parameters, and most importantly provide meaningful estimates for their errors. Some of the above problems can be dealt with in statistical inversion. The Bayesian a priori information can sometimes be used to constrain underdetermined cases where the data alone does not suffice to find a solution (e.g., two-observation methods). Also, the Gaussian assumption for the noise is typically built-in in inverse techniques, as in the least-squares analysis. As shown by Carpino et al. (2003), residual distributions of astrometric observations can deviate from Gaussian ones, and Paper VI states one example of the critical effect of the false assumption. Adopting non-Gaussian noise statistics has been put forward also earlier (Muinonen and Bowell, 1993), but its potential has not really been explored in practice.

For models that are not deterministic, an additional noise p.d.f. can be introduced to take into account errors in the adopted model. On the whole, techniques that solve the complete parameter p.d.f.'s are "safest" when the distribution of parameters is very complicated, even to the extent that it might be uninterpretable. The p.d.f. gives an unambiguous answer to the inverse problem, even in the most ill-posed cases where the answer may be that the model parameters cannot be well estimated with the data.

Some other advantages of the Bayesian approach are the sequential use of data, i.e., combining information from several experiments. The a posteriori p.d.f. of a previous measurement can be adopted as the a priori for the next. Or the inference from measurements of different kinds can be combined to a single a posteriori p.d.f., e.g., radar observations can be used as a priori for optical astrometry, or vice versa.

4.4.3 Numerical techniques

Statistical orbital ranging

The technique of statistical orbital ranging (Ranging) is intended for initial orbit computation, that is, when the observational data is exiguous–consisting of small numbers of observations and/or short observational arcs–and the phase-space of orbital parameters is not yet well-constrained (p.d.f. not localized). Ranging maps the orbital element p.d.f. in topocentric spherical coordinates (ρ, α, δ), a coordinate space which is well-constrained even for exiguous data because it is naturally partly coincident with the observation space (α, δ), the extent of which is connected to the observational accuracy and cannot thus be overly extended.

In Ranging, two observation dates (here A and B) are chosen from the complete observation set. The corresponding topocentric distances (or ranges $\rho_{\rm A}$ and $\rho_{\rm B}$), as well as the R.A. ($\alpha_{\rm A}$ and $\alpha_{\rm B}$) and Dec. ($\delta_{\rm A}$ and $\delta_{\rm B}$) angles are MC sampled using intervals subject to iteration, resulting in altogether 12 interval boundary parameters. Explicitly,

$$\left\{\begin{array}{lcl}\rho_{\rm A} & \in & [\rho_{\rm A}^-, \rho_{\rm A}^+],\\ \alpha_{\rm A} & \in & [\alpha_{\rm A}^-, \alpha_{\rm A}^+],\\ \delta_{\rm A} & \in & [\delta_{\rm A}^-, \delta_{\rm A}^+],\end{array}\right. \qquad \left\{\begin{array}{lcl}\rho_{\rm B} & \in & [\rho_{\rm A} + \rho_{\rm B}^-, \rho_{\rm A} + \rho_{\rm B}^+],\\ \alpha_{\rm B} & \in & [\alpha_{\rm B}^-, \alpha_{\rm B}^+],\\ \delta_{\rm B} & \in & [\delta_{\rm B}^-, \delta_{\rm B}^+].\end{array}\right. \tag{15}$$

As a starting point for the angular intervals, one may utilize the one-dimensional 3σ variation interval based on assumed standard deviation σ of the noise p.d.f., i.e., the accuracy of the observations. For the range intervals, an educated guess of the dynamical class of the object based on its coordinate motion–taking typical values for NEO, MBO and TNO–provides the first values. Note that it is computationally efficient to generate $\rho_{\rm B}$ based on $\rho_{\rm A}$ generated at an earlier stage. The final boundary values $\rho_{\rm A,B}^{\pm}$ must be carefully chosen/iterated so as to secure the coverage of the entire relevant interval in $\rho_{\rm A,B}$. Once the two sets of spherical coordinates have been generated, the two corresponding Cartesian positions $(X_{\rm A}, Y_{\rm A}, Z_{\rm A})^T$ and $(X_{\rm B}, Y_{\rm B}, Z_{\rm B})^T$ lead to an unambiguous set of orbital elements $\boldsymbol{P}$, based on well-established techniques in celestial mechanics (Paper I; Danby, 1992).

The set of trial orbital elements $\boldsymbol{P}$ is included in the set of sample orbital elements if and only if it produces an acceptable fit to the entire set of observations, that is, with the help of Eq. (13),

$$\begin{aligned}&\exp\Big[-\frac{1}{2}(\chi^2(\boldsymbol{P}) - \chi^2(\boldsymbol{P}_{\rm ref}))+\\ &\ln\sqrt{\det\Sigma^{-1}(\boldsymbol{P})} - \ln\sqrt{\det\Sigma^{-1}(\boldsymbol{P}_{\rm ref})}\Big] \geq c_{\rm min},\end{aligned} \tag{16}$$

where $c_{\rm min}$ is the level of acceptance and $\boldsymbol{P}_{\rm ref}$ refers to the best-fit orbital solution available, constantly updated during the iterative computation. The acceptance criterion thus becomes analogous to the $\Delta\chi^2$ criterion for Gaussian p.d.f.'s. In addition, the residuals of the individual observations must not exceed a given threshold, e.g., 3 arcsec.

