FINITARY ABSTRACT ELEMENTARY CLASSES

MEERI KESÄLÄ

Academic dissertation

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in Auditorium 6 of the Main Building of the University, on December 19th, 2006, at 12 a.m.

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HELSINKI 2006

Cover design Iris Tomaszewski
ISBN 952-92-1357-3 (Paperback)
ISBN 952-10-3564-1 (PDF)
Helsinki University Printing House
Helsinki 2006

Acknowledgements

I would like to express my sincere gratitude to my advisor Tapani Hyttinen for guiding me into the world of model theory and for the insightful ideas which are the basis of this work. His ideas and advice have carried me through even the most vicious constructions. I am also grateful to Olivier Lessmann and Andrés Villaveces with whom I have been able to discuss these topics and learn more. The logic group of Helsinki has been an inspiring community to work in, not only for its expertise in logic and international connections, but also its encouraging atmosphere. I want to thank professors John Baldwin and Lauri Hella for reading the manuscripts, and for their valuable comments on them.

For financial support I thank the graduate school MALJA and the Department of Mathematics and Statistics of the University of Helsinki.

I would like to thank my fellow students, especially Åsa Hirvonen and Saara Lehto, for making the math department a fun place to work at. I wish to thank my parents Marja-Leena and Juha Viljanen for their continuous encouragement and support. I am grateful to my husband Eero for his love and patience.

Helsinki, November 2006

Meeri Kesälä

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Introduction

In this doctoral thesis we introduce finitary abstract elementary classes, a non-elementary framework of model theory. These classes are a special case of abstract elementary classes (AEC), introduced by Saharon Shelah [26] in the 1980's. We have collected a set of properties for classes of structures, which enables us to develop a 'geometric' approach to stability theory, including an independence calculus, in a very general framework. The novelty is the property of *finite character*, which enables to use *weak type* as a notion of type. The thesis consists of three independent papers. All three papers are joint work with Tapani Hyttinen.

I Independence in finitary abstract elementary classes,
Tapani Hyttinen and Meeri Kesälä.

II Categoricity transfer in simple finitary abstract elementary classes,
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III Superstability in simple finitary AEC,
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Paper I will appear in the journal Annals of Pure and Applied Logic. The first versions of these papers were written at the following times: Paper I during the years 2004 and 2005, Paper II in autumn 2005 and Paper III in spring 2006. Part of the material in Paper I is presented in the author's licentiate thesis Independence in Local Abstract Elementary Classes, 2005. All the work for this thesis was done at the University of Helsinki and the author was supported by the graduate school MALJA.

During the last decades of the twentieth century, the focus in model theory moved from the study of syntactical questions to the study of structural properties of classes of models of a theory. An *elementary class* $Mod(T)$ is the collection of models of similarity type τ satisfying the axioms of a complete theory T in elementary logic with vocabulary τ. The work by Morley and Shelah played an essential role in this transformation.

The Löwenheim-Skolem theorem of elementary logic says that a theory with infinite models has models in every infinite cardinality above the size of the vocabulary. Hence we cannot hope that the theory could describe an infinite model up to isomorphism. But can the theory describe a model uniquely in a fixed cardinality? We say that a class $Mod(T)$ is λ-categorical, if it has only one model of cardinality λ, up to isomorphism. A famous theorem of by Michael D. Morley [21] says that if the class of models of theory in a countable elementary language is λ-categorical in *some* uncountable cardinal λ, it is categorical in *every* uncountable cardinal. The proof of this theorem introduced useful methods for classifying structures such as *ranks* and counting the number of *types* in a structure. Shelah developed the theory further by introducing a wide collection of tools, such as a general notion of independence and a concept of a strong type, see the book [23]. On the basis of the number of types of tuples over a set of a fixed size, we can divide theories into different classes, so called $\aleph_0$-stable, superstable, stable or unstable theories. Shelah's Main Gap Theorem introduces a dividing line for classes of models of a countable and complete theory: it says that there are either the maximal number of models very hard to distinguish from each other, or a relatively small number of relatively easily distinguishable models in each cardinality $\aleph_\alpha$. The proof of this theorem uses stability theory and properties of the independence calculus.

Many natural classes of structures in mathematics are not axiomatizable by means of elementary logic. In order to generalize classification theory to a wider range of classes, many *non-elementary* frameworks have been introduced. An important one is classes of structures definable in the language $L_{\omega_1\omega}$, where countable conjunctions and disjunctions are allowed. Shelah defined *excellent classes* in [25]. There he studies a class of models of a $L_{\omega_1\omega}$-sentence, which has good amalgamation properties. Another extensively studied context is *homogeneous classes* [22] where one studies elementary substructures of a big homogeneous model. A characteristic for the non-elementary contexts is the failure of compactness, which makes the classification task more difficult. However, both in the excellent and the homogeneous framework there are good analogues to Morley's theorem. Also analogues to the Main Gap have been studied; see [7] for excellent classes by Grossberg and Hart and [17] for a homogeneous framework by Hyttinen and Shelah. Both contexts of homogeneous classes and of excellent classes have been used for applications in concrete classes; see for example Berenstein and Buechler [5], Mekler and Shelah [20] or Zilber [30], [29].

