Diamonds on large cardinals

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Academic dissertation

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Helsinki, November 21, 2003
Alex Hellsten

Contents

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Introduction

The thesis consists of the following two independent papers of which the second is published as a monograph. The first paper is joint work with Tapani Hyttinen and Saharon Shelah.

I. Potential isomorphism and semi-proper trees. Fundamenta Mathematicae, 175(2):127–142, 2002. *Alex Hellsten, Tapani Hyttinen, and Saharon Shelah.*

II. Diamonds on large cardinals. Annales Academiæ Scientiarum Fennicæ Mathematica Dissertationes, 134, 48 pages, 2003. *Alex Hellsten.*

In the first paper we study a strong notion of potential isomorphism. Suppose we are given two structures of cardinality κ. We consider generic extensions that preserve stationary subsets of κ and do not add new subsets of cardinality less than κ. The given structures are said to be potentially isomorphic, if they are isomorphic in some generic extension of the type described above. Other similar notions have been studied in literature [10, 7, 1, 8, 5], where different limitations on the method of extending the universe are considered.

In the following, a (λ, κ)-tree is a tree such that every branch has length less than κ and every element has less than λ immediate successors. We say that a (λ, κ)-tree T is weakly semi-proper if there exists a forcing notion P that adds a κ-branch to T, but preserves stationary subsets of κ and adds no sets of cardinality less than κ. If T itself, regarded as a forcing notion, has the properties of P mentioned above, then we say that T is strongly semi-proper or just semi-proper.

We notice that there is a strong connection between the existence of weakly semi-proper (κ^+, κ)-trees and the existence of potentially isomorphic models of cardinality κ for a given complete theory.

We show that the assumption $2^{\aleph_0} < 2^{\aleph_1}$ implies the existence of a semi-proper $(\aleph_2, \aleph_1)$-tree and that and that GCH implies the existence of a semi-proper (κ^{++}, κ^+)-tree for every infinite successor cardinal κ.

In the negative direction we show that it is consistent relative to a super-compact cardinal that there are no weakly semi-proper $(\infty, \aleph_1)$-trees. We also prove that it is consistent relative to a weakly compact cardinal that there are no weakly semi-proper $(\aleph_3, \aleph_2)$-trees, by showing that such trees do not exist in the Mitchell model.

For the latter result it is crucial that we deal with (λ, κ)-trees where $\lambda \leq 2^\kappa$. In ZFC we prove that there exists a semi-proper $((2^\kappa)^+, \kappa)$-tree for every regular $\kappa > \aleph_1$.

In the second paper the main focus is on the following combinatorial principle called weakly compact diamond: There exists a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ such that

$$\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$$

is a weakly compact subset of κ for every set A. Here κ is a fixed cardinal which is weakly compact if the principle holds. The weakly compact subsets of κ are the sets of positive measure with respect to the weakly compact ideal, i.e. the ideal generated by the sets $\{\alpha \in \kappa : \langle V_\alpha, \in, U \cap V_\alpha\rangle \models \neg\phi\}$ where $U \subseteq V_\kappa$ and ϕ is a Π^1_1-sentence such that $\langle V_\kappa, \in, U\rangle \models \phi$. A cardinal is weakly compact in the standard sense if and only if it is weakly compact as a subset of itself. The weakly compact diamond was defined independently by Sun [12] and Shelah [11].

Since the weakly compact ideal is normal, weakly compact diamond on κ is a strengthening of the regular diamond principle $\diamondsuit_\kappa$. In fact it implies $\diamondsuit_\kappa(E)$ where E is the set of all regular cardinals below κ. Thus by a result of Hauser [6] on $\diamondsuit_\kappa(E)$ it is consistent that weakly compact diamond fails at a weakly compact cardinal.

We also study normal ideals over regular cardinals in general. We define the concept of an n-club where n is a natural number, and note that they provide a characterisation of the Π^1_n-ideals. A version of a 1-club was first introduced by Sun [12]. The Π^1_n-ideals where originally studied by Levy [9] among the other ideals in the so called Levy hierarchy. We show that one can use forcing to shoot a 1-club through a weakly compact subset of κ

while preserving the weak compactness κ. This works as a generalisation of the idea presented in [4] of shooting a club through a stationary subset of $\aleph_1$.

The research leading to the second paper was initiated by an eventually successful effort to provide proofs of the following three facts that were stated in [11] without proof: Weakly compact diamond holds for every measurable cardinal, holds in the constructible universe for every weakly compact cardinal, and can be obtained through forcing. The proofs of the first two of the above facts presented in the second paper were discovered independently of Sun's work [12]. They are slightly more general than the proofs given in [12].

In fact weakly compact diamond holds for almost ineffable cardinals, so measurability is an unnecessary strong assumption. Baumgartner [2, 3] has studied the ideals connected to ineffable and almost ineffable cardinals in greater detail. We give a compact presentation of some of Baumgartner's results and study the connections to diamond principles in suitable generality.

Forcing arguments are presented by which weakly compact diamond holds in many generic extensions. Then a forcing notion is defined that kills weakly compact diamond on a weakly compact set while preserving it on the complement.

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