Orbital sampling in volumes of variation

For moderately observed asteroids, that is, for observational time arcs and numbers of observations that result in well-constrained but nonlinear p.d.f.'s, the orbital-element p.d.f. can be efficiently sampled in the element phase-space. In the volume-of-variation (VoV) technique, we choose a mapping parameter $P_{\rm m}$ (one of the six orbital elements, for example), march through its full interval of variation and, by correcting differentially for the remaining five orbital elements, compute a discrete set of local maximum-p.d.f. points.

Guided by these local linear approximations, we introduce MC sampling in the phase-space volume for a fully nonlinear treatment.

Sample orbits are drawn from the rigorous orbital-element p.d.f. with the help of the local linear approximations. First, we specify the variation interval for the mapping element with the help of the covariance matrix Σ derived in the global linear approximation (cf. Sect 4.3.2) and emphasize that the variation interval must be subject to iteration. For example, one may utilize the one-dimensional 3σ variation interval as given by the linear approximation so that

$$P_{\mathrm{m}} \in [P_{\mathrm{m,ls}} - 3\sigma_{\mathrm{m}}, P_{\mathrm{m,ls}} + 3\sigma_{\mathrm{m}}], \tag{17}$$

where $P_{\mathrm{m,ls}}$ is the global least-squares value for the mapping element. Second, the remaining elements are sampled with the help of the local intervals of variation so that

$$\boldsymbol{P}' = \boldsymbol{P}'_{\mathrm{ls}}(P_{\mathrm{m}}) + \sum_{j=1}^{5}(1-2r_j) \cdot \sqrt{\Delta\tilde{\chi}^2 \lambda'_j(P_{\mathrm{m}})} S'(P_{\mathrm{m,ls}}) \boldsymbol{X}'_j(P_{\mathrm{m}}), \tag{18}$$

where $r_j \in (0,1)$ $(j = 1, \ldots, 5)$ are independent uniform random deviates and $\Delta\tilde{\chi}^2$ is a scaling parameter to be iterated so that the entire orbit solution space is covered and the final results have converged. Initially, one may start with $\Delta\tilde{\chi}^2 = 11.3$ and slowly increase its value. $S'(P_{\mathrm{m,ls}})$ designates the single standard-deviation matrix used throughout the interval of the mapping parameter, which allows a straightforward debiasing of the sample orbits at the end of the computation. Here, $S'(P_{\mathrm{m,ls}})$ is the S' matrix evaluated at the global least-squares value of the mapping element P_{m}. Finally, $\lambda'_j(P_{\mathrm{m}})$ and $\boldsymbol{X}'_j(P_{\mathrm{m}})$ are the eigenvalue and eigenvector of the orbital-element correlation matrix, respectively.

In practice, once the parameters for the variation intervals in Eqs. 17 and 18 have been fixed, a value for the mapping element is then obtained from uniform sampling over the mapping interval and the remaining elements are generated by interpolating their variation interval based on the precomputed map. Finally, as in Ranging, the trial orbit is accepted if it produces an acceptable fit to the observations.

4.5 Other advances

Several two-observation methods based on random or systematic variation of topocentric coordinates or motions–resembling the Ranging technique just described–have been put forward recently. Typically, the line-of-sight components are varied since they are not determined from astrometric observations. McNaught (1999) and Tholen and Whiteley (2000) derive sets of orbits from a single Cartesian position and velocity by varying the topocentric range and range rate. The technique by McNaught is not readily applicable to uncertainty estimation but that of Tholen and Whiteley provides a statistical treatment by incorporating weights for the computed orbits. Although using the difference of two angular positions to derive the angular motion works in the favor of removing the systematic observational errors, it sets limits to the timing of the two observations to allow accurate estimation of the motion.

Subsequently, Goldader and Alcock (2003) described a Ranging-like technique that Monte Carlo maps the possible orbits in topocentric coordinates. As that of Bernstein and Khushalani (2000, see below), their technique is designed for transneptunian objects and makes use of their nearly linear motion. Compared to Ranging, they have restricted the used data to only two observations and they do not provide any probabilistic interpretation, although the techniques are congruent in mapping the extent of the orbital uncertainty.

Milani et al. (2004, 2005b) have recently taken a geometric approach to uncertainty estimation. For what they term ”too short arc”, they make a linear fit to the angular observations and represent the observations with the average observation and angular rates. By eliminating other than solar-bound orbits, the possible interval for the unknown parameters, range and range-rate, is constrained and the boundary of the uncertainty region sampled with ”virtual asteroids”.

Targeted for TNOs, Bernstein and Khushalani (2000) devised a linearized orbit–fitting procedure in which accelerations are treated as perturbations to the inertial motions of distant objects. The method can be used to produce ephemerides and uncertainty ellipses, even for short–arc orbits.

Semilinear approximations. While orbit improvement is today routinely carried out via least-squares, the limits of the linear approximation have also been intensively explored. The differential correction in Eq. (10) regularly converges for well-observed objects, but problems are known to arise for poorly observed single-apparition objects. Already Muinonen and Bowell (1993) in their description of the statistical orbit computation problem offered a MC technique for assessing mildly non-Gaussian orbital-element p.d.f.'s.

