ON GAMES ON NON-WELLFOUNDED SETS AND STATIONARY SETS

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1. Introduction

In this thesis we study a few games related to non-wellfounded and stationary sets. Games have turned out to be an important tool in mathematical logic ranging from semantic games defining the truth of a sentence in a given logic to for example games on real numbers whose determinacies have important effects on the consistency of certain large cardinal assumptions.

The equality of non-wellfounded sets can be determined by a so called bisimulation game already used to identify processes in theoretical computer science and possible world models for modal logic. Here we present a game to classify non-wellfounded sets according to their branching structure. We also study games on stationary sets moving back to classical wellfounded set theory. The Banach-Mazur game, also called the ideal game, is connected to precipitousness of the corresponding ideal. The pressing down game is played on regressive functions defined on stationary sets, and it has applications in model theory to the determinacy of the Ehrenfeucht-Fraïssé game. We introduce the relevant concepts later in this section.

This thesis is divided into three independent sections with the following papers:

(i) *On Non-wellfounded Sets as Fixed Points of Substitutions*, by Tapani Hyttinen and Matti Pauna, published in Notre Dame Journal of Symbolic Logic, vol. 42
(ii) *A Domain Order Over Non-wellfounded Sets*, by Matti Pauna
(iii) *The Banach Mazur and the Pressing Down Games are Different*, by Jakob Kellner, Matti Pauna, and Saharon Shelah, accepted to Journal of Symbolic Logic.

1.1. **Non-wellfounded sets.** Non-wellfounded sets have infinite descending membership sequences, which makes them counterintuitive in view of the iterative concept, which requires that the members of a set have to be formed before the set itself. For instance, for the set $\Omega = \{\Omega\}$ this kind of process of forming the members is impossible. These kinds of sets contradict the foundation axiom usually used in set theory and therefore non-wellfounded sets have not been studied systematically until recently. However, non-wellfounded sets have found applications in mathematical logic, computer science, and in philosophy, which has started an active research in the area.

In 1917 Dimitry Mirimanoff developed the concept of wellfounded and non-wellfounded sets. In 1920–1930 foundation axiom, which states that all sets are wellfounded, was added to the axioms of set theory by Zermelo, and proved relatively independent from the other axioms by von Neumann.

Non-wellfounded sets have been studied by several authors during 1920–1980, but until Peter Aczel's seminal work [1] no systematic study has taken place.

Aczel develops the theory of non-wellfounded sets, formulates and proves consistent his anti foundation axiom, *AFA*, and studies the central concept of bisimulation. In [2] Barwise and Moss further develop the theory of non-wellfounded sets, coinduction, corecursion, and monotone operators.

Aczel's key notion is a graph picturing a set meaning that there is a set whose membership relation is the same as the graph relation. Graphs that have infinite descending chains will therefore produce non-wellfounded sets. Certain graphs are identified with the bisimulation relation giving the equality condition of non-wellfounded sets. Bisimulation can be viewed as a game where two players try to move along the edges of the graphs. At each round the players are on a certain node of the graphs (at the beginning both are in some appointed starting nodes of the two graphs) and first player I moves an edge from either of the graphs. Player II has to respond with an edge from the other graph. If player II cannot make a move, she loses, meaning that the two graphs are not bisimilar. In terms of sets this means that II has reached an empty set while on the other set there are still members and thus the sets are different.

The anti foundation axiom states that every graph has a unique set that it pictures. In [2] the approach of defining non-wellfounded sets uses equation systems which describe the elements of a set. For example, the equation system

$$\begin{aligned} x &= \{y\} \\ y &= \{x\} \end{aligned}$$

defines a non-wellfounded set which is equal by bisimulation to the set $\Omega = \{\Omega\}$.

In the first part of this thesis we will use both of these approaches of producing non-wellfounded sets and present a generalization of the latter in which the non-wellfounded sets are obtained as fixed points of substitutions. By substitution we mean an arbitrary function f and a fixed point of it would be a function g such that, if $f(x)$ is a set, then $g(x) = \{g(y) \mid y \in f(x)\}$. (As in [2] we work in set theory with urelements which are objects that can be members of sets but themselves do not have any members.) For instance, if $f(x) = x$, then for the fixed point g of it, we would have that $g(x) =$ the non-wellfounded set $\Omega = \{\Omega\}$.

We show that set theory without the foundation axiom and including the anti foundation axiom, *ZFA*, implies that every function has a fixed point. Also as a corollary we determine for which functions f there is a function g such that $g = g \star f$, where $\star$ is the substitution operator defined in [2].

We also aim to describe a certain hierarchy of non-wellfounded sets. First we define a game that describes the branching structure of the sets, and from this we can define a rank of a non-wellfounded set. Formulated in terms of equation systems, in the game, denoted by G_α, the player I chooses an infinite descending sequence of indeterminates (same as a descending infinite chain in a graph). The second player chooses a natural number n and an ordinal $\beta < \alpha$ used as a clock. Then II has to play again a different descending sequence starting below the nth element of the previous sequence. After that I again chooses a natural number and an ordinal $\gamma < \beta$, and so on. We say that I wins the game if she is able to respond to II's moves until II has no more moves left, because the ordinal clock has ran out. Also an infinite version, G_∞, of this game is defined without the ordinal clock.

Now according to this game, the non-wellfounded sets can be divided into classes V_{afa}^{α} which contains all the sets for which II wins G_α in the corresponding equation system. For example, V_{afa}^{0} consists of the wellfounded sets. We let V_{afa}^{∞} consist of the sets where I wins the game G_∞ in the corresponding equation system. It is shown that the classes V_{afa}^{α} form an increasing system of models of set theory together with modified AFA_α which is the anti foundation axiom restricted to equation systems for which the player II has a winning strategy.

In the second part concerning non-wellfounded sets, we study the hereditarily finite sets HF_1 as a domain. Domains are approximation structures, where a partial order $\sqsubseteq$ defines the approximation relation. In this setting, we are going to approximate the sets in HF_1 by their wellfounded parts. We will form the inverse limit of these approximations obtaining the class of all inverse limits, $\widehat{HF}$, which is a domain and also an ultra metric space. This is closely connected with the work of Maurice Boffa [3].

More specifically, we define a partial ordering $\sqsubseteq$ on the class of all non-wellfounded sets. That ordering is defined as a kind of end extension in the tree pictures of sets. A tree picture is a graph that is a tree obtained by unwinding an arbitrary graph. Then the domain, D, of non-wellfounded sets is obtained by taking all inverse limits of $\sqsubseteq$–increasing sequences. This produces a subclass of the universe and we can show that all hereditarily finite sets belong to D.

1.2. **Stationary sets.** In the third part we work in classical set theory with the foundation axiom and study the Banach-Mazur game originally introduced to study properties of topological spaces and later generalized into Boolean algebras, where the players alternately play a descending sequence of elements of the Boolean algebra. Thomas Jech [8] used it to characterize distributivity properties of Boolean algebras. When played on the Boolean algebra consisting of equivalence classes of stationary sets module the non-stationary ideal of a given uncountable cardinal the game is usually called ideal game. In [4] it is shown that the corresponding ideal of non-stationary sets is precipitous if and only if the player I has not a winning strategy in the ideal game of length ω. Precipitousness is an important property since it allows one to form a wellfounded ultrapower of the universe modulo the generic ultrafilter obtained by forcing with the stationary sets. It is shown in [12] that even the non-stationary ideal on $\aleph_1$ can be precipitous.

In [7] the pressing down game, PD, is introduced. In this game, at each round the player I plays a regressive function and II plays a stationary set where the function is constant forming a descending sequence of length ω_1 of stationary sets.

Using this game, it can be shown to be consistent that the Ehrenfeucht-Fraïssé game is determined on models of size at most $\aleph_2$.

It is easy to see that if II wins BM, then she wins also PD. In this thesis we show that it is consistent that the reverse implication does not hold. We show that in $V = L[U]$, where U is a normal ultrafilter, the player I wins BM and we also present a forcing construction which gives the player I a winning strategy in BM.

2. On non–wellfounded sets as fixed points of substitutions

2.1. **Introduction and definitions.** In Aczel [1] and in Barwise and Moss [2] non–wellfounded sets and the Anti-Foundation Axiom (*AFA*) have been studied. The non–wellfounded sets are modelled by equations. In the equations we use urelements and the class of all urelements is denoted by $\mathcal{U}$. Urelements are not vital for the theory but often they are convenient, see e.g. [2], section 11. We recall the definitions of a flat system of equations and a solution to it from [2]:

Definition 2.1. .

(i) A *flat system of equations* is a triple (X, A, f) where X and A are sets of urelements, $X \cap A = \emptyset$, and $f : X \to \mathcal{P}(X \cup A)$ is a function.

(ii) A *solution* to a flat system of equations (X, A, f) is a function g such that $\mathrm{dom}(g) = X$, and for all $x \in X$, $g(x) = \{g(y) \mid y \in f(x) \cap X\} \cup (f(x) \cap A)$.

The idea is that X is the set of indeterminates of the equations and A is the set of "constants". The equations are understood as $x = f(x)$, for $x \in X$. For example, let $A = \{a\}$, $X = \{x\}$, and $f(x) = \{a, x\}$, then (X, A, f) is a flat system of equations. The solution to this system is a function g such that $g(x) = \{a, g(x)\}$. The Anti-Foundation Axiom, *AFA*, says that every flat system of equations has a unique solution.

Substitution operations $\mathrm{sub}(s, b)$ are also studied in [2]. The operation $\mathrm{sub}(s, b)$ means that in b all x are substituted by $s(x)$. We recall the definition of a substitution from [2]. If $A \subseteq \mathcal{U}$, then $V_{afa}[A]$ is the class of all sets x such that $\mathrm{support}(x) \subseteq A$, where $\mathrm{support}(x)$ is defined to be $TC(x) \cap \mathcal{U}$. So $V_{afa}[\mathcal{U}]$ is the class of all sets.

Definition 2.2. *Substitution* is a function s such that $\mathrm{dom}(s) \subset \mathcal{U}$. *The substitution operation* is the operation sub such that the domain of sub consists of a class of pairs $\langle s, b \rangle$ where s is a substitution and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$ such that the following conditions hold

(i) If $x \in \mathrm{dom}(s)$, then $\mathrm{sub}(s, x) = s(x)$.

(ii) If $x \in \mathcal{U} - \mathrm{dom}(s)$, then $\mathrm{sub}(s, x) = x$.

(iii) For all sets b, $\mathrm{sub}(s, b) = \{\mathrm{sub}(s, p) \mid p \in b\}$.

In [2] it is shown that there is a unique substitution operation $\mathrm{sub}(s, b)$ defined for all substitutions s and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. As a corollary to our theory of substitution fixed points, we obtain the same result, see Corollary 2.18. Next we recall the definition of a composition of substitutions from [2].

Definition 2.3. The substitution operation $\mathrm{sub}(s, b)$ is also denoted by $b[s]$, and $[s]$ is the operation mapping each set or urelement b to $b[s]$. A substitution s is *proper* if for all $x \in \mathrm{dom}(s)$, $s(x) \in V_{afa}[\mathcal{U}]$ whenever $s(x) \neq x$. If s and t are substitutions, then $t \star s$ is the substitution whose domain is $\mathrm{dom}(s)$ and for every $x \in \mathrm{dom}(s)$, $(t \star s)(x) = s(x)[t]$.

It is shown in [2] that we can state the *AFA* axiom in terms of substitution. *AFA* is equivalent to the assertion that for every proper substitution e there is a unique proper substitution s such that $s = s \star e$.

We remark here first that if f is a substitution, not necessarily proper, then a substitution s such that $s = s \star f$ is not necessarily unique for the following reason: Let a and b be distinct urelements. Let $f(a) = b, f(b) = a$. Let $u \in \mathcal{U} - \{a, b\}$ and $s(a) = s(b) = u$. Then $s = s \star f$.

Second, if f is such that the domain of f contains sets, then s does not necessarily exist. For example, let a and b be distinct urelements, and let $f(a) = \{a\}$, $f(\{a\}) = b$. If s is such that $s = s \star f$, then $s(a) = (s \star f)(a) = f(a)[s] = \mathrm{sub}(s, f(a)) = \mathrm{sub}(s, \{a\}) = s(\{a\})$ by Definition 2.3. Then by (ii) of Definition 2.2, $s(\{a\}) = f(\{a\})[s] = \mathrm{sub}(s, f(\{a\})) = \mathrm{sub}(s, b) = b$. But by (iii) $s(a) = \mathrm{sub}(s, \{a\}) = \{\mathrm{sub}(s, a)\} = \{s(a)\}$, hence $s(a) = \Omega$, where Ω is the unique non–wellfounded set x such that $x = \{x\}$.

As a corollary of our theory of substitution fixed points, we will show that for every substitution f there is g such that $g = g \star f$, see Lemma 2.16.

2.2. **Fixed points approach.** Next we study the non–wellfounded sets as fixed points of substitutions. Here we generalize the equation systems to arbitrary functions. The solutions are then defined in terms of substitution. The fixed points are further generalizations of the solutions. This approach works well also in the situation without urelements. First we introduce some notation.

Definition 2.4. .

(i) With every function f we associate a class function f^* defined as follows. If $x \in \mathrm{dom}(f)$, then $f^*(x) = f(x)$, otherwise $f^*(x) = x$.

(ii) If f and g are functions, then by $f[g]$ we mean a function such that $\mathrm{dom}(f[g]) = \mathrm{dom}(g)$ and $f[g](x)$ is defined as follows. If $f^*(x)$ is an urelement, then $f[g](x) = f^*(x)$ and otherwise $f[g](x) = \{g^*(y) \mid y \in f^*(x)\}$.

(iii) For all functions f and g, we say that g is a *solution* to f $(S(g, f))$, if $\mathrm{dom}(g) = \mathrm{dom}(f)$ and $g = f[g]$.

From the above we see that in [2] a solution to a flat system (X, A, f) of equations is defined so that g is the solution to the system iff $S(g, f)$ holds. So, in a sense, the solution to a flat system (X, A, f) of equations is obtained if in $f(x)$ all elements y from $f(x) \cap \mathrm{dom}(f)$ are replaced by $f(y)$. Then all elements z in $f(y) \cap \mathrm{dom}(f)$ are replaced by $f(z)$ and so on. So the solutions are some kind of restricted substitution–fixed points of the function from the system. In fact, in [2] it is shown that we get an equivalent theory if instead of equations we study substitution (cf. above).

Because of the urelements, we define $\bigcup X = \bigcup\{x \mid x \in X$ and x is not an urelement$\}$.

Next we introduce the concept of a fixed point and the Fixed Point Axiom.

Definition 2.5. .

(i) We say that g is a *fixed point* of f, ($FP(g,f)$), if $\mathrm{dom}(f) \cup \bigcup f^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g)$ and $g = f[g]$ (where $f^*[\mathrm{dom}(g)] = \{f^*(y) \mid y \in \mathrm{dom}(g)\}$).

(ii) We say that a function f is *generating* if for all $x \in \mathrm{dom}(f)$ the following holds: if $f(x)$ is an urelement, then $x = f(x)$. We say that a generating f is *basic* if $\mathrm{dom}(f) \subseteq \mathcal{U}$.

(iii) The Fixed Point Axiom (FPA) is the following: every function has a fixed point.

Note that if $FP(g,f)$ holds and $f^*(x)$ is not an urelement, then $g(x) = \{g(y) \mid y \in f^*(x)\}$. The following example shows the difference between solutions and fixed points: Let x be an urelement, $\mathrm{dom}(f) = \{x\}$ and $f(x) = (\emptyset, x)$. Then f itself is the solution to f but if g is a fixed point of f, then $g(x) = (\emptyset, g(x))$. Also following the notation from [2], if f is basic and $FP(g,f)$ holds, then for all $x \in \mathrm{dom}(g)$, if $f(x)$ is not an urelement, then $g(x) = \mathrm{sub}(g, f(x))$. Note that the basic functions are the same as the proper substitutions in [2].

Example 2.6. Assume ZFC. Let X be a set and $f : X \to \mathcal{P}(X)$ be such that $f(x) = x \cap X$. Then f has a (unique) fixed point g (such that $\mathrm{dom}(g) = \mathrm{dom}(f)$), namely the Mostowski collapse of X.

In [2] it is also shown that in the presence of the axiom AFA, bisimulation characterizes identity:

By $TC(x)$ we mean the transitive closure of x and in the case x is an urelement, $TC(x)$ is defined to be $\emptyset$.

Definition 2.7. .

(i) We write $B(x,y)$ if there is $B \subseteq TC(\{x\}) \times TC(\{y\})$ such that
  (a) $(x,y) \in B$,
  (b) if $(a,b) \in B$ and $c \in a$, then there is $d \in b$ such that $(c,d) \in B$,
  (c) if $(a,b) \in B$ and $d \in b$, then there is $c \in a$ such that $(c,d) \in B$,
  (d) if $(a,b) \in B$, then a is an urelement iff b is an urelement and if they are urelements, then $a = b$.

  We call this kind of a relation B a *bisimulation relation between x and y.*

(ii) We let the *Strong Extensionality Axiom* (SEA) be the following axiom:

$$\forall x, y(B(x,y) \to x = y).$$

The axiom system ZFC^{-2} consists of pairing, union, power set, infinity, collection, separation, and choice, together with the Axiom of Urelements: $\forall p \forall q(\mathcal{U}(p) \to q \notin p)$, and the Axiom of Plenitude of Urelements: for every set S there is an injective function $f : S \to \mathcal{U}$ whose image $f[S]$ is disjoint from S. So the list of the axioms of ZFC^{-2} is the same as that in page 28 of [2] excluding Extensionality and replacing Strong Plenitude by Plenitude of Urelements. ZFA means $ZFC^{-2}+$ Extensionality $+AFA$. By ZFC^+ we mean $ZFC^{-2} + SEA + FPA$. So the difference between *ZFA* and *ZFC*$^+$ is that we have replaced equations by substitution and uniqueness of the solutions by *SEA*.

We feel that our axiom system follows the lines of the axiom systems of [1] in that the axioms for the existence and the uniqueness of the solutions are separated. Also we think that this approach is a bit more set theoretical in nature, since the Fixed Point Axiom refers to functions instead of graphs or equation systems.

We start by showing that *ZFA* and *ZFC*$^+$ are equivalent. Especially we show that *ZFA* implies that every function has a fixed point. Then we show that the fixed points of the basic functions are fixed points of themselves, thus the name fixed point. For all functions this does not hold. Finally, we study the following question: Do we need to assume the existence of all solutions to the flat systems of equations to get all fixed points? We show that the answer is (essentially) yes.

2.3. **Equivalence of** *ZFA* **and** *ZFC*$^+$**.** Item (ii) in the following Lemma is [2] Exercise 7.3 and item (i) is well–known.

Lemma 2.8. .

(i) $ZFC^{-2} \vdash \forall x, y \notin \mathcal{U}(\forall z(z \in x \leftrightarrow z \in y) \to B(x,y))$. *In particular,* $ZFC^{-2} \vdash \forall x, y(x = y \to B(x,y))$.

(ii) $ZFC \vdash \forall x, y(B(x,y) \to x = y)$.

Proof.

(i): Let B consist of (x,y) together with $(a,b) \in TC(x) \times TC(y)$, such that $a = b$. To show that B is a bisimulation between x and y, let $z \in x$. Then by the assumption, $z \in y$ also. By the definition of B, $(z,z) \in B$ and hence B is a bisimulation between x and y. Especially, if $x = y$ and $x, y \notin \mathcal{U}$, then $\forall z(z \in x \leftrightarrow z \in y)$, and by the above, we can construct a bisimulation between x and y. If $x = y$ and $x, y \in \mathcal{U}$, then $\{(x,y)\}$ is a bisimulation between x and y.

(ii): By $\in$–induction: Assume the claim for all $x' \in x$ and that for some set y, $B(x,y)$ holds. This means that for every $x' \in x$ there is $y' \in y$ such that $B(x',y')$ holds. This is so because if B is the bisimulation between x and y, then $B \restriction (TC(\{x'\}) \times TC(\{y'\}))$ is a bisimulation between x' and y'. But this means that for every $x' \in x$ there is $y' \in y$ such that $x' = y'$, i.e. $x' \in y$. Similarly, if $y' \in y$, then there is $x' \in x$, such that $y' = x'$, i.e. $y' \in x$. By extensionality, $x = y$. □

The following lemma is essentially proved in [2].

Lemma 2.9. *Assume* $ZFC^{-2} + SEA$.

(i) $\forall x, y \notin \mathcal{U}(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$ *i.e. the Extensionality Axiom holds.*

(ii) For all functions f*, if* $S(g,f)$ *and* $S(h,f)$ *hold, then* $g = h$.

(iii) For all functions f*, if* $FP(g,f)$ *and* $FP(h,f)$ *hold and* $A = \mathrm{dom}(g) \cap \mathrm{dom}(h)$*, then* $g \restriction A = h \restriction A$.

Notice that since the Extensionality Axiom holds in $ZFC^{-2} + SEA$, also the functions of the form $f[g]$ are well–defined and thus $S(g, f)$ and $FP(g, f)$ are well–defined.

Proof.

(i): Let B consist of (x, y) together with $(a, b) \in TC(x) \times TC(y)$, such that $a = b$. Then B is a bisimulation between x and y. This is so because if $a \in x$, then $a \in y$ also, and therefore $(a, a) \in B$. If $(a, a) \in B$, and $a \in \mathcal{U}$, then the condition (d) holds. If $a \notin \mathcal{U}$, and $c \in a$, then $(c, c) \in B$ and we are done.

(ii): Let B consist of $(g(a), h(a))$ together with $(x, y) \in TC(g(a)) \times TC(h(a))$ such that either

(1) $x = y$

or

(2) there is $z \in \text{dom}(f)$ such that $x = g(z)$ and $y = h(z)$.

To show that B is a bisimulation, let $z \in \text{dom}(f)$ and $(g(z), h(z)) \in B$. If $g(z)$ is an urelement, then $g(z) = f[g](z) = f^*(z) = f[h](z) = h(z)$.

Assume that $g(z)$ in not an urelement. Then $g(z) = f[g](z) = \{g^*(y) \mid y \in f^*(z)\}$. Let $c \in g(z)$. So $c = g^*(y)$ for some $y \in f^*(z)$. But because also $h(z) = \{h^*(y) \mid y \in f^*(z)\}$, we have that $h^*(y) \in h(z)$. Now if $y \notin \text{dom}(f)$, then $y \notin \text{dom}(g) = \text{dom}(h)$. Therefore $g^*(y) = h^*(y) = y$ and $(y, y) \in B$. If $y \in \text{dom}(f)$, then $(g(y), h(y)) \in B$.

(iii): Let B consist of $(g(a), h(a))$ together with $(x, y) \in TC(g(a)) \times TC(h(a))$ for which there is $z \in \bigcup f^*[A]$ such that $x = g(z)$ and $y = h(z)$.

Assume that $(g(z), h(z)) \in B$ and $f^*(z)$ is not an urelement. So $g(z) = \{g(y) \mid y \in f^*(z)\}$. Now let $y \in \bigcup f^*[A]$, then $y \in A$ and therefore if $g(y) \in g(z)$, then also $h(y) \in h(z)$ and $(g(y), h(y)) \in B$. $\square$

Lemma 2.10. *Assume $ZFC^{-2} + SEA$.*

(i) Assume $FP(g, f)$ holds and let $x \in \text{dom}(g)$. If $TC(f^*(x)) \cap \text{dom}(f) = \emptyset$, then $g(x) = f^*(x)$.

(ii) If $FP(g, f)$ holds and for all $x \in \text{dom}(f)$, $TC(f(x) - \text{dom}(f)) \cap \text{dom}(f) = \emptyset$, then $S(g \restriction \text{dom}(f), f)$ holds.

Proof.