The independence calculus

Baldwin lists in the book [2] the essential properties of a notion of independence $\downarrow$. We call such properties an independence calculus. These properties hold for the notion based on *nonforking* in stable first-order theories. If the theory T is superstable or $\aleph_0$-stable, we have in addition that $\kappa(T) = \aleph_0$ in local character below. We write $\bar{a} \downarrow_A B$ and say that the type $\mathrm{tp}(\bar{a}/B \cup A)$ is independent over A.

(1) **Invariance:** If f is an isomorphism and $\bar{a} \downarrow_A B$, then $f(\bar{a}) \downarrow_{f(A)} f(B)$.
(2) **Monotonicity:** If $A \subset B \subset C \subset D$ and $\bar{a} \downarrow^s_A D$, then $\bar{a} \downarrow^s_B C$.
(3) **Transitivity:** Let $B \subset C \subset D$. If $\bar{a} \downarrow_B C$ and $\bar{a} \downarrow_C D$, then $\bar{a} \downarrow_B D$.
(4) **Symmetry:** If $\bar{a} \downarrow_A \bar{b}$, then $\bar{b} \downarrow_A \bar{a}$.
(5) **Extension:** For any tuple $\bar{a}$ and $A \subset B$ there is a type $\mathrm{tp}(\bar{b}/B)$ extending $\mathrm{tp}(\bar{a}/A)$ such that $\bar{b} \downarrow_A B$.
(6) **Finite character:** If $\bar{a} \not\downarrow_A B$ and $A \subset B$, there is a formula $\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}/B)$ such that no type containing $\phi(\bar{x}, \bar{b})$ is independent over A.
(7) **Local character:** There is a cardinal $\kappa(T)$ such that for any $\bar{a}$ and B there is $A \subset B$ such that $|A| < \kappa(T)$ and $\bar{a} \downarrow_A B$.
(8) **Reflexivity:** If $A \subset B$, $\bar{b} \in B \setminus A$ and $\mathrm{tp}(\bar{b}/A)$ is not algebraic, then $\bar{b} \not\downarrow_A B$.
(9) **Stationarity:** Assume that $\mathscr{A}$ is a model, $\mathrm{tp}(\bar{a}/\mathscr{A}) = \mathrm{tp}(\bar{b}/\mathscr{A})$, $\bar{a} \downarrow_{\mathscr{A}} B$ and $\bar{b} \downarrow_{\mathscr{A}} B$. Then $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{b}/B)$.

It might be useful to have at least some of the above properties for independence or some restricted forms of the properties. See for example Shelah [24] for the study of simple and unstable first-order theories.

Abstract elementary classes

Shelah suggested in [26] *abstract elementary classes* (AEC) as a platform to study model theoretic concepts in a more general setting. He studies a class $\mathbb{K}$ of structures in a fixed similarity type τ, but does not define any specific language. Instead he gives axioms for $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$, where $\preccurlyeq_{\mathbb{K}}$ is a relation between the models of the class. The class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is for example assumed to be closed under isomorphisms, behave well with respect to $\preccurlyeq_{\mathbb{K}}$-increasing chains and have a downward Löwenheim-Skolem number $\mathrm{LS}(\mathbb{K})$. If B is a subset of a structure $\mathscr{A} \in \mathbb{K}$, there is a $\preccurlyeq_{\mathbb{K}}$-elementary substructure of $\mathscr{A}$ containing B of cardinality at most $\mathrm{LS}(\mathbb{K})+|B|$. This context generalizes an elementary class $(Mod(T), \preccurlyeq)$, where $\preccurlyeq$ is the elementary substructure relation.

In his Presentation Theorem Shelah showed that such a class can be presented as a class of reducts of models in an elementary class omitting a set of types. This enables us to use the method of Ehrenfeucht-Mostowski models in the study of AEC's and gives a Hanf number depending on the Löwenheim-Skolem number of the class. If an abstract elementary class has models of cardinality greater or equal to the Hanf number, it has arbitrarily large models. Shelah also stated a conjecture: There is a cardinal κ such that if an abstract elementary class is categorical in one cardinal above κ, then it is categorical in all cardinals above κ.