Using the linear approximation, Muinonen et al. (1994) carried out orbital uncertainty analysis for the more than 10,000 single–apparition asteroids known at the time. Their analysis captured the dramatic increase of orbital uncertainties for short–arc orbits. Subsequently, studying the covariance matrix via eigenvalues Muinonen et al. (1997) found that there exists a bound, as a function of observational arc and number of observations, outside which the linear approximation can be applied, and inside which nonlinearity dominates the inversion.

Milani (1999) discussed the linear approximation using six–dimensional confidence boundaries. The ultimate simplification is a one–dimensional line of variations along the principal eigenvector of the covariance matrix (Muinonen 1996), a precursor to the more general one–dimensional curves of variation used in semilinear approximations. However, as pointed out by Muinonen et al. (1997) and Milani (1999), there are risks in applying the line–of–variation methods.

A cascade of one–dimensional semilinear approximations follows from the notion that the complete differential correction procedure for six orbital elements is replaced, after fixing a single orbital element (mapping parameter; for example, the semimajor axis or perihelion distance; cf. Bowell et al. 1993 and Muinonen et al. 1997), by an incomplete one for five orbital elements. Varying the mapping parameter and repeating the algorithm allows one to obtain a one–dimensional, nonlinear curve of variation, following along the ridge of the a posteriori probability density in the six-dimensional phase space of the orbital elements.

The incomplete differential correction procedure has been used by several researchers over the decades, typically as an intermediate phase in the case of convergence problems. Only Milani (1999), in what he terms the multiple–solution technique (see also Milani et al., 2005a), has systematically explored its practical implementation. In Milani's technique, the mapping parameter is the step along the principal eigenvector of the covariance matrix computed in the linear approximation. However, the covariance matrix and the eigenvector are recomputed after each step, allowing efficient tracking of the probability–density ridge. Because of the nonlinearity of the inverse problem for short–arc asteroids, Milani's multiple–solution technique has turned out to be particularly successful in many of the applications such as asteroid identification and impact monitoring described, e.g., in Milani et al. (1999, 2002, 2005a). The technique is attractive and efficient because of its simplicity.

Aiming towards a fully nonlinear assessment, the line-of-variation approach can be generalized to account for the remaining dimensions of the inverse problem. In a recent work, Chesley (2005) introduces what can be called a plane-of-variation technique where a two-dimensional plane of orbital elements (range and range rate) is obtained using local linear approximations for the remaining four orbital parameters. The technique resembles the VoV-technique described in Paper V and Sect. 4.4.3 but, instead of being fully nonlinear, it is restricted by the linear approximation in four dimensions.

5 Summary of papers

5.1 Paper I

Statistical ranging of asteroid orbits

In the paper, we introduce a new inverse method for initial orbit computation of asteroids. We start with a review of the previously developed methods, and discuss their applicability to single-apparition asteroids, the target group of the new method.

The foundations of the method of statistical orbital ranging are in statistical inverse theory, and its Bayesian formulation for orbit computation is therefore summarized before the method itself is described. Instead of single estimates, Ranging characterizes the probability density function of the orbital elements, which contains the complete solution to the inverse problem, using sample orbits. As illustrated with several examples of main-belt and near-Earth asteroids, the orbital-element probability density can be highly complicated for short-arc objects, strongly implying that the solutions from the traditional methods for preliminary orbits are not unique. Also the techniques relying on the linear approximation can severely fail in describing the orbital uncertainties in terms of covariance matrices.

Several implications of the ambiguities of short-arc orbital solutions are put forward. To begin with, the classification of a newly discovered object based on a single orbit can be misleading: main-belt objects with only a few observations were shown to have high probabilities for being near-Earth asteroids. Instead, a probabilistic approach should be adopted based on techniques such as the one described in the paper. Next, in ephemeris prediction, the Bayesian inference was shown to be efficient when estimating sky-plane uncertainties for recovery attempts of otherwise lost objects. Also, the potential of the current technique for collision probability studies, as well as for unveiling systematic errors in observations, was proposed. The need of numerical integrations was pointed out as a possible black spot of this class of techniques that are based on MC simulations, although two-body approximation was found adequate for all the case-studies presented in the paper.

5.2 Paper II

Collision probabilities for Earth-crossing asteroids using orbital ranging

We illustrate the applicability of Ranging to collision probability computations with a case study of 1998 OX_4, an Earth-crossing asteroid for which non-zero collision estimates had been computed but which was lost at the time of the study.[9] Several improvements are presented in both theoretical and numerical techniques. The formulation of the statistical inverse theory is completed by introducing a fundamental regularization to the inverse problem in terms of a noninformative a priori probability density. The regularization secures the invariance of the statistical model in case the probability densities are transformed from one parameter set to another. In numerical techniques, an optimized version of Ranging is introduced which is considerably faster, and makes the automation of the technique more feasible. An MC technique for collision probability computation

[9] The dedicated negative observation campaigns aimed at eliminating the collision solutions failed, but 1998 OX_4 was serendipitously recovered as 2002 PJ_{34} in August 6, 2002 by NEAT.

is established using a rigorous orbital-element probability density available from Ranging, which makes it currently the only six-dimensional technique applicable to short-arc objects. For 1998 OX_4, the fully nonlinear collision probability analysis is carried out, and the computed estimates are compared to those published by others. We confirm the earlier impact intervals, and find the collision probabilities to be in accordance with other published values within two orders of magnitude, which is a satisfactory result.