(i): We may assume that $f^*(x)$ is not an urelement, since if $f^*(x)$ is an urelement, then $g(x) = f[g](x) = f^*(x)$ by the assumption that $FP(g, f)$ holds. Let $A = TC(f^*(x)) \cap \text{dom}(g)$. Because for all $y \in A$, $y = f^*(y) \subseteq \text{dom}(g)$, A is transitively closed. Since $g(x) = \{g(y) \mid y \in f^*(x)\}$, it is enough to show that for all $y \in f^*(x)$, $g(y) = y$. Since $A \cap \text{dom}(f) = \emptyset$, $FP(g \restriction A, id_A)$ holds. Since A is transitively closed, $S(g \restriction A, id_A)$ holds. So by Lemma 2.9 (ii), it is enough to show that $S(id_A, id_A)$ holds, but this is clear.

(ii): Assume that $f(x)$ is not an urelement. By (i), $g(x) = \{g(y) \mid y \in f(x)\} = \{g(y) \mid y \in f(x) - \text{dom}(f)\} \cup \{g(y) \mid y \in f(x) \cap \text{dom}(f)\} = \{y \mid y \in f(x) - \text{dom}(f)\} \cup \{g(y) \mid y \in f(x) \cap \text{dom}(f)\} = f[g \restriction \text{dom}(f)](x)$. $\square$

Corollary 2.11. Assume $\mathrm{dom}(f) \subseteq \mathcal{U}$ and for all $x \in \mathrm{dom}(f)$, $f(x) \subseteq \mathcal{U}$. If $FP(g, f)$ holds, then $S(g \restriction \mathrm{dom}(f), f)$ holds.

Proof. Let f be as in the assumption, then $TC(f(x) - \mathrm{dom}(f)) \cap \mathrm{dom}(f) = (f(x) - \mathrm{dom}(f)) \cap \mathrm{dom}(f) = \emptyset$. Hence, by Lemma 2.10 (ii), $S(g \restriction \mathrm{dom}(f), f)$ holds. □

Lemma 2.12. *Assume ZFC^{-2}+ extensionality. Assume that f is a generating function, $A \subseteq \mathrm{dom}(f)$, for all $x \in \mathrm{dom}(f) - A$, $f(x)$ is an urelement, and $FP(g, f \restriction A)$ holds. If h is a function such that $\mathrm{dom}(h) = \mathrm{dom}(g) \cup \mathrm{dom}(f)$, $h \restriction \mathrm{dom}(g) = g$, and for all $x \in \mathrm{dom}(f) - \mathrm{dom}(g)$, $h(x) = f(x)$, then $FP(h, f)$ holds.*

Proof. If $x \in \mathrm{dom}(f) - A$, then $f(x) = x \in \mathcal{U}$, because f is generating. Hence for all x, $f^*(x) = (f \restriction A)^*(x)$. Because $\bigcup f^*[\mathrm{dom}(h)] = \bigcup f^*[\mathrm{dom}(g) \cup \mathrm{dom}(f)] = \bigcup f^*[\mathrm{dom}(g) \cup (\mathrm{dom}(f) - A)] = \bigcup f^*[\mathrm{dom}(g)] = \bigcup (f \restriction A)^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g) \subseteq \mathrm{dom}(h)$, we have that $\mathrm{dom}(f) \cup \bigcup f^*[\mathrm{dom}(h)] \subseteq \mathrm{dom}(h)$.

Assume $x \in \mathrm{dom}(h)$ and $f^*(x) \notin \mathcal{U}$. Then $x \in \mathrm{dom}(g)$ and $f[h](x) = \{h(y) \mid y \in f^*(x)\} = \{g(y) \mid y \in (f \restriction A)^*(x)\} = g(x) = h(x)$, because $f^*(x) = (f \restriction A)^*(x) \subseteq \mathrm{dom}(g)$. Assume that $f^*(x) \in \mathcal{U}$. If $x \notin \mathrm{dom}(g)$, then $h(x) = f(x) = x$. If $x \in \mathrm{dom}(g)$, then $h(x) = g(x) = f^*(x)$. So we have shown that $h = f[h]$. □.

The following Lemma is proved for basic functions in [2] (cf. Theorem 8.5).

Lemma 2.13. *Assume ZFA. For all generating f there is g such that $FP(g, f)$ holds.*

Proof. Let f be a generating function. We show that f has a fixed point. By Lemma 2.12, we may assume that for all $x \in \mathrm{dom}(f)$, $f(x)$ is not an urelement. Choose a transitively closed A so that $\mathrm{dom}(f) \cup \bigcup \mathrm{rng}(f) \subseteq A$. Then $\bigcup f^*[A] \subseteq A$. Choose a one-one function h so that $\mathrm{dom}(h) = B = (A - \mathcal{U}) \cup \mathrm{dom}(f)$, $\mathrm{rng}(h) \subseteq \mathcal{U}$, $h(y) = y$ if $y \in \mathrm{dom}(f) \cap \mathcal{U}$ and $\mathrm{rng}(h) \cap A = \mathrm{dom}(f) \cap \mathcal{U}$. Define f' so that $\mathrm{dom}(f') = \mathrm{rng}(h)$ and for all $x \in B$, $f'(h(x)) = \{h^*(y) \mid y \in f^*(x)\}$. Then $(\mathrm{rng}(h), (A \cap \mathcal{U}) - \mathrm{rng}(h), f')$ is a flat system of equations. Let g' be such that $S(g', f')$ holds and let g be such that $\mathrm{dom}(g) = A$, $g \restriction B = g' \circ h$ and $g \restriction A - B = id_{A-B}$. We show that g is a fixed point of f. We have already shown that $\mathrm{dom}(f) \cup f^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g)$. So it is enough to show that for all $x \in A$, $g(x) = f[g](x)$. If $x \notin B$, then $g(x) = x$ and $x \notin \mathrm{dom}(f)$. So $f^*(x) = x$ is an urelement and we have that $f[g](x) = f^*(x) = x = g(x)$.

Assume that $x \in B$. Then $f'(h(x))$ is not an urelement and so $g(x) = g'(h(x)) = \{g'(y) \mid y \in f'(h(x))\} =$

$$\{g'(y) \mid y \in f'(h(x)) - \mathrm{rng}(h)\} \cup \{g'(y) \mid y \in f'(h(x)) \cap \mathrm{rng}(h)\} \tag{1}$$

Now $f'(h(x)) - \mathrm{rng}(h) = \{h^*(z) \mid z \in f^*(x)\} - \mathrm{rng}(h) = \{z \mid z \in f^*(x) - B\} = f^*(x) - B$. If $y \in f^*(x) - B = f'(h(x)) - \mathrm{rng}(h)$, then $y \notin \mathrm{dom}(f')$, since $\mathrm{rng}(h) = \mathrm{dom}(f')$. Because $f^*(x) \subseteq A$, we have that $f^*(x) - B \subseteq \mathcal{U}$, so $y \in \mathcal{U}$.

Hence $g'(y) = f'[g'](y) = (f')^*(y) = y$. On the other hand $f'(h(x)) \cap \mathrm{rng}(h) = \{h^*(y) \mid y \in f^*(x)\} \cap \mathrm{rng}(h) = \{h(y) \mid y \in f^*(x) \cap B\}$. Thus we have that (1) is equal to $\{y \mid y \in f^*(x) - B\} \cup \{g'(y) \mid y \in \{h(z) \mid z \in f^*(x) \cap B\}\} = \{g(y) \mid y \in f^*(x) - B\} \cup \{g'(h(z)) \mid z \in f^*(x) \cap B\} = \{g(y) \mid y \in f^*(x)\}$. □

Lemma 2.14. *Assume ZFC*$^{-2}$ *+ SEA. If every generating function has a fixed point, then every function has a fixed point.*

Proof. Let f be a function. Let A be a transitively closed set such that $\mathrm{dom}(f) \cup \mathrm{rng}(f) \subseteq A$. Let B be the set of those $x \in A$ such that $f^*(x) \neq x$ is an urelement and let C be the set of those $x \in A$ such that $f^*(x) = x$ is an urelement. Let h be a one-one function such that $\mathrm{dom}(h) = B$ and $\mathrm{rng}(h) \subseteq \mathcal{U} - A$. Define f' so that $\mathrm{dom}(f') = A - B$ and for all $x \in \mathrm{dom}(f')$, if $x \in C$, then $f'(x) = x$ and otherwise $f'(x) = \{h^*(y) \mid y \in f^*(x)\}$. Then f' is generating and so by Lemma 2.13, it has a fixed point g'. Let $D = g'[\mathrm{dom}(f')] - \mathcal{U}$ and define f'' so that $\mathrm{dom}(f'') = D$ and for all $x \in \mathrm{dom}(f'')$, $f''(x) = \{h'(y) \mid y \in x\}$, where $h'(y) = y$, if $y \notin \mathrm{rng}(h)$ and otherwise $h'(y) = f(h^{-1}(y))$. Then f'' is generating and let g'' be a fixed point of f''. We define g so that $\mathrm{dom}(g) = A$, for all $x \in \mathrm{dom}(g)$, if $f^*(x)$ is an urelement, then $g(x) = f^*(x)$ and otherwise, $g(x) = g''(g'(x))$. We show that g is a fixed point of f.

Since $\mathrm{rng}(f) \subseteq A$ and A is transitively closed, $\bigcup f^*[A] \subseteq A$. Also $\mathrm{dom}(f) \subseteq A$. So it is enough to prove that for all $x \in A$, $g(x) = f[g](x)$. If $x \in B \cup C$, the claim is clear. So assume $x \in A - (B \cup C)$. Then $g(x) = g''(g'(x)) = \{g''(y) \mid y \in f''(g'(x))\} = \{g''(h'(y)) \mid y \in g'(x)\} = \{g''(h'(g'(y))) \mid y \in f'(x)\} = \{g''(h'(g'(h^*(y)))) \mid y \in f^*(x)\}$. We have several cases:

1. $y \in C$: Then $h^*(y) = y$, $g'(y) = (f')^*(y) = y$, $h'(y) = y$ and $g''(y) = (f'')^*(y) = y$. Also $g(y) = y$ and so $g''(h'(g'(h^*(y)))) = g(y)$.
2. $y \in B$: Then $(f')^*(h^*(y)) = h^*(y)$ and so $g'(h^*(y)) = h^*(y)$. Furthermore $h'(h^*(y)) = f(y)$ and since $f(y) \notin \mathrm{dom}(f'')$, $g''(f(y)) = f(y)$. So $g''(h'(g'(h^*(y)))) = f(y) = g(y)$.
3. $y \in A - (B \cup C)$: Clearly $h^*(y) = y$ and $g'(y)$ is a set. So $h'(g'(y)) = g'(y)$. Then $g''(h'(g'(h^*(y)))) = g''(g'(y)) = g(y)$.

By 1-3, $g(x) = \{g''(h'(g'(h^*(y)))) \mid y \in f^*(x)\} = \{g(y) \mid y \in f^*(x)\} = f[g](x)$. □

Corollary 2.15. *ZFC*$^+$ is equivalent to *ZFA*.

Proof. Assume *ZFC*$^+$ and that (X, A, f) is a flat system of equations. Then $f(x) \subseteq \mathcal{U}$ for every $x \in \mathrm{dom}(f)$. Let g be such that $FP(g, f)$ holds. Then by Corollary 2.11, $S(g \restriction \mathrm{dom}(f), f)$ holds. Thus *AFA* holds. The Extensionality Axiom follows from Lemma 2.9 (i) and hence *ZFA* holds.

Assume *ZFA*. Then by Lemmas 2.13 and 2.14, every function has a fixed point. So FPA holds. By Theorem 7.3 of [2], the Strong Extensionality Axiom holds in *ZFA*. Hence *ZFC*$^+$ holds. □

Lemma 2.16. *If f is a substitution, then there is a function g such that $g = g \star f$.*

Proof. Let $u \in \mathcal{U}$ be such that $u \notin \mathrm{dom}(f)$. We define f' as follows. Let $\mathrm{dom}(f') = \mathrm{dom}(f)$. If $f(x)$ is not an urelement, or $f(x) = x$, or $f(x) \in \mathcal{U} - \mathrm{dom}(f)$, then let $f'(x) = f(x)$. If there is $n < \omega$ such that $\forall m < n : f^m(x) \in \mathcal{U}$ and $f^m(x) \neq f^{m-1}(x)$ but $f^n(x) \notin \mathcal{U}$, or $f^n(x) = f^{n-1}(x)$, or $f^n(x) \in \mathcal{U} - \mathrm{dom}(f)$, then let $f'(x) = f^n(x)$. Otherwise let $f'(x) = u$.

Let g' be such that $FP(g', f')$ holds and let $g = g' \restriction \mathrm{dom}(f)$. We show that $g = g \star f$.

If $f(x) \in \mathrm{dom}(f)$ is an urelement, then by the definition of f', we have that $f'(f(x)) = f'(x)$. Now if $f'(x) \in \mathcal{U}$, $g(x) = f'(x) = f'(f(x)) = g(f(x)) = \mathrm{sub}(g, f(x))$. If $f'(x) \notin \mathcal{U}$, $g(x) = \{g'(y) \mid y \in f'(x)\} = \{g'(y) \mid y \in f'(f(x))\} = g(f(x)) = \mathrm{sub}(g, f(x))$. If $f(x) \in \mathcal{U} - \mathrm{dom}(f)$, then $g^*(x) = f'(x) = f(x) = \mathrm{sub}(g, f(x))$. So we have shown that if $f(x)$ is an urelement, then $g(x) = g(f(x))$, hence we need to show that for all $x \in \mathrm{dom}(f)$, if $f(x) \notin \mathcal{U}$, then $g'(x) = \mathrm{sub}(g, f(x))$. For this, we define a bisimulation B so that $(a, b) \in B$ iff there is a $y \in \mathrm{dom}(g')$ such that $a = g'(y)$ and $b = \mathrm{sub}(g, f^*(y))$, or $a = b \in \mathcal{U} \cap \mathrm{dom}(g')$, or $a = g'(y) = b$.

To show that B is bisimulation, let $(g'(y), \mathrm{sub}(g, f^*(y))) \in B$ for some $y \in \mathrm{dom}(g')$. We have several cases:

1. $y \in \mathcal{U} - \mathrm{dom}(f)$: Then $(f')^*(y) = f^*(y) = y$ so $g'(y) = y = \mathrm{sub}(g, f^*(y))$.
2. $y \in \mathrm{dom}(f)$ and $f(y) \in \mathcal{U}$: As above we have that $g'(y) = g(f(y)) = \mathrm{sub}(g, f^*(y))$.
3. $y \in \mathrm{dom}(f)$, $f(y) \notin \mathcal{U}$: Because $f(y) \notin \mathcal{U}$, we have that $f(y) = f'(y)$, so

$$\begin{aligned} g'(y) &= \{g'(z) \mid z \in f'(y)\} \\ \mathrm{sub}(g, f^*(y)) &= \{\mathrm{sub}(g, z) \mid z \in f(y)\}. \end{aligned}$$

Assume $z \in f(y)$, so $z \in \mathrm{dom}(g')$. If $z \notin \mathrm{dom}(f)$, then $(g'(z), \mathrm{sub}(g, f^*(z))) \in B$. If $z \in \mathrm{dom}(f)$, then $\mathrm{sub}(g, z) = g(z) = g'(z)$ and $(g'(z), g'(z)) \in B$. □

For a class function F, $FP(G, F)$ is defined as for the set functions. We show that under ZFC^+ also the class functions have fixed points.

Lemma 2.17. *Assume ZFC^+. Let $F : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$ be a definable class function. Then there exists a unique definable class function $G : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$ such that $FP(G, F)$ holds.*

Proof. Let $x \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. If $F(x) \in \mathcal{U}$, then let $G(x) = F(x)$. Otherwise we define $G(x)$ as follows. Let $A_0 = TC(\{x\})$, $A_{n+1} = A_n \cup TC(F[A_n])$, and $A(x) = \bigcup_{n<\omega} A_n$. Then $A(x)$ is transitively closed and $F[A(x)] \subseteq A(x)$. Now let g be a function such that $FP(g, F \restriction A(x))$ holds and define $G(x) = g(x)$.

We show that $g(y)$ does not depend on the choice of $A(x)$ as long as $y \in A(x)$ and $F[A(x)] \subseteq A(x)$. Let A and A' be transitively closed sets such that $y \in A$, $F[A] \subseteq A$, and $F[A'] \subseteq A'$. Let g and g' be such that $FP(g, F \restriction A)$ and $FP(g', F \restriction$

A') hold. Let $C = A \cap A'$, then C is transitively closed, $y \in C$, and $F[C] \subseteq C$. We show that $g \restriction C = g' \restriction C$ from which the claim follows. So let $z \in C$ and let B consist of $(g(z), g'(z))$ together with $(a, b) \in TC(g(z)) \times TC(g'(z))$ such that either $a = b$ or there is $c \in C$ such that $a = g(c)$ and $b = g'(c)$. To show that B is a bisimulation between $g(z)$ and $g'(z)$, let $c \in C$ and $(g(c), g'(c)) \in B$. Assume that $F(c)$ is not an urelement. Then $g(c) = \{g(d) \mid d \in (F \restriction C)(c)\}$. Now if $d \in (F \restriction C)(c)$, then $d \in C$ and so $(g(d), g'(d)) \in B$. So B is a bisimulation and hence for all $z \in C$, $g(z) = g'(z)$.

We show that G is a fixed point of F. Let $x \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. If $F(x) \in \mathcal{U}$, then $G(x) = F[G](x) = F(x)$. If $F(x) \notin \mathcal{U}$, let $A(x)$ be the transitively closed set such that $x \in A(x)$ and $F[A(x)] \subseteq A(x)$. Let g be such that $FP(g, F \restriction A(x))$ holds. Then $G(x) = g(x) = \{g(y) \mid y \in (F \restriction A(x))^*(x)\} = \{G(y) \mid y \in F(x)\}$, because $A(x)$ is transitively closed and $F[A(x)] \subseteq A(x)$. □

As a corollary we have the Theorem 8.1 of [2].

Corollary 2.18. There is a unique operation $\mathrm{sub}(s, b)$ as in Definition 2.2 defined for all substitutions s and sets b.

Proof. Assume s is a substitution. Define a class function F by $F(x) = x$ if $x \notin \mathrm{dom}(s)$ and $F(x) = s(x)$ otherwise. By the above lemma, let G be such that $FP(G, F)$ holds. We claim that $G(x) = \mathrm{sub}(s, x)$ for all x.

If x is an urelement, then $G(x) = F(x) = \mathrm{sub}(s, x)$. Let x be a set. Define the relation B by $(a, b) \in B$ iff $a = G(y)$ and $b = \mathrm{sub}(s, y)$ for some $y \in \cup TC(\{x\})$. We show that B is a bisimulation between $G(x)$ and $\mathrm{sub}(s, x)$. The case for urelements is as in the above, so let $(G(y), \mathrm{sub}(s, y)) \in B$ where $G(y)$ is a set. Then $G(y) = \{G(z) \mid z \in F(y)\}$ and $\mathrm{sub}(s, y) = \{\mathrm{sub}(s, z) \mid z \in y\}$. Because $y \notin \mathrm{dom}(s)$, $F(y) = y$ and we see that the bisimulation can be continued. □

We finish this section by showing that a fixed point of a fixed point of a basic f is a fixed point of f. Thus the name fixed point.

Lemma 2.19. *Assume ZFC$^+$. For every function f there is a function g such that $FP(g, f)$ holds and* $\mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$.

Proof. Let f be a function and g' such that $FP(g', f)$. We define inductively functions f_n and g_n for $n < \omega$ as follows. Let $f_0 = f$ and $g_0 = g'$.

Let $A_n = \mathrm{dom}(g_n) \cup \mathrm{rng}(g_n) \cup \bigcup \mathrm{rng}(g_n)$ and $\mathrm{dom}(f_{n+1}) = \mathrm{dom}(f_n) \cup A_n$. Define $f_{n+1}(x) = f_n(x)$, if $x \in \mathrm{dom}(f_n)$, and otherwise $f_{n+1}(x) = x$. Let g_{n+1} be such that $FP(g_{n+1}, f_{n+1})$ holds.

Because for every n, $\mathrm{dom}(f_n) \subseteq \mathrm{dom}(g_{n+1})$, we have that

$$\mathrm{rng}(g_n) \cup \bigcup \mathrm{rng}(g_n) \subseteq \mathrm{dom}(g_{n+1}). \tag{2}$$

From the definition of f_n if follows that for all n, $f_n \subseteq f_{n+1}$ and also $f \subseteq f_n$. Clearly if $x \notin \mathrm{dom}(f)$, then $f_n(x) = x$, hence for every x and n, $f_n^*(x) = f^*(x)$.

It is clear that $\mathrm{dom}(f) \subseteq \mathrm{dom}(g_n)$. Also $f^*[\mathrm{dom}(g_n)] = f[\mathrm{dom}(g_n) \cap \mathrm{dom}(f)] \cup (\mathrm{dom}(g_n) - \mathrm{dom}(f)) = f_n^*[\mathrm{dom}(g_n)]$. Hence $\bigcup f^*[\mathrm{dom}(g_n)] \subseteq \bigcup f_n^*[\mathrm{dom}(g_n)] \subseteq \mathrm{dom}(g_n)$. If $x \in \mathrm{dom}(g_n)$ and $f^*(x)$ is an urelement, then $g_n(x) = f_n^*(x) = f^*(x)$. If $f^*(x)$ is not an urelement, then $g_n(x) = \{g_n(y) \mid y \in f_n^*(x)\} = \{g_n(y) \mid y \in f^*(x)\}$. Thus $FP(g_n, f)$.

Now because for every $n < \omega$, $FP(g_n, f)$, $FP(g_{n+1}, f)$, and $\mathrm{dom}(g_n) \subseteq \mathrm{dom}(g_{n+1})$, we have by Lemma 2.9 (iii) that $g_n \subseteq g_{n+1}$. So we can define $g = \bigcup_{n<\omega} g_n$. By (2) we have that $\mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$.

Finally, we show that $FP(g, f)$ holds. Because for all n, $FP(g_n, f)$ holds, we have that $\mathrm{dom}(f) \cup \bigcup f^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g)$. Let $x \in \mathrm{dom}(g)$. Then for some n, $x \in \mathrm{dom}(g_n)$. If $f^*(x) \in \mathcal{U}$, then $g(x) = g_n(x) = f^*(x)$. Otherwise $g(x) = g_n(x) = \{g_n(y) \mid y \in f_n^*(x)\} = \{g(y) \mid y \in f^*(x)\} = f[g](x)$. □

Lemma 2.20. *Assume $ZFC^{-2} + SEA$, f is basic, $FP(g, f)$ holds and* $\mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$. *Then for all $x \in \mathrm{dom}(g)$, $g(g(x)) = g(x)$ and if $g(x) \notin \mathcal{U}$, then $g(x) = \{g(y) \mid y \in g(x)\}$. In particular, $FP(g, g)$ and $S(g, g)$ hold.*

Proof. Let $a \in \mathrm{dom}(g)$. Let B consist of $(g(a), g(g(a)))$ together with $(x, y) \in TC(g(a)) \times TC(g(g(a)))$ such that either $x = y$ or there is $z \in \mathrm{dom}(g)$ such that $x = g(z)$ and $y = g(g(z))$.

We show that B is a bisimulation between $g(a)$ and $g(g(a))$. Assume that $z \in \mathrm{dom}(g)$, $f^*(z) \notin \mathcal{U}$, and $(g(z), g(g(z))) \in B$.

Let $y \in g(z) = \{g(w) \mid w \in f^*(z)\}$. So $y = g(w)$ for some $w \in f^*(z)$. Because $g(w) \in \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$, $g(g(w))$ is defined. So $(g(w), g(g(w))) \in B$.

Let $y \in g(g(z)) = \{g(w) \mid w \in f^*(g(z))\}$. So $y = g(w)$ for some $w \in f^*(g(z)) \subseteq \mathrm{dom}(g)$. Thus $(g(w), g(g(w))) \in B$.

For the second claim, assume that $g(x) \notin \mathcal{U}$. So $g(x) \notin \mathrm{dom}(f)$ and thus $g(x) = g(g(x)) = \{g(y) \mid y \in g(x)\} = g[g](x)$. Thus $S(g, g)$ holds. Because $\mathrm{dom}(g) \cup \bigcup g^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g) \cup \mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$, we have that $FP(g, g)$ holds. □

The assumption that f is basic is needed in Lemma 2.20:

Example 2.21. Assume ZFC^+ (the first example works also in ZFC).