Galois types

If we want to generalize more model theoretic tools to abstract elementary classes, we might have to isolate the needed properties from elementary model theory as new axioms for the class. For example, there is a problem of defining a good notion of type. The most popular procedure is to assume the amalgamation property and use a notion of Galois type. Galois types generalize the usual notion of types, defined as sets of formulas. The concept is due to Shelah, but the name Galois type was introduced by Grossberg [6]. Another common practice is to assume the class to have arbitrarily large models, amalgamation and joint embedding properties. Then we can use the construction by Jónsson and Fraïssé [18] to build a universal and model-homogeneous *monster model* $\mathfrak{M} \in \mathbb{K}$. Two tuples in $\mathfrak{M}$ have the same Galois type over a model $\mathscr{A}$, if there is an automorphism of the monster model mapping $\bar{a}$ to $\bar{b}$ and fixing $\mathscr{A}$ pointwise.

In [27], Shelah shows that in the framework above there is a cardinal H_2 called the second Hanf number[1], such that if $\mathrm{H}_2 < \lambda \leq \kappa$ and the class is categorical in the successor cardinal κ^+, then it is categorical in λ. Shelah showed that categoricity above H_2 implies some good behaviour for Galois types, which enables the transfer of categoricity. Grossberg and VanDieren [10] [11] [12] isolated the notion of *tameness* of Galois types as the required property for categoricity transfer. Let $\mathrm{tp}^g(\bar{a}/\mathscr{A})$ denote the Galois type of a tuple $\bar{a}$ in the monster model over an $\preccurlyeq_{\mathbb{K}}$-elementary submodel $\mathscr{A}$. A class is said to be tame in a cardinal χ, if for all models $\mathscr{A}$ such that

$$\mathrm{tp}^g(\bar{a}/\mathscr{A}) \neq \mathrm{tp}^g(\bar{b}/\mathscr{A}),$$

there is $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{A}$ of size at most χ such that

$$\mathrm{tp}^g(\bar{a}/\mathscr{B}) \neq \mathrm{tp}^g(\bar{b}/\mathscr{B}).$$

We say that a class is tame, if it is tame in $\mathrm{LS}(\mathbb{K})$. The context of [10] is an abstract elementary class with amalgamation, joint embedding, arbitrary large models and tameness in some cardinal χ. Then categoricity in some $\kappa^+ > \max\{\chi, \mathrm{LS}(\mathbb{K})^+\}$ implies categoricity in all $\lambda \geq \kappa^+$. Lessmann [19] showed that categoricity can be

[1] Baldwin [1] improves this result by replacing H_2 with the Hanf number.

transferred upwards also from $\aleph_1 > \max\{\chi, \mathrm{LS}(\mathbb{K})\}$. Both these results and the result of Shelah's require the categoricity cardinal to be a successor.

Independence in abstract elementary classes

Several authors besides Shelah, including Baldwin, Grossberg, Kolesnikov, Lessmann, VanDieren and Villaveces have studied abstract elementary classes. John Baldwin has collected much of the current research in his book [1]. Also in the paper [3] he has listed several open questions of the field. Among these questions is to study an independence calculus and find the 'correct' notion of superstability for AEC's. Some notions of independence have been introduced for AEC's, see Shelah [27] or Grossberg [6], but the analogue of the full independence calculus in elementary classes has not been achieved in the most general context. Many examples of AEC's, including excellent classes and homogeneous classes, admit a notion of independence. There are also several frameworks of AEC's with a abstract notion of independence, where the definition is not specified but only axioms for the independence calculus are given; see for example Shelah [28], Grossberg and Kolesnikov [8] or Grossberg and Lessmann [9].

In saying that a tuple $\bar{a}$ is independent of a set B over a set C, written $\bar{a} \downarrow_C B$, we mean roughly that the set B does not give more information about $\bar{a}$ than C does. In the following examples of AEC's the 'natural' notion of independence agrees with a model-theoretic notion in a suitable framework. Example 1 is elementary and for Example 2 in an homogeneous context see Berenstein and Buechler [5].

Example 1 (Field of complex numbers $\mathbb{C}$)**.** *Consider the class of algebraically closed fields of characteristic zero and take as the notion $\preccurlyeq_{\mathbb{K}}$ the subfield relation. Denote by $acl(A)$ the algebraic closure of a subset A of an algebraically closed field. Then for any subsets $C \subset B$, $\bar{a} \downarrow_C B$ iff*

$$acl(\bar{a} \cup C) \cap acl(B) \subset acl(C).$$

Example 2 (Hilbert spaces)**.** *Consider the class of all normed vector spaces over the reals which can be completed to a Hilbert space. We take as the notion $\preccurlyeq_{\mathbb{K}}$ the linear subspace relation. Denote by $\bar{B}$ the closed subspace generated by B and by $P_{\bar{B}}(\bar{a})$ the orthogonal projection of the tuple $\bar{a}$ to the space $\bar{B}$. Let $A \subset B$ be subsets of a Hilbert space. Then $\bar{a} \downarrow_A B$ iff*

$$P_{\bar{B}}(\bar{a}) \in \bar{A}.$$

That is, when we write $\bar{a} = \bar{a}_A + \bar{a}_\perp$, where $\bar{a}_A \in \bar{A}$ and $\bar{a}_\perp$ is in the orthocomplement of $\bar{A}$, then $\bar{a} \downarrow_A B$ if and only if $\bar{a}_\perp$ is orthogonal to $\bar{B}$.