5.3 Paper III

Orbit computation for transneptunian objects

Paper III presents a systematic application of Ranging to the entire population of transneptunian objects. The aim was to improve our knowledge of the orbital distribution of the known population, and to see if the current picture of the dynamical nature of the region was correct. Most of the earlier studies, even the detailed dynamical ones, have been based on the MPC database which only gives a single orbit per object. Since 50 % of the known objects had been observed over one apparition only, corresponding to an orbital period coverage of less than 1 %, analyses based on a single orbit must be in vain. This notion is also supported by our earlier studies of poorly observed objects in Paper I. In Paper III, we conclude that Ranging turned out to be highly applicable to these distant objects. In addition to solving the short-arc problem, a straightforward linkage was performed for objects observed over several apparitions. The computed orbital-element distributions are the basis for more detailed dynamical studies of the known population, such as a more definitive dynamical classification scheme. We show that although the orbital-element p.d.f.'s for short-arc TNOs are complicated, their projection to sky-plane ephemeris results typically in linear although extended p.d.f.'s. The reason is that the motion of the object as seen from the Earth during groundbased observations is nearly linear. However, recovering these faint objects is challenging, and the need to use large telescopes mandates accurate ephemeris prediction. We propose the use of dynamical filtering to reduce the sky-plane search region. This work also set the scene for the development of a Web-based TNO ephemeris prediction service[10] (Virtanen et al., 2003; Granvik et al., 2003). The ephemeris filtering was also put to practice in observing programmes aiming at recovery and follow-up of TNOs and carried out at the Nordic Optical Telescope in La Palma and at the European Southern Observatory in La Silla during 2002-2004. In a follow-up paper (Tancredi et al. 2005, in preparation), we return to the dynamical classification question and use the updated orbital element p.d.f.'s as a starting point for a study of objects on peculiar orbits.

5.4 Paper IV

Asteroid orbit computation

We discuss the status of the problem of asteroid orbit computation, and review the recent advances in techniques, from late-1980's to date. One of the major factors that was identified to have contributed also to this field is the development of the World Wide Web. It has enabled the nearly real-time interplay between the observers and orbit computers

[10]see http://asteroid.lowell.edu

through several Web-based services for both software and databases made available by different institutes and researchers, the pioneer being the Minor Planet Center. The latest decade has been the starting point for a statistical era of orbit computation, since all modern inverse methods provide uncertainty analysis as part of the solution. We also demonstrate the application of Ranging to the analysis of systematic errors as well as to double solutions. Finally, an equivalence between observations and orbital elements is foreseen to take place within the next decade, a prophecy that will need to be confirmed.

5.5 Paper V

Asteroid orbits using phase-space volumes of variation

We present a new nonlinear inverse technique for moderate, or transitional observational data. The volume-of-variation technique complements Ranging for exiguous data and the least-squares technique for extensive data. VoV sampling is based on local linear approximations which are used as guide for the nonlinear MC sampling of the orbital-element phase space. The call for such a technique was already put forward by Muinonen and Bowell (1993) who brought up the non-Gaussian (i.e., nonlinear) characteristics of the orbital-element p.d.f.'s. Our preliminary studies had pointed out that the evolution of orbital uncertainties for increasing observational arcs is, in fact, highly nonlinear across a rather narrow transition regime. We termed the phenomenon phase transition, and also noted that the existence of such an effect suggests that different computational methods could be used across the transition regime. In Paper V, we illustrate the applicability of the new technique over the phase-transition regime, and demonstrate how it tackles the nonlinear features of the orbital-element p.d.f.'s.

5.6 Paper VI

Time evolution of orbital uncertainties for the impactor candidate 2004 $\mathbf{AS_1}$

We analyse the case of asteroid 2004 AS_1, which was suggested to have a significant risk for an Earth impact within 48 hours of the discovery. In particular, we demonstrate the implications of noisy data in initial orbit computation, and discuss outlier detection in exiguous data. We confirm the drastic first prediction, and also that it was due to the discordance of the discovery night observations. We explore the possibility to have been able to deduce the poor quality of the data at the time. Although outlier detection did not succeed in the case of 2004 AS_1, we propose to develop a Ranging-based algorithm to detect discordant observations even from exiguous data. We return to the question raised in Paper II, that is, the importance of proper regularization of probability densities. Securing the invariance is particularly important in collision probability studies of which drastic conclusions from the human point of view might be due.

5.7 Author's contribution

In numerical techniques, the author of this thesis has developed the Ranging technique further from the original idea in Muinonen (1999), in particular by completing the probabilistic modeling, such as the use of a priori distributions and derivation of end-products such as NEO probabilities (Paper I). The author initiated and carried through the automation of Ranging for more practical application (Papers II and III). Also the implementation of both Ranging and VoV-techniques into Fortran95 has been the responsibility of the author (together with M. Granvik, T. Laakso and K. Muinonen). In all the papers, the author is responsible for the application of the orbit computation techniques to the example cases of the selected asteroids as well as for illustrating and describing the results. Papers I, III, and VI (first author) were written by the author with the exception of Section 2.2 and part of Section 3.2 in Paper III concerning the stability studies which were provided by G. Tancredi. In Papers II, IV, and V the author was responsible for the interpretation and writing of the results from the practical computations, in Paper IV only for the Ranging application.