(i) We define sets e^n, $n < \omega$, so that $e^0 = \emptyset$ and $e^{n+1} = \{e^n\}$. Let f be such that $f(e^3) = e^2$, $f(e^2) = \{e^0, e^1\}$ and for $n < 2$, $f(e^n) = e^n$. Then $FP(f, f)$ holds, but $f(f(e^3)) = \{e^0, e^1\} \neq f(e^3)$

(ii) Let Ω be such that $\Omega = \{\Omega\}$. Define f so that $f(\emptyset) = \{\emptyset\}$ and $f(\Omega) = \emptyset$. Let g be such that $FP(g, f)$ holds: Then since $g(x) = \{g(y) \mid y \in f^*(x)\}$, $g(\emptyset) = \Omega$ and $g(\Omega) = \emptyset$. But $\{g(y) \mid y \in g(\emptyset)\} = \{\emptyset\} \neq g(\emptyset)$, so it is not the case that $FP(g, g)$.

2.4. **A model in which not all equations have solutions.** We now turn to the question: Do we need to assume that all flat systems of equations have solutions in order to get all fixed points. First we show how to construct a model of set theory from a given transitive class of non–wellfounded sets.

Definition 2.22. Let $C \subset V_{afa}[\mathcal{U}] \cup \mathcal{U}$ be a transitive class.

$$\mathbf{C}(C) = \{x \in V_{afa}[\mathcal{U}] \cup \mathcal{U} \mid \text{there is no sequence } x_i, i < \omega \text{ such that } x_0 \in TC(x) \text{ and } \forall i, x_{i+1} \in x_i, x_i \notin C\}.$$

Intuitively this class is the same as the following class V'': Let $V'_0 = C \cup \mathcal{U}$, $V'_{\alpha+1} = \{x \mid x \subseteq V'_\alpha\}$, $V'_\beta = \bigcup_{\alpha<\beta} V'_\alpha$, when β is a limit ordinal, and let $V'' = \bigcup\{V'_\alpha \mid \alpha \text{ is an ordinal}\}$.

Lemma 2.23. *Assume that $C \subset V_{afa}[\mathcal{U}] \cup \mathcal{U}$ is a transitive class and $V' = \mathbf{C}(C)$, then $V' \models ZFC^{-2} + SEA$.*

Proof. Now V' is a transitive class, so the Axioms of Extensionality and Strong Extensionality hold in V'. If x is a subset of V', then clearly $x \in V'$. Hence the Power Set Axiom holds in V'.

The Axiom of Urelements, $\forall p \forall q(\mathcal{U}(p) \rightarrow \neg(q \in p))$ holds in V'. The Pairing and Union Axioms also hold in V'. Because $\omega \in V'$, we have that $\emptyset$ and the successor operation are absolute for V', V' satisfies the Axiom of Infinity.

For the Collection Axiom it is enough to show that for each formula $\phi(x, y, A, w_1, \ldots, w_n)$ and each $A, w_1, \ldots, w_n \in V'$, if $\forall x \in A \exists! y \in V' \phi^{V'}(x, y, A, w_1, \ldots, w_n)$, then $\exists Y \in V'(\{y \mid \exists x \in A, \phi^{V'}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y)$. So let $Y = \{y \in V' \mid \exists x \in A, \phi^{V'}(x, y, A, w_1, \ldots, w_n)\}$. Then $Y \subset V'$ and hence $Y \in V'$.

Since for every $z \in V'$, $P(z) \subseteq V'$, we have that V' satisfies the Separation Axiom. For the Axiom of Choice, we can show that if $x \in V'$ and x can be well–ordered, then $(x \text{ can be well–ordered})^{V'}$: If $R \subseteq x \times x$ well–orders x, then since $x \times x \in V'$ we have that $R \in V'$. The formula "R totally orders x" is absolute for V'. For well–ordering we have to check that $(\forall y \phi(y, x, R))^{V'}$, where $\phi(y, x, R)$ is

$$y \subseteq x \wedge y \neq \emptyset \rightarrow \exists z \in y \forall w \in y((w, z) \notin R).$$

Now ϕ is absolute for V' so it is enough to show that $\forall y \in V' \phi(y, x, R)$, which follows since R well–orders x. Thus the Axiom of Choice holds in V'.

For the Axiom of Plenitude, which is: $\forall S \notin \mathcal{U}(\exists f : S \rightarrow \mathcal{U}$ such that f is injective and $f[S] \cap S = \emptyset)$, let S be a set in V'. Let $g : S \rightarrow \mathcal{U}$ be an injection in $V_{afa}[\mathcal{U}]$. Then also $g \in V'$, because V' is closed under the power set operation. We have shown that $V' \models ZFC^{-2} + SEA$. □

Lemma 2.24. *Assume that $C \subset V_{afa}[\mathcal{U}]$ is a transitive class and there exist x_i, for $i < \omega$ such that $x_{i+1} \in TC(x_i)$ and $x_i \notin C$. If $V' = \mathbf{C}(C)$, then $V' \models ZFC^{-2} + SEA$ and $V' \not\models AFA$.*

Proof. We define the canonical flat system of equations for x_0 as follows. Let h be an injection such that $\mathrm{dom}(h) = TC(x_0)$, $\mathrm{rng}(h) \subseteq \mathcal{U}$ and if $a \in TC(x_0) \cap \mathcal{U}$, then $h(a) = a$. Let $A = TC(x_0) \cap \mathcal{U}$, $a_0 \in \mathcal{U} - \mathrm{rng}(h)$, $X = (\mathrm{rng}(h) \cup \{a_0\}) - A$. Define f in X such that $f(a_0) = \{h(y) \mid y \in x_0\}$ and $f(h(z)) = \{h(y) \mid y \in z\}$ for $z \in \mathrm{dom}(h)$. So f is a system of equations which belongs to V' and it was constructed so that for the solution g to f, we have that $g(a_0) = x_0$.

Now x_0 can not be in V', because of the definition of V'. Because being a solution to a flat system of equations is an absolute property for V', we have that f has no solution in V'. □

Next we introduce the notion of a flat $\mathcal{P}$–coalgebra from [2], which corresponds to a flat system of equations with no atoms.

Definition 2.25. A flat $\mathcal{P}$–coalgebra is a pair (X, f) such that $X \subseteq \mathcal{U}$ is a set of urelements and $f : X \to \mathcal{P}(X)$. A function g is a *substitution solution* to the flat $\mathcal{P}$–coalgebra if $FP(g, f)$ holds.

It is shown in [2] that if g is a substitution solution to a flat $\mathcal{P}$–coalgebra, then $\mathrm{rng}(g) \subseteq \mathcal{P}^*$, where $\mathcal{P}^*$ is the greatest fixed point of the operator $\mathcal{P}$ and it is equal to the class of all pure sets $V_{afa}[\emptyset]$. (In case there are no urelements, then $\mathcal{P}^*$ is the whole universe and the example below does not hold anymore, but see the next section.)

Example 2.26. Assume $ZFC^{-2} + SEA$. The following does not imply *AFA*: Every flat $\mathcal{P}$–coalgebra has a substitution solution.

Proof. Let $C = V_{afa}[\emptyset]$. If (X, f) is a flat $\mathcal{P}$–coalgebra and g its solution, then $\mathrm{rng}(g) \subseteq C$. If we let $V' = \mathbf{C}(C)$, then $\mathrm{dom}(g) \in V'$ and hence $g \in V'$, because V' is closed under the power set operation. Because being a flat $\mathcal{P}$–coalgebra and a solution to it are absolute properties for V', we have that in V' every flat $\mathcal{P}$–coalgebra has a solution.

Let x be an urelement and $f(x) = \{a, x\}$, where $a \in \mathcal{U}$ and $a \neq x$. Then f is an equation system and it has a solution g in $V_{afa}[\mathcal{U}]$. Then $g(x) = \{a, g(x)\} \notin C$, and we get the x_i's as required in Lemma 2.24, by setting $x_i = g(x)$. Hence by Lemma 2.24, *AFA* does not hold in V'. □

Γ–coalgebras can be seen as systems of equations (see [2] section 16) and if we restrict our interest to flat Γ–coalgebras, then the class of solutions can be seen as the final Γ–coalgebra. One may wonder if the same can be done (e.g. by fixed points) for a larger class of Γ–coalgebras than the flat ones. This does not seem to be the case or at least a much deeper understanding of non–wellfounded sets is needed. The crucial property of the flat Γ–coalgebras (X, e) is that X is new for Γ (i.e. for all substitutions t and sets a, $\Gamma(a[t]) = \Gamma(a)[t]$), this forces X to be flat in the usual sense of the word. And without something like this the theory does not work. E.g. the crucial Lemma 16.1 in [2] fails:

Let $\Gamma(X) = \mathcal{P}(\mathcal{P}(X) - \{\emptyset\})$. This is a monotone and proper operator, i.e. if $X \subseteq Y$, then $\Gamma(X) \subseteq \Gamma(Y)$, and for all sets a, $\Gamma(a) \subseteq V_{afa}[\mathcal{U}]$. Let $X = \{\emptyset, \{\emptyset\}\}$, and let $e(\emptyset) = \{\{\emptyset\}\}$ and $e(\{\emptyset\}) = \emptyset$. Then (X, e) is a Γ–coalgebra that is not flat. If s is a solution to e, then $s(\emptyset) = \{\emptyset\}$ and $s(\{\emptyset\}) = \emptyset$ but $\{\emptyset\} \notin \Gamma^*$. Hence s is not a Γ–morphism of (X, e) into (Γ^*, id).

2.5. **Classifying non–wellfounded sets.** Here we have a fixed class of urelements, $\mathcal{U}$, and all sets can contain urelements as their members. So from now on we denote by V_{afa} the class $V_{afa}[\mathcal{U}] \cup \mathcal{U}$ of [2]. But as in the above, urelements are not vital in here. We regard arbitrary functions as equation systems and when we speak of the indeterminates of the equation systems, we mean the elements of their domain.

We define a series of classes of equation systems E_α, $\alpha \in \mathbf{ON}$ in increasing complexity. From these equation systems we obtain a series of classes of non–wellfounded sets,

$$V^0_{afa} \subset V^1_{afa} \subset \cdots \subset V^\alpha_{afa} \subset \cdots ,$$

so that $V^{\alpha+1}_{afa} \not\subset V^\alpha_{afa}$. We also define the *rank* of a non–wellfounded set x as the least α such that $x \in V^\alpha_{afa}$.

The non–wellfounded sets become more complicated in the series $V^0_{afa} \subset V^1_{afa} \subset \cdots$ according to the branching structure of the non–wellfounded sets. V^0_{afa} is the class of wellfounded sets. In V^1_{afa} there are sets, which can be described as either Ω and sets that can be obtained from it by standard set theoretical operations or sets which have a non-wellfounded $\in$–sequence of length ω such that going down this sequence one has ω chances to branch out of that sequence. But in V^1_{afa} after branching the sets are wellfounded. In V^2_{afa} there are sets in which there are ω chances to branch to sets in which there are again ω chances to branch into sets in which there are only finite number of possibilities to branch. So the rank tells how many times it is possible to branch arbitrarily deep.

A non–wellfounded set of rank ω has elements of arbitrarily high rank below ω. In a non–wellfounded set of rank $\omega + 1$ one can find a non–wellfounded $\in$–sequence in which there are ω chances to branch into sets of rank ω. And so on in the higher degrees.

There is also the possibility that this branching process goes on arbitrarily long. In this case we say that the rank is ∞. First we need to characterize the different classes of equation systems. This is done in game theoretic terms.

Definition 2.27. Let f be a system of equations. A sequence $\vec{x} = \langle x_i \mid i < \omega\rangle$, where $x_i \in \mathrm{dom}(f), i < \omega$, of indeterminates of f is called *descending* if for all $i < \omega$, $x_{i+1} \in f(x_i)$.

We describe a game $G_\alpha(\mathcal{E})$, where $\alpha \in \mathbf{ON}$, that is played on a given system of equations f as follows. There are two players, black and white. First the black player chooses a descending sequence $\vec{x}$ of indeterminates. Then white chooses an ordinal $\alpha_0 < \alpha$ and a natural number $n < \omega$. Black must respond with a descending sequence of indeterminates $\vec{y}$ such that for some $m \geq n$, $y_0 \in f(x_m)$, and $y_0 \neq x_{m+1}$. So $\vec{y}$ branches out of $\vec{x}$. Then again white chooses an ordinal $\alpha_2 < \alpha_1$ and a natural number and so on.

The length of this game is the number of pairs of moves by black and white. This length is finite, since there are no infinite descending sequences of ordinals.

We say that black has a winning strategy in the game $G_\alpha(f)$ if she is able to respond to white's moves until white has no more moves. White wins otherwise, that is if black is not able to respond with a descending sequence of indeterminates to white's move.

There is also a game of infinite length. In $G_\infty(f)$ the white player does not choose ordinals, only indices. Hence the length of this game is ω.

More formally, we say that a move of the white player is a pair (α, n), where α is an ordinal and $n < \omega$. We use the projection function $\pi_2(\alpha, n) = n$ to get the second coordinate of the pair (α, n). In $G_\alpha(f)$ we say that a sequence $\vec{w}$ is a *legal* sequence of white's moves of length k if

$$\vec{w} = \langle (\alpha_i, n_i) \mid i < k \rangle, \ \forall i \in \omega (n_i < \omega), \text{ and } \alpha > \alpha_0 > \alpha_1 > \cdots > \alpha_{k-1}$$

We define the winning strategy σ for black as follows.

Definition 2.28. Let f be a system of equations and α an ordinal. A *winning strategy* for the black player in the game $G_\alpha(f)$ is a function σ of two arguments, a natural number k and a legal sequence $\vec{w}$ of white's moves of length k, that satisfies the following conditions:

(i) $\sigma(0, \emptyset) = \vec{x}$, where $\vec{x}$ is a descending sequence of the indeterminates of f

(ii) $\sigma(k+1, \vec{w}) = \vec{y}$, where $\vec{y}$ is a descending sequence of indeterminates such that the following holds. Denote by $\vec{x}$ the previous move of black, i.e. $\sigma(k, \vec{w} \restriction k)$ and denote by n white's last move, i.e. $\pi_2(w_k)$. We require from $\vec{y}$ that $\exists m \geq n (y_0 \in f(x_m)$ and $y_0 \neq x_{m+1})$.

We say that the black player *wins* a game if she has a winning strategy. The white player wins, if the black player does not win.

We may also define a similar game played on non–wellfounded sets, $G_\alpha(x)$. We say that a sequence $\langle x_i \mid i < \omega \rangle$ is a non–wellfounded sequence, if for all $i < \omega$, $x_{i+1} \in x_i$. If we replace in the above definitions the system of equations f by a set x and the descending sequences of indeterminates by non–wellfounded sequences, then we have the corresponding definition for sets. For sets we also require that $\sigma(0, \emptyset)$ is a non–wellfounded sequence starting from x.

If white wins the game $G_0(x)$, then x is well–founded. If white wins $G_1(x)$, and black wins $G_0(x)$, then in $TC(x)$ there are non–wellfounded sets but no sets in which we can branch two times as described above. Also, if black wins G_α, then black wins G_β for all $\alpha \leq \beta$ and if white wins G_α, then white wins G_β for all $\beta \geq \alpha$.

Definition 2.29. .

(i) $E_\alpha = \{f \mid \text{white wins } G_\alpha(f)\}$

(ii) $E_\infty = \{f \mid \text{black wins } G_\infty(f)\}$,

(iii) $V^\alpha_{afa} = \{x \mid \text{white wins } G_\alpha(x)\}$

(iv) $V^\infty_{afa} = \{x \mid \text{black wins } G_\infty(x)\}$

(v) AFA_α is the statement that all the systems of equations in E_α have solutions.

From the preceding definition it follows that if x is a set and f is its canonical system of equations, then $x \in V^\alpha_{afa}$ iff $f \in E_\alpha$. Also black player's winning strategy in the game on sets can be straightforwardly converted into a winning strategy in the game on equation systems. Thus all the solutions to the equation systems from E_α are in V^α_{afa}.

The solution set x to an equation system f does not always have the same rank as f. For example define an equation system f such that $\mathrm{dom}(f) = \{u_\eta \in \mathcal{U} \mid \eta \in 2^{<\omega}\}$, where u_η's are distinct, by $f(u_\eta) = \{u_{\eta^\frown\{0\}}, u_{\eta^\frown\{1\}}\}$. Then $f \notin E_\alpha$ for all α but the solution set of f is Ω, by exercise 7.1 of [2], and $\Omega \in V^1_{afa}$.

Definition 2.30. The *non–wellfoundedness* rank of a set x, denoted by $\mathrm{nwfrank}(x)$ is the least α such that $x \in V^\alpha_{afa}$, if there is such and ∞ otherwise.

Note that for x such that $\mathrm{nwfrank}(x) \in \mathbf{ON}$ we have that $\mathrm{nwfrank}(x) = \min\{\alpha \mid \text{white wins } G_\alpha(x)\} = \sup\{\alpha + 1 \mid \text{black wins } G_\alpha(x)\}$. We also have that if $x \in y$, then $\mathrm{nwfrank}(x) \leq \mathrm{nwfrank}(y)$ but not necessarily $\mathrm{nwfrank}(x) < \mathrm{nwfrank}(y)$. In fact, Marshall and Schwarze [18] have shown that it is not possible to define a rank function r such that if $x \in y$, then $r(x) < r(y)$, in set theory without the Foundation Axiom. Another notion of rank for non–wellfounded sets, defined using modal logic, appears in [2], section 11.

Lemma 2.31. *Black wins $G_\alpha(x)$ iff there is a non–wellfounded sequence $\vec{x}$ starting from x such that for all $\beta < \alpha$ the set*

$$A_\beta = \{i < \omega \mid \exists y \in x_i (y \neq x_{i+1} \text{ and black wins } G_\beta(y))\}$$

is an unbounded subset of ω.

Proof. Assume that black has a winning strategy σ in the game $G_\alpha(x)$. Let $\vec{x} = \sigma(0, \emptyset)$. Let $n < \omega$ and $\beta < \alpha$. If we let (β, n) be the first move of the white player, then $\sigma(1, (\beta, n))$ is a non–wellfounded sequence $\vec{y}$ such that $\exists i \geq n (y_0 \in x_i$ and $y_0 \neq x_{i+1})$ by the definition of a winning strategy. The winning strategy σ' for black in the game $G_\beta(y_0)$ is defined by the following equations

$$\begin{aligned} \sigma'(0, \emptyset) &= \sigma(1, (\beta, n)) \\ \sigma'(m + 1, \vec{w}) &= \sigma(m + 2, (\beta, n)^\frown\vec{w}). \end{aligned}$$

Hence A_β is unbounded in ω.

Assume on the other hand that there is a non–wellfounded sequence $\vec{x}$ starting from x and satisfying the condition. We prove that black has a winning strategy σ in the game $G_\alpha(x)$. Let $\sigma(0, \emptyset) = \vec{x}$. Let $(\beta, n) \in \alpha \times \omega$ be white's first move. Since A_β is unbounded in ω there is $i \geq n$ such that $\exists y \in x_i(y \neq x_{i+1}$ and black wins $G_\beta(y)$ with winning strategy $\sigma')$. Then let

$$\begin{aligned} \sigma(1, (\beta, n)) &= \sigma'(0, \emptyset) \\ \sigma(m + 2, (\beta, n)^\frown\vec{w}) &= \sigma'(m + 1, \vec{w}). \end{aligned}$$

□

Corollary 2.32. nwfrank$(x) \geq \alpha$ iff for all $\beta < \alpha$ there is a non–wellfounded sequence $\vec{x}$ starting from x such that

$$\{i < \omega \mid \exists y \in x_i (y \neq x_{i+1} \text{ and } \mathrm{nwfrank}(y) \geq \beta)\}$$

is an unbounded subset of ω.

By Corollary 2.32 we could have also defined the classes V^{α}_{afa} via the concept of nwfrank, by letting wellfounded sets have rank 0. So we have that $V^{\alpha}_{afa} = \{x \mid \mathrm{nwfrank}(x) \leq \alpha\}$.

Theorem 2.33. $V^{\alpha}_{afa} \models ZFC^{-2} + SEA + AFA_{\alpha}$.

Proof. Let $V' = \mathbf{C}(V^{\alpha}_{afa})$. We claim that $V^{\alpha}_{afa} = V'$ from which the conclusion follows. Since all the solutions to the equation systems from E_{α} are in V^{α}_{afa}, $V^{\alpha}_{afa} \models AFA_{\alpha}$.

By the definition of V', we have that $V^{\alpha}_{afa} \subseteq V'$. We show that $V' \subseteq V^{\alpha}_{afa}$ by showing that if nwfrank$(x) > \alpha$, then $x \notin V'$.

Assume towards a contradiction that there is $x \in V'$ for which nwfrank$(x) > \alpha$. So the white player does not have a winning strategy in $G_{\alpha}(x)$ and this means that the black player has. From this it follows by Lemma 2.31, that there is a non–wellfounded sequence $\vec{x}$ starting from x such that for all $\beta < \alpha$ the set

$$A_{\beta} = \{i < \omega \mid \exists y \in x_i (y \neq x_{i+1} \text{ and black wins } G_{\beta}(y))\}$$

is an unbounded subset of ω.

Let $i < \omega$. Black wins $G_{\beta}(x_i)$ for all $\beta < \alpha$, because $\vec{x} \restriction [i, \omega]$ is now a non–wellfounded sequence where black wins. So by Lemma 2.31, black wins $G_{\alpha}(x_i)$. Hence nwfrank$(x_i) > \alpha$, and so $x_i \notin V^{\alpha}_{afa}$ which violates the definition of $\mathbf{C}(V^{\alpha}_{afa})$.
□

From the preceding proof we can extract the following corollaries:

Corollary 2.34. If $\alpha < \gamma$, then $V^{\alpha}_{afa} \subsetneq V^{\gamma}_{afa}$.

Proof. If nwfrank$(x) = \gamma > \alpha$, then by the proof of the theorem, we have that $x \notin V^{\alpha}_{afa}$.

We construct an example of a set x for which nwfrank$(x) = \gamma$ as follows.

Let $X = \{u_{\alpha} \mid \alpha \leq \gamma\}$ be a set of distinct urelements. Let f be such that dom$(f) = X$ and for $\alpha \leq \gamma$, let

$$f(u_{\alpha}) = \begin{cases} \{u_{\beta} \mid \beta < \alpha\} & \text{if } \alpha \text{ is a limit,} \\ \{u_{\alpha}, u_{\alpha-1}\} & \text{if } \alpha \text{ is a successor,} \\ \emptyset & \text{if } \alpha = 0. \end{cases}$$

Let g be the solution to f and let $x = g(u_{\gamma})$. We show by induction that nwfrank$(g(u_{\alpha})) = \alpha$ for $\alpha \leq \gamma$. It is clear that nwfrank$(g(u_0)) = 0$.

Assume the claim for α. By the construction, there is a non–wellfounded sequence $\vec{x} = \langle g(u_{\alpha+1}), g(u_{\alpha+1}), \ldots \rangle$ starting from $g(u_{\alpha+1})$. Let $i < \omega$. Then $g(u_\alpha) \in x_i$ and $\text{nwfrank}(g(u_\alpha)) = \alpha$, hence by Corollary 2.32, $\text{nwfrank}(g(u_{\alpha+1})) \geq \alpha+1$. We show that white wins $G_{\alpha+1}(g(u_{\alpha+1}))$, whence $\text{nwfrank}(g(u_{\alpha+1})) = \alpha+1$. For the first move black has to choose $\vec{x}$ (other choices would be worse). But black can not win $G_{\alpha+1}(g(u_{\alpha+1}))$ in $\vec{x}$ by Lemma 2.31, because for all $i < \omega$, it holds that there is no $y \in x_i$ such that black wins $G_\alpha(y)$.

