BELTRAMI OPERATORS AND MICROSTRUCTURE

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Academic dissertation

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List of included papers

This thesis consists of an introductory part and the following articles:

[1]**Quasiregular mappings and Young measures**
Kari Astala and Daniel Faraco.
To appear in *Proc. Roy. Soc. Edinburgh Sect. A*

[2] **Milton's conjecture on the regularity of isotropic equations**
Daniel Faraco
Submitted to *Ann. Inst. H. Poincaré Anal. Non Linéaire*

[3] **Tartar conjecture and Beltrami Operators**
Daniel Faraco
Preprint 323, Department of Mathematics,
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In the introductory part these articles will be referred to as [1], [2], [3], whereas other references will be referred using the initials of the authors, e.g, [Ah], [As].

1. INTRODUCTION

In the modern mathematical theory of materials science a pressing topic is the understanding of the presence of very fine microstructures in various physical models. For example, one can place into this general framework several aspects of theories such as those of composites of solids, optimal design and phase transitions in crystals.

In many of these situations, the existence of microstructures and some of their properties can be understood by analyzing the behaviour of sequences of mappings $\{f_j\} \in W^{1,p}(\Omega, \mathbb{R}^m)$ such that

$$\lim_{j\to\infty} \int_\Omega \operatorname{dist}_E^p(Df_j(x))dx = 0. \tag{1.1}$$

In this model, E is a closed subset of the space of matrices $\mathbf{M^{n\times m}}$, $p \in [1, \infty]$, $\Omega \subset \mathbb{R}^n$ is a smooth domain and the function $\operatorname{dist}_E : \mathbf{M^{n\times m}} \to \mathbb{R}$ denotes the distance from E, i.e, $\operatorname{dist}_E(A) = \inf_{P\in E} |P-A|$. The crucial parameters are the closed set E and the exponent p, both depending on the particular case we are investigating. The canonical example of this situation is the Variational theory of phase transitions developed by Ball and James [BJ]. In their model, $p = \infty$ and the set E is a union of orthogonal wells, i.e, $E = SO(n) \cup SO(n)H_1 \cup ...SO(n)H_j$, where $SO(n)$ stands for the group of rotations and $H_1, ..H_j$ are diagonal matrices with positive determinant. Other examples arise in the theory of nematic elastomers [DD], in G-closure problems, and in optimal design [Ta1],[F],[BP]. It should be emphasized that in the last two examples the associated set E is unbounded.

The presence of microstructure and its properties are reflected on the kind of constraint that (1.1) imposes on the sequence $\{Df_j\}$. For example, when (1.1) implies the compactness of $\{Df_j\}$ in $L^p(\Omega)$ there is no microstructure. If this is not the case, the features of the microstructure and its influence in the macroscopical scale corresponds to the oscillating behaviour of the sequences $\{Df_j\}$, which satisfy (1.1) but converges only in the appropriate weak topology.

It goes back to Tartar [Ta1] in the setting of homogenization and to Ball and James [BJ] [BJ1] in their Variational approach to martensitic phase transitions that an excellent devise to analyze sequences converging weakly but not strongly are the so-called (Gradient) Young measures (see also [Mü] and the references therein to find more information about the relation of Young measures with microstructures). The formal definition of these mathematical objects is given in [1, Section 3] see also [B1], but one can think of Young measures as generalized functions in the following way. Let $\{u_j\}$ be a sequence of weakly convergent functions say, in $L^1(\Omega)$. Then, $\{\nu_x\}_{x\in\Omega}$, the Young measure generated by $\{u_j\}$, stores information on the oscillating behaviour of

u_j. Loosely speaking, for a Borel set $E \subset \mathbb{R}^m$, $\nu_x(E)$ gives the probability that $u_j(y)$ takes values "arbitrarily close to E"when y is in $B(x,r)$, in the limits $j \to \infty$ and $r \to \infty$. The class of $W^{1,p}$-Gradient Young measures are those Young measures for which the generating sequence $\{u_j\}$ equals to $\{Df_j\}$, where $\{f_j\}$ is a sequence weakly convergent in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p \in [1,\infty]$. We will use GYM for Gradient Young measures and *generating sequence* for $\{Df_j\}$. Young measures with no spatial dependence, i.e, $\nu_x = \nu$ for almost every x are called homogeneous. The Lebesgue differentiation theorem guarantees that local properties of GYMs follow from the properties of Homogeneous GYMs. Furthermore, a covering argument shows that the domain Ω plays no role in the study of homogeneous GYMs. Therefore, we will use the notation $\mathcal{H}^p(E)$ to denote the set of Homogeneous $W^{1,p}$-GYMs supported in E.