Finitary classes

In this thesis we introduce a context of finitary classes, with the relation $\preccurlyeq_{\mathbb{K}}$ having *finite character*. Let $\mathrm{tp}^g(\bar{a}/\emptyset, \mathscr{A})$ denote the Galois type of a tuple $\bar{a}$ in some model $\mathscr{A} \in \mathbb{K}$ over the empty set. We define finite character as the property that if $\mathscr{A}, \mathscr{B} \in \mathbb{K}$, $\mathscr{A} \subseteq \mathscr{B}$ and for each finite tuple $\bar{a} \in \mathscr{A}$ we have that

$$\mathrm{tp}^g(\bar{a}/\emptyset, \mathscr{A}) = \mathrm{tp}^g(\bar{b}/\emptyset, \mathscr{B}),$$

then $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$. Finite character is not a form of compactness, since it describes the relation between two models *in* the class, and is not a way to construct new models. If the relation $\preccurlyeq_{\mathbb{K}}$ is defined syntactically in a logic L which has only finitely many free variables in a formula, then it has finite character. Both the excellent and homogeneous classes belong to this framework.

We say that a notion of type has finite character when any two tuples have the same type over a set A if and only if they have the same type over each finite subset of A. In the elementary context the notion of Galois type agrees with that of syntactic type, and thus has finite character. This is essential in many constructions of elementary classification theory and enables us to define a notion of independence based on dependency of fine sets. We want to study particulary this kind of finite dependencies and take as the notion of type that of *weak type*, which has finite character by definition.

The two main themes in all three papers in the thesis are 'How good a control can we have on the behaviour of Galois types?' and 'Can we build a good analogue for the independence calculus in elementary classes?' For the first question the answer remains unsatisfactory, since we often have to *assume* tameness for Galois types. The study of the second question is more rewarding.

I Independence in finitary abstract elementary classes

In the first paper we define finitary classes to be abstract elementary classes $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ with countable Löwenheim-Skolem number, arbitrarily large models, disjoint amalgamation, prime model and finite character. Disjoint amalgamation and prime model are slightly stronger versions of amalgamation and joint embedding. By the Jónsson-Fraïssé construction we can build a monster model. This model has an expansion in the elementary class implied by Shelah's Presentation theorem. Using the assumptions of disjoint amalgamation and prime model we can make the expansion to be a *homogeneous* model. The stronger versions of amalgamation are only needed for this, and we use the homogeneous expansion to gain good control over indiscernible sequences. We define weak type of finite tuples as follows:

$$\mathrm{tp}^w(\bar{a}/A) = \mathrm{tp}^w(\bar{b}/A) \text{ iff } \mathrm{tp}^g(\bar{a}/B) = \mathrm{tp}^g(\bar{b}/B) \text{ for all finite } B \subset A.$$

In the two first papers we restrict the study on the $\aleph_0$-stable case. We define $\aleph_0$-stability with respect to weak types, but are able to show that this notion is equivalent to the notion for Galois types. We prove as Theorem 3.12 the following:

Theorem 3. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is finitary and* $\aleph_0$*-stable with respect to weak types. Let* $\mathscr{A}$ *be a countable model,* $\bar{a}$ *and* $\bar{b}$ *finite tuples and* $\mathrm{tp}^w(\bar{a}/\mathscr{A}) = \mathrm{tp}^w(\bar{b}/\mathscr{A})$*. Then also* $\mathrm{tp}^g(\bar{a}/\mathscr{A}) = \mathrm{tp}^g(\bar{b}/\mathscr{A})$*.*

The proof for the theorem is a primary model construction. It follows that under $\aleph_0$-stability and $\aleph_0$-tameness the notions of weak type and Galois type agree over all models of $\mathbb{K}$.

We define two different notions of independence. The first is denoted by $\downarrow^s$ and the definition is based on *splitting* over finite sets. We are able to show several properties for $\downarrow^s$ over $\aleph_0$-*saturated models* using the finite character of $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ and $\aleph_0$-stability. To gain the full picture including symmetry, we need to assume *extension property for splitting.* This assumption is needed throughout the paper, and we show that it is implied either from tameness or categoricity above the Hanf number. The properties are listed in Theorem 3.17 and Corollaries 4.14 and 4.21 of Paper I. Reflexivity is an easy consequence of the definition and it is not mentioned in the paper, but we list it here for completeness.