6 Simulations for spacebased surveys

The asteroid orbital inversion is a key task in the Gaia project. Gaia is the astrometric cornerstone mission of the European Space Agency (ESA) scheduled for launch in mid-2011. Gaia will survey the entire sky down to a limiting magnitude V = 20 mag with unprecedented accuracy of, e.g., 10 microarcseconds at V = 15 mag. This is an accuracy several orders of magnitude better than with current groundbased surveys, and this improvement will have a major impact on accuracy of the orbital elements. Moreover, the entire problem of orbit computation will be affected (Muinonen et al., 2005; Virtanen and Muinonen, 2005): the astrometric implications of the finite size and irregular shape of the asteroids need to be modeled, because in measurements of these accuracies, the photocenter and the center of mass of an asteroid do no longer coincide. Due to the intertwining of the asteroid dynamical evolution and physical properties, the global solution for asteroid orbits from Gaia data will only be ready at the end of the survey. The full statistical inverse problem encompasses solving for the sizes, shapes, and masses (also for perturbing objects) as well as relativistic effects simultaneously for large numbers of asteroids.

In simulations for Gaia by Muinonen and Virtanen (2002), a nonlinear collapse was seen in the orbital uncertainties as a function of the improving accuracy of astrometric observations (Fig. 5). This phase transition effect can also be recognized in the time evolution of the orbital uncertainty: there is a threshold for the length of the observational arc (and the number of observations) over which the orbital-element distributions nonlinearly evolve from extended to well-constrained ones. The existence of such an effect suggests that different computational methods could be used to assess the uncertainties before, at, and after the transition. Paper V of the present thesis focuses on *transitional* observational data, and the decribed phase-space sampling technique promises to be applicable across the phase transition regime.

The case-studies in Virtanen and Muinonen (2005) further suggest that the phase transition for high-precision MBO data may take place already at the discovery moment. For Gaia, the results are encouraging, because the simulated asteroid observations have turned out to very sparse in time, implying that weeks or months may pass before the discovery data is replenished and the initial orbit improved.

NERO (Near-Earth objects Radiometric Observatory) is one of the six ESA studies for possible missions dedicated to near-Earth objects. The general concept is a small satellite equipped with both a detector for visible wavelengths and an array for thermal IR measurements. NERO would address two of the major objectives of current NEO science, namely the physical characterization of the objects and the discovery of NEOs which are difficult to detect from ground, the objects residing entirely inside the orbit of the Earth. In Cellino et al. (2005), the initial orbit computation for objects observed at small solar elongations was studied, a matter of importance with a space mission because no follow-up from ground can be effectively performed. The results from our simulations with Ranging indicate that the uncertainties in the distances of the newly detected objects at the epochs of observations should be sufficiently well constrained as to allow the observers to apply the usual technique of reduction of radiometric data. The successful data reduction in turn implies that NERO should be capable of obtaining sizes and albedos for the discovered objects.