The canonical tools to understand Gradient Young measures are functions F for which the integral functional

$$I(\varphi) = \int_\Omega F(D\varphi)$$

is sequentially weakly lower semicontinuous in the appropriate Sobolev space . However, provided that F satisfies suitable growth conditions, the necessary and sufficient condition guaranteeing the lower semicontinuity of I is the mysterious notion of quasiconvexity (see the monographs [Mü], [P1] and the original article of Kinderlehrer and Pedregal [KP]). Unfortunately the quasiconvexity of a given function is very hard and often impossible to verify. Thus, in practice one needs to search for tamer conditions. Weakly continuous functions, such as the determinant, subdeterminants or div-curl products have proved to be both effective and easy to handle tools in restricting and clarifying the properties of Gradient Young measures. At the heart of this circle of ideas are the notions of polyconvexity and compensated compactness (see [B],[D],[Ta1]). Nevertheless, in spite of their effectiveness, a simple example in which the set E consists only of three matrices shows that the above notions do not always suffice when dealing with GYM (see [Mü, Section 2.4]).

This thesis studies the interplay between the theory of Gradient Young measures and planar quasiconformal and quasiregular mappings. The relation stems from the pioneering work of Nesi [N], were quasiconformal techniques where related to G-closure problems, see also the subsequent articles of Astala and Miettinen [AM] and Astala and Nesi [AN].

In our work the one to one correspondence between planar quasiconformal mappings and linear elliptic PDEs is fundamental. Quasiregular mappings are related with elliptic PDEs in all dimensions, but the Beltrami equation

$$\begin{aligned} &\overline{\partial} f(z) = \mu(z)\partial f(z), \\ &\mu(z) \in L^\infty(\mathbb{C}), \quad \|\mu\|_\infty < 1, \end{aligned} \tag{1.2}$$

bestows many special properties to planar quasiconformal (quasiregular) mappings. Every quasiregular mapping solves a Beltrami equation and conversely, by the measurable Riemann mapping theorem [Ah], every Beltrami equation is solved by a quasiconformal mapping. The linearity of the Beltrami equation invites to use the powerful machinery of singular integral operators. In [AIS], Astala Iwaniec and Saksman used this theory together with Astala theorem on the area distortion of quasiconformal mappings to establish the precise range of invertibility of the so-called Beltrami operators.

Since many of our results rely on appropriate versions of Astala-Iwaniec-Saksman Theorem we quote the canonical one for illustration.

Theorem 1.1. *Let $\mu_1, \mu_2 \in L^\infty(\mathbb{C})$ be such that $k = \||\mu_1| + |\mu_2|\|_\infty < 1$. Denote $p = 1 + 1/k$ and $p' = 1 + k$. Then the Beltrami operator*

$$I - \mu_1 S - \mu_2 \overline{S} \tag{1.3}$$

and its transpose $I - S\mu_1 - \overline{S}\mu_2$ are invertible on $L^q(\mathbb{C})$ for all $q \in (p', p)$.

Here S stands for the Beurling-Ahlfors transform, see [1, Section 2] for the definition and some of its properties. For other versions and generalizations of Theorem 1.1 see [3, Theorem 2.2, Corollary 2.3]. The other fundamental tool coming from quasiconformal geometry is Astala's theorem on the area distortion of quasiconformal mappings and the subsequent reverse Hölder inequalities [As].

This thesis consists of three papers. In [1] and in most of [3] we exploit the invertibility of the Beltrami operators, Astala's theorem and other results concerning planar quasiconformal mappings to obtain restrictions and information about the set of GYM supported in $\mathbf{M}^{\mathbf{2}\times\mathbf{2}}$. In [2] we combine an special kind of GYMs called p-laminates with Beltrami operators to study the regularity of the isotropic equation, i.e,

$$\operatorname{div}\big(\rho(z)\nabla u(z)\big) = 0 \quad \text{in } Q, \tag{1.4}$$

where Q is a cube in the plane $\mathbb{R}^2$, $u \in W^{1,2}(Q, \mathbb{R})$ and $\rho \in L^\infty(Q, [\frac{1}{K}, K])$ is real valued.

We give a rigorous proof of an old conjecture of Milton [Mi] concerning the integrability of the gradients of solution to isotropic equations.