Theorem 4. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an* $\aleph_0$*-stable finitary AEC. Then* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has a notion of splitting with the following properties:*

1. ***Invariance:*** *If* f *is an automorphism of the monster model* $\mathfrak{M}$*,* $\bar{a} \downarrow^s_A B$ *if and only if* $f(\bar{a}) \downarrow^s_{f(A)} f(B)$*.*
2. ***Monotonicity:*** *If* $A \subset B \subset C \subset D$ *and* $\bar{a} \downarrow^s_A D$*, then* $\bar{a} \downarrow^s_B C$*.*
3. ***Transitivity:*** *If* $A \subset \mathscr{B} \subset C$ *and* $\mathscr{B}$ *is an* $\aleph_0$*-saturated model, then* $\bar{a} \downarrow^s_A C$ *if and only if* $\bar{a} \downarrow^s_A \mathscr{B}$ *and* $\bar{a} \downarrow^s_{\mathscr{B}} C$*.*
4. ***Countable extension:*** *Let* $\mathscr{A} \subset B$ *be countable and let* $\mathscr{A}$ $\aleph_0$*-saturated model. For each* $\bar{a}$ *there is* $\bar{b}$ *realizing* $\mathrm{tp}^w(\bar{a}/\mathscr{A})$ *such that* $\bar{b} \downarrow^s_{\mathscr{A}} B$*. Moreover, if* $\mathrm{tp}^w(\bar{a}/\mathscr{A})$ *does not split over some finite subset* $E \subset \mathscr{A}$*, then* $\mathrm{tp}^w(\bar{b}/B)$ *does not split over* E*.*
5. ***Finite character:*** *Let* $\mathscr{A}$ *be an* $\aleph_0$*-saturated model and* $\mathscr{A} \subset B$*. Then* $\bar{a} \downarrow^s_{\mathscr{A}} B$ *if and only if* $\bar{a} \downarrow^s_{\mathscr{A}} B_0$ *for every finite* $B_0 \subset B$*.*
6. ***Local character:*** *For each model* $\mathscr{A}$ *and finite sequence* $\bar{a}$ *there is a finite* $E \subset \mathscr{A}$ *such that* $\bar{a} \downarrow^s_E \mathscr{A}$*.*
7. ***Reflexivity:*** *Let* $\mathscr{A}$ *be an* $\aleph_0$*-saturated model. Then for all tuples* $\bar{a} \notin \mathscr{A}$*,* $\bar{a} \not\downarrow^s_{\mathscr{A}} \bar{a}$*.*
8. ***Stationarity:*** *Assume* $\mathscr{A}$ *is an* $\aleph_0$*-saturated model and* $A \subset B$*. If* $\mathrm{tp}^w(\bar{a}/\mathscr{A}) = \mathrm{tp}^w(\bar{b}/\mathscr{A})$*,* $\bar{a} \downarrow^s_{\mathscr{A}} B$ *and* $\bar{b} \downarrow^s_{\mathscr{A}} B$*, then* $\mathrm{tp}^w(\bar{a}/B) = \mathrm{tp}^w(\bar{b}/B)$*.*

Furthermore, if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is in addition tame* or *categorical above the Hanf number:*

8. ***Extension:*** *Let* $\mathscr{A}$ *be an* $\aleph_0$*-saturated model and* $\mathscr{A} \subset B$*. For each* $\bar{a}$ *there is* $\bar{b}$ *realizing* $\mathrm{tp}^w(\bar{a}/\mathscr{A})$ *such that* $\bar{b} \downarrow^s_{\mathscr{A}} B$*. Moreover, if* $\mathrm{tp}^w(\bar{a}/\mathscr{A})$ *does not split over some finite subset* E*, then* $\mathrm{tp}^w(\bar{b}/B)$ *does not split over* E*.*
9. ***Symmetry:*** *Let* $\mathscr{A}$ *be an* $\aleph_0$*-saturated model. Then* $\bar{a} \downarrow^s_{\mathscr{A}} \bar{b}$ *if and only if* $\bar{b} \downarrow^s_{\mathscr{A}} \bar{a}$*.*

We define weak λ-stability and weak saturation with respect to weak types. We show that $\aleph_0$-stability implies weak stability in each cardinality in finitary classes. If we assume also the extension property 4(8), we are able to show that weakly saturated models exist in every cardinality, by showing that the union of weakly saturated models is weakly saturated. We gain the analogous results for Galois types assuming tameness.