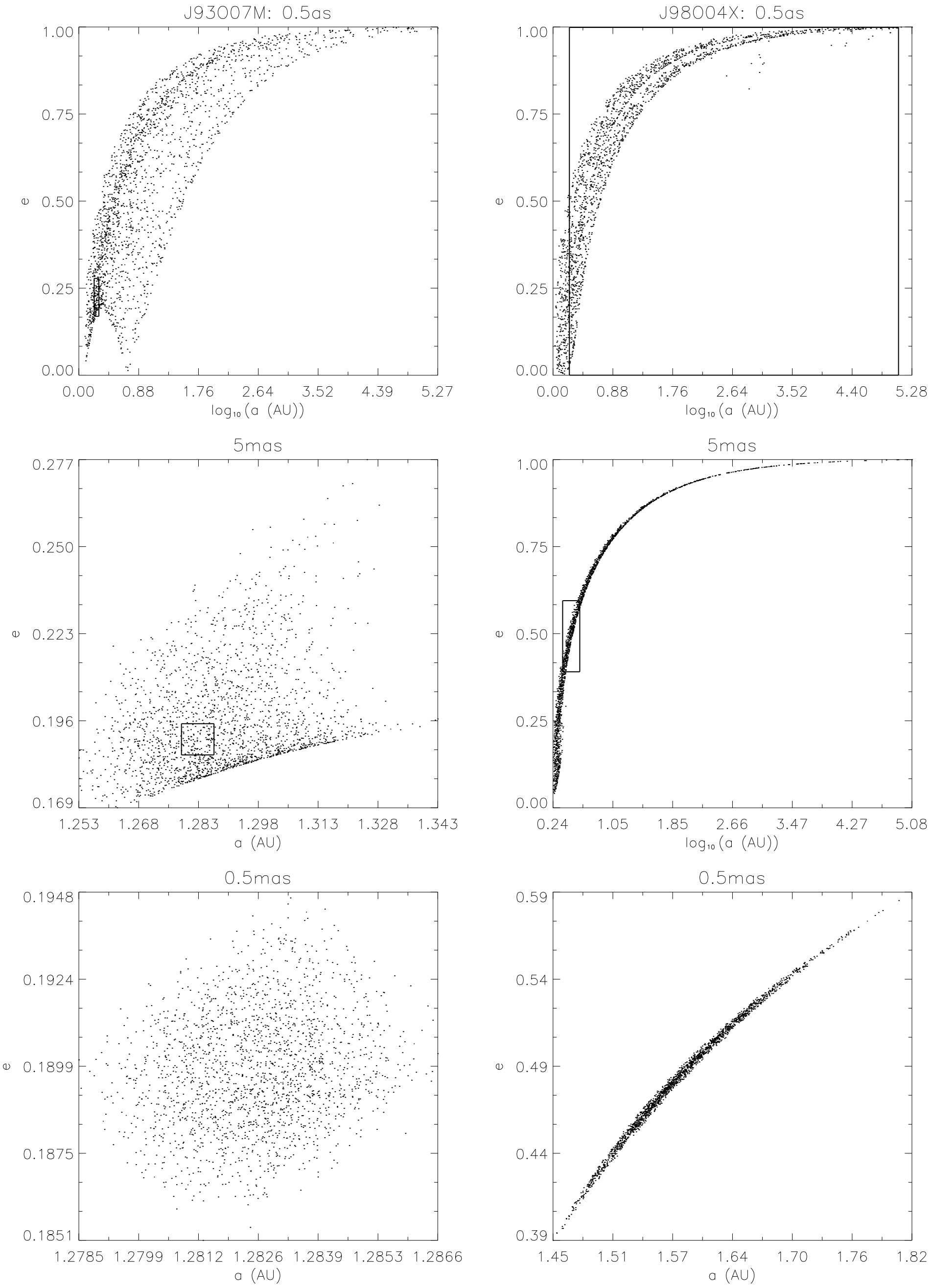

Figure 5: Nonlinear collapse in the extent of the orbital distributions with improving observational accuracy for the near-Earth asteroids 1993 OM_7 (left) and 1998 OX_4 (right). From top to bottom, we show the extent of the marginal probability densities for $\sigma = 0.5$ as, 5 mas, and 0.5 mas. The boxes in the upper plots indicate the extents of the corresponding plotting windows below. The results indicated that there is a threshold value for the astrometric accuracy below which the orbital-element probability density becomes well-confined even for exiguous data. From Muinonen and Virtanen (2002).

7 Conclusions and future prospects

The aim of the present thesis has been to provide a complete solution for the problem of asteroid orbital inversion in terms of statistical methods. In particular, rigorous solutions are called for in initial orbit computation, where the asteroid impact risk evaluation mandates a firm understanding of the possible orbits for the newly discovered objects. Two nonlinear statistical techniques based on Monte Carlo sampling of the orbital-element probability density have been presented. The technique of statistical orbital ranging is tailored for exiguous observational data, for which the orbital-element p.d.f.'s are shown to be complicated, for example, having multiple maxima. Such ambiguities in orbital inversion can be resolved with Ranging, the solution of which is the rigorous orbital-element p.d.f., which is mathematically well-defined even if the parameters it describes are not.

The Volume-of-Variation technique complements Ranging for exiguous data and the least-squares technique for extensive data. It can be used to assess the nonlinearities in the phase transition from complicated orbital-element p.d.f.'s to well-constrained ones. Both techniques have been shown to be readily applicable to ephemeris predictions in observing programmes run at the Nordic Optical Telescope in La Palma, Spain, and at the European Southern Observatory in La Silla, Chile.

The criticism often expressed against the computationally demanding Monte Carlo methods can be partially answered with the results from the large-scale application. Nevertheless, the Monte Carlo based methods are most suited for case studies where rigour is called for.

While the presented inverse techniques together with the standard least-squares analysis complete the spectrum by providing a continuum of techniques to meet the needs imposed by the variety of available asteroid observational data, several optimizations are due. First, the optimal application area of each technique in terms of both the observations and the dynamically different target groups needs to be found out. Fine-tuning is also called for to find the optimal parameters to be used within each technique (Cartesian, Keplerian, topocentric spherical coordinates/coordinate rates). Finally, prospects for future work include several topics such as analysis of TNOs on peculiar orbits (Tancredi et al., 2005), linking of short-arc objects (Granvik and Muinonen, 2005), population study for the poorly observed sungrazing comets, and the outlier detection from exiguous data.

References

Barucci, M. A., D. P. Cruikshank, S. Mottola, M. Lazzarin 2002. Physical properties of Trojans and Centaur asteroids. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 273. The University of Arizona Press, Tuscon.

Bate, R. R., D. D. Mueller, and J. E. White 1971. *Fundamentals of Astrodynamics.* Dover Pub. Inc., New York.

Bernstein, G. M., and B. Khushalani 2000. Orbit fitting and uncertainties for Kuiper belt objects. *Astron. J.* **120**, 3323–3332.

Bottke, W. F. Jr., D. Vokrouchlický, D. P. Rubincam, and M. Brož 2002a. The effect of Yarkovsky thermal forces on the dynamical evolution of asteroids and meteoroids. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 395. The University of Arizona Press, Tuscon.

Bottke, W. F. Jr, et al. 2002b. Debiased orbital and absolute magnitude distribution of the near-Earth objects. *Icarus* **156**, 399–433.

Bowell, E., B. A. Skiff, and L. H. Wasserman 1990. Orbital information from asteroid motion vectors. In *Asteroids, Comets, and Meteors III* (C.-I. Lagerkvist, H. Rickman, B. A. Lindblad, and M. Lindgren, Eds.), pp. 19–24. Uppsala Universitet, Uppsala, Sweden.