The last section in [3] is devoted to constructing counterexamples and it uses ideas from [2] and from the theory of very weak quasiregular mappings.

2. Quasiconformal geometry as a restriction for microstructure

A linear mapping $f(z) = Az$ is K-quasiconformal if and only if the matrix A belongs to the set

$$Q_2(K) = \{A \in \mathbf{M^{2\times 2}} : \|A\|^2 \leq K \det(A)\},$$

where $1 \leq K < \infty$ and $\|A\|$ stands for the operator norm of A. Thus, $Q_2(K)$ is called the set of quasiconformal matrices. One can define K-quasiregular mappings as those mappings $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ whose Jacobian derivative $Df(z)$ lies in $Q_2(K)$ for a.e. $x \in \Omega$. If the mapping is also a homeomorphism it is said to be K-quasiconformal.

It is natural to expect that $W^{1,2}$-GYMs supported in $Q_2(K)$ are blessed with many of the nice features of quasiregular mappings. We called these measures quasiregular Gradient Young measures. The paper [1] is devoted to the study of this type of GYMs. Often, the properties of a class of Gradient Young measures follow from choosing the appropriate generating sequence (A prototype of this situation is the so-called Zhang lemma [Z],[Mü1]). As a first result we use the invertibility of Beltrami operators to prove that quasiregular GYMs can be generated by quasiregular mappings.

Theorem 2.1. [1, Theorem 1.2] *Suppose that $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary. Assume also that $p > \frac{2K}{K+1}$. Then a $W^{1,p}$-GYM $\{\nu_z\}_{z\in\Omega}$ is supported in $Q_2(K)$ if and only if it can be generated by a sequence of (gradients of) K-quasiregular mappings.*

If we restrict our attention to Homogeneous GYMs we can choose a more regular generating sequence. A simple and yet useful consequence of the homogeneity of a GYM ν is that any generating sequence converges weakly to a constant matrix, namely the centre of mass of ν. This fact and the factorization theorem for planar quasiregular mappings enable us to prove that

Theorem 2.2. [1, Theorem 1.5] *Let ν be an homogeneous $W^{1,p}$-GYM with support contained in $Q_2(K)$. Assume $p > \frac{2K}{K+1}$. Then there are $K-$ quasiconformal mappings $F_j : \mathbf{R}^2 \to \mathbf{R}^2$ such that the sequence $\{DF_j|_{\mathbb{D}}\}_{j\in\mathbb{N}}$ generates ν.*

For simplicity in the theorem we have as the underlying domain the unit disk $\mathbb{D}$. The proof uses normal families arguments for both the holomorphic and the quasiconformal factors in the decomposition of the generating sequence of quasiregular mappings obtained in Theorem 2.1.

Next we obtain from Astala's theorem sharp estimates for the measure of the set where the Jacobian of a quasiconformal mapping is small ([1, Theorem 2.4]). In turn, this yields a 0-1 law for Homogeneous quasiregular GYMs. Since zero sets of Jacobians of quasiregular mappings have measure zero the result is also true for general quasiregular GYMs.

Theorem 2.3. (The 0-1 Law for Quasiregular $W^{1,p}$-GYM 's.)[1, Theorem 1.3] *Let* $\Omega \subset \mathbf{R}^2$ *and let* $\{\nu_z\}_{z\in\Omega}$ *be a* $W^{1,p}$*-gradient Young measure supported in* $Q_2(K)$ *for some* $K < \infty$*. Assume that* $p > \frac{2K}{K+1}$*. Then either* $\nu_z(\{0\}) = 0$ *for almost every* $z \in \Omega$*, or* $\nu_z = \delta_0$ *for almost every* $z \in \Omega$*.*

It is interesting to observe that to prove Theorem 2.3 one needs estimates of the size of the level sets of the Jacobian of a quasiconformal mapping. It is not enough knowing that the zero set has measure zero. We do not know if any of these theorems hold in higher dimensions. In higher dimensions, a more technical proof yields the 0-1 law provided that Theorem 2.1 holds. However, since we do not have Measurable Riemann Mapping Theorem in the space, a different approach should be used.

As a corollary of the 0-1 law we obtain a new proof of Šverák's three matrices problem [S], [S1]. As we pointed out before, the study of weakly continuous quantities, such as the determinant, is not enough to deal with this situation.

Another consequence of Theorem 2.2 is that quasiregular GYMs share with genuine quasiregular mappings higher integrability properties.