The second notion of independence, denoted with the symbol $\downarrow$, is based on *Lascar splitting.* Lascar splitting is a version of strong splitting in elementary classes, which generalizes to the non-elementary context. We also define Lascar strong type of a tuple $\bar{a}$ in the monster model over a set A, written $\mathrm{Lstp}(\bar{a}/A)$, such that for any n-tuples $\bar{a}$ and $\bar{b}$,

$$\mathrm{Lstp}(\bar{a}/A) = \mathrm{Lstp}(\bar{b}/A)$$

if $(\bar{a}, \bar{b}) \in E$ for each A-invariant equivalence relation E of n-tuples with a bounded number of classes. For the notion $\downarrow$ we get all the usual properties of the independence calculus over *sets*, assuming $\aleph_0$-stability, the extension property and *simplicity.* The study on Lascar splitting and simplicity is an analogue to the similar study in excellent classes, see Hyttinen and Lessmann [15]. Simplicity is defined as the notion $\downarrow$ having *local character* for arbitrary sets (see below), but it is enough to assume local character for finite sets only, as is done in Paper III. Without simplicity there might not be *any* notion of independence with these properties over sets. Shelah has provided such an example, see Hyttinen and Lessmann [16]. The following properties are listed in Paper I as Theorem 6.5, except reflexivity, which follows from Lemma 5.38(b).

Theorem 5. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is finitary, simple, stable in* $\aleph_0$ *and has the extension property. Then,* $\downarrow$ *satisfies the following properties:*

(1) ***Invariance:*** *If* $A \downarrow_C B$*, then* $f(A) \downarrow_{f(C)} f(B)$ *for any automorphism* f *of the monster model.*
(2) ***Monotonicity:*** *If* $A \downarrow_B D$ *and* $B \subset C \subset D$ *then* $A \downarrow_C D$ *and* $A \downarrow_B C$*.*
(3) ***Transitivity:*** *Let* $B \subset C \subset D$*. If* $A \downarrow_B C$ *and* $A \downarrow_C D$*, then* $A \downarrow_B D$*.*
(4) ***Symmetry:*** $A \downarrow_C B$ *if and only if* $B \downarrow_C A$*.*

(5) ***Extension:*** *For any* $\bar{a}$ *and* $C \subset B$ *there is* $\bar{b}$ *such that* $\mathrm{tp}^w(\bar{b}/C) = \mathrm{tp}^w(\bar{a}/C)$ *and* $\bar{b} \downarrow_C B$.

(6) *For any finite* C, $\bar{a}$ *and* B *containing* C, *there is* $\bar{b}$ *such that* $\mathrm{Lstp}(\bar{b}/C) = \mathrm{Lstp}(\bar{a}/C)$ *and* $\bar{b} \downarrow_C B$.

(7) ***Finite character:*** $A \downarrow_C B$ *if and only if* $\bar{a} \downarrow_C \bar{b}$ *for every finite* $\bar{a} \in A$ *and* $\bar{b} \in B$.

(8) ***Local character:*** *For any finite* $\bar{a}$ *and any* B *there exists a finite* $E \subset B$ *such that* $\bar{a} \downarrow_E B$.

(9) ***Reflexivity:*** *For each* $\bar{a}$ *and* C *such that* $\mathrm{tp}^w(\bar{a}/C)$ *is not bounded,* $\bar{a} \not\downarrow_C \bar{a}$.

(10) ***Stationarity over*** $\aleph_0$***-saturated models:*** *Let* $\mathscr{A}$ *be an* $\aleph_0$*-saturated model. If* $\mathrm{tp}^w(\bar{a}/\mathscr{A}) = \mathrm{tp}^w(\bar{b}/\mathscr{A})$, $\bar{a} \downarrow_{\mathscr{A}} B$ *and* $\bar{b} \downarrow_{\mathscr{A}} B$, *then* $\mathrm{tp}^w(\bar{a}/B) = \mathrm{tp}^w(\bar{b}/B)$.

(11) ***Stationarity of Lascar strong types:*** *If* $\mathrm{Lstp}(\bar{a}/C) = \mathrm{Lstp}(\bar{b}/C)$, $\bar{a} \downarrow_C B$ *and* $\bar{b} \downarrow_C B$, *then* $\mathrm{tp}^w(\bar{a}/B) = \mathrm{tp}^w(\bar{b}/B)$.

We also define U-rank in $\aleph_0$-stable finitary classes with the extension property, and show that finite U-rank implies simplicity. First we define inductively the U-rank of a tuple $\bar{a}$ over an $\aleph_0$-saturated countable model $\mathscr{A}$, written $U(\bar{a}/\mathscr{A})$. Always $U(\bar{a}/\mathscr{A}) \geq 0$ and $U(\bar{a}/\mathscr{A}) \geq \alpha + 1$ if there is a countable $\aleph_0$-saturated $\mathscr{B} \supseteq \mathscr{A}$ such that $U(\bar{a}/\mathscr{B}) \geq \alpha$ and $\bar{a} \not\downarrow^s_{\mathscr{A}} \mathscr{B}$. The U-rank over an arbitrary $\aleph_0$-saturated model is defined as a minimum of U-ranks over countable $\aleph_0$-saturated submodels. Finally we generalize the definition to ranks over arbitrary sets as is done by Hyttinen and Lessmann in [15].