Bowell, E., L. H. Wasserman, K. Muinonen, R. H. McNaught, and R. M. West 1993. A search for the lost asteroid (719) Albert. *Bull. Amer. Astron. Soc.* 25, 1118.

Box, G. E. P., and G. C. Tiao 1973.*Bayesian inference in statistical analysis.* Addison-Wesley Publ., Reading, MA.

Brouwer, D., and G. M. Clemence 1961. *Methods of Celestial Mechanics.* Academic Press, New York, NY.

Cappellari, J. O., C. E. Velez, and A. J. Fuchs 1976. *Mathematical theory of the Goddard Trajectory Determination System.* Goddard Space Flight Center Report X-582-76-77.

Carpino, M., A. Milani, and S.R. Chesley 2003. Error statistics of asteroid optical astrometric observations. *Icarus* **166**, 248–270.

Cellino, A., R. Somma, L. Tommasi, R. Paolinetti, K. Muinonen, J. Virtanen, E. F. Tedesco, M. Delbò 2005. NERO: general concept of a Near-Earth objects Radiometric Observatory. In *Proceedings of the 35th Scientific Assembly of COSPAR* (Paris, France, 18–25 July 2004). Submitted.

Chesley, S. R. 2005. Very short arc orbit determination: the case of 2004 FU_{162}. In *Dynamics of Populations of Planetary Systems*, Proceedings IAU Colloquium No. 197, (Z. Knežević and A. Milani, Eds.), Cambridge Univ. Press, Cambridge, UK, pp. 255–258.

Chesley, S. R., and A. Milani 1999. Nonlinear methods for the propagation of orbital uncertainty. Paper 99-418 in AAS/AIAA Space Flight Mechanics Meeting, 16-19 August 1999, Girkwood, Alaska.

Chodas, P. W., and D. K. Yeomans 1996. The orbital motion and impact circumstances of comet Shoemaker-Levy 9. In *The Collision of Comet Shoemaker-Levy 9 and Jupiter* (K. S. Noll, P. D. Feldman, and H. A. Weaver, Eds.), pp. 1–30. Cambridge University Press.

Danby, J. M. A. 1992. *Fundamentals of Celestial Mechanics*. Willmann-Bell, Inc., Richmond.

Dubyago, A. D. 1961. *The Determination of Orbits*. New York, NY.

Gauss C. F. 1809. *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections*. Dover, New York, 326 pp., 1963.

Goldader, J. D., and C. Alcock 2003. Constraining recovery observations for trans-neptunian objects with poorly known orbits. *PASP* **115**, 1330–1339.

Granvik, M., and K. Muinonen 2005. Asteroid identification at discovery. *Icarus*, in press.

Granvik, M. J. Virtanen, K. Muinonen, E. Bowell, B. Koehn, and G. Tancredi 2003. Transneptunian object ephemeris service (TNOEPH). In *First Decadal Review of the Edgeworth-Kuiper belt* (John K. Davies and Luis H. Barrera, Eds.), Kluwer Academic, Dordrecht, Netherlands, pp. 73–78 (Reprinted from Earth, Moon, and Planets, Volume 92, Nos. 1-4).

Herget, P. 1965. Computation of preliminary orbits. *Astron. J.* **70**, 1–3.

Jedicke, R., J. Larsen, and T. Spahr 2002. Observational selection effects in asteroid surveys. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 71. The University of Arizona Press, Tuscon.

Jeffreys H. 1946. An invariant form for the prior probability in estimation problems. Proceedings of the Royal Statistical Society of London, Series A. 186, 453–461.

Karttunen, H. 1996. *Vanhin tiede: tähtitiedettä kivikaudesta kuulentoihin.* Gummerus Kirjapaino Oy, Jyväskylä.

Karttunen, H., P. Kröger, H. Oja, M. Poutanen, K. J. Donner 2003. *Fundamentals astronomy.* Springer-Verlag, Berlin Heidelberg.

Kristensen, L. K. 1992. The identification problem in asteroid surveys. *Astron. Astrophys.* **262**, 606–612.

Kristensen, L. K. 1995. Orbit determination by four observations. *Astron. Nachr.* **316**, 261–266.

Kristensen, L. K., and R. M. West 1989. On the lost minor planet (719) Albert. *Astron. Astrophys.* **218**, 317–324.

Lehtinen, M. S. 1988. On statistical inversion theory. In *Theory and Applications of Inverse Problems* (H. Haario, Ed.), pp. 46–57. Pitman Research Notes in Mathematical Series 167.

Marsden, B. G. 1985. Initial orbit determination – The pragmatist's point of view. *Astron. J.* **90**, 1541–1547.

Marsden, B. G. 1991. The computation of orbits in indeterminate and uncertain cases. *Astron. J.* **102**, 1539–1552.

McNaught, R. H. 1999. Letter to the Editor: Unusual object 1998 DK_{36}. *J. British Astron. Assoc.* **109**, 299.

Menke, W. 1989. *Geophysical Data Analysis: Discrete Inverse Theory.* Academic Press, New York.

Milani, A. 1999. The asteroid identification problem. I. Recovery of lost asteroids. *Icarus* **137**, 269–292.

Milani, A., S. R. Chesley, and G. B. Valsecchi 1999. *Astron. Astrophys.* **346**, L65

Milani, A., S. R. Chesley, P. W. Chodas, G. B. Valsecchi 2002. Asteroid close approaches: analysis and potential impact detection. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 55. The University of Arizona Press, Tuscon.