Corollary 2.4. [1, Corollary 1.4] *Suppose that* $p > \frac{2K}{K+1}$ *and that* ν *is a homogeneous* $W^{1,p}$*-GYM supported in* $Q_2(K)$*. Then*

$$\Big(\int_{\mathbf{M^{2\times 2}}} |\lambda|^s d\nu(\lambda)\Big)^{\frac{1}{s}} \le C(s,K)|\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda)| < \infty \quad \forall \quad s < \frac{2K}{K-1}.$$

Observe that this corollary implies a highly non-convex constraint in the possible values of the centre of mass $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu$ in terms of the moments of ν. It would be of interest to investigate if this constraint is useful in the computation of quasiconvex hulls (see below and [3] for more on quasiconvex hulls).

In [3] we need to extend the above higher integrability result to sets which we call asymptotically K-quasiconformal. The definition is as follows:

Definition 2.5. A set $E \subset \mathbf{M^{2\times 2}}$ is said to be asymptotically K-quasiconformal if

$$\lim_{M\to\infty} \sup\{\frac{\|A\|^2}{\det A}, A \in E \setminus B(0,M)\} \le K \tag{2.1}$$

and

$$\lim_{M\to\infty} \inf\{\det A, A \in E \setminus B(0,M)\} > 0. \tag{2.2}$$

We give a different use of Beltrami operators, this time in collaboration with the so-called Measurable Selection Theorem, to obtain that the higher integrability result remains true in this situation. In addition, we find other nice properties of asymptotically quasiconformal sets. Namely, we prove coercivity results for the distance function and show that one can characterize the so-called p-quasiconvex hull of a set E asymptotically quasiconformal in terms of GYMs. The definition of p-quasiconvex hull is given in [3, Definition 1.3]. Quasiconvex hulls often reflect the macroscopical impact of the microstructure. We refrain ourselves from being more specific and refer to the existing literature (see for example [Mü] or [P1]).

Theorem 2.6. [3, Theorem 3.6] *Let $E \subset \mathbf{M^{2\times 2}}$ be a set asymptotically K-quasiconformal and let $\frac{2K}{K+1} < q, \quad p < \frac{2K}{K-1}$. Then E enjoys the following properties:*

i) *For every $A \in \mathbf{M^{2\times 2}}$ there exists a constant $C(K, A)$ such that for every $\varphi \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$*

$$\int_{\mathbb{D}} |D\phi(z)|^p dz \le C(K, A)\Big(1 + \int_{\mathbb{D}} dist_E^p(A + D\phi(z))dz\Big).$$

ii) $E^{qc,p} = \{A \in \mathbf{M^{2\times 2}} : A = \int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\}$.

iii) $\mathcal{H}^q(E) = \mathcal{H}^p(E)$.

The remaining part of [3] is devoted to study sets $E \subset \mathbf{M^{2\times 2}}$ that forbid microstructure. That is, sets E for which $\mathcal{H}^p(E)$ is trivial. A well-known necessary condition is the lack of rank-one connections, i.e,

$$\det(A - B) \neq 0$$

for every $A, B \in E$. Šverák realized that for compact sets

$$\det(A - B) > 0 \tag{2.3}$$

is also a sufficient condition independently of p, [S2]. In the series of papers [Z1],[Z2],[Z3] and [Z4], Zhang studied the cases where E is unbounded and realized that the situation is more delicate there. He needed to assume (2.3) to be satisfied in a strong form. Namely, his sets in addition to (2.3) satisfy that the quantity,

$$K_E = \sup\Big\{\frac{\|A - B\|^2}{\det(A - B)} : (A, B) \in E \times E\Big\} \tag{2.4}$$

is finite. If we fix $B = 0$ we are again in the set of K-quasiconformal matrices. Therefore, in this work, a closed set E such that K_E is finite is called K-quasiconformal at every point. In [Z2], Zhang proved that in the unbounded case $\mathcal{H}^p$ is trivial only if $p > p(K_E)$, where $p(K_E)$ depends only on K_E. However the precise relation between p and K remained open.

