On almost sure elimination of generalized quantifiers

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Academic dissertation

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Summary

First-order logic has a rather limited expressive power. For instance, apart from some trivial cases, there is no first-order sentence which is true on every structure over a fixed vocabulary if and only if the structure is rigid, that is, it has only one automorphism. Generalized quantifiers provide convenient ways for extending logics. This line of research was initiated by Mostowski [17] who studied first-order logic augmented with cardinality quantifiers such as "there are infinitely many elements". Lindström [11] defined a general class of quantifiers by associating with every property of structures a quantifier in a natural way. Tarski [19] founded another interesting study to strengthen first-order logic by allowing infinitely long expressions.

Zero-one and convergence laws provide a method for analyzing the expressive powers of logics on finite structures. The *zero-one law* of a logic means that the probabilities of all sentences on random structures of a given finite size converge to zero or one as the size approaches infinity. If the probabilities converge, but not necessarily to zero or one, then the logic has the *convergence law.* The very first zero-one law for first-order logic was proved by Glebskii, Kogan, Liogon'kii, and Talanov [4] and, independently, by Fagin [2]. Both of these proofs actually show that first-order logic has *almost sure quantifier elimination.* This means that, for every formula $\varphi(\bar{x})$ of first-order logic, there is a quantifier-free formula $\theta(\bar{x})$ of first-order logic so that the probability of the sentence $\forall\bar{x}[\varphi(\bar{x}) \leftrightarrow \theta(\bar{x})]$ converges to one as the size of structures approaches infinity. (Here $\bar{x}$ may also be the empty sequence and it is assumed that first-order logic has a quantifier-free everywhere true sentence and its negation.) Almost sure quantifier elimination implies the zero-one law if the vocabulary of random structures does not have constant symbols. The more general question of almost sure equivalence of logics is studied in Hella, Kolaitis, and Luosto [6]. Zero-one and convergence laws are known to hold in

several cases. For first-order logic, some very notable results can be found in Shelah and Spencer [18] and Łuczak and Spencer [12]. For least fixed point logic, some interesting results can be found, for example, in Tyszkiewicz [20]. Zero-one laws for the logic $\mathcal{L}^{\omega}_{\infty\omega}$ can be found, for example, in Kolaitis and Vardi [10] and Lynch [16].

There have been only few zero-one laws for logics with generalized quantifiers. The only published results which I know can be found in Dawar and Grädel [1]. On random graphs, Dawar and Grädel [1] investigated almost sure quantifier elimination and zero-one laws of first-order logic augmented with some generalized quantifiers expressing graph properties such as rigidity. Results for some restricted classes of sentences with generalized quantifiers can be found in Fayolle et al. [3] and Knyazev [8]. However, zero-one laws are very interesting on logics with generalized quantifiers because non-definability results for such logics are often difficult to obtain by using other methods and generalized quantifiers have been actively studied on finite structures in recent years. For example, in the context of descriptive complexity theory, some very important results can be found in Hella [5]. This motivated me to establish a new powerful method for proving almost sure quantifier elimination and zero-one laws for logics with generalized quantifiers.

This doctoral thesis is consisting of the following two papers.

(i) On probabilistic elimination of generalized quantifiers.

(ii) On almost sure elimination of numerical quantifiers.

The first paper has both more methodological results and a wider class of applications while the second paper is focused on numerical quantifiers. The approach is slightly different in the second paper, but the methods are equivalent to the corresponding one in the first paper.

Definitions

In this thesis *(generalized) quantifiers* mean the following. They are also called *Lindström quantifiers,* and the definition is equivalent to the original one in Lindström [11]. Let $\bar{r} = (r_1, \ldots, r_m)$ be a finite sequence of numbers in $\mathbb{N}_+ = \{1, 2, \ldots\}$. A structure $\mathfrak{A}$ is of type $\bar{r}$ if it is of the form $\mathfrak{A} = (A, P_1, \ldots, P_m)$, where A is the universe and $P_i \subseteq A^{r_i}$ for each $1 \leqslant i \leqslant m$. A quantifier $\mathsf{Q}_{\mathcal{K}}$ of type $\bar{r}$ is associated with every class $\mathcal{K}$ of structures of type $\bar{r}$, which is closed under isomorphisms. The set of formulas of the logic $\mathcal{L}_{\omega\omega}(\mathsf{Q}_{\mathcal{K}})$ is defined as for first-order logic $\mathcal{L}_{\omega\omega}$ with the additional rule:

if ψ_i is a formula and $\bar{y}_i$ is an r_i-tuple of distinct variable symbols for each $1 \leqslant i \leqslant m$, then $\mathsf{Q}_{\mathcal{K}}\bar{y}_1, \ldots, \bar{y}_m(\psi_1, \ldots, \psi_m)$ is also a formula.