Assume the claim for $\beta < \alpha$. By the construction, $g(u_\alpha) = \{g(u_\beta) \mid \beta < \alpha\}$. For every $\beta < \alpha$, $\text{nwfrank}(g(u_\beta)) = \beta$ so $\text{nwfrank}(g(u_\alpha)) \geq \alpha$. We show that white wins $G_\alpha(g(u_\alpha))$. Black has to choose some non–wellfounded sequence starting from $g(u_\alpha)$, say $\vec{x} = \langle g(u_\beta), g(u_\beta), \ldots \rangle$. Then white chooses some ordinal γ such that $\beta < \gamma < \alpha$. Now black cannot win $G_\gamma(g(u_\beta))$, because $\text{nwfrank}(g(u_\beta)) = \beta < \gamma$. □

Corollary 2.35. If $\alpha < \gamma$, then $V^\alpha_{afa} \not\models AFA_\gamma$.

Proof. If we let f be the canonical equation system for a set x such that $\text{nwfrank}(x) = \gamma$, then $f \in E_\gamma$. But f does not have a solution in V^α_{afa}. □

Next we show that all the AFA_α axioms together with AFA_∞ imply AFA. But note that $\forall\alpha AFA_\alpha \not\vdash AFA$.

Lemma 2.36. $\vdash AFA \leftrightarrow (AFA_\infty \wedge \forall\alpha\, AFA_\alpha)$.

Proof. Let f be an arbitrary system of equations, and assume that the white player does not win $G_\alpha(f)$ for any α. We show that then black wins $G_\infty(f)$. For a descending sequence of indeterminates $\vec{u}$ of f, let

$$r(\vec{u}) = \sup\{\alpha \mid \text{black wins } G_\alpha(f) \text{ where the first move of black is } \vec{u}\}.$$

There is an ordinal α such that if $\vec{u}$ is a descending sequence of indeterminates of f and $r(\vec{u}) \geq \alpha$, then $r(\vec{u}) = \infty$. This is so because otherwise the set $\{\vec{u} \mid \vec{u}$ is a descending sequence of indeterminates of $f\}$, and hence f, would be a proper class.

We describe a winning strategy for black in the game $G_\infty(f)$ as follows. There is a descending sequence of indeterminates $\vec{u}_0$ of f such that $r(\vec{u}_0) \geq \alpha$ since otherwise we could take

$$\gamma = \sup\{r(\vec{u}) \mid \vec{u} \text{ from } f \text{ such that } r(\vec{u}) \neq \infty\}$$

and white would win $G_{\gamma+1}(f)$. Let $\vec{u}_0$ be the first move of black. Let n be the first move of white. Because $r(\vec{u}_0) \geq \alpha$, then by the above, $r(\vec{u}_0) = \infty$. So there is a descending sequence $\vec{u}_1$ such that it branches out of $\vec{u}_0$ below n and $r(\vec{u}_1) \geq \alpha$. So this way we can continue the game arbitrarily long. □

By the previous lemma, we also see that $V^\infty_{afa} \cup \{x \mid \exists\alpha(x \in V^\alpha_{afa})\} = V_{afa}$.

3. A domain order over non–wellfounded sets

In this section we construct a domain structure on a subclass of $V_{afa}[\mathcal{U}]$. The approximation order $\sqsubseteq$ is based on a certain kind of tree extension on canonical tree pictures of sets.

3.1. **Introduction.** Domain theory can be seen as a theory of approximation and it is used widely in computer science to model various programming languages and abstract data types. On the other hand, non–wellfounded sets, studied in Aczel [1] and in Barwise and Moss [2], have found applications in computer science. There are connections between these two theories. In an appendix to his book [1], Peter Aczel writes:

> A natural way to try to understand non–well–founded sets is to view them as limits, in some sense, of their well–founded approximations. This approach is inspired by Scott's theory of domains, but it cannot be done in any simple minded way, as I found out.

In [20], Mislove, Moss, and Oles show how to construct a domain structure on HF_1, the set of all hereditarily finite sets, which may be non–wellfounded. Their construction involves initial continuous algebras. Here we try to define a domain ordering directly by giving a set–theoretic definition in terms of certain kinds of tree extensions.

In [3], Boffa describes a way to approximate non–wellfounded sets by their wellfounded components by an inverse limit construction. The approximating sets are taken from the class of all wellfounded and hereditarily finite sets, HF_0. We show that this actually defines a domain ordering in the appropriate domain completion, $\widehat{HF_0}$.

This approach differs from the one in [20] in that it does not use protosets. Lindström [16] has also studied non–wellfounded sets as the inverse limits of wellfounded sets in constructive set theory.

3.2. **Domains.** We give the basic definitions of domains. As a reference to domain theory we use Stoltenberg–Hansen, Lindström, and Griffor [22]. Let $(D, \sqsubseteq)$ be a partially ordered set. A set $A \subseteq D$ is called directed, if $A \neq \emptyset$ and for all $a_1, a_2 \in A$ there is $a \in A$ such that $a_1 \sqsubseteq a$ and $a_2 \sqsubseteq a$. We denote by $\bigsqcup A$ the least upper bound of A, if it exists.

Definition 3.1. Let $D = (D, \sqsubseteq, \bot)$ be a partially ordered set with a least element $\bot$. Then D is called a *complete partial order* if $\bigsqcup A$ exists for all directed $A \subseteq D$.

In domains we have an abstract notion of finiteness, or compactness as it is called, by the following definition.

Definition 3.2. Let D be a complete partial order. An element $a \in D$ is said to be *compact*, if whenever $A \subseteq D$ is a directed set and $a \sqsubseteq \bigsqcup A$, then there is some $x \in A$ such that $a \sqsubseteq x$. We say that D is *algebraic* if for each $x \in D$, the set $\text{approx}(x) = \{a \sqsubseteq x \mid a \text{ is compact}\}$ is directed and $x = \bigsqcup \text{approx}(x)$.

We add one more condition to obtain the definition of Scott–Ershov domains. Two elements a and b of D are called consistent if there is c such that $a \sqsubseteq c$ and $b \sqsubseteq c$.

Definition 3.3. An algebraic complete partial order D is called a *domain* if for any two consistent compact elements a and b, $a \sqcup b$ exists.

Domains can be obtained also by so called ideal completions of conditional upper semi lattices.

Definition 3.4. A partial order $P = (P, \sqsubseteq, \bot)$ with least element $\bot$ is a *conditional upper semi lattice with least element* (abbreviated cusl) if whenever $\{a, b\}$ is consistent in P, then $a \sqcup b$ exists in P.

We cite the following well–known representation theorem for domains from [22]. The notion of an ideal can be defined for domains in the usual way. The principal ideal generated by an element a of a domain is denoted by $[a] = \{b \mid b \sqsubseteq a\}$.

Theorem 3.5. Let P be a cusl and let $\overline{P} = \{I \subseteq P \mid I \text{ an ideal}\}$. Then the structure $\overline{P} = (\overline{P}, \subseteq, [\bot])$ is a domain. Furthermore, the compact elements of $\overline{P}$, denoted by $\overline{P}_c$ are precisely the principal ideals. Finally, the map $\iota : P \to \overline{P}_c$ defined by $\iota(a) = [a]$ is an isomorphism.

3.3. **Non–wellfounded sets.** Here we follow the lines of [1] in formulating the Anti-Foundation Axiom, *AFA*. But we use urelements as in [2]. Although they are not vital to the theory, they will simplify a few things in some applications. We denote the class of all urelements by $\mathcal{U}$. (Officially there is a predicate $\mathcal{U}(x)$ in the language and axiom, $\forall x \forall y(\mathcal{U}(x) \to y \notin x)$, and an axiom stating that for any set there is an equipollent set of urelements. We denote $\mathcal{U}(x)$ also by $x \in \mathcal{U}$. The Axiom of Extensionality is also adjusted to hold only on sets.)

We work in the set theory ZFC^- which is like ordinary Zermelo–Fraenkel set theory, but the Foundation Axiom is excluded. Non–wellfounded sets are modeled by directed graphs, which consist of nodes and edges between the nodes. For a graph G we denote the set of its edges by G_E and its vertices by G_V.

We require that the graphs are pointed, i.e. there is a distinguished point, and accessible, i.e. every node is accessible from the point. We abbreviate accessible pointed graph by apg. We may assign sets of urelements to the nodes of an apg with a labelling function f such that $\mathrm{dom}(f) = G_E$ and $f(x) \subset \mathcal{U}$ for all $x \in \mathrm{dom}(f)$.

Definition 3.6. Let G be an apg and f its labelling. A *decoration* of G is a function d such that $d(n) = \{d(n') \mid (n, n') \in G_V\} \cup f(n)$.

The *AFA* axiom now says that every apg has a unique decoration. So to an apg G we may assign a set x such that $x = d(n)$, where d is the unique decoration of G and n is the distinguished point of G. We also say then that G is a picture of the set x. In the next section we show how to assign to a set a canonical tree picture of it.

The uniqueness condition in the *AFA* axiom is equivalent to the strong extensionality axiom which states that bisimulation characterizes identity. We recall the definition of bisimulation.

Definition 3.7. We write $B(x, y)$ if there is a relation $B \subseteq (\{x\} \cup TC(x)) \times (\{y\} \cup TC(y))$ such that

(i) $(x, y) \in B$,
(ii) if $(a, b) \in B$ and $c \in a$, then there is $d \in b$ such that $(c, d) \in B$,
(iii) if $(a, b) \in B$ and $d \in b$, then there is $c \in a$ such that $(c, d) \in B$,
(iv) if $(a, b) \in B$, then a is an urelement iff b is an urelement and if they are urelements, then $a = b$.

We call this kind of a relation B a *bisimulation* relation between x and y.

3.4. **A partial order of non–wellfounded sets.** We are going to define a domain ordering on a subclass of non–wellfounded sets. Here we first define a partial order $\sqsubseteq$ on all non–wellfounded sets.

When trying to see non–wellfounded sets as limits of wellfounded sets one idea could be to see them as some kind of substitution limits, or fixed points as in Hyttinen and Pauna [6]. But then the ordering does not come out as a domain ordering.

In order to construct the partial order, we see sets as trees, called *canonical tree pictures*. For a set a, we can define the canonical tree picture of a, denoted by $T(a)$ as in [1]. The graph relation $(n, n') \in G_V$ is from now on to be denoted by $n \to n'$.

Definition 3.8. Let a be a set. Then $T(a)$ consists of all finite sequences $\langle a_i : i < k \rangle$ such that $a_0 = a$, and for all $i < k - 1$, $a_{i+1} \in a_i$. For t and t' in $T(a)$, we let $t \to t'$ (i.e. $(t, t') \in T(a)_V$) iff t' is obtained from t by adding one element to t.

Here we have implicitly the labelling f of a tree picture T by urelements, so that if the last element of a node t is an urelement u, then we require that f assigns to the predecessor of t the urelement u. Formally, $t = t'^\frown u \in T$ iff $u \in f(t')$. If we drop the condition that $a_{i+1} \in a_i$, then we only say that T is a *tree picture*. By *AFA*, understood as in [1], every tree picture is a picture of a unique set.

Definition 3.9. Let T be a tree picture and $t \in T$.

(i) $\to^*$ is the transitive and reflexive closure of $\to$,
(ii) $\ln(t) = |\{t' \mid t' \to^* t\}|$ is the length of t,
(iii) t_n, where $n \leq \ln(t)$, is the nth element of t,
(iv) $\mathrm{last}(t) = t_{\ln(t)-1}$ is the last element of t,
(v) $t_T = \{t' \in T \mid t \to t'\}$ is the set of immediate successors of t,
(vi) $Tt = \{t' \in T \mid t \to^* t'\}$ is the subtree of T whose root is t.

We define an ordering on tree pictures as a certain kind of end extension. We say that a node t is a leaf node when there is no $t' \leftarrow t$. So either the last element of $t \in T$ is the empty set or an urelement when T is a canonical tree picture.

Definition 3.10. Let a and b be sets. Then $a \sqsubseteq b$ if there is a partial surjection $f : T(b) \to T(a)$ such that for all $t \in \mathrm{dom}(f)$

(i) for $t, t' \in \mathrm{dom}(f)$, $t \to t'$ iff $f(t) \to f(t')$,
(ii) $f(\langle b \rangle) = \langle a \rangle$,
(iii) if $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(t) = \mathrm{last}(f(t))$,
(iv) if $t', t'' \leftarrow t$ and $t' \in \mathrm{dom}(f)$, then $t'' \in \mathrm{dom}(f)$,
(v) if $\mathrm{last}(f(t)) = \emptyset$ and there is $t' \in \mathrm{dom}(f)$ such that $\mathrm{ln}(t') > \mathrm{ln}(t)$, then $\mathrm{last}(t) = \emptyset$.

In what follows, we mean by an *epimorphism* a function that satisfies the previous definition. For a set a we define its height $\mathrm{ht}(a) = \mathrm{ht}(T(a)) = \sup\{\mathrm{ln}(t) \mid t \in T(a)\}$.

Lemma 3.11. *Assume $a \sqsubseteq b$ and $f : T(b) \to T(a)$ is an epimorphism witnessing this. Then*

(i) $\mathrm{ht}(\mathrm{dom}(f)) = \mathrm{ht}(a) \leq \mathrm{ht}(b)$*,*
(ii) $\mathrm{dom}(f) = T(b) \upharpoonright \mathrm{ht}(a)$*,*
(iii) if $\mathrm{ht}(a) = \mathrm{ht}(b)$*, then* $a = b$*.*

Proof. Here we understand $\mathrm{dom}(f)$ as the subtree of $T(b)$ where f is defined.

(i) This follows from the fact that f is surjective and for all $t \in \mathrm{dom}(f)$, $\mathrm{ln}(t) = \mathrm{ln}(f(t))$.

(ii) We show the claim by induction on $t \in T(b)$, where $\mathrm{ln}(t) \leq \mathrm{ht}(T(a))$. First, the root, $b \in \mathrm{dom}(f)$. Assume $t \in \mathrm{dom}(f)$ and $t' \leftarrow t$. If there is $h' \leftarrow f(t)$, then $\mathrm{ln}(t) < \mathrm{ht}(T(a))$ and for some $t'' \leftarrow t$, $f(t'') = h'$, because f is surjective and respects $\to$. Now by the condition (iv) of Definition 3.10, also $t' \in \mathrm{dom}(f)$.

Assume then that $f(t)$ is a leaf node. If $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(t) \in \mathcal{U}$. Assume $\mathrm{last}(f(t)) = \emptyset$. Now if for some $h \in T(a)$, $\mathrm{ln}(h) > \mathrm{ln}(f(t))$, then there is, by (i), some $t'' \in \mathrm{dom}(f)$ such that $\mathrm{ln}(h) = \mathrm{ln}(t'')$. But then, by (v) of Definition 3.10, $\mathrm{last}(t) = \emptyset$. If for all $h \in T(a)$, $\mathrm{ln}(h) \leq \mathrm{ln}(f(t))$, then $f(t)$ is maximal in $T(a)$, so $\mathrm{ln}(f(t)) = \mathrm{ln}(t) = \mathrm{ht}(T(a))$. Hence $\mathrm{ln}(t') > \mathrm{ht}(T(a))$ and so $t' \notin \mathrm{dom}(f)$.

(iii) Let $B = \{(\mathrm{last}(f(t)), \mathrm{last}(t)) \mid t \in T(b)\}$. We show that B is a bisimulation between a and b. First, we have that $(a, b) \in B$. Assume $(\mathrm{last}(f(t)), \mathrm{last}(t)) \in B$. Let $x \in \mathrm{last}(f(t))$, then there is $h' \in T(a)$ such that $\mathrm{last}(h') = x$ and $h' \leftarrow f(t)$. By (ii), there is some $t' \in T(b)$ such that $f(t') = h'$. Because $f(t') = h' \leftarrow f(t)$ we have that $t' \leftarrow t$. Hence $\mathrm{last}(t') \in \mathrm{last}(t)$ and $(\mathrm{last}(f(t'), \mathrm{last}(t'))) \in B$.

Let $y \in \mathrm{last}(t)$, so $y = \mathrm{last}(t')$ for some $t' \leftarrow t$. Now by (ii), $t' \in \mathrm{dom}(f)$. So $f(t) \to f(t')$ and $(\mathrm{last}(f(t')), \mathrm{last}(t')) \in B$. Assume $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(t) = \mathrm{last}(f(t))$. If $\mathrm{last}(t) \in \mathcal{U}$, then also $\mathrm{last}(f(t)) \in \mathcal{U}$ and hence $\mathrm{last}(f(t)) = \mathrm{last}(t)$.
□

By the previous lemma, we see that all strictly $\sqsubseteq$–increasing sequences are at most of length ω.

Lemma 3.12. *For all sets a, b, and c we have*

(i) $a \sqsubseteq a$,

(ii) if $a \sqsubseteq b$ *and* $b \sqsubseteq a$, *then* $a = b$,

(iii) if $a \sqsubseteq b$ *and* $b \sqsubseteq c$, *then* $a \sqsubseteq c$.

Proof. (i) The identity $id_{T(a)} : T(a) \to T(a)$ is an epimorphism.

(ii) If $f : T(b) \to T(a)$ and $g : T(a) \to T(b)$, then $\mathrm{ht}(a) \leq \mathrm{ht}(b)$ and $\mathrm{ht}(b) \leq \mathrm{ht}(a)$, hence $\mathrm{ht}(a) = \mathrm{ht}(b)$. So by (iii) of Lemma 3.11, $a = b$.

(iii) Assume $f : T(b) \to T(a)$ and $g : T(c) \to T(b)$ are epimorphisms. Define $h : T(c) \to T(a)$ by $h(t) = f(g(t))$, if $g(t)$ and $f(g(t))$ are defined. We show that h is an epimorphism. First we have that $h(c) = a$ and that it respects $\to$. Assume $t' \in T(a)$. Then for some $t'' \in T(b)$, $f(t'') = t'$, but also for some $t \in T(c)$, $g(t) = t''$, so $h(t) = f(g(t)) = t'$.

If $h(t) = f(g(t)) \in \mathcal{U}$, then $\mathrm{last}(f(g(t))) = \mathrm{last}(g(t)) = \mathrm{last}(t)$. Assume that $\mathrm{last}(f(g(t))) = \emptyset$ and there is some $t' \in \mathrm{dom}(h)$ such that $\mathrm{ln}(t') > \mathrm{ln}(t)$. So $t' \in \mathrm{dom}(g)$ and $g(t') \in \mathrm{dom}(f)$. Now $\mathrm{ln}(g(t')) > \mathrm{ln}(g(t))$, so $\mathrm{last}(g(t)) = \emptyset$ and hence $\mathrm{last}(t) = \emptyset$. □

Lemma 3.13. *Assume $f : T(b) \to T(a)$ is a partial epimorphism. If $n \leq \mathrm{ht}(b)$, then $f \restriction (T(b) \restriction n) : T(b) \restriction n \to T(a) \restriction n$ is also a partial epimorphism. In particular, if $n \leq \mathrm{ht}(a)$, then $T(b) \restriction n$ and $T(a) \restriction n$ picture the same sets.*

Proof. If $n \geq \mathrm{ht}(a)$, then $f \restriction (T(b) \restriction n) = f$. If $n \leq \mathrm{ht}(a)$, then we see by the definition of a partial epimorphism, that $f \restriction (T(b) \restriction n)$ is a partial epimorphism. The last remark follows from (iii) of Lemma 3.11. □

Corollary 3.14. The following are equivalent.

(i) $a \sqsubseteq b$,

(ii) $T(a)$ and $T(b) \restriction \mathrm{ht}(a)$ are pictures of the same sets,

(iii) $T(b) \restriction \mathrm{ht}(a)$ is a picture of a.

□

We have now obtained that epimorphisms are unique.

Corollary 3.15. If f and f' are partial epimorphisms from $T(b)$ to $T(a)$, then $f = f'$.

Proof. By Lemma 3.11 (ii), $\mathrm{dom}(f) = T(b) \restriction \mathrm{ht}(a) = \mathrm{dom}(f')$. Let $B = \{(f(t), f'(t)) \mid t \in \mathrm{dom}(f)\}$. We show that B is a bisimulation relation on $T(a)$, from which the claim follows. If $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(f(t)) = \mathrm{last}(t) = \mathrm{last}(f(t'))$. Let $f(t) \to s$. Because f is surjective, there is some $t' \in \mathrm{dom}(f)$ such that $f(t') = s \leftarrow f(t)$. Hence $t \to t'$, so $f'(t) \to f'(t')$ and $(f(t'), f'(t')) \in B$. So B is a bisimulation on $T(a)$. □

Lemma 3.16. *Assume $a \sqsubseteq c$ and $b \sqsubseteq c$. Then*

(i) if $\mathrm{ht}(a) = \mathrm{ht}(b)$*, then* $a = b$*,*

(ii) if $\mathrm{ht}(a) < \mathrm{ht}(b)$*, then* $a \sqsubseteq b$*.*

Proof. Let $f_a : T(c) \to T(a)$ and $f_b : T(c) \to T(b)$ be the partial epimorphisms.

(i) Since $\mathrm{ht}(a) = \mathrm{ht}(b)$, we have that $\mathrm{dom}(f_a) = T \restriction \mathrm{ht}(a) = \mathrm{dom}(f_b)$ by (iv) of Lemma 3.11. Define a bisimulation B between a and b as

$$B = \{(\mathrm{last}(f_a(t)), \mathrm{last}(f_b(t))) \mid t \in \mathrm{dom}(f_a)\}.$$

First, $(a, b) \in B$. Assume $(\mathrm{last}(f_a(t)), \mathrm{last}(f_b(t))) \in B$ and $x \in \mathrm{last}(f_a(t))$, i.e. $s \leftarrow f_a(t)$ for some s such that $\mathrm{last}(s) = x$. There is some $t' \in \mathrm{dom}(f_a)$ such that $f_a(t') = s$. Then $t \to t'$ and $f_a(t) \to f_a(t')$. Also, $t' \in \mathrm{dom}(f_b)$ and $f_b(t) \to f_b(t')$. So we have that $\mathrm{last}(f_b(t')) \in \mathrm{last}(f_b(t))$ and $(\mathrm{last}(f_a(t')), \mathrm{last}(f_b(t'))) \in B$, hence B is a bisimulation between a and b.

(ii) Now $T(c) \restriction \mathrm{ht}(a)$ and $T(a)$ are pictures of the set a, by Lemma 3.13. But then $T(b) \restriction \mathrm{ht}(a)$ is also a picture of a. So by Corollary 3.14, $a \sqsubseteq b$. □

Corollary 3.17. If a and b are consistent, then $a \sqcup b$ exists.

Proof. Actually, the predecessors of a set are linearly ordered, by Lemma 3.16, so either $a \sqsubseteq b$ or $b \sqsubseteq a$, or equivalently $a \sqcup b = b$ or $a \sqcup b = a$. □

The previous corollary states the property of conditional closedness for domains in the class of all non–wellfounded sets. Below we are going to show how to restrict the class to obtain a domain.

3.5. **Inverse limits of projective sequences.** Here we show how to obtain limits of $\sqsubseteq$–increasing sequences of wellfounded sets. We first need a lemma stating that epimorphisms can always be composed to yield a unique epimorphism.

Lemma 3.18. *If $f_{cb} : T(c) \to T(b)$ and $f_{ba} : T(b) \to T(a)$ are partial epimorphisms, then* $\mathrm{dom}(f_{ba}) \subseteq ran(f_{cb})$ *and $f_{ca} = f_{ba} \circ f_{cb}$ is the unique epimorphism from $T(c)$ to $T(a)$.*

Proof. Now $\mathrm{ht}(c) \geq \mathrm{ht}(b) \geq \mathrm{ht}(a)$ and $\mathrm{dom}(f_{ba}) = T(b) \restriction \mathrm{ht}(a) \subseteq T(b) = ran(f_{cb})$. So $f_{ba}(f_{cb}(t))$ is defined iff $t \in T(c) \restriction \mathrm{ht}(a)$. Since $f_{ba} \circ f_{cb}$ is a partial epimorphism, cf. proof of Lemma 3.12 (iii), it is unique by Corollary 3.15. □.