We find yet another use of Beltrami operators to present a new proof of the results of Šverák and Zhang. We also manage to find the right relation between p and K. The theorem is as follows:

Theorem 2.7. [3, Theorem 4.1] *Let $K > 1$ and let $E \subset \mathbf{M}^{\mathbf{2}\times\mathbf{2}}$ be K-quasiconformal at every point in the sense of (2.4). Let $\frac{2K}{K+1} < p$. Then E has the following properties:*

i) *$\mathcal{H}^p(E)$ is trivial.*
ii) *$E^{p,qc} = E$.*
iii) *There exists a constant $C(p,k)$ such that*

$$dist_E^p(A) \leq C(p,k)\, Qdist_E^p(A).$$

The function Qdist_E^p denotes the quasiconvexification of dist_E^p, see [3, Definition 1.4] for the definition of quasiconvexification and [D] for the importance of this notion in the study of relaxation in the Calculus of Variations. The theorem is shown to be sharp by constructing suitable examples in [3, Section 5].

We treat also the case of sets satisfying (2.3) everywhere and quasiconformal at infinity. In this case we use the weak continuity of the determinant and the higher integrability result proved in Theorem 2.6 to obtain

Theorem 2.8. [3, Theorem 4.3] *Let $E \subset \mathbf{M}^{\mathbf{2}\times\mathbf{2}}$ be an asymptotically K-quasiconformal set such that*

$$\det(A - B) > 0, \text{ for all } A, B \in E. \tag{2.5}$$

Then for every $q > \frac{2K}{K+1}$,

1) *$\mathcal{H}^q(E)$ is trivial.*
2) *$E^{q,qc} = E$.*

A similar theorem was proved by Zhang in [Z4] for the case $q = 2$ using a different method. As a final remark in this section we quote

Proposition 2.9. [3, Proposition 4.5] *Let $\{H_i\}_{i=1}^n$ with $H_i : CO_+(2) \to CO_-(2)$ be a collection of k- Lipschitz functions with $k < 1$. Denote by $E_i \subset \mathbf{M}^{\mathbf{2}\times\mathbf{2}}$ the graph of H_i. Let $q > 1+k$ and $E = \cup_{i=1}^n E_i$. Then every $\nu \in \mathcal{H}^q(E)$ can be generated by gradients $\{Dg_j\}$ such that $Dg_j(z) \in E$ for a.e. z.*

Several versions of this proposition have found applications in the literature. See for example the next section, the formulation of G-closure problems in [F] and the explicit quasiconvexifications of functionals obtained in [BP], in particular the proof of [BP, Theorem 1.2] for general GYMs.

3. Unbounded laminates and regularity of elliptic equations

The paper [2] is devoted to constructing counterexamples to the regularity of isotropic equations. We prove the following theorem,

Theorem 3.1. [2, Theorem 1.1] *Let $K > 1$. There exist sequences of functions $\{\rho_j\} \in L^\infty(Q, \{K, \frac{1}{K}\})$ and $\{u_j\} \in W^{1,2}(Q, \mathbb{R})$ with $\|u_j\|_{W^{1,2}} \leq 1$, such that*

$$\mathrm{div}(\rho_j(z)\nabla u_j(z)) = 0, \qquad a.e\ z \in Q, \tag{3.1}$$

and for every compact set R of positive measure contained in Q

$$\lim_{j\to\infty} \int_R |\nabla u_j(z)|^{\frac{2K}{K-1}} dz = \infty.$$

This theorem is surprising since in 1972 Piccinini and Spagnolo proved that every solution to an isotropic equation (3.1) is Hölder continuous with exponent $\frac{4}{\pi}\arctan(\frac{1}{K})$. Let us compare with the more general class of equations

$$\mathrm{div}(\sigma(z)\nabla u(z)) = 0, \qquad z \in Q, \tag{3.2}$$

where $\sigma(z) \in \mathbf{M^{2\times 2}}$ with $\sigma(z) = \sigma(z)^t$ and $\frac{1}{K}|\xi|^2 \leq \langle \xi, \sigma(z)\xi\rangle \leq K|\xi|^2$ for every $\xi \in \mathbb{R}^2$ and a.e.z in Q. Then, it goes back to Morrey that the solutions are locally $\frac{1}{K}$ Hölder continuous. On the other hand, it follows from the work of Astala, Leonetti and Nesi, [As], [LN] that the threshold for the integrability of the gradient of solutions to (3.2) is $\frac{2K}{K-1}$. Hence, we observe that for solutions to (3.2), the Hölder regularity is the one implied by the integrability of the gradient, and the Sobolev embedding (which says that $W^{1,p}_{loc}$ is embedded in the Hölder space $C^{0,1-\frac{2}{p}}_{loc}$). Arguing by analogy, one would conjecture that in the isotropic case the threshold for the integrability of the gradient would be $\frac{2}{1-\frac{4}{\pi}\arctan(\frac{1}{K})}$. However, in 1986 Milton suggested that the value $\frac{2K}{K-1}$ was extremal in the isotropic case and proposed that the coefficients ρ_j in Theorem 3.1 should be arranged in a lamination pattern where infinite scales are involved, see [Mi]. Nevertheless, a mathematical proof was lacking and the question appear again in several discussions and papers [LN], [AN].