Finite character is essential both for the independence calculus in Theorems 4 and 5 and the proof of Theorem 3. The framework of this paper is further studied by Hyttinen in [14].

II Categoricity transfer in simple finitary abstract elementary classes

In the second paper we introduce a weaker set of axioms for finitary classes, and show that all the main results of Paper I hold also with the weaker assumptions. We assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an abstract elementary class with a countable Löwenheim-Skolem number, arbitrarily large models, finite character, amalgamation and joint embedding. The use of the homogeneous expansion of the monster model is replaced by a finer study on Ehrenfeucht-Mostowski models. We also refine the study of Paper I on U-rank and equivalents of the extension property.

We define a notion of a constructible model called an f-primary model. Such models exist over any set by simplicity. We also need to assume $\aleph_0$-stability and the extension property for splitting, but we show that simplicity and weak categoricity in any uncountable cardinal imply both of these. We say that the class $\mathbb{K}$ is *weakly λ-categorical* if all models of size λ are weakly saturated. We use f-primary models to prove the following in Theorem 4.11. Here $(\mathbb{K})^\omega$ is the class of $\aleph_0$-saturated models of $\mathbb{K}$.

Theorem 6. *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple finitary AEC and weakly categorical in some uncountable cardinal κ. Then*

1. *$((\mathbb{K})^\omega, \preccurlyeq_{\mathbb{K}})$ is weakly categorical in each uncountable κ and*
2. *$(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is weakly categorical in each λ such that $\lambda \geq \min\{\kappa, \mathrm{Hanf}\}$.*

We denote by Galois saturation the saturation respect to Galois types. Under tameness the notions of weakly saturated and Galois saturated agree, and we have that any two Galois saturated models of equal cardinality are isomorphic. Thus the previous theorem gives a categoricity transfer result for $\aleph_0$-tame simple finitary classes, with no restrictions on the cofinality of the categoricity cardinal.

Corollary 7. *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple tame finitary AEC categorical in some uncountable κ. Then*

1. *$((\mathbb{K})^\omega, \preccurlyeq_{\mathbb{K}})$ is categorical in each uncountable κ and*
2. *$(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is categorical in each λ such that $\lambda \geq \min\{\kappa, \mathrm{Hanf}\}$.*

An example introduced by Hart and Shelah [13] and further studied by Baldwin and Kolesnikov [4], shows that tameness is necessary for the categoricity transfer. The example shows that for each finite $k > 0$ there is a finitary class which is categorical in the cardinals $\aleph_0, ...\aleph_k$, but fails categoricity in $\aleph_{k+1}$.

III Superstability in simple finitary AEC

In Paper III the definition of a finitary class is as in Paper II. We introduce a notion of superstability, which denies the existence of infinite 'forking chains'. If $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an elementary class, the notion coincides with the usual notion. We show that we gain the independence calculus for $\downarrow$ over finite sets and arbitrary models, assuming both simplicity and superstability. If we add another assumption called the Tarski-Vaught property, we gain the independence calculus over all sets. This result improves also the result of Paper II, since simple and $\aleph_0$-stable finitary classes are superstable and have the Tarski-Vaught property. We do not need to assume the extension property for splitting. The full independence calculus is stated in Theorem 3.13 of Paper III.

Theorem 8. *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple, superstable, finitary AEC with the Tarski-Vaught property. Then the relation $\downarrow$ has the following properties.*

(1) ***Invariance:*** *If* $A \downarrow_C B$ *and* f *is an automorphism of the monster model, then* $f(A) \downarrow_{f(C)} f(B)$.
(2) ***Monotonicity:*** *If* $A \downarrow_B D$ *and* $B \subset C \subset D$ *then* $A \downarrow_C D$ *and* $A \downarrow_B C$.
(3) ***Transitivity:*** *Let* $B \subset C \subset D$. *If* $A \downarrow_B C$ *and* $A \downarrow_C D$, *then* $A \downarrow_B D$.
(4) ***Symmetry:*** $A \downarrow_C B$ *if and only if* $B \downarrow_C A$.
(5) ***Extension:*** *For any* $\bar{a}$ *and* $C \subset B$ *there is* $\bar{b}$ *such that* $\mathrm{Lstp}^w(\bar{b}/C) = \mathrm{Lstp}^w(\bar{a}/C)$ *and* $\bar{b} \downarrow_C B$.
(6) ***Finite character:*** $A \downarrow_C B$ *if and only if* $\bar{a} \downarrow_C \bar{b}$ *for every finite* $\bar{a} \in A$ *and* $\bar{b} \in B$.
(7) ***Local character:*** *For any finite* $\bar{a}$ *and any* B *there exists a finite* $E \subset B$ *such that* $\bar{a} \downarrow_E B$.
(8) ***Reflexivity:*** *If* $\mathrm{tp}^w(\bar{a}/A)$ *is not bounded, then* $\bar{a} \not\downarrow_A \bar{a}$.
(9) ***Stationarity:*** *If* $\mathrm{Lstp}^w(\bar{a}/C) = \mathrm{Lstp}^w(\bar{b}/C)$, $\bar{a} \downarrow_C B$ *and* $\bar{b} \downarrow_C B$, *then* $\mathrm{Lstp}^w(\bar{a}/B) = \mathrm{Lstp}^w(\bar{b}/B)$.