Assume that $\vec{a} = \langle a_i \mid i < \omega \rangle$ is a strictly $\sqsubseteq$-increasing sequence of sets. So for all $i < \omega$, $\mathrm{ht}(a_i) < \omega$, and there is an epimorphism $f_{i+1,i} : T(a_{i+1}) \to T(a_i)$. We show next how to construct an upper bound for $\vec{a}$. The idea is to construct the inverse limit of the projective sequence $(T(a_{i+1}), f_{i+1,i})_{0 \leq i < \omega}$.

Let

$$T = \{(t_n) \in \prod_{n=k}^{\omega} T(a_n) \mid k < \omega, f_{n+1,n}(t_{n+1}) = t_n \text{ and } (t_k \notin \mathrm{dom}(f_{k,k-1}) \text{ or } k = 0)\}.$$

Let $(t_n)_{k\le n<\omega} \to (t'_n)_{k'\le n<\omega}$ iff $k' \ge k$ and $t_n \to t'_n$ for all $n \ge k'$. Note that if $t_k \notin \mathrm{dom}(f_{k,k-1})$, then $t_k \notin \mathrm{dom}(f_{k,k'})$ for any k' such that $0 \le k' < k$, since in that case $f_{k,k'} = f_{k-1,k'} \circ f_{k,k-1}$.

We show first that if $(t_n)_{k\le n<\omega} \to (t'_n)_{k'\le n<\omega}$, then $k' = k$ or $k' = k+1$. First assume that t_k is a leaf node in $T(a_k)$, so $k' > k$. We show that in this case, we have that $k' = k+1$. Assume $k' > k+1$ and k'' is such that $k < k'' < k'$. Then $t_{k''}$ is a leaf node in $T(a_{k''})$ because otherwise $t_{k''} \to f_{k',k''}(t'_{k'})$, since $t_{k'} \to t'_{k'}$ and $f_{k',k''}(t_{k'}) = t_{k''}$, hence $t'_{k'} \in \mathrm{dom}(f_{k',k''})$, a contradiction. But because $\vec{a}$ was strictly increasing, there is a node $t \in T(a_{k'})$ such that $\ln(t) > \ln(t_{k''})$. Since $t_{k''}$ is a leaf, $\mathrm{last}(f_{k',k''}t_{k'}) = \emptyset$, and hence also $\mathrm{last}(t_{k'}) = \emptyset$. But on the other hand, $t_{k'} \to t'_{k'}$, a contradiction.

Assume then that t_k is not a leaf node. So some $t'' \leftarrow t_k$ in $T(a_k)$. Now if $k' > k$, then $f_{k',k}(t_{k'}) = t_k$ and $t_{k'} \to t'_{k'}$ imply that $f_{k',k}(t'_{k'}) = t''$, which is a contradiction, since $t'_{k'} \notin \mathrm{dom}(f_{k',k})$. Hence $k' = k$. So we have shown in all that $k' = k+1$ if and only if t_k is a leaf node in $T(a_k)$. In this case the extension to t_k comes from the next tree $T(a_{k+1})$.

Next we show that T is strongly extensional in the sense that it has no two bisimilar subtrees starting from the same node. So assume that there is a node $\vec{t} \in T$ and its immediate successors $\vec{t^1}$ and $\vec{t^2}$ such that $T\vec{t^1}$ and $T\vec{t^2}$ are bisimilar subtrees of T. Let B be the bisimulation. Let $\pi_n : T \to T(a_n)$, $n < \omega$ be the projection mapping, i.e.

$$\pi_n((t_i)_{i\ge k}) = \begin{cases} \text{undefined} & \text{if } n < k, \\ t_n & \text{otherwise.} \end{cases}$$

By the above we have that the first index of both $\vec{t^1}$ and $\vec{t^2}$ is the same, say k. We show first that for every $n \ge k$, and $t \in T(a_n)\pi_n(\vec{t^1})$ there is $\vec{t'} \in T\vec{t^1}$ such that $\pi_n(\vec{t'}) = t$ by induction on the nodes of the tree. So assume the claim holds for $\vec{t'}$ and let $t''_n \leftarrow \pi_n(\vec{t'})$ in $T(a_n)$. We build a sequence $\vec{t''}$ around t''_n such that $\vec{t''} \leftarrow \vec{t'}$, so $\vec{t''} \in T\vec{t^1}$. Let $m > n$. Because $f_{mn}(t'_m) = t'_n$ and $t''_n \leftarrow t'_n$, there is some $t''_m \in T(a_m)$ such that $f_{mn}(t''_m) = t''_n$ and $t''_m \leftarrow t'_m$. Let $m \le n$. Assume we have found t''_m. If $t_m \notin \mathrm{dom}(f_{m,m-1})$, then by the above, $m \le k+1$. Otherwise let $t''_{m-1} = f_{m,m-1}(t''_m) \leftarrow t'_{m-1}$.

We claim that for every $n \ge k$, $T(a_n)\pi_n(\vec{t^1})$ and $T(a_n)\pi_n(\vec{t^2})$ are bisimilar subtrees of $T(a_n)$, hence $\pi_n(\vec{t^1}) = \pi_n(\vec{t^2})$, because they have a common immediate predecessor. Let

$$B_n = \{(\pi_n(\vec{h^1}), \pi_n(\vec{h^2})) \mid (\vec{h^1}, \vec{h^2}) \in B\}.$$

Assume $(\pi_n(\vec{h^1}), \pi_n(\vec{h^2})) \in B_n$ and $h \leftarrow \pi_n(\vec{h^1})$, so by the above, there is some $\vec{h'} \in T\vec{t^1}$ such that $\pi_n(\vec{h'}) = h$ and $\vec{h'} \leftarrow \vec{h^1}$. Since B is a bisimulation, there is $\vec{h''} \leftarrow \vec{h^2}$ such that $(\vec{h'}, \vec{h''}) \in B$. Hence $\pi_n(\vec{h''}) \leftarrow \pi_n(\vec{h^2})$ and $(\pi_n(\vec{h'}), \pi_n(\vec{h''})) \in B_n$. So B_n is a bisimulation between $T(a_n)\pi_n(\vec{t^1})$ and $T(a_n)\pi_n(\vec{t^2})$.

Hence $\pi_n(\vec{t^1}) = \pi_n(\vec{t^2})$ for all $n \geq k$, and so $\vec{t^1} = \vec{t^2}$. From this it follows that if we take a to be the unique set pictured by T, then $T(a)$ and T are isomorphic. It is clear that a is an upper bound for all a_i, $i < \omega$, since $\pi_i : T \to T(a_i)$ is in fact an epimorphism. We call the above construction of a the *inverse limit* of $\langle a_i \mid i < \omega \rangle$, and denote it by $a = \text{inv lim}_{i<\omega} a_i$. Also we denote $T = \text{inv lim}_{i<\omega} T(a_i)$.

3.6. **A domain of non–wellfounded sets.** Now let

$$\begin{aligned} C &= \{a \mid \text{ht}(a) < \omega\}, \\ D &= \{\text{inv lim}_{i<\omega} \vec{b} \mid \vec{b} \text{ is an increasing sequence in } C\}. \end{aligned}$$

We have shown that C is a conditional upper semi lattice. We show that D is isomorphic to the domain completion $\overline{C}$ of the class of compact elements C.

Lemma 3.19. *$\overline{C} \cong D$.*

Proof. Assume that $I \subseteq C$ is an ideal. If there is some $a \in I$ such that for all $a' \in I$, $a' \sqsubseteq a$, then I is the principal ideal generated by a. Assume that for all $a \in I$ there is $a' \in I$ such that $a \sqsubset a'$. So I is infinite and contains a strictly increasing sequence, $(a_i)_{i<\omega}$. Now for all $a \in I$ there is some n such that $a \sqsubseteq a_n$. Now if we assign $\text{inv lim}_{i<\omega} a_i$ to I then we have the isomorphism from $\overline{C}$ to D. □

There is a canonical way to obtain a limiting sequence of sets, a_n, $n < \omega$, for any set a. But we need a lemma first.

Lemma 3.20. *Let T be a tree picture. There is an equivalence relation $\sim$ on T such that $T/\sim$ is isomorphic to $T(a)$, where a is the set pictured by T, and a surjective homomorphism $\eta : T \to T/\sim$ such that if $t \sim t'$, then $\eta(t) = \eta(t')$.*

Proof. Let $\sim$ be an equivalence relation on T defined as $t_1 \sim t_2$ iff there is $t \to t_1, t_2$ and the subtrees Tt_1 and Tt_2 are bisimilar. Let $\eta(t) = \{t' \in T \mid t \sim t'\}$. Define $\eta(t) \to \eta(t')$ if there is $t'' \sim t'$ such that $t \to t''$. If $t_1 \sim t_2$ and $\eta(t_1) \to \eta(t')$, then for some $t'' \sim t'$, $t_1 \to t''$. Because $t_1 \sim t_2$, there is $t_2' \sim t'' \sim t'$ and $t_2 \to t_2'$, so $\eta(t_2) \to \eta(t')$.

Let $T' = T/\sim$. Because T' is reduced by bisimulation, it is isomorphic to the canonical tree picture of the set a. We have also shown that $\eta : T \to T'$ is a surjective homomorphism. (It is also an epimorphism in the sense of Definition 3.10.) □

Lemma 3.21. *Let a be a set. Then there is a sequence of sets $a_0 \sqsubseteq a_1 \sqsubseteq \cdots \sqsubseteq a$ such that* $\text{approx}(a) = \{a_i \mid i < \omega\}$.

Proof. Let $T(a)$ be the canonical tree picture of a. Let $T_n = T(a) \restriction n$, and a_n be the set pictured by T_n. There is a partial isomorphism $h_n : T(a) \to T_n$, namely $id_{T(a)\restriction n}$.

Let $T_n' = T_n/\sim$ and let $g_n : T_n' \to T(a_n)$ be the isomorphism from the previous lemma. Let $f_n = g_n \circ \eta \circ h_n : T(a) \to T(a_n)$, where $\eta : T_n \to T_n'$ is the homomorphism from the proof of the previous lemma. We show that f_n is a partial epimorphism.

Assume $\mathrm{last}(f_n(t)) = \emptyset$ and there is $t' \in \mathrm{dom}(f_n)$ such that $\mathrm{ln}(t') > \mathrm{ln}(t)$. Assume that $\mathrm{last}(t) \neq \emptyset$, i.e. there is some $t'' \leftarrow t$. Because $\mathrm{ln}(t'') \leq \mathrm{ln}(t') \leq n$, we have that $t'' \in \mathrm{dom}(id_{T(a)\restriction n})$. But then $\eta(t'') \leftarrow \eta(t)$ in T'_n and $\mathrm{last}(f_n(t)) \neq \emptyset$. This proves the condition (v) of being a partial epimorphism. The other conditions are clear. □

Definition 3.22. Let a be a set. The increasing sequence $a_1 \sqsubseteq a_2 \sqsubseteq \cdots$ of the proof of the previous lemma is called the canonical limiting sequence of the set a.

Because D is a domain, we have that if $x \in D$, then $x = \bigsqcup \mathrm{approx}(x)$, i.e. x is the limit of its canonical limiting sequence.

3.7. **Bisimulation in HF_1.** Recall that bisimulation characterizes the identity for the non–wellfounded sets. Bisimulation can be also approximated by the length of how deep one sees. At limit stages it is required that two sets are equal in all the approximating bisimulations. In HF_1 it is enough to approximate only to length ω. As a corollary we can prove that $HF_1 \subseteq D$. This is made precise below.

Bisimulation can also be seen as a game played between two players, $\forall$ and $\exists$, and on two graphs or on two sets, a and b. The rules for this bisimulation game, $BG(a, b)$, are as follows. First the player $\forall$ chooses one of the sets a or b and an element x_1 of that chosen set. Then the player $\exists$ has to respond with an element, y_1 of the other set. Following that, $\forall$ chooses an element, x_2, from either x_1 or y_1 and $\exists$ responds with an element, y_2, from the other set. If $\forall$ moves an urelement, then $\exists$ has to respond with the same urelement from the other set. This way the game continues.

The player $\forall$ wins if $\exists$ is not able to respond with an element at some point of the game. Otherwise $\exists$ wins, i.e. the game continues arbitrarily long or $\forall$ is not able to move. A winning strategy for either of the players in the game $BG(a, b)$ is a function $\sigma : (TC(a) \cup TC(b))^{<\omega} \to TC(a) \cup TC(b)$ such that following that strategy the player wins the game. That is, given any legal sequence of moves, σ tells the next move in the game.

We may also restrict the length of the game $BG(a, b)$, i.e. the number of moves by $\forall$, obtaining games $BG_n(a, b)$. Similarly we can define the winning strategies σ_n by letting their domain be $(TC(a) \cup TC(b))^{\leq n}$. When $\exists$ wins $BG_n(a, b)$ we denote this also by $a \sim_n b$. It is shown in [2], cf. Theorem 12.6., that $\exists$ wins $BG(a, b)$ if and only if a and b are bisimilar.

Note that $\sim_n$ is an equivalence relation for every $n < \omega$. We begin with a characterization of the $\sqsubseteq$ ordering with the restricted length game G_n.

Lemma 3.23. *Let a and b be sets. Then $a \sqsubseteq b$ iff $a \sim_{\mathrm{ht}(a)} b$.*

Proof. Let $T(a)$ and $T(b)$ be the canonical tree pictures of a and b respectively. By Corollary 3.14, $a \sqsubseteq b$ iff $T(a)$ and $T(b) \restriction \mathrm{ht}(a)$ picture the same sets iff $a \sim_{\mathrm{ht}(a)} b$. □

Definition 3.24. Let T be a tree and $t \in T$. We denote by $[t]^n$ the subtree Tt restricted to length n and reduced by bisimulation, i.e. $[t]^n = (Tt \restriction n)/\sim$.

We are going to prove that if $\exists$ wins the game $BG_n(a, b)$ for all $n < \omega$, where a is a hereditarily finite set, then $a = b$. For this we need a lemma providing us with a "uniform" set of winning strategies in the games $BG_n(a, b)$, $n < \omega$.

Lemma 3.25. *Let T and T' be trees such that $T \sim_n T'$ for all $n < \omega$. Then there are following kind of winning strategies σ_n, $n < \omega$ for the player $\exists$ in the games $BG_n(T, T')$, $n < \omega$.*

Assume $t \in T \cup T'$ and S is a maximal set of immediate successors of t such that $t' \sim_n t''$ for all $t', t'' \in S$ and $n < \omega$. For all $i, j < \omega$, if $\vec{s} \in \mathrm{dom}(\sigma_i)$, $\vec{s'} \in \mathrm{dom}(\sigma_j)$, $\sigma_i(\vec{s}) \in S$, and $\sigma_j(\vec{s'}) \in S$, then $\sigma_i(\vec{s}) = \sigma_j(\vec{s'})$.

Proof. Let $n < \omega$. Because $T \sim_n T'$, let σ'_n be a winning strategy for $\exists$ in $BG_n(T, T')$. We construct the winning strategy σ_n as follows. Let us well–order the nodes of T and T' and let t and S be as above. Now if $\sigma'_n(\vec{s}) \in S$, let $\sigma_n(\vec{s})$ be the least $t' \in S$. Otherwise let $\sigma_n(\vec{s}) = \sigma'_n(\vec{s})$. This is a winning strategy, since $t \sim_n t'$ for all $t, t' \in S$. □

Recall that $HF_1 = \{x \mid \forall y \in TC(x) \cup \{x\}(|y| < \omega)\}$ is the class of all hereditarily finite sets.

Lemma 3.26. *Assume $a \in HF_1$, b is a set and for all $n < \omega$, $\exists$ wins the game $BG_n(a, b)$, then $\exists$ wins the game $BG(a, b)$.*

Proof. Let σ_n, $n < \omega$, be winning strategies for $\exists$ in $BG_n(a, b)$ that also satisfy the conditions of the previous lemma. We define inductively on the sequence of moves the winning strategy σ for $\exists$ in $BG(a, b)$. Let $\vec{s}$ be a sequence of moves of length n. Assume first that $\mathrm{last}(\vec{s}) \in T(b)$, then there is an infinite set $X(\vec{s}) \subseteq X(\vec{s} \restriction n-1) \subseteq \omega$ such that $\sigma_p(\vec{s}) = \sigma_{p'}(\vec{s})$ for all $p, p' \in X(\vec{s})$, because there is only a finite number of possible moves for $\exists$ in any node of $T(a)$. Then we define $\sigma(\vec{s}) = \sigma_p(\vec{s})$ for some $p \in X(\vec{s})$.

Assume then that $t = \mathrm{last}(\vec{s}) \in T(a)$. Let $t' \in T(b)$ be the node from which $\exists$ has to choose the corresponding node for t. If t' has finitely many successors, then we can do as above. Assume t' has infinitely many successors. There are two cases.

1°: The number of successors of the root node in the tree $[t']^n$ increases as n increases. Denote by r the predecessor of t. So $\exists$ wins $BG_n(T(a)r, T(b)t')$ for all $n < \omega$. But at some $n < \omega$, the root of $[t']^n$ has more successors than the root of $[r]^n$ since $T(a)$ is hereditarily finite. Hence $[r]^n \not\cong [t']^n$ and as these trees are reduced by bisimulation, $[r]^n \not\sim [t']^n$, i.e. $r \not\sim_n t'$. Thus $\exists$ does not win $BG_n(T(a)r, T(b)t')$, a contradiction.

2°: There is $k < \omega$ such that the root of $[t']^n$ has at most k immediate successors for all $n < \omega$. Let S_n be the set of $\sim_n$ equivalence classes of the immediate successors of the root of $[t']^n$. By the fact that if for some $n < \omega$ and $t'_1, t'_2 \leftarrow t'$, $t'_1 \not\sim_n t'_2$ implies $t'_1 \not\sim_m t'_2$ for any $m \geq n$, we have that S_n becomes finer as n increases. Since the root of $[t']^n$ has at most k immediate successors for all $n < \omega$, there is some $n < \omega$, such that $S_n = S_m$ for all $m \geq n$. Let us denote $S' = S_m$ for any such $m \geq n$ and let $S = \bigcup S' \cap \bigcup_{i<\omega} ran(\sigma_i)$. The previous lemma guarantees that $S = \{t'_1, \ldots, t'_k\}$ is a finite set of representatives of the equivalence classes in S'.

Thus we can find an infinite set $X(\vec{s}) \subseteq X(\vec{s} \restriction (n-1))$ such that $\sigma_p(\vec{s}) = \sigma_q(\vec{s})$ for all $p, q \in X(\vec{s})$. So we can define $\sigma(\vec{s}) = \sigma_p(\vec{s})$ for some $p \in X(\vec{s})$. □

Note that from the previous lemma it follows that then also $b \in HF_1$ and $a = b$.

Lemma 3.27. *Assume a is a set. If b is an upper bound for* approx(a)*, then $\exists$ wins $BG_n(a, b)$ for all $n < \omega$.*

Proof. Let $T(a)$ be the canonical tree picture for a, and a_i, $i < \omega$ be its canonical limiting sequence. Because $a_n \sqsubseteq b$ for all $n < \omega$, by Corollary 3.14 (ii), it follows that $T(a_n)$ and $T(b) \restriction n$ picture the same set, i.e. they are bisimilar. So $\exists$ wins $BG_n(a, b)$ for all $n < \omega$. □

So we have now achieved that all the hereditarily finite sets can be approximated in the ordering $\sqsubseteq$.

Corollary 3.28. For any hereditarily finite set a, $a = \bigsqcup \text{approx}(a)$. Thus $HF_1 \subseteq D$.

Proof. Assume $a \in HF_1$ is a set. Let $T(a)$ be the canonical tree picture for a, and a_i, $i < \omega$ be its canonical limiting sequence. Let $T = \text{inv lim}_{i<\omega} T(a_i)$. Because T is an upper bound for all $T(a_i)$, $\exists$ wins the game $BG(T(a), T)$, by Lemma 3.26. So $T(a)$ and T are bisimilar and hence picture the same sets. Similarly for any other upper bound b for a, we have that $b = a$. Hence $a = \bigsqcup \text{approx}(a)$. □

The next example shows that there are sets that cannot be approximated in the ordering $\sqsubseteq$.

Example 3.29. There is a set a such that $a \neq \text{inv lim}_{i<\omega} a_i$, where a_i, $i < \omega$ is the canonical limiting sequence of a.

Proof. We define sets x_i, $i < \omega$ as follows: Let $x_0 = \emptyset$, and $x_{i+1} = \{x_i\}$. Let $a = \{x_i \mid i < \omega\}$ and let a_i, $i < \omega$ be its canonical limiting sequence. So $a_i = \{x_n \mid n \leq i\}$. Let us consider $T = \text{inv lim}_{i<\omega} T(a_i)$ and $T(a)$. Let b be the unique set pictured by T. We show that b is non–wellfounded. Let $f_n : T(a) \to T(a_n)$, for $n < \omega$, be the epimorphisms.

Let t be the root of $T(a)$. Let $t_{x_i} \leftarrow t$ be the unique node in $T(a)$ such that $d(t_{x_i}) = x_i$, where d is the unique decoration of T. Then let $t^j_{x_i}$, where $j \leq i$, be the unique node such that $t^j_{x_i} \leftarrow^* t_{x_i}$ and $\ln(t^j_{x_i}) = j+1$. The node can be chosen uniquely, since the successors of every t_{x_i} are linearly ordered.

Let $\vec{t_i} = \langle f_n(t^i_{x_n}) \mid n \geq i\rangle$. Now $f_n(t^i_{x_n}) \to f_n(t^{i+1}_{x_n})$ for all $n \geq i+1$, so $\vec{t_i} \in T$ and $\vec{t_i} \to \vec{t}_{i+1}$, for all $i < \omega$. Let c_i be the set assigned to $\vec{t_i}$ by the decoration of T. Then $c_{i+1} \in c_i$ for all $i < \omega$, and so c_i, $i < \omega$ is a non–wellfounded sequence in b. Because a is wellfounded, $a \neq b$. So approx(a) has two upper bounds, namely a and b, which are incomparable. □

The previous example shows also that $\exists$ having a winning strategy in $BG_n(a, b)$ for all $n < \omega$ does not imply that $\exists$ has a winning strategy in $BG(a, b)$. Next we consider when a set a is the same as $\bar{a}$.

3.8. **A characterization of the sets in D.** The domain D consists of those sets that can be approximated in the ordering $\sqsubseteq$. We show that there is another condition characterizing this.

Let T be a tree and $t \in T$. The notation $t \to T'$, where T' is a tree means that for some $t' \leftarrow t$, $Tt' \sim T'$, i.e. t has an immediate successor such that the tree beginning from that successor is bisimilar to T'.

Definition 3.30. Let a be a set. We say that a is inv lim–closed if there is no $t \in T(a)$ and its immediate successors t_i, $i < \omega$ such that $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$ and $t \not\to \text{inv}\lim_{i<\omega}[t_i]^i$.

Definition 3.31. Let a be a set and a_i, $i < \omega$ its canonical limiting sequence. Define $\bar{a}$ to be the unique set pictured by the tree $\text{inv}\lim_{i<\omega} T(a_i)$, or equivalently, $\bar{a} = \text{inv}\lim_{i<\omega} a_i$.

Theorem 3.32. Let a be a set, a_i, $i < \omega$ its canonical limiting sequence, $f_i : T(a) \to T(a_i)$ be the partial epimorphisms, and let $T = \text{inv}\lim_{i<\omega} T(a_i)$. The following are equivalent:

(i) the function $f : T(a) \to T$, $f(t) = (f_i(t))_{i \geq \ln(t)}$, is an epimorphism,
(ii) $a = \bar{a}$,
(iii) a is inv lim–closed.

Proof. Note that T is a picture of $\bar{a}$. We may assume that $\text{ht}(a) = \omega$ since otherwise the claim is clear.

(i) $\to$ (ii): Because $\text{ht}(T(a)) = \text{ht}(T)$, and there is a partial epimorphism between them, we have by Corollary 3.14, that $a = \bar{a}$.