Free and bound variable symbols are defined in the obvious way and a formula with no free variable symbols is a sentence. I use $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ as distinct variable symbols and notation like $\bar{x} = (x_1, \ldots, x_m)$ for sequences of distinct variable symbols. The notation $\varphi(x_1, \ldots, x_m)$ and $\psi(\bar{y})$ for formulas φ and ψ mean that the free variable symbols are among $x_1, \ldots, x_m$ and among the components of $\bar{y}$ respectively.

The semantics of the quantifier $\mathsf{Q}_{\mathcal{K}}$ is defined as follows. Suppose that the free variable symbols of a formula ψ_i are among the components of $\bar{x}_i$ and $\bar{y}_i$. For every structure $\mathfrak{A}$ and interpretation $\bar{a}_i$ of $\bar{x}_i$, let

$$\mathfrak{A} \models \mathsf{Q}_{\mathcal{K}}\bar{y}_1, \ldots, \bar{y}_m\Big(\psi_1(\bar{a}_1, \bar{y}_1), \ldots, \psi_m(\bar{a}_m, \bar{y}_m)\Big) \quad \Leftrightarrow \\ \Big(A, \psi_1^{\mathfrak{A}, \bar{a}_1}, \ldots, \psi_m^{\mathfrak{A}, \bar{a}_m}\Big) \in \mathcal{K},$$

where $\psi_i^{\mathfrak{A}, \bar{a}_i} = \{\bar{b} \in A^{r_i} : \mathfrak{A} \models \psi_i(\bar{a}_i, \bar{b})\}$. The *arity* of the quantifier $\mathsf{Q}_{\mathcal{K}}$ of type $(r_1, \ldots, r_m)$ is $\max\{r_i : 1 \leqslant i \leqslant m\}$. If $m = 1$, $\mathsf{Q}_{\mathcal{K}}$ is a *simple* r_1*-ary quantifier.* The existential and universal quantifiers may be viewed as simple unary quantifiers. Quantifiers are often identified with the defining classes.

The logic $\mathcal{L}_{\omega\omega}(\mathcal{Q})$, where $\mathcal{Q}$ is a collection of quantifiers, can be defined in a similar way. The logic $\mathcal{L}^k_{\omega\omega}(\mathcal{Q})$, $k \in \mathbb{N}_+$, is as $\mathcal{L}_{\omega\omega}(\mathcal{Q})$ but every formula has at most k variable symbols (bound or free). The logic $\mathcal{L}^k_{\infty\omega}(\mathcal{Q})$, $k \in \mathbb{N}_+$, is defined as $\mathcal{L}^k_{\omega\omega}(\mathcal{Q})$ but disjunctions and conjunctions are allowed over any set of formulas, provided that at most k variable symbols (bound or free) occur in the formulas. Further, $\mathcal{L}^\omega_{\infty\omega}(\mathcal{Q})$ is the union of the logics $\mathcal{L}^k_{\infty\omega}(\mathcal{Q})$, $k \in \mathbb{N}_+$. The logic $\mathcal{L}^k_{\infty\omega}$ is the same as $\mathcal{L}^k_{\infty\omega}(\emptyset)$.

With every quantifier Q of type $(r_1, \ldots, r_m)$ and $v \in \mathbb{N}_+$, a quantifier of type $(vr_1, \ldots, vr_m)$ is associated as follows:

$${}^v\mathsf{Q} = \Big\{(A, P_1, \ldots, P_m) \,:\, (A^v, P_1, \ldots, P_m) \in \mathsf{Q}\Big\},$$

where in $(A, P_1, \ldots, P_m)$ the relation P_i is viewed as an vr_i-ary relation over A and in $(A^v, P_1, \ldots, P_m)$ it is viewed as an r_i-ary relation over A^v. The quantifier ${}^v\mathsf{Q}$ is called the *v-vectorization* of Q. The size of a set S is denoted by $|S|$. A quantifier Q of type $(r_1, \ldots, r_m)$ is *numerical* if $(A, P_1, \ldots, P_m) \in \mathsf{Q}$ with $|P_i| = |P_i'|$ and $P_i' \subseteq A^{r_i}$ for each $1 \leqslant i \leqslant m$ imply that $(A, P_1', \ldots, P_m') \in \mathsf{Q}$. Further information on generalized quantifiers can be found, for example, in [9].