We present a proof of Milton's conjecture by instead of looking at the equations, considering the properties characterizing the solutions. The main idea goes as follows. Firstly, we recall that u solves an isotropic equation with $\rho \in \{K, \frac{1}{K}\}$ if and only if u is the real part of a K-quasiregular mapping f such that Df lies in the set E given by

$$E = \left\{ \begin{pmatrix} X \\ J\rho X \end{pmatrix} : \quad X \in \mathbb{R}^2,\ \rho \in \{K, \frac{1}{K}\} \text{ and } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{3.3}$$

Next, we build a Gradient Young measure ν such that $\nu \in \mathcal{H}^p(E)$ for every $p < \frac{2K}{K-1}$ but

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^{\frac{2K}{K-1}} d\nu(\lambda) = \infty. \tag{3.4}$$

Then, we consider a sequence $\{f_j\} \in W^{1,2}(Q, \mathbb{R}^2)$ such that $\{Df_j\}$ generates ν. It is not hard to see that (3.4) implies that $\lim_{j\to\infty} \int_R |Df_j|^{\frac{2K}{K-1}} = \infty$ for every compact set R with positive measure. Unfortunately, the sequence $\{Df_j\}$ does not need to lie in E but only to converge to E in L^p, in the sense of (1.1). To solve this difficulty we use again Beltrami operators. An adequate version of Proposition 2.9 yields another sequence $g_j \in W^{1,2}(Q, \mathbb{R}^2)$ such that $Dg_j(z)$ lies in E for almost every z and

$$\lim_{j\to\infty} \int_Q |Df_j - Dg_j|^p dx = 0$$

for every $p < \frac{2K}{K-1}$. In turn, the real parts of $\{g_j\}$ are shown to satisfy the claim of Theorem 3.1.

Let us conclude this introduction with some words about the construction of ν. The classical examples of Homogeneous GYMs which do not come directly from the gradient of a given Sobolev function are the so-called laminates, see [P2], [K] or [2, Section 3]. In the literature the interest is so far restricted to compactly supported laminates. We present a way of constructing laminates with unbounded support such that their integrability is easy to compute. Motivated by the figures, see [2, Figure 3] and [3, Figure 3], we call these laminates staircase laminates. They can be constructed for many other subclasses of elliptic equations of divergence type. However when Beltrami operators are not available it is not clear how to push the generating sequence to the appropriate closed set E supporting the laminate. Even more challenging is to decide if these laminates are extremal for those classes of equations for which the corresponding Weyl's exponents and thresholds for integrability remain unknown.

References

[1] Astala, K; Faraco, D. Quasiregular mappings and Young measures, Proc. Roy. Soc. Edinburgh Sect. A (To appear).

[2] Faraco, D. Milton's conjecture on the regularity of isotropic equations. Submitted to Ann. Inst. H. Poincaré Anal. Non Linéaire.

[3] Faraco, D. Tartar's conjecture and Beltrami Operators. Preprint, University of Helsinki, 2002.

[Ah] Ahlfors, L.V. *Lectures on quasiconformal mappings*, Van Nostrand, Princenton 1966.

[As] Astala, K. Area distortion of quasiconformal mappings, Acta math., 173, (1994), 37-60.

[AIS] Astala, K; Iwaniec, T; Saksman, E. Beltrami operators in the plane. Duke Math. J. 107 (2001), no. 1, 27–56.

[AM] Astala, K; Miettinen, M. On quasiconformal mappings and 2-dimensional G-closure problems. Arch. Rational Mech. Anal. 143 (1998), no. 3, 207–240.

[AN] Astala, K; Nesi, V. Composites and quasiconformal mappings: New optimal bounds in two dimensions. Calc. Var. Partial Differential Equations (To appear)

[B] Ball, J.M. Convexity conditions and existence theorems in non linear elasticity. Arch. Rat. Mech. Anal. 63 (1978), 337-403.

[B1] Ball, J.M. A version of the fundamental theorem for Young measures. Proc of conference on PDE and continuum models of phase transitions, Nice, 1988, M. Rascle, D.Serre, M. Slemrod (Eds). Springer-