(ii) $\to$ (iii): Assume $a = \bar{a}$. Then there is an isomorphism $g : T \to T(a)$, because $T = \text{inv}\lim_{i<\omega} T(a_i)$ and $T(a)$ picture the same set a and T is already reduced by bisimulation in the sense of Lemma 3.20. Assume $t \in T(a)$, $t_i \leftarrow t$, and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$ for $i < \omega$. Let $T' = \text{inv}\lim_{i<\omega}[t_i]^i$ and let $f'_{i+1,i} : [t_{i+1}]^{i+1} \to [t_i]^i$, $i < \omega$, be the epimorphisms. We show that $t \to T'$. Let $k = \ln(t) + 1$.

First we show that there is a natural way to embed the subtree $[t_i]^i$ into $T(a_{k+i})$. The idea is to find the corresponding node for t_i in $T(a_{k+i})$ and then map $[t_i]^i$ surjectively onto that node's successors. Let us call this kind of embedding a *canonical embedding*.

We build the embedding $h_i : [t_i]^i \to T(a_{k+i})$ inductively as follows: Let $\eta : (Tt_i \upharpoonright i) \to [t_i]^i$ be the $\sim$–epimorphism from Lemma 3.20. Let $h_i(t_i) = f_{k+i}(t_i) \in T(a_{k+i})$. Assume $h_i(t)$ is defined and $t \to t'$. Let $t'' \in (Tt_i \upharpoonright i)$ be such that $\eta(t'') = t'$. Define $h_i(t') = f_{k+i}(t'')$. If there is another $s \in (Tt_i \upharpoonright i)$ such that $\eta(s) = t'$, then $s \sim_{i-\ln(s)} t''$ and hence also $f_{k+i}(s) = f_{k+i}(t'')$, so h_i is well–defined. If $h_i(t) = h_i(t')$ then $f_{k+i}(s) = f_{k+i}(s')$ for the corresponding s and s'. But then also $t = \eta(s) = \eta(s') = t'$ and we have that h_i is an injection. We show that it is also a homomorphism. Assume $t \to t'$, and let s and s' be such that $\eta(s) = t$ and $\eta(s') = t'$. Then $\eta(s) \to \eta(s')$, and since η is a homomorphism, $s \to s'$. But then $h_i(t) = f_{k+i}(s) \to f_{k+i}(s') = h_i(t')$. Similarly to the other direction.

We have that $f'_{i+1,i}(t_{i+1}) = t_i$ because t_{i+1} is the root of $[t_{i+1}]^{i+1}$ and t_i is the root of $[t_i]^i$ and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$. We also have that $f_{k+i+1,k+i}(f_{k+i+1}(t_{i+1})) = f_{k+i}(t_{i+1})$, because $f_{k+i+1,k+i} \circ f_{k+i+1} : T(a) \to T(a_{k+i})$ is an epimorphism and hence equal to f_{k+i} since epimorphisms are unique. Furthermore $f_{k+i}(t_{i+1}) = f_{k+i}(t_i)$, because t_{i+1} and t_i have a common immediate predecessor and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$ means that $t_i \sim_i t_{i+1}$. So $f_{k+i+1,k+i}(h_{i+1}(t_{i+1})) = f_{k+i+1,k+i}(f_{k+i+1}(t_{i+1})) = f_{k+i}(t_{i+1}) = f_{k+i}(t_i) = h_i(t_i) = h_i(f'_{i+1,i}(t_{i+1}))$. And so on for all other $t \in [t_{i+1}]^{i+1}$. So the following diagram commutes:

$$
\begin{array}{ccc}
T(a_{k+i}) & \xleftarrow{f_{k+i+1,k+i}} & T(a_{k+i+1}) \\
{\scriptstyle h_i}\uparrow & & \uparrow{\scriptstyle h_{i+1}} \\
[t_i]^i & \xleftarrow[f'_{i+1,i}]{} & [t_{i+1}]^{i+1}
\end{array}
$$

To show that $t \to T'$, we are going to construct an embedding $g' : T' \to T$ such that $g'(T')$ becomes a subtree of $g^{-1}(t)$, and furthermore g' is a surjection onto the successors of $g'(r)$ where r is the root of T'. So let $\vec{s} = (s_i)_{i \geq k'} \in T'$, where k' is the length of $\vec{s}$ in T'. We have that for all $i \geq k'$, $f'_{i+1,i}(s_{i+1}) = s_i$. Because the above diagram commutes, we have that $f_{k+i+1,k+i}(h_{i+1}(s_{i+1})) = h_i(f'_{i+1,i}(s_{i+1})) = h_i(s_i)$ for all $i \geq k'$. Furthermore, $h_{k'}(s_{k'}) \notin \mathrm{dom}(f_{k'+k,k'+k-1})$ since otherwise $s_{k'} \in \mathrm{dom}(f'_{k',k'-1})$. So let $g'(\vec{s}) = (h_i(s_i))_{i \geq k'} \in T$ but then $g(g'(\vec{s})) \in T(a)$. Hence we have shown that $g(g'(T')) \subseteq T(a)$. Also, $f_i(t) \to f_i(t_i)$ for $i < \omega$, and so $t \to T'$.

(iii) $\to$ (i): Assume a is inv lim–closed. We show that $f : T(a) \to T$, $f(t) = (f_i(t))_{i \geq \ln(t)}$ is an epimorphism by showing that it is surjective. So let $\vec{t} \in T$. We can write $\vec{t} = (f_i(t_i))_{i \geq k}$, where $t_i \in T(a)$, and $f_{i+1,i}(f_{i+1}(t_{i+1})) = f_i(t_i)$ for all $i \geq k$. We show that there is some $s \in T(a)$ such that $f_i(s) = f_i(t_i)$ for all $i \geq k$.

We may assume that there is an infinite number of different t_i's, otherwise the claim follows immediately. There is some $t' \in T(a)$ and an infinite number of its immediate successors t'_i, $i \geq k$, such that for every $i \geq k$, $t'_i \to^* t_i$. For some $i, i{+}1$, it may happen that $t'_i = t'_{i+1}$, but in that case also $f_{i+1,i}(f_{i+1}(t'_{i+1})) = f_i(t'_i)$, because $f_{i+1,i}$ as a homomorphism preserves predecessors. We can find an infinite number of different t'_i's because there are only finitely many levels above t_i's.

So for all $i \geq k$, $f_{i+1,i}(f_{i+1}(t'_{i+1})) = f_i(t'_i)$, hence $[t'_i]^{l+i} \sqsubseteq [t'_{i+1}]^{l+i+1}$, where $l = \mathrm{ht}(t_i) - \mathrm{ht}(t'_i)$ for some (any) $i \geq k$. Because a is inv lim–closed, we have that $t' \to T'$ where $T' = \text{inv}\lim_{i \geq k}[t'_i]^{l+i}$. Then $[t'_i]^{l+i} \sqsubseteq T'$ for all $i \geq k$. Let $h_i : [t'_i]^{l+i} \to T(a_{k'+i})$ be the canonical embedding, where $k' = \ln(t_i)$ for any $i \geq k$. Because for every $i \geq k$, $t'_i \to^* t_i$, there are $s_i \in [t'_i]^{l+i}$, $i \geq k$, such that $h_i(s_i) = f_{k'+i}(t_i)$. Let $f'_{i+1,i} : [t'_{i+1}]^{l+i+1} \to [t'_i]^{l+i}$, $i < \omega$ be the epimorphisms. As above, we have that the following diagram commutes:

$$\begin{array}{ccc} T(a_{k'+i}) & \xleftarrow{f_{k'+i+1,k'+i}} & T(a_{k'+i+1}) \\ {\scriptstyle h_i}\uparrow & & \uparrow{\scriptstyle h_{i+1}} \\ [t'_i]^{l+i} & \xleftarrow[f'_{i+1,i}]{} & [t'_{i+1}]^{l+i+1} \end{array}$$

Hence $h_i(f'_{i+1,i}(s_{i+1})) = f_{k'+i+1,k'+i}(h_{i+1}(s_{i+1})) = f_{k'+i+1,k'+i}(f_{k'+i+1}(t_{i+1})) = f_i(t_i) = h_i(s_i)$, and so $f'_{i+1,i}(s_{i+1}) = s_i$, because h_i is injective. Hence $s = (s_i)_{i \geq k} \in T'$, and $f'_i(s) = s_i$ for all $i \geq k$, where $f'_i : T' \to [t'_i]^{i+l}$, $i \geq k$ are the epimorphisms. There is a canonical embedding $h : T' \to T(a)$ such that the following diagram commutes:

$$\begin{array}{ccc} T' & \xrightarrow{h} & T(a) \\ {\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle f_{k'+i}} \\ [t'_i]^{i+l} & \xrightarrow[h_i]{} & T(a_{k'+i}) \end{array}$$

Hence we have that $f_{k'+i}(h(s)) = h_i(f'_i(s)) = h_i(s_i) = f_{k'+i}(t_i)$, for all $i \geq k$. So $f(h(s)) = (f_i(t_i))_{i \geq k}$, and we have shown that f is a surjection. It is straightforward to see that it is also an epimorphism. □

We next show that in the case of pure sets, Corollary 3.28 is the best we can have. A set is called pure if its transitive closure contains no urelements.

Lemma 3.33. *Assume that a is a pure, well-founded, and infinite set. Then $a \notin D$.*

Proof. Let a_i, $i < \omega$ be the canonical limiting sequence of a, and let $f_{i+1,i} : T(a_{i+1}) \to T(a_i)$ be the epimorphisms. Also let $f_i : T(a) \to T(a_i)$ be the epimorphisms witnessing $a_i \sqsubseteq a$, for $i < \omega$. We are going to show that a is not inv lim–closed.

For every $i < \omega$, a_i is a finite set since there are only a finite number of pure sets of height $i-1$. Let t be the root of $T(a)$. We are going to find $t_i \leftarrow t$, $i < \omega$ such that $\mathrm{ht}[t_i]^i = i$ and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$, i.e. $f_{i+1,i}(f_{i+1}(t_{i+1})) = f_i(t_i)$. We show the claim by induction on $i < \omega$, but we require also that for every $i < \omega$, the set

$$A_i = \{t' \in T(a) \mid f_{i+1,i}(f_{i+1}(t')) = f_i(t_i)\}$$

is infinite.

Let $t_0 \leftarrow t$ be arbitrary. The choice can be arbitrary since $[t_0]^0$ pictures the empty set. Assume that t_i which satisfies the above conditions has been found. Let A_i be the infinite set guaranteed by the induction condition. The set $B = \{f_{i+1}(t') \in T(a_{i+1}) \mid t' \in A\}$ is on the other hand finite, since a_{i+1} is finite. But the set $B' = \{t'' \in T(a) \mid f_{i+2,i+1}(f_{t+2}(t'')) \in B\}$ is infinite since a was infinite. Hence for some $t_{i+1} \leftarrow t$, there is an infinite number of $t'' \in B'$ such that $f_{i+2,i+1}(f_{i+2}(t'')) = f_{i+1}(t_{i+1})$. Gather those into a set A_{i+1}. From this it also follows that $\mathrm{ht}([t_{i+1}]^{i+1}) = i+1$. So this proves the induction step.

Now we have the strictly increasing infinite sequence t_i, $i < \omega$. We show that $T(a)$ is not inv lim–closed. Assume towards a contradiction that $T(a)$ is inv lim–closed. We show that then a is non–wellfounded.

We build a non–wellfounded sequence s_i, $i < \omega$ of nodes in $T(a)$. Let r be the root of $T(a)$. We have that $(t_i)_{i \geq 0} \in T'$. By Theorem 3.32, there is some $t' \leftarrow r$ such that $f_i(t') = f_i(t_i)$ for all $i \geq 0$. Let $s_0 = t'$. Now if s_0 has finitely many immediate successors $\{t'_1, \ldots, t'_l\}$, then for some $1 \leq l' \leq l$ there are infinitely many t'' such that $f_i(t'') = f_i(t'_{l'})$. Then let $s_1 = t'_{l'}$.

On the other hand, if s_0 has infinitely many immediate successors, then we can do as above, i.e. find infinitely many immediate successors t'_i, $i < \omega$, of s_0 that form an increasing sequence. Because $T(a)$ is assumed to be inv lim–closed, there is some $t' \leftarrow s_0$ such that $f_i(t'_i) = f_i(t')$ for all $i \geq \ln(s_0)$. Then let $s_1 = t'$. This way we can continue infinitely long finding a sequence $r \rightarrow s_0 \rightarrow s_1 \rightarrow \cdots$. This shows that a is non–wellfounded, which is a contradiction. □

Lemma 3.34. *Let a be a set.*

(i) If $a \subseteq D$, then $a \subseteq \bar{a}$,
(ii) $\bar{\bar{a}} = \bar{a}$,
(iii) $a \in D$ iff $a = \bar{a}$.

Proof. (i) Let r be the root of $T(a)$ and let $\vec{r}$ be the root of $T(\bar{a})$. Let $x \in a$ and let $t \leftarrow r$ be the unique node such that $T(a)t$ pictures x. Now for every $t' \in T(a)t$, let $\vec{t'} = (f_i(t'))_{i \geq k}$. Then $\vec{t'} \in T(\bar{a})$. We also have that $T(a)t$ and $T(\bar{a})\vec{t}$ are isomorphic, since $a \subseteq D$, and $\vec{t} \leftarrow \vec{r}$. From this it follows that $x \in \bar{a}$.

(ii) Let a_i, $i < \omega$ be the approximation sequence of $\bar{a}$. But then a_i, $i < \omega$ is also the approximation sequence for a. Hence both $\bar{\bar{a}}$ and $\bar{a}$ are decorations of the same tree $\mathrm{inv\,lim}_{i<\omega} T(a_i)$. □

(iii) If $a \in D$, then a is of the form inv $\lim_{i<\omega} \vec{b}$, where $\vec{b}$ is an increasing sequence of wellfounded sets. But we have that $\vec{b}$ is cofinal in $\vec{a}$, where $\vec{a}$ is the canonical limiting sequence of a. Hence $a = \bar{a}$.

If $a = \bar{a} =$ inv $\lim \vec{a}$, then a is a limit of wellfounded sets. □

3.9. **The axioms of** ***ZFA*** **in** D**.** Although D does not satisfy some important axioms of *ZFA*, it satisfies some of them. Since D has some resemblance to *HF*, which satisfies *ZFC*− infinity, it is somewhat interesting to study this question.

When considering the axioms of *ZFA*, we are going to use the bisimulation games as well as the definition of inv lim–closedness.

To prove that D is extensional, we show that D is transitive. So assume $x \in D$, and $y \in x$. So x is inv lim–closed. If y were not inv lim–closed, then x would not be either, since $T(y)$ is a subtree of $T(x)$. Thus the axiom of strong extensionality holds in D. The axiom of urelements, $\forall x \forall y(\mathcal{U}(x) \rightarrow y \notin x)$ holds in D.

Let us consider pairing. Assume $x, y \in D$. We immediately see that $\{x, y\}$ is inv lim–closed, since there are no new strictly increasing infinite sequences in $\{x, y\}$ which were not already either in x or in y. Hence $\{x, y\}$ is inv lim–closed, so $\{x, y\} \in D$. The axiom of choice also holds in D.

We show that the union axiom fails in D. For $i < \omega$, let x_i be as in the example 3.29 and let u_i be an urelement such that if $i \neq j$, then $u_i \neq u_j$. Let $y_i = \{u_i, x_0, \ldots, x_i\}$ and let $a = \{y_i \mid i < \omega\}$. We have that $a \in D$ since the sets y_i, $i < \omega$ do not form an increasing sequence. On the other hand $\bigcup a \notin D$, since $x_i \in \bigcup a$, and $x_i \sqsubseteq x_{i+1}$ for every $i < \omega$, but inv $\lim_{i<\omega} x_i \notin \bigcup a$. Hence $\bigcup a$ is not inv lim–closed and therefore cannot belong to D.

Next we show the infinity axiom. Let us consider the unique set x such that $x = \omega \cup \{x\}$. We are going to show that $x = \bar{\omega}$ from which it follows that $\bar{x} = \bar{\bar{\omega}} = \bar{\omega} = x$, and thus $x \in D$. Moreover, x is an inductive set and thus the infinity axiom will hold.

For every $n \in \omega$, $\bar{n} = n$, since $\mathrm{ht}(n) < \omega$. We describe the winning strategy for $\exists$ in $BG(T(\bar{\omega}), T(x))$. For $y \in x$, let t_y be the node $\langle xy \rangle$ in $T(x)$. Let a_i, $i < \omega$ be the canonical limiting sequence for ω, and let $f_i : T(\omega) \rightarrow T(a_i)$ be the epimorphisms. We view inv $\lim_{i<\omega} T(a_i)$ and $T(\bar{\omega})$ as the same trees.

Let r be the root of $T(x)$. First if $\forall$ chooses some $t_n \leftarrow r$, then let $\exists$ respond with $\vec{t} = (f_i(t_n))_{i \geq 0} \in$ inv $\lim_{i<\omega} T(a_i)$. $\exists$ wins in this case since $T(n)$ and $T(\bar{\omega})\vec{t}$ are pictures of the set n. If $\forall$ chooses $t_x \leftarrow r$, then let $\exists$ choose $(f_i(t_i))_{i \geq 0} \in T(\bar{\omega})$. After this $\forall$ and $\exists$ are in the same position as in the beginning.

Assume $\forall$ chooses the first move $\vec{t}$ from $T(\bar{\omega})$. There are two cases: First if there is some $j < \omega$ and $n < \omega$ such that $f_i(t_n)$ appears in the sequence $\vec{t}$ for all $i \geq j$, then let $\exists$ choose $t_n \leftarrow r$. In this case $\vec{t} = (f_i(t_n))_{i<\omega}$ and hence $\exists$ wins since $T(\bar{\omega})\vec{t}$ and $T(x)t_n$ both picture the set n. Second, if there are no such j and n, then $\vec{t} = (f_n(t_n))_{n \geq 0}$. This is so because $f_{n+1,n}(f_{n+1}(t_{n+1})) = f_n(t_n)$ and $f_{n+1}(t_{n+1}) \in \mathrm{dom}(f_{n+1,n})$ for all $n < \omega$, where $f_{n+1,n} : T(a_{n+1}) \rightarrow T(a_n)$

are the epimorphisms. In this case, let $\exists$ choose $t_x \leftarrow r$. Again after this move $\forall$ and $\exists$ are in a similar position as in the beginning. Hence $\exists$ wins the game $BG(T(\bar{\omega}), T(x))$. The set x is inductive, i.e. if $y \in x$, then $y \cup \{y\} \in x$. Since $x \in D$, D satisfies the axiom of infinity. Note that from the above, it also follows that $\omega \notin D$, since $\omega \neq x = \bar{\omega}$.

The separation axiom fails in D, because we can define the natural numbers from $\bar{\omega}$ by $\omega = \{n \in \bar{\omega} \mid n \neq \bar{\omega}\}$, and $\omega \notin D$ as we saw above. Next we show the collection axiom. Assume $x \in D$ and for every $y \in x$, there is $z \in D$ such that $\phi(x, y, z)$. Let $a = \{z \in D \mid \exists y \in x \phi(x, y, z)\}$. So $a \subseteq D$. But then $a \subseteq \bar{a} \in D$, by Lemma 3.34 (i).

Let us consider the power set axiom. We need to show that $y = \mathcal{P}(x) \cap D \in D$ for every $x \in D$. We show that $y = \bar{y}$ from which the claim follows. For that, we are going to show that y is inv lim–closed. Assume $y_i \in y$, and $[y_i]^i$ is an increasing sequence of sets. Let $z_i \in y_i$, $i < \omega$ be an increasing sequence. But because $z_i \in x$, then $\text{inv}\lim_{i<\omega}[z_i]^i \in x$, since x is inv lim–closed. Hence $\text{inv}\lim_{i<\omega}[y_i]^i \subseteq x$ and y is inv lim–closed.

Considering AFA, we can reformulate it to deal only with trees, such that the tree and the inverse limit of its approximations are bisimilar. Restricted to that class of graphs, $AFA_{\text{inv lim}}$ holds in D.

So we have that $D \models ZFC^{-2} + SEA + AFA_{\text{inv lim}} -$ Separation $-$ Union.

3.10. **Comparison to Boffa's work.** Next we discuss briefly the earlier construction of Boffa [3] of the non–wellfounded sets as limits of their wellfounded approximations. The goal in [3] is not to show that this construction produces a domain structure and so the ordering $\sqsubseteq$ is not explicitly defined. Also the urelements were not assumed.

Definition 3.35. Let i be a natural number.

(i) $HF[i] = \{x \mid \text{ht}(x) < i\}$,
(ii) $HF = \bigcup_{i<\omega} HF[i]$,
(iii) $x[i]$, the ith *approximation* of x, is the set which decorates the tree obtained by restricting the canonical tree picture of x to height i.

We have that $x[0] = \emptyset$, $x[i+1] = \{y[i] \mid y \in x\}$. So the canonical limiting sequence of a set x is the same as $\langle x[i] \mid i < \omega\rangle$. Now we obtain a sequence of finite sets

$$HF[1] \overset{f_0}{\leftarrow} H[2] \overset{f_1}{\leftarrow} \cdots$$

where f_i is the function such that $f_i(x) = x[i]$. The inverse limit $\widehat{HF}$ consists of the limits $\langle x[i] \mid i < \omega\rangle$ and the $\in$–relation is defined as before.

Proposition 3.36. $V_{afa}[\emptyset] \cap D = \widehat{HF}$.

Proof. It is immediate that $\widehat{HF} \subseteq V_{afa}[\emptyset] \cap D$. Let $x \in V_{afa}[\emptyset] \cap D$. Since $TC(x)$ does not contain urelements, every $x[i]$ is finite, and hence in $HF[i]$. Thus $x = \text{inv}\lim_{i<\omega} x[i] \in \widehat{HF}$. □

Boffa mentions that topologically viewed, $\widehat{HF}$ is a compact and totally disconnected space. HF is open and dense in $\widehat{HF}$. All finitely branching graphs have decorations in $\widehat{HF}$. When there are no urelements, the construction above coincides with that of Boffa. Recall that $V_{afa}[A]$, where $A \subseteq \mathcal{U}$, is the class of all sets whose transitive closure may contain only the urelements listed in A.

Note that now $\sqsubseteq$ can be defined as $x \sqsubseteq y$ iff $y[\mathrm{ht}(x)] = x$. When there are no urelements, the Domain D as a topological space actually looks very much like the Cantor space 2^ω. As it is known, $|D| = 2^{\aleph_0}$. We can readily define an ultra metric on D. Let $x, y \in D$. Then let $d(x, y) = 0$, if $x = y$, and $d(x, y) = 2^{-n}$, if $x \neq y$, where n is the least number such that $x[n] \neq y[n]$.

It is easy to see that d is an ultra metric on D.

3.11. **Open problems.** A problem left open in this study is to generalize the ordering $\sqsubseteq$ to the class of all non–wellfounded sets so that the result is a domain. One possibility is to try to take longer approximation sequences $(a_\alpha)_{\alpha<\gamma}$ for some ordinal $\gamma \geq \omega$.

4. The Banach Mazur and Pressing Down Games Are Different

4.1. **Introduction.** We set $E^\kappa_\theta = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \theta\}$. Let S be a stationary set. We investigate two games, each played by players called "empty" and "nonempty". Empty has the first move.

In the Banach Mazur game $\mathrm{BM}(S)$ of length θ, the players choose decreasing stationary subsets of S. Empty wins, if at some $\alpha < \theta$ the intersection of these sets is nonstationary. (Exact definitions are give in the next section.)

In the pressing down game $\mathrm{PD}(S)$, empty cannot choose a stationary subset of the moves so far, but only a regressive function. Nonempty chooses a homogeneous stationary subset.

So it is at least as hard for nonempty to win BM as to win PD.

BM can be really harder than PD. This follows from well known facts about precipitous ideals (cf. 4.5 for a more detailed explanation): Nonempty can never win $\mathrm{BM}_{\leq\omega}(\omega_2)$, but it is consistent (relative to a measurable) that nonempty wins $\mathrm{PD}_{<\omega_1}(\omega_2)$. The reason is the following: In BM, empty can first choose $E^{\omega_2}_\omega$, and empty always wins on this set. However in PD, it is enough for nonempty to win on $E^{\omega_2}_{\omega_1}$, which is consistent. In a certain way this is "cheating", since nonempty wins PD on $E^{\omega_2}_{\omega_1}$ but looses BM on the disjoint set $E^{\omega_2}_\omega$, and the difference arises because empty has the first move in BM.