Two tuples $\bar{a}$ and $\bar{b}$ have the same *weak Lascar strong type* over a set A, written

$$\mathrm{Lstp}^w(\bar{a}/A) = \mathrm{Lstp}^w(\bar{b}/A),$$

if $\mathrm{Lstp}(\bar{a}/B) = \mathrm{Lstp}(\bar{b}/B)$ for each finite $B \subset A$. We show that with the assumptions above, the equivalence of weak Lascar strong types implies the equivalence of Galois types over any *countable set*. Again $\aleph_0$-tameness generalizes this result to types over arbitrary models, and we are able to determine the Galois type by finitary means. Furthermore, we are able to apply stationarity of Theorem 8(9) not only to gain equivalence of weak Lascar strong types but also to gain equivalence of Galois types.

We define that a model $\mathscr{A}$ is *a-saturated* if all Lascar strong types over finite subsets are realized in $\mathscr{A}$. The following is Theorem 3.21 of Paper III.

Theorem 9. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is tame, simple, superstable, finitary AEC with the Tarski-Vaught -property. If* $\mathscr{A}$ *is an a-saturated model, then the following are equivalent:*

(1) $\mathrm{Lstp}^w(\bar{a}/\mathscr{A}) = \mathrm{Lstp}^w(\bar{b}/\mathscr{A})$
(2) $\mathrm{tp}^g(\bar{a}/\mathscr{A}) = \mathrm{tp}^g(\bar{b}/\mathscr{A})$
(3) $\mathrm{tp}^w(\bar{a}/\mathscr{A}) = \mathrm{tp}^w(\bar{b}/\mathscr{A})$.

We define a concept of *a-categoricity* in a cardinal κ as the property that there is only one a-saturated model of size κ, up to isomorphism. We show that superstability is implied by a-categoricity in a cardinal κ above the Hanf number with $cf(\kappa) > \omega$. As an application, we prove an a-categoricity transfer result using a-primary models.

Theorem 10. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, tame finitary AEC with the Tarski-Vaught property. If* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a-categorical in some* $\kappa \geq \mathrm{Hanf}$ *with uncountable cofinality, it is a-categorical in any* $\kappa \geq \mathrm{Hanf}$.

As part of the proof we show that under simplicity and superstability, in all large enough cardinalities there is a model $\mathscr{A}$ such that all weak Lascar strong types over subsets of size $< |\mathscr{A}|$ are realized in $\mathscr{A}$. Again this is done by showing that an arbitrary union of such models has the same property.

The notion of superstability is tailored for simple, finitary classes and the results rely heavily on these properties. Several proofs use trees or other constructions of finite sets and they cannot be applied in a context without finite character. The question about a notion of superstability for general AEC remains open. However, this framework can be thought as a generalization of the context of excellent classes beyond $\aleph_0$-stability.

In conclusion, it seems that finitary classes provide a good platform for generalizing the theory of independence to a non-elementary context, and give many reasons for further study. We have studied the superstable case, but one could try to study the theory assuming only weak stability, and maybe simplicity. One might try to formulate a stability hierarchy theorem for weak types. Also one could try to find a classification for finitary classes with some analogue of the Main Gap theorem.

One other direction is to analyze further the $\aleph_0$-stable case and some context 'near exellence'. The notion of a primary model is important in excellent classes. We introduce several notions of primary models for finitary classes, but we are not able to show similar good properties for these notions. Assuming $\aleph_0$-stability and $\aleph_0$-tameness we could try to prove for example *uniqueness* for some notion of primary model. Also there are some contexts where $\aleph_0$-tameness has been proved from the existence of a well-behaved notion of independence and a notion of amalgamation over independent $\mathscr{P}^-(n)$-diagrams, see Grossberg and Kolesnikov [8]. Could we find *'a notion of excellence'* for finitary classes?

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