Milani, A., G. F. Gronchi, M. de'Michieli Vitturi, and Z. Knezević 2004. Orbit determination with very short arcs. I Admissible regions. *Cel. Mech. Dyn. Astron.* **90**, 59–87.

Milani, A., M. E. Sansaturio, G. Tommei, O. Arratia, and S. R. Chesley 2005a. Multiple solutions for asteroid orbits: computational procedure and applications. *Astron. Astrophys.* **431**, 729–746.

Milani, A., G. F. Gronchi, Z. Knezević, M. E. Sansaturio, O. Arratia 2005b. Orbit determination with very short arcs. II Identifications. *Icarus*, submitted.

Millis, R. L. et al. 2002. The Deep Ecliptic Survey. *Astron. J.* **123**, 2083.

Muinonen, K. 1996. Orbital covariance eigenproblem for asteroids and comets. *Mon. Not. R. Astron. Soc.* **280**, 1235–1238.

Muinonen, K. 1999. Asteroid and comet encounters with the Earth. In *The Dynamics of Small Bodies in the Solar System* (B. A. Steves and A. E. Roy, Eds.), pp. 127–158. Kluwer Academic Publishers, Dordrecht.

Muinonen, K., and E. Bowell 1993. Asteroid orbit determination using Bayesian probabilities. *Icarus* **104**, 255–279.

Muinonen, K., and J. Virtanen 2002. Computation of orbits for near-Earth objects from high-precision astrometry. In *Proceedings of the International Workshop on Collaboration and Coordination Among NEO Observers and Orbit Computers*, pp. 105–113.

Muinonen, K., E. Bowell, and L. H. Wasserman 1994. Orbital uncertainties of single-apparition asteroids. *Planet. Space Sci.* **42**, 307–313.

Muinonen, K., A. Milani, and E. Bowell 1997. Determination of initial eigenorbits for asteroids. In *Dynamics and Astrometry of Natural and Artificial Celestial Bodies* (I. M. Wytrzyszczak, J. H. Lieske and R. A. Feldman, Eds.), pp. 191–198. Kluwer Academic Publishers, Dordrecht.

Muinonen, K., J. Virtanen, M. Granvik, and T. Laakso 2005. Asteroid orbits with Gaia: statistical inversion and identification. *Proceedings of the symposium Three Dimensional Universe with Gaia* (ESA SP-576), in 223–230.

Orlov, B. A. 1939. Determination of the preliminary orbit of a minor planet by two observations. *Pulkovo Obs. Circ.* No. 26, 55–63.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery 1994. *Numerical Recipes in FORTRAN, The Art of Scientific Computing.* Cambridge University

Press.

Sagan, C. and A. Druyan 1997. *Comet.* Headline Book Publishing.

Sitarski, G. 1983. Effects of general relativity in the motion of minor planets and coments. *Acta Astronomica*, Vol. 33, No. 2, 295–304.

Stokes, G. H., J. B. Evans, and S. M. Larson. 2002. Near-Earth asteroid search programmes. In *Asteroids III* (W. Bottke, A. Cellino, P. Paolicchi, and R. P. Binzel, Eds.), pp. 45. The University of Arizona Press, Tuscon.

Tancredi, G., J. Virtanen, and K. Muinonen 2005. Peculiar orbits in the Centaur and transneptunian populations. In preparation.

Tholen, D., and R. J. Whiteley 2000. Short arc orbit solutions and ephemeris computations: application to NEOs. *Bull. Amer. Astron. Soc.* **32**, 1018.

Väisälä, Y. 1939. Eine einfache Methode der Bahnbestimmung. *Mitt. Sternw. Univ. Turku*, Suomalainen Tiedeakatemia, **1**, 1-32.

Yeomans, D. K., S. J. Ostro, and P. W. Chodas 1987. Radar astrometry of near-Earth asteroids. *Astron. J.* **94**, 189-200.

Yeomans, D. K., P. W. Chodas, M. S. Keesey, S. J. Ostro, J. F. Chandler, and I. I. Shapiro 1992. Asteroid and comet orbits using radar data. *Astron. J.* **103**, 303-317.

Vallinkoski, M. 1988. Statistics of incoherent scatter multiparameter fits. *Journal of Atmospheric and Terrestrial Physics*, Vol. 50, No. 9, 839–851.

Vallinkoski, M., and M. S. Lehtinen 1990a. Parameter mixing errors within a measuring volume with applications to incoherent scatter. *Journal of Atmospheric and Terrestrial Physics*, Vol. 52, No. 6-8, 665–674.

Vallinkoski, M., and M. S. Lehtinen 1990b. The effect of a priori knowledge on parameter estimation errors with applications to incoherent scatter. *Journal of Atmospheric and Terrestrial Physics*, Vol. 52, No. 6-8, 675–685.

Virtanen, J., and K. Muinonen 2005. Asteroid orbits with Gaia: application of computational techniques to simulations. *Proceedings of the symposium Three Dimensional Universe with Gaia* (ESA SP-576), 325–328.

Virtanen, J., K. Muinonen, G. Tancredi, and E. Bowell 2003. Ephemeris uncertainties using statistical ranging: recovery and follow-up of short-arc transneptunian objects. *Proceedings of the Conference Asteroids, Comets, and Meteors III* (ESA-SP-500), pp. 465–469.