So a better question is: Can nonempty win $\mathrm{PD}(S)$ but loose $\mathrm{BM}(S)$ even if nonempty gets the first move,* e.g. on $S = E^{\omega_2}_{\omega_1}$?†

We show that this is indeed the case:

Theorem 4.1. *It is consistent relative to a measurable that for $\theta = \aleph_1$ and $S = E^{\theta^+}_\theta$, nonempty wins $PD_{<\omega_1}(S)$ but not $BM_{\leq\omega}(S)$, even if nonempty gets the first move.*

The same holds for $\theta = \aleph_n$ (for $n \in \omega$) etc.

Various aspects of these and related games have been studied for a long time.

Note that in this paper we consider the games on sets, i.e. a move is an element of the powerset of κ minus the (nonstationary) ideal. A popular (closely related but not always equivalent) variant is to consider games on a Boolean algebra B: Moves are elements of B, in our case B would be the powerset of κ *modulo* the ideal.

Also note that in Banach Mazur games of length greater than ω, it is relevant which player moves first at limit stages (in our definition this is the empty player). Of course it is also important who moves first at stage 0 (in this paper again the empty player), but the difference here comes down to a simple density effect (cf. 4.1.4).

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*Which is equivalent to: nonempty does not win $\mathrm{BM}_{\leq\omega}(S')$ for any stationary $S' \subseteq S$.

†$S = E^{\omega_2}_{\omega_1}$ is the simplest possible example, since empty always wins PD if every element of S has cofinality ω, cf. 4.4.2.

The Banach Mazur BM game has been investigated e.g. in [8] or [23]. It is closely related to the so-called "ideal game" and to precipitous ideals, cf. Theorem 4.4 and [13], [4], or [12]. BM is also related to the "cut & choose game" of [9].

The pressing down game is related to the Ehrenfeucht-Fraïssé game in model theory, cf. [19] or [7], and has applications in set theory as well [17].

Other related games have been studied e.g. in [11] or [21].

We thank Jouko Väänänen for asking about Theorem 4.1 and for pointing out Theorem 4.6.

4.2. **Banach Mazur, pressing down, and precipitous ideals.** Let κ and θ be regular, $\theta < \kappa$.

We set $E^\kappa_\theta = \{\alpha \in \kappa : \operatorname{cf}(\alpha) = \theta\}$. $\mathcal{E}^\kappa_\theta$ is the family of stationary subsets of E^κ_θ. Analogously for $E^\kappa_{>\theta}$ etc.

Instead of "the empty player has a winning strategy for the game G" we just say "empty wins G" (as opposed to: empty wins a specific run of the game).

$\mathcal{I}$ denotes a fine, normal ideal on κ. (I.e. every $\alpha \in \kappa$ is in $\mathcal{I}$. Together with normal this implies that $\mathcal{I}$ is $< \kappa$-complete.)

A set $S \subseteq \kappa$ is called $\mathcal{I}$-positive if $S \notin \mathcal{I}$.

Definition 4.2. Let κ be regular, and $S \subseteq \kappa$ an $\mathcal{I}$-positive set.

- $\mathrm{BM}_{<\zeta}(\mathcal{I}, S)$, the *Banach Mazur game* of length ζ starting with S, is played as follows:

  At stage 0, empty plays an $\mathcal{I}$-positive $S_0 \subseteq S$, nonempty plays $T_0 \subseteq S_0$. At stage $\alpha < \zeta$, empty plays an $\mathcal{I}$-positive $S_\alpha \subseteq \bigcap_{\beta<\alpha} S_\beta$ (if possible), and nonempty plays some $T_\alpha \subseteq S_\alpha$.

  Empty wins the run, if $\bigcap_{\beta<\alpha} S_\beta \in \mathcal{I}$ at any stage $\alpha < \zeta$. Otherwise nonempty wins.

  (For nonempty to win a run, it is not necessary that $\bigcap_{\beta<\zeta} S_\beta$ is $\mathcal{I}$-positive or even just nonempty.)
- $\mathrm{BM}_{\leq\omega}(\mathcal{I}, S)$ is $\mathrm{BM}_{<\omega+1}(\mathcal{I}, S)$. (So empty wins the run iff $\bigcap_{n<\omega} S_n \in \mathcal{I}$, i.e. the game is naturally equivalent to one of length ω.)
- $\mathrm{PD}_{<\zeta}(\mathcal{I}, S)$, the *pressing down game* of length ζ starting with S, is played as follows:

  At stage $\alpha < \zeta$, empty plays a regressive function $f_\alpha : \kappa \to \kappa$, and nonempty plays some f_α-homogeneous $T_\alpha \subseteq S \cap \bigcap_{\beta<\alpha} T_\beta$.

  Empty wins the run, if $T_\alpha \in \mathcal{I}$ for any $\alpha < \zeta$. Otherwise, nonempty wins.
- $\mathrm{PD}_{\leq\omega}(\mathcal{I}, S)$ is $\mathrm{PD}_{<\omega+1}(\mathcal{I}, S)$. (I.e. empty wins the run iff $S \cap \bigcap_{n\in\omega} T_n \in \mathcal{I}$.)
- $\mathrm{BM}_{<\zeta}(S)$ is $\mathrm{BM}_{<\zeta}(\mathrm{NS}, S)$, and $\mathrm{PD}_{<\zeta}(S)$ is $\mathrm{PD}_{<\zeta}(\mathrm{NS}, S)$ (where NS denotes the nonstationary ideal).

$\mathrm{PD}_{<\theta}$ could equivalently be defined such that nonempty chooses at stage α some $\beta_\alpha \in \kappa$, and empty wins the run if $S \cap \bigcap_{\zeta<\alpha} f^{-1}(\beta_\zeta) \in \mathcal{I}$ for some $\alpha < \theta$.

The following is trivial:

Facts 4.1. *(i) Assume $S \subseteq T$.*

- *If empty wins $BM_{<\zeta}(\mathcal{I}, S)$, then empty wins $BM_{<\zeta}(\mathcal{I}, T)$.*
- *If nonempty wins $BM_{<\zeta}(\mathcal{I}, T)$, then nonempty wins $BM_{<\zeta}(\mathcal{I}, S)$.*
- *If empty wins $PD_{<\zeta}(\mathcal{I}, T)$, then empty wins $PD_{<\zeta}(\mathcal{I}, S)$.*
- *If nonempty wins $PD_{<\zeta}(\mathcal{I}, S)$, then nonempty wins $PD_{<\zeta}(\mathcal{I}, T)$.*

(ii) Assume that $\mathcal{I} \subseteq \mathcal{J}$, and that $\mathcal{J}$ is also fine and normal.

- *If empty wins $PD_{<\zeta}(\mathcal{I}, S)$, then empty wins $PD_{<\zeta}(\mathcal{J}, S)$.*
- *If nonempty wins $PD_{<\zeta}(\mathcal{J}, S)$, then nonempty wins $PD_{<\zeta}(\mathcal{I}, S)$.*

(iii) In particular, if nonempty wins $PD_{<\zeta}(\mathcal{I}, S)$, then nonempty wins $PD_{<\zeta}(S)$.

(iv) Let BM' be the variant of BM where nonempty gets the first move (at stage 0 only). The difference between BM and BM' is a simple density effect:

- *Empty wins $BM'_{<\zeta}(\mathcal{I}, S)$ iff empty wins $BM_{<\zeta}(\mathcal{I}, S')$ for all positive $S' \subseteq S$ iff empty has a winning strategy for BM with S as first move.*
- *Empty wins $BM_{<\zeta}(\mathcal{I}, S)$ iff empty wins $BM'_{<\zeta}(\mathcal{I}, S')$ for some positive $S' \subseteq S$.*
- *Nonempty wins $BM'_{<\zeta}(\mathcal{I}, S)$ iff nonempty wins $BM_{<\zeta}(\mathcal{I}, S')$ for some positive $S' \subseteq S$.*

(v) Assume that S is $\mathcal{I}$-positive, and let $\mathcal{I}_S$ be generated by $\mathcal{I} \cup \{\kappa \setminus S\}$. Then $A \in \mathcal{I}_S$ iff $A \cap S \in \mathcal{I}$, and empty wins $BM_{<\theta}(\mathcal{I}, S)$ iff empty wins $BM_{<\theta}(\mathcal{I}_S, \kappa)$. The same holds for PD or the ideal game (defined below), and for player nonempty instead of player empty.

(For 3, use that $\mathcal{I}$ is normal, which implies $\mathrm{NS} \subseteq \mathcal{I}$.)

We will use the following definitions and facts concerning precipitous ideals, as introduced by Jech and Prikry [13]. We will usually refer to Jech's *Millennium Edition* [10] for details.

Definition 4.3. Let $\mathcal{I}$ be a fine, normal ideal on κ.

- Let V be an inner model of W. $U \in W$ is called a *normal V-ultrafilter* if the following holds:
  - If $A \in U$, then $A \in V$ and A is a subset of κ.
  - $\alpha \notin U$ for all $\alpha \in \kappa$, and $\kappa \in U$.
  - If $A, B \in V$ are subsets of κ, $A \subseteq B$ and $A \in U$, then $B \in U$.
  - If $A \in V$ is a subset of κ, then either $A \in U$ or $\kappa \setminus A \in U$.
  - If $f \in V$ is a regressive function on $A \in U$, then f is constant on some $B \in U$.

  (Note that we do not require iterability or amenability.)

- A normal V-ultrafilter U is *wellfounded*, if the ultrapower of V modulo U is wellfounded. In this case the transitive collapse of the ultrapower is denoted by $\mathrm{Ult}_U(V)$.
- For a $<\kappa$-complete ideal $\mathcal{I}$, let $P_{\mathcal{I}}$ be the family of $\mathcal{I}$-positive sets ordered by inclusion. $P_{\mathcal{I}}$ forces that the generic filter G is a V-ultrafilter (cf. [10, 22.13]). An ideal $\mathcal{I}$ is called *precipitous*, if it is κ-complete and $P_{\mathcal{I}}$ forces that G is wellfounded.
- The ideal game on $\mathcal{I}$ is played just like $\mathrm{BM}_{\leq\omega}(\mathcal{I},\kappa)$, but empty wins iff $\bigcap_{n\in\omega} S_n$ is empty (as opposed to "in $\mathcal{I}$").

So if empty wins the ideal game, then empty wins $\mathrm{BM}_{\leq\omega}(\mathcal{I},\kappa)$. And if nonempty wins $\mathrm{BM}_{\leq\omega}(\mathcal{I},\kappa)$, then nonempty wins the ideal game.

Theorem 4.4. *Let $\mathcal{I}$ be a fine, normal ideal on κ.*

(i) (Jech, cf. [10, 22.21]*) $\mathcal{I}$ is not precipitous iff empty wins the ideal game. So in this case empty also wins $BM_{\leq\omega}(\mathcal{I},\kappa)$.*

(ii) (cf. [4]*) If $\mathcal{I}$ is such that E^κ_ω is $\mathcal{I}$-positive, then nonempty cannot win the ideal game, and empty wins*[‡] *$PD_{\leq\omega}(\mathcal{I},E^\kappa_\omega)$ and therefore also $BM_{\leq\omega}(\mathcal{I},\kappa)$.*

(iii) (Jech, Prikry [12], *cf.* [10, 22.33]*) If $\mathcal{I}$ is precipitous, then κ is measurable in an inner model.*

(iv) (Laver, cf. [4] *or* [10, 22.33]*) Assume that U is a normal measure on κ. Let $\aleph_1\leq\theta<\kappa$ be regular and let $Q = Levy(\theta,<\kappa)$ be the Levy collapse (cf. Lemma 4.18). In $V[G_Q]$, let $\mathcal{F}$ be the filter generated by U and $\mathcal{I}$ the corresponding ideal. Then $\mathcal{I}$ is fine and normal, and the family of $\mathcal{I}$-positive sets has a $<\theta$-closed dense subfamily.*

So in particular in $V[G_Q]$ nonempty wins $BM_{<\theta}(\mathcal{I},S)$ for all $\mathcal{I}$-positive sets S (nonempty just has to pick sets from the dense subfamily), and therefore that nonempty wins $PD_{<\theta}(S)$ (cf. 4.1.3).

(v) (Magidor [12], *penultimate paragraph) One can modify this forcing to get a $<\theta$-closed dense subset of $\mathcal{E}^{\theta^+}_\theta$.*

So in particular, $\mathcal{E}^{\theta^+}_\theta$ can be precipitous.

Mitchell [12] showed that even for $\theta=\aleph_0$, $\mathrm{Levy}(\theta,<\kappa)$ gives a precipitous ideal on $\theta^+=\omega_1$ (and with Magidor's extension, NS_{ω_1} can be made precipitous). So the ideal game is interesting on ω_1, but our games are not:

Corollary 4.5. *(i) Empty always wins $PD_{\leq\omega}(S)$ and $BM_{\leq\omega}(S)$ for $S\subseteq\omega_1$.*

(ii) It is equiconsistent with a measurable that nonempty wins $BM_{<\theta}(E^{\theta^+}_\theta)$ for e.g. $\theta=\aleph_1$, $\theta=\aleph_2$, $\theta=\aleph^+_{\aleph_7}$ etc.

(iii) The following is consistent relative to a measurable: Nonempty wins $PD_{<\theta}(\theta^+)$ but not $BM_{\leq\omega}(\theta^+)$ for e.g. $\theta=\omega_1$.

[‡]There is even a fixed sequence of winning moves for empty: For every $\alpha\in E^\kappa_\omega$ let $(\alpha_n)_{n\in\omega}$ be a normal sequence in α. As move n, empty plays the function that maps α to α_n. If β and β' are both in $\bigcap_{n\in\omega} T_n$, then $\beta_n=\beta'_n$ for all n and therefore $\beta=\beta'$.

Proof. (1) is just 4.4.2, and (2) follows from 4.4.3–4.

(3) Let κ be measurable, and Levy-collapse κ to θ^+. According to 4.4.2, nonempty wins $\mathrm{PD}_{<\omega_1}(S)$ for all $S \in U$, in particular for $S = \theta^+$. However, empty wins $\mathrm{BM}_{\leq\omega}(\theta^+)$ (by playing $E^{\theta^+}_\omega$). □

In the rest of the paper will deal with the proof of Theorem 4.1.

4.3. **Overview of the proof.** We assume that κ is measurable, and that $\omega < \theta < \kappa$ is regular.

Step 1. We construct models M satisfying:

($*$) κ is measurable and player empty wins $\mathrm{BM}_{\leq\omega}(S)$ for every stationary S.

We present two constructions, showing that ($*$) is true in $L[U]$ as well as compatible with larger cardinals:

(i) The inner model $L[U]$, Section 4.4:
Let D be a normal measure on κ, and set $U = D \cap L[D]$. Then in $L[U]$, (the dual ideal of) U is the only normal precipitous ideal on κ. In particular, $L[U]$ satisfies ($*$).

(ii) Forcing ($*$), Section 4.5:
(α) We construct a partial order $R(\kappa)$ forcing that empty wins $\mathrm{BM}_{\leq\omega}(S)$ for all S. However, $R(\kappa)$ does not preserve measurability of κ.
(β) We use $R(\kappa)$ to force ($*$) while preserving e.g. supercompactness.

Step 2. Now we look at the Levy-collapse Q that collapses κ to θ^+.

In Section 4.6 we will see: If in $V[G_Q]$, nonempty wins $\mathrm{BM}_{\leq\omega}(\dot S)$ for some $\dot S \in \mathcal{E}^\kappa_\theta$, then in V nonempty wins $\mathrm{BM}_{\leq\omega}(\tilde S)$ for some $\tilde S \in \mathcal{E}^\kappa_{\geq\theta}$.

So if we start with V satisfying ($*$) of Step 1, then Q forces:

- Nonempty does not win $\mathrm{BM}_{\leq\omega}(\dot S)$ for any stationary $\dot S \subseteq E^\kappa_\theta$. Equivalently: Nonempty does not win $\mathrm{BM}_{\leq\omega}(E^\kappa_\theta)$, even if nonempty gets the first move.
- Nonempty wins $\mathrm{PD}_{<\theta}(E^\kappa_\theta)$. This follows from 4.4.4: Nonempty wins $\mathrm{PD}_{<\theta}(S)$ for all $S \in U$, and $E^\kappa_\theta = (E^\kappa_{\geq\theta})^V \in U$.

4.4. U **is the only normal, precipitous ideal in** $L[U]$**.** If $V = L$, then there are no normal, precipitous ideals (recall that a precipitous ideal implies a measurable in an inner model). Using Kunen's results on iterated ultrapowers, it is easy to relativize this to $L[U]$:

Theorem 4.6. *Assume $V = L[U]$, where U is a normal measure on κ. Then the dual ideal of U is the only normal, precipitous ideal on κ.*

In particular, NS_κ is nowhere precipitous, and empty wins $BM_{\leq\omega}(S)$ for any stationary $S \subseteq \kappa$.

Remark: Much deeper results by Jech and later Gitik show that e.g.

($\star$) κ is measurable and either E^κ_λ or $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is precipitous implies more than a measurable (in an inner model) [5, Sect. 5], so ($\star$) fails not only in $L[U]$ but also in any other universe without "larger inner-model-cardinals". However, it is not clear to us whether the same hold e.g. for

($\star'$) κ is measurable and $\mathrm{NS}_\kappa \restriction S$ is precipitous for some S.

Back to the proof of Theorem 4.6.

If empty does not win $\mathrm{BM}_{\leq\omega}(S)$, then empty does not win the ideal game starting with S, and empty does not win the ideal game on the ideal NS_S defined in 4.1.5. That means that NS_S is precipitous. But NS_S can never be equal to the dual of U, a contradiction. (S can be partitioned into disjoint positive subsets, but U is an ultrafilter). So it is enough to show that the dual ideal of U is the only normal, precipitous ideal.

If $\mathcal{I}$ is a normal, precipitous ideal, then $P_{\mathcal{I}}$ forces that the generic filter G is a normal, wellfounded V-ultrafilter (cf. [10, 22.13]). So it is enough to show that in any forcing extension, U is the only normal, wellfounded V-ultrafilter on κ. We will do this in Lemma 4.8.

If $U \in L[U]$ and $L[U]$ thinks that U is a normal ultrafilter on κ, then we call the pair $(L[U], U)$ a κ-model.

If D is a normal ultrafilter on κ, and $U = D \cap L[D]$, then $(L[U], U)$ is a κ-model.

We will use the following results of Kunen [14], cited as Theorem 19.14 and Lemma 19.16 in [10]:

Lemma 4.7. *(i) For every ordinal κ there is at most one κ-model.*

(ii) Assume $\kappa < \lambda$ are ordinals, $(L[U], U)$ is the κ-model and $(L[W], W)$ the λ-model. Then $(L[W], W)$ is an iterated ultrapower of $(L[U], U)$, in particular: There is an elementary embedding $i : L[U] \to L[W]$ definable in $L[U]$ such that $W = i(U)$.

(iii) Assume that
- *$(L[U], U)$ is the κ-model,*
- *A is a set of ordinals of size at least κ^+,*
- *θ is a cardinal such that $A \cup \{U\} \subset L_\theta[U]$, and*
- *$X \subseteq \kappa$ is in $L[U]$.*

Then there is a formula φ, ordinals $\alpha_i < \kappa$ and $\gamma_i \in A$ such that in $L_\theta[U]$, X is defined by $\varphi(X, \alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_m, U)$.

(That means that in $L[U]$ there is exactly one y satisfying $\varphi(y, \alpha_1, \dots)$, and $y = X$.)

Lemma 4.8. *Assume $V = L[U]$, where U is a normal ultrafilter on κ. Let V' be a forcing extension of V, and $G \in V'$ a normal, wellfounded V-ultrafilter on κ. Then $G = U$.*

Proof. In V', let $j : V \to \mathrm{Ult}_G(V)$ be elementary. Set $\lambda = j(\kappa) > \kappa$ and $W = j[U]$. So $\mathrm{Ult}_G(V)$ is the λ-model $L[W]$.

In V, we can define a function $J : \mathbf{ON} \to \mathbf{ON}$ such that in V', $J(\alpha)$ is a cardinal greater than $(\alpha^\kappa)^{+V'}$. (After all, V' is just a forcing extension of V.) So

$J(\alpha)$ is greater than both $i(\alpha)$ and $j(\alpha)$. In V, let $\mathcal{C}$ be the class of ordinals that are ω-limits of iterations of J, i.e. $\alpha \in \mathcal{C}$ if $\alpha = \sup(\alpha_0, J(\alpha_0), J(J(\alpha_0)), \dots)$. If $\alpha \in C$, then $i(\alpha) = j(\alpha) = \alpha$, since

$$\begin{aligned} i(\alpha) &= \sup(i(\alpha_0), i(J(\alpha_0)), i(J(J(\alpha_0))), \dots) \\ &\leq \sup(J(\alpha_0), J(J(\alpha_0)), J(J(J(\alpha_0))), \dots) = \alpha. \end{aligned}$$

Also, each $\alpha \in \mathcal{C}$ is a cardinal in V', since it is a supremum of cardinals.

In V', pick a set A of κ^+ many members of $\mathcal{C}$, and $\theta \in \mathcal{C}$ such that and $A \cup \{U\} \subseteq L_\theta[U]$. Pick any $X \subseteq \kappa$. Then in $L[U]$, X is defined by

$$L_\theta[U] \vDash \varphi(X, \vec{\alpha}, \vec{\gamma}, U).$$

Let k be either i or j. Then by elementarity, in $L[W]$ $k(X)$ is the set Y such that

$$L_\theta[W] \vDash \varphi(Y, \vec{\alpha}, \vec{\gamma}, W),$$

since $W = k(U)$ and $k(\beta) = \beta$ for all $\beta \in \kappa \cup A \cup \{\theta\}$.

Therefore $i(X) = j(X) = Y$. So $X \in G$ iff $\kappa \in j(X) = i(X)$ iff $X \in U$, since both G and U are normal. □

4.5. **Forcing empty to win.** As in the last section, we construct a universe with in which empty wins $\mathrm{BM}_{\leq\omega}(S)$ for every stationary $S \subseteq \kappa$, this time using forcing. This shows that the assumption is also compatible with e.g. κ supercompact.

4.5.1. *The basic forcing.*

Assumption 4.9. κ is inaccessible and $2^\kappa = \kappa^+$.

We will define the $< \kappa$-support iteration $(P_\alpha, Q_\alpha)_{\alpha<\kappa^+}$ and show:

Lemma 4.10. *P_{κ^+} forces: Empty has a winning strategy for $BM_{\leq\omega}(\kappa)$ where empty's first move is κ. P_{κ^+} is κ^+-cc and has a dense subforcing P'_{κ^+} which is $< \kappa$-directed-closed and of size κ^+.*

We use two basic forcings (maybe more exactly: forcing-definitions) in the iteration:

- If $S \subseteq \kappa$ is stationary, then Cohen(S) adds a Cohen subset of S. Conditions are functions $f : \zeta \to \{0, 1\}$ with $\zeta < \kappa$ successor such that $\{\xi < \zeta : f(\xi) = 1\}$ is a subset of S. ζ is called height of f. Cohen(S) is ordered by inclusion.

  This forcing adds the generic set $S' = \{\zeta < \kappa : (\exists f \in G) f(\zeta) = 1\} \subset S$.
- If $\lambda \leq \kappa^+$, and $(S_i)_{i<\lambda}$ is a family of stationary sets, then Club($(S_i)_{i<\lambda}$) consists of $f : (\zeta \times u) \to \{0, 1\}$, $\zeta < \kappa$ successor, $u \subseteq \lambda$, $|u| < \kappa$ such that $\{\xi < \zeta : f(\xi, i) = 1\}$ is a closed subset of S_i. ζ is called height of f, u domain of f. Club($(S_i)_{i<\lambda}$) is ordered by inclusion.