The probabilistic quantifier elimination technique of this thesis can be used with all sequences μ_n^{d}, $n \in \mathbb{N}_+$, of discrete probability measures of structures, where n is the size of structures. In most applications I consider probability distributions of random structures which are defined as follows. Let the finite vocabulary τ consist of finitary relation symbols and let $\mathfrak{A}$ be a random structure of size n. For every relation symbol R of the vocabulary, let the probability of $\mathfrak{A} \models R(\bar{a})$ be $p_R(n)$ with these events mutually independent over all $\bar{a} \in A^{\#(R)}$ and $R \in \tau$, where $\#(R)$ is the arity of R. The function p_R is called the *atomic probability* of R. If p_R is the same function for all $R \in \tau$, it is denoted by p_{ato} and called the *atomic probability.* For random graphs, the *edge probability* p_{edg} is defined in the similar manner but $E(x,x)$ is never true and $E(x,y) \leftrightarrow E(y,x)$ is always true, where E is the edge relation. Finally, if the probability of a property φ of structures converges to one as the size of structures approaches infinity, then *almost all* structures are called to have the property φ.

A survey of the results

I shall first describe the idea of a new quantifier elimination technique for the logic $\mathcal{L}^k_{\infty\omega}$, where $k \in \mathbb{N}_+$. It is easy to extend the idea for all logics of the form $\mathcal{L}^k_{\infty\omega}(\mathcal{Q})$. The new quantifier elimination technique is the basis of this doctoral thesis.

Let the vocabulary be fixed and let $\mathcal{K}$ be a class of structures. A *complete quantifier-free formula* $\chi(\bar{x})$ is a quantifier-free formula of $\mathcal{L}^\omega_{\infty\omega}$ which fixes the truth value of each atomic formula $\psi(\bar{x})$. Suppose that, for every complete quantifier-free formula $\chi(\bar{x})$ of $\mathcal{L}^k_{\infty\omega}$ and every quantifier-free formula $\theta(\bar{x}, y)$ of $\mathcal{L}^k_{\infty\omega}$, either

$$\forall\bar{x}[\chi(\bar{x}) \rightarrow \exists y \theta(\bar{x}, y)] \quad \text{holds on all structures of } \mathcal{K} \text{ or}$$
$$\forall\bar{x}[\chi(\bar{x}) \rightarrow \neg\exists y \theta(\bar{x}, y)] \quad \text{holds on all structures of } \mathcal{K}.$$

Then it is easy to see that every formula of $\mathcal{L}^k_{\infty\omega}$ is equivalent to a quantifier-free formula of $\mathcal{L}^k_{\infty\omega}$ over $\mathcal{K}$. Further, the probabilities of sentences of the above form are often easy to estimate. So here is a technique for proving probabilistic elimination of quantifiers. In many cases this simple new technique turns out to be a powerful way to prove almost sure quantifier elimination and zero-one laws. Furthermore, almost sure quantifier elimination can be used to prove convergence laws, as it is shown in this thesis.

When the above technique is used to prove almost sure quantifier elimination, it actually shows that there is a class $\mathcal{K}$ of structures such that almost all structures are in $\mathcal{K}$ and that the logic has quantifier elimination over $\mathcal{K}$, that is, for every formula $\varphi(\bar{x})$, there is a quantifier-free formula $\theta(\bar{x})$ such that $\forall\bar{x}[\varphi(\bar{x}) \leftrightarrow \theta(\bar{x})]$ holds on every structure of $\mathcal{K}$. I call such quantifier elimination *almost sure strong quantifier elimination* to distinguish it from the case where the probabilities of the sentences $\forall\bar{x}[\varphi(\bar{x}) \leftrightarrow \theta(\bar{x})]$ are considered separately. In this thesis it is shown that almost sure strong quantifier elimination coincide with almost sure quantifier elimination for logics of the form $\mathcal{L}^k_{\infty\omega}(\mathcal{Q})$, where $k \in \mathbb{N}_+$, if the vocabulary of random structures does not have constant symbols.

As the first application of the new almost sure quantifier elimination technique, I give a practical criterion for a finite set $\mathcal{Q}$ of simple unary quantifiers such that the logic $\mathcal{L}^\omega_{\infty\omega}(\mathcal{Q})$ has the zero-one law for constant atomic probabilities. I also show that the logic $\mathcal{L}^\omega_{\infty\omega}$ has the zero-one law for atomic probabilities which satisfy

$$n^{-\alpha} \leqslant p_{\mathrm{ato}}(n) \leqslant 1 - n^{-\alpha} \quad \text{for every } \alpha > 0 \tag{1}$$

for all sufficiently large n. This result extends the zero-one law of Kolaitis and Vardi [10] for the logic $\mathcal{L}^\omega_{\infty\omega}$ with the constant atomic probability 1/2. Further, I show that this more general result also follows from a closer analysis of the proofs of the very first zero-one law of Glebskii et al. [4] and Fagin [2]. Lynch [15, 16] proved that, if the edge probability p_{edg} satisfies $n^{-\alpha} \leqslant p_{\mathrm{edg}}(n) \leqslant 1 - n^{-\alpha}$ for some $0 < \alpha < 1/(k-1)$ with $k \in \{2, 3, \ldots\}$ for all sufficiently large n, then the logic $\mathcal{L}^k_{\infty\omega}$ has the zero-one law for random graphs. This result is generalized for random structures.

I show that even the logic $\mathcal{L}^\omega_{\infty\omega}(\mathcal{Q}_{\mathrm{rig}})$, where $\mathcal{Q}_{\mathrm{rig}}$ is the collection of all rigidity quantifiers, has the zero-one law for atomic probabilities which satisfy Condition (1). This result extends the zero-one law of Dawar and Grädel [1] for the logic $\mathcal{L}_{\omega\omega}(\mathsf{Q}^2_{\mathrm{rig}})$, where $\mathsf{Q}^2_{\mathrm{rig}}$ is the simple binary rigidity quantifier, on random graphs with the constant edge probability 1/2.

The *Härtig quantifier* I is defined by the class $\{(A, P_1, P_2) : P_1, P_2 \subseteq A$ and $|P_1| = |P_2|\}$ and the *Rescher quantifier* R is defined by the class $\{(A, P_1, P_2) : P_1, P_2 \subseteq A$ and $|P_1| \leqslant |P_2|\}$. Luosto [13] left an open question: is there, for every $v \in \mathbb{N}_+$, a sentence of $\mathcal{L}_{\omega\omega}({}^{v+1}\mathsf{I})$ which is not equivalent to any sentence of $\mathcal{L}_{\omega\omega}({}^{v}\mathsf{I})$. This question is answered affirmatively by using the new technique. A similar result holds also for the Rescher quantifier.

Random structures, which have so-called built-in permutations, are also considered. The results for such cases extend a zero-one law of Lynch [14].

The second paper of this doctoral thesis is focused on proving almost sure quantifier elimination and zero-one laws for logics of the form $\mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q})$, where $\mathcal{Q}$ is a properly chosen collection of simple numerical quantifiers. For instance, let γ_1 and γ_2 be constants in the interval $]i, i+1[$ for some $i \in \mathbb{N}$ and let $\mathcal{Q}_{\gamma_1,\gamma_2}$ be the collection of all quantifiers defined by the classes

$$\mathcal{K}_{m,g} = \Big\{(A, P) : P \subseteq A^m \text{ and } |P| \geqslant g(|A|)\Big\},$$

where $m \in \mathbb{N}_+$ and g is any function $g : \mathbb{N}_+ \longrightarrow \mathbb{R}$ such that $n^{\gamma_1} \leqslant g(n) \leqslant n^{\gamma_2}$ for all $n \in \mathbb{N}_+$. Then the results show that the logic $\mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q}_{\gamma_1,\gamma_2})$ has the zero-one law for atomic probabilities which satisfy Condition (1). Note that the quantifier which is defined by the class $\mathcal{K}_{m,g}$ express that "there are at least $g(n)$ m-tuples of elements" on structures of size n.

In this thesis, the new almost sure quantifier elimination technique is not used for proving convergence laws. However, such applications can be found in [7], where convergence laws are proved on very sparse random structures. These results extend the convergence laws of Lynch [16] on very sparse random graphs.

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