The following is well known:

Lemma 4.11. *Cohen(S) is $<\kappa$-closed and forces that the generic Cohen subset $S' \subseteq S$ is stationary.*

So Cohen(S) is a well-behaved forcing, adding a generic stationary subset of S. Club($(S_i)_{i<\lambda}$) adds unbounded closed subsets of each S_i. Other than that it is not clear why this forcing should e.g. preserve the regularity of κ (and it will generally not be σ-closed). However, we will shoot clubs only through complements of Cohen-generics which we added previously, and this will simplify matters considerably.

The P_α will add more and more moves to our winning strategy.

Set $D = \{\delta < \kappa^+ : \delta \text{ limit}\}$ (D stands for "destroy").

Set $\mathcal{T} = (\kappa^+)^{<\omega}$, a tree ordered by inclusion. (Let us call the order $\preceq_\mathcal{T}$.) Find a bijection $i : \mathcal{T} \to \kappa^+ \setminus D$ so that $s \preceq_\mathcal{T} t$ implies $i(s) \leq i(t)$. Let M be the image of i, i.e. $\kappa^+ = D \cup M$. (M stands for "moves".) i defines a tree-order $\preceq_M$ on M such that $\alpha \preceq_M \beta$ implies $\alpha \leq \beta$. Tree-order means that for $\alpha \in M$, the set of $\preceq_M$-predecessors of M is finite and totally ordered by $\preceq_M$. This defines for $\alpha \in M$ the sequence $\alpha_0 \preceq_M \alpha_1 \preceq_M \cdots \preceq_M \alpha_m \preceq_M \alpha$ of predecessors.

For $\delta \in D$, we can look at all infinite branches through $M \cap \delta$. Some of them will be "new", i.e. not in $M \cap \gamma$ for any $\gamma \in D \cap \delta$. Let λ_δ be the number of these new branches, i.e. $0 \leq \lambda_\delta \leq 2^\kappa = \kappa^+$.

We define Q_α by induction on α, and assume that at stage α (i.e. after forcing with P_α) we have already defined a partial strategy. (For increasing α, the partial strategy will increase, i.e. it will know responses to more initial segments of runs of the game.) We will see that P_α forces $2^\kappa = \kappa^+$. This allows us to use some simple book-keeping to pick at stage α some $T_\alpha \subseteq \kappa$ such that every $T \subseteq \kappa$ in $\bigcup_{\beta<\kappa^+} V[G_\beta]$ appears as some T_ζ. In more detail:

Fix an enumeration $(\tilde{T}_{\alpha,\gamma})_{\gamma\in\kappa^+}$ of all (P_α-names for) subsets of κ. Fix a $\psi : M \to \kappa^+ \times \kappa^+$ such that $\psi(\alpha) = (\beta, \gamma)$ implies $\beta \leq \alpha$, and such that for all $\alpha \in M$ and $\beta, \gamma \in \kappa^+$ there is an immediate $\preceq_M$-successor α' of α such that $\psi(\alpha') = (\beta, \gamma)$. For $\psi(\alpha) = (\beta, \gamma)$, set $T_\alpha = \tilde{T}_{\beta,i}$ if it satisfies some additional assumption ($*$) (see below), otherwise pick some arbitrary T_α satisfying ($*$).

We work in $V[G_\alpha]$ to define Q_α:

- $\alpha \in M$, with the predecessors $0 = \alpha_0 < \alpha_1 \cdots < \alpha_m < \alpha$. By induction we know that at stage α_m
  - we dealt with the sequence $x_{\alpha_m} = (\kappa, T_{\alpha_1}, S_{\alpha_1}, T_{\alpha_2}, \ldots, S_{\alpha_{m-1}}, T_{\alpha_m})$, which is played according to empty's partial strategy (at stage α_m),
  - we defined Q_{α_m} to be Cohen(T_{α_m}), adding the generic set S_{α_m},
  - this S_{α_m} was added to the partial strategy as response to x_{α_m}.

  Now we use the book-keeping described above to pick T_α satisfying:

  ($*$) $T_\alpha \subset S_{\alpha_m}$ is stationary, and the partial strategy is not (at stage α) already defined on $x_\alpha = x_{\alpha_m}{}^\frown(S_{\alpha_m}, T_\alpha)$.

  Then we set $Q_\alpha = \text{Cohen}(T_\alpha)$, and add the Q_α-generic $S_\alpha \in V[G_{\alpha+1}]$ to the partial strategy as response to x_α.

- $\alpha \in D$. In V, there are $0 \le \lambda_\alpha \le \kappa^+$ many new branches b_i. (All old branches have already been dealt with in the previous D-stages.) For each new branch $b_i = (\alpha_0^i < \alpha_1^i < \dots)$, we set $S^i = \bigcap_{n\in\omega} S_{\alpha_n^i}$, and we set $Q_\alpha = \text{Club}((\kappa \setminus S^i)_{i\in\lambda_\alpha})$.

So empty always responds to nonempty's move T with a Cohen subset of T, and the intersection of an ω-sequence of moves according to the strategy is made non-stationary.

We will show:

Lemma 4.12. *P_{κ^+} does not add any new countable sequences of ordinals, forces that κ is regular and that the Q_α-generic S_α (i.e. empty's move) is stationary for all $\alpha \in M$.*

We will prove this Lemma later. Then the rest follows easily:

Lemma 4.13. *P_{κ^+} forces that the partial strategy is a winning strategy for player empty in the game $BM_{\le\omega}(\kappa)$, using κ as first move.*

Proof. At the final limit stage, P_{κ^+} does not add any new subsets of κ, nor any countable sequences of such subsets. (In particular, there are only κ^+ many.) Work in $V[G_{\kappa^+}]$.

We first show that the partial strategy is a strategy: Assume towards a contradiction that there is some minimal $m \ge 0$ and a sequence $x = (\kappa, T'_1, S'_1, T'_2, S'_2, \dots, S'_m, T'_{m+1})$ such that x is a valid initial sequence of a run played according to the partial strategy, but we do not have a respond to x. So S'_m was added as response to $x \restriction 2m$, at some stage $\alpha \in M$, i.e. α has predecessors $\alpha_0 < \dots < \alpha_m$, and $T'_i = T_{\alpha_i}$ and $S'_i = S_{\alpha_i}$ for $i < m$, and $S'_m = S_\alpha$. T'_{m+1} appears in some V_β for $\beta < \kappa^+$, i.e. $T'_{m+1} = \tilde{T}_{\beta,i}$ for some $i < \kappa^+$. Then there is some $\alpha' \in M$ such that $\alpha' > \beta$ is immediate $\preceq_M$-successor of α and such that $\psi(\alpha') = (\beta, i)$. So at stage α' we add to the partial strategy $S_{\alpha'}$ as response to x (unless we already added a response at an earlier stage), a contradiction.

Now we show that the strategy is actually a winning strategy: Let $y = (\kappa, T'_1, S'_1, T'_2, S'_2, \dots)$ be an infinite run of the game such that nonempty uses the partial strategy. Then $x \restriction 2n$ corresponds to an element of M for every n, and x defines a branch b through M. $b \in V$, since P_{κ^+} does not add new countable sequences of ordinals. Let $\alpha \in D$ be minimal so that $x \restriction 2n < \alpha$ for all n. Then in the D-stage α, the stationarity of $\bigcap_{n\in\omega} S'_n$ was destroyed, i.e. empty wins the run x. $\square$

We now define the dense subset of P_α:

Definition 4.14. $p \in P'_\alpha$ if $p \in P_\alpha$ and there are (in V) a successor ordinal $\epsilon(p) < \kappa$, $(f_\alpha)_{\alpha\in\text{dom}(p)}$ and $(u_\alpha)_{\alpha\in\text{dom}(p)\cap D}$ such that:

- If $\alpha \in M$, then $f_\alpha : \epsilon(p) \to \{0, 1\}$.
- If $\alpha \in D$, then $u_\alpha \subseteq \lambda_\alpha$, $|u_\alpha| < \kappa$, and $f_\alpha : \epsilon(p) \times u_\alpha \to \{0, 1\}$.

- Moreover, for $\alpha \in D$, u_α consists exactly of the new branches through $\mathrm{dom}(p) \cap \alpha \cap M$.
- $p \restriction \alpha \Vdash p(\alpha) = f_\alpha$.

So a $p \in P'_\alpha$ corresponds to a "rectangular" matrix with entries in $\{0, 1\}$. Of course only some of these matrices are conditions of P_α and therefore in P'_α.

Lemma 4.15. *(i) P'_α is ordered by extension. (I.e. if $p, q \in P'_\alpha$, then $q \leq p$ iff q (as Matrix) extends p.)*
(ii) $P'_\alpha \subseteq P_\alpha$ is a dense subset.
(iii) P'_α is $< \kappa$-directed-closed, in particular P_α does not add any new sequences of length $< \kappa$ nor does it destroy stationarity of any subset of κ.

Proof. (1) should be clear.

(3) Assume all p_i are pairwise compatible. We construct a condition q by putting an additional row on top of $\bigcup p_i$ (and filling up at indices where new branches might have to be added). So we set

- $\mathrm{dom}(q) = \bigcup \mathrm{dom}(p_i)$.
- $\epsilon(q) = \bigcup \epsilon(p_i) + 1$.
- For $\alpha \in \mathrm{dom}(q) \cap M$, we put 0 on top, i.e. $q_\alpha(\epsilon(q) - 1) = 0$.
- For $\alpha \in \mathrm{dom}(q) \cap D$, and $i \in \bigcup \mathrm{dom}(p_i(\alpha))$, set $q_\alpha(\epsilon(q) - 1, i) = 1$.
- For $\alpha \in \mathrm{dom}(q) \cap D$, if i is a new branch through $M \cap \mathrm{dom}(q) \cap \alpha$ and not in $\bigcup \mathrm{dom}(p_i(\alpha))$, set $q_\alpha(\xi, i) = 0$ for all $\xi < \epsilon(q)$.

Why can we do that? If $\alpha \in M$, whether the bookkeeping says that $\epsilon(q) - 1 \in T_\alpha$ or not, we can of course always choose to not put it into S_α (i.e. set $q_\alpha(\epsilon(q) - 1) = 0$). Then for $\alpha \in D$, $\epsilon(q) - 1$ will definitely not be in the intersection along the branch i, so we can put it into the complement.

(2) By induction on α. Assume $p \in P_\alpha$.

$\alpha = \beta + 1$ is a successor. We know that P_β does not add any new $< \kappa$ sequences of ordinals, so we can strengthen $p \restriction \beta$ to a $q \in P'_\beta$ which decides $f = p(\beta) \in V$. Without loss of generality $\epsilon(q) \geq \mathrm{height}(f)$, and we can enlarge f up to $\epsilon(q)$ by adding values 0 (note that $\mathrm{height}(f) < \kappa$ is a successor, so we do not get problems with closedness when adding 0). And again, we also add values for the required "new branches" if necessary.

If α is a limit of cofinality $\geq \kappa$, then $p \in P_\beta$ for some $\beta < \alpha$, so there is nothing to do.

Let α be a limit of cofinality $< \kappa$, i.e. $(\alpha_i)_{i \in \lambda}$ is an increasing cofinal sequence in α, $\lambda < \kappa$. Using (2), define a sequence $p_i \in P'_{\alpha_i}$ such that $p_i < p_j \wedge p \restriction \alpha_i$ for all $j < i$, then use (3). □

How does the quotient forcing $P^\alpha_{\kappa^+}$ (i.e. P_{κ^+}/G_α) behave compared to P_{κ^+}?

- Assume $\alpha \in D$. In $V[G_\alpha]$, Q_α shoots a club through the complement of the (probably) stationary set $\bigcap_{i \in \omega} S^i$. In particular, Q_α cannot have a $< \kappa$-closed subset.

- Nevertheless, $P_\alpha * Q_\alpha$ has a $< \kappa$-closed subset (and preserves stationarity).
- So if we factor P_{κ^+} at some $\alpha \in D$, the remaining $P^\alpha_{\kappa^+}$ will look very different from P_{κ^+}.
- However, if we factor P_{κ^+} at $\alpha \in M$, $P^\alpha_{\kappa^+}$ will be more or less the same as $P^\alpha_{\kappa^+}$ (just with a slightly different bookkeeping).

In particular, we get:

Lemma 4.16. *If $\alpha \in M$, then the quotient $P^\alpha_{\kappa^+}$ will have a dense $< \kappa$-closed subset (and therefore it will not collapse stationary sets).*

(The proof is the same as for the last lemma.)

Note that for this result it was necessary to collapse the new branches as soon as they appear. If we wait with that, then (looking at the rest of the forcing from some stage $\alpha \in M$) we shoot clubs through stationary sets that already exist in the ground model, and things get more complicated.

Now we can easily prove Lemma 4.12:

Proof of Lemma 4.12. In stage $\alpha \in M$, nonempty's previous move S_{α_m} is still stationary (by induction), the bookkeeping chooses a stationary subset T_{α_m} of this move, and we add S_α as Cohen-generic subset of T_{α_m}. So according to Lemma 4.11, S_α is stationary at stage $\alpha+1$, i.e. in $V[G_{\alpha+1}]$. But since $\alpha+1 \in M$, the rest of the forcing, $P^{\alpha+1}_{\kappa^+}$, is $< \kappa$-closed and does not destroy stationarity of S_α. □

4.5.2. *Preserving Measurability.* We can use the following theorem of Laver [15], generalizing an idea of Silver: If κ is supercompact, then there is a forcing extension in which κ is supercompact and every $< \kappa$-directed closed forcing preserves the supercompactness. Note that we can also get $2^\kappa = \kappa^+$ which such a forcing.

Corollary 4.17. *If κ is supercompact, we can force that κ remains supercompact and that empty wins $BM_{\leq\omega}(S)$ for all stationary $S \subseteq \kappa$.*

Remark: It is possible, but not obvious that we can also start with κ just measurable and preserve measurability. It is at least likely that it is enough to start with strong to get measurable. Much has been published on such constructions, starting with Silver's proof for violating GCH at a measurable (as outlined in [10, 21.4]).

4.6. **The Levy collapse.** We show that after collapsing κ to θ^+, nonempty still has no winning strategy in BM.

Assume that κ is inaccessible, $\theta < \kappa$ regular, and let $Q = \text{Levy}(\theta, < \kappa)$ be the Levy collapse of κ to θ^+: A condition $q \in Q$ is a function defined on a subset of $\kappa \times \theta$, such that $|\text{dom}(q)| < \theta$ and $q(\alpha, \xi) < \alpha$ for $\alpha > 1, (\alpha, \xi) \in \text{dom}(q)$ and $q(\alpha, \xi) = 0$ for $\alpha \in \{0, 1\}$.

Given $\alpha < \kappa$, define $Q_\alpha = \{q : \text{dom}(q) \subseteq \alpha \times \theta\}$ and $\pi_\alpha : Q \to Q_\alpha$ by $q \mapsto q \restriction (\alpha \times \theta)$.

The following is well known (see e.g. [10, 15.22] for a proof):

Lemma 4.18.
- *Q is κ-cc and $<\theta$-closed.*
- *In particular, Q preserves stationarity of subsets of κ: If p forces that $\dot{C} \subseteq \kappa$ is club, then there is a $C' \subseteq \kappa$ club and a $q \leq p$ forcing that $C' \subseteq \dot{C}$.*
- *If $q \Vdash p \in G$, then $q \leq p$ (i.e. $\leq^*$ is the same as $\leq$).*

We will use the following simple consequence of Fodor's Lemma (similar to a Δ-system Lemma):

Lemma 4.19. *Assume that $p \in Q$ and $S \in \mathcal{E}^\kappa_{\geq\theta}$. If $\{q_\alpha \mid \alpha \in S\}$ is a sequence of conditions in Q, $q_\alpha < p$, then there is a $\beta < \kappa$, a $q \in Q_\beta$ and a stationary $S' \subseteq S$, such that $q \leq p$ and $\pi_\alpha(q_\alpha) = q$ for all $\alpha \in S'$.*

Proof. For $q \in Q$ set $\operatorname{dom}^\kappa(q) = \{\alpha \in \kappa : (\exists \zeta \in \theta)\, (\alpha, \zeta) \in \operatorname{dom}(q)\}$. For $\alpha \in S$ set $f(\alpha) = \sup(\operatorname{dom}^\kappa(q_\alpha) \cap \alpha)$. f is regressive, since $|\operatorname{dom}^\kappa(q_\alpha)| < \theta$ and $\operatorname{cf}(\alpha) \geq \theta$. By the pressing down lemma there is a $\beta < \kappa$ such that $T = f^{-1}(\beta) \subseteq S$ is stationary.

For $\alpha \in T$, set $h(\alpha) = \pi_{\beta+1}(q_\alpha)$. The range of h is of size at most $|\beta \times \theta|^{<\theta} < \kappa$. So there is a stationary $S' \subseteq T$ such that h is constant on S', say q. If $\alpha \in S'$, then $\sup(\operatorname{dom}^\kappa(q_\alpha) \cap \alpha) = \beta$, therefore $\pi_\alpha(q_\alpha) = \pi_{\beta+1}(q_\alpha) = q$.

Pick $\alpha \in S'$ such that $\alpha > \sup(\operatorname{dom}^\kappa(p))$. $q_\alpha \leq p$, so $q = \pi_\alpha(q_\alpha) \leq \pi_\alpha(p) = p$. □

Lemma 4.20. *Assume that*
- *κ is strongly inaccessible, $\theta < \kappa$ regular, $\mu \leq \theta$.*
- *$Q = Levy(\theta, <\kappa)$,*
- *$\dot{S}$ is a Q-name for an element of $\mathcal{E}^\kappa_\theta$,*
- *$\tilde{p} \in Q$ forces that $\dot{F}$ is a winning strategy of nonempty in $BM_{<\mu}(\dot{S})$.*

Then in V, nonempty wins $BM_{<\mu}(\tilde{S})$ for some $\tilde{S} \in E^\kappa_{\geq\theta}$.

If $\dot{S}$ is a standard name for $T \in (E^\kappa_{\geq\theta})^V$, then we can set $S = T$.

Proof. First assume that $\dot{S}$ is a standard name.

For a run of $\mathrm{BM}_{<\mu}(S)$, we let A_ε and B_ε denote the εth moves of empty and nonempty. We will construct by induction on $\varepsilon < \mu$ a strategy for empty, including not only the moves B_ε, but also Q-names $\dot{A}'_\varepsilon, \dot{B}'_\varepsilon$, and Q-conditions $p_\varepsilon, \langle p^\varepsilon_\alpha \mid \alpha \in B_\varepsilon\rangle$, such that the following holds:
- $p_\varepsilon \leq p_\xi$ and $p^\varepsilon_\alpha \leq p^\xi_\alpha$ for $\xi < \varepsilon$.
- p_ε forces that $(\dot{A}'_\xi, \dot{B}'_\xi)_{\xi \leq \varepsilon}$ is an initial segment of a run of $\mathrm{BM}_{<\mu}(\dot{S})$ in which nonempty uses the strategy $\dot{F}$.
- $p_\varepsilon \Vdash \dot{A}'_\varepsilon \subseteq A_\varepsilon$.
- For $\alpha \in B_\varepsilon$, $\pi_\alpha(p^\varepsilon_\alpha) = p_\varepsilon$ (in particular $p^\varepsilon_\alpha \leq p_\varepsilon$), and $p^\varepsilon_\alpha \Vdash$ "$\alpha \in \dot{B}'_\varepsilon$".

Assume that we have already constructed these objects for all $\xi < \varepsilon$.

In limit stages ε, we first have to make sure that $\bigcap_{\xi<\varepsilon} B_\xi$ is stationary (otherwise nonempty has already lost). Pick a q stronger than each p_ξ for $\xi < \varepsilon$. (This is possible since Q is $<\theta$-closed.) Then q forces that $\bigcap_{\xi<\varepsilon} B_\xi = \bigcap_{\xi<\varepsilon} A_\xi \supseteq \bigcap_{\xi<\varepsilon} \dot{A}'_\xi$ and that $(\dot{A}'_\xi, \dot{B}'_\xi)_{\xi\leq\varepsilon}$ is a valid initial segment of a run where nonempty uses the strategy, in particular $\bigcap_{\xi<\varepsilon} \dot{A}'_\xi$ is stationary.

So now ε can be a successor or a limit, and empty plays the stationary set $A_\varepsilon \subseteq \bigcap_{\xi<\varepsilon} B_\xi$. (That implies that p^ξ_α is defined for all $\alpha \in A_\varepsilon$ and $\xi < \varepsilon$.)

- Define the εth move of empty in $V[G_Q]$ to be

  $$\dot{A}'_\varepsilon = \{\alpha \in A_\varepsilon : (\forall \xi < \varepsilon)\, p^\xi_\alpha \in G_Q\},$$

  and pick $\tilde{p}_\varepsilon \leq p_\xi$ for $\xi < \varepsilon$ (for $\varepsilon = 0$, pick $\tilde{p}_0 = \tilde{p}$).

  $\tilde{p}_\varepsilon$ forces that $\dot{A}'_\varepsilon \subseteq \bigcap_{\xi<\varepsilon} \dot{B}'_\xi$, since p^ξ_α forces that $\alpha \in \dot{B}'_\xi$. $\tilde{p}_\varepsilon$ also forces that $\dot{A}'_\varepsilon$ is stationary:
  Otherwise there is a $C \subseteq \kappa$ club and a $q \leq \tilde{p}_\varepsilon$ forcing that $C \cap \dot{A}_\varepsilon$ is empty (cf. 4.18). $q \in Q_\beta$ for some $\beta < \kappa$. Pick $\alpha \in (C \cap A_\varepsilon) \setminus (\beta+1)$. For $\xi < \varepsilon$, $\pi_\alpha(p^\xi_\alpha) = p_\xi \geq q$, and $q \in Q_\beta$, so q and p^ξ_α are compatible. Moreover, the conditions $(q \cup p^\xi_\alpha)_{\xi\in\varepsilon}$ are decreasing, so there is a common lower bound q' forcing that $p^\xi_\alpha \in G_Q$ for all ξ, i.e. that $\alpha \in \dot{A}'_\varepsilon$, a contradiction.
- Given $\dot{A}'_\varepsilon$, we define $\dot{B}'_\varepsilon$ as the response according to the strategy $\dot{F}$.
- Now we show how to obtain the next move of nonempty, B_ε, (in the ground model), as well as p^ε_α for $\alpha \in B_\varepsilon$. B_ε of course has to be a subset of the stationary set S defined by

  $$S = \{\alpha \in A_\varepsilon \mid \tilde{p}_\varepsilon \not\Vdash \alpha \notin \dot{B}'_\varepsilon\}.$$

  For each $\alpha \in S$, pick some $p^\varepsilon_\alpha \leq \tilde{p}_\varepsilon$ forcing that $\alpha \in \dot{B}'_\varepsilon$. By the definition of $\dot{A}'_\varepsilon$ and since $\tilde{p}_\varepsilon \Vdash \dot{B}'_\varepsilon \subseteq \dot{A}'_\varepsilon$, we get

  $$p^\varepsilon_\alpha \Vdash (\forall \xi < \varepsilon)\, p^\xi_\alpha \in G_Q,$$

  which means that for $\alpha \in S$ and $\xi < \varepsilon$, $p^\varepsilon_\alpha \leq p^\xi_\alpha$.

  Now we apply Lemma 4.19 (for $p = \tilde{p}_\varepsilon$). This gives us $S' \subseteq S$ and $q \leq \tilde{p}_\varepsilon$. We set $B_\varepsilon = S'$ and $p_\varepsilon = q$.

If $\dot{S}$ is not a standard name, set

$$S^0 = \{\alpha \in E^\kappa_{\geq\theta} : \tilde{p} \not\Vdash \alpha \notin \dot{S}\}$$

As above, for each $\alpha \in S_0$, pick a $\tilde{p}^{-1}_\alpha \leq \tilde{p}$ forcing that $\alpha \in \dot{S}$, and choose a stationary $\tilde{S} \subseteq S^0$ according to Lemma 4.19. Now repeat the proof, starting the sequence (p_ε) and (p^ε_α) already at $\varepsilon = -1$. $\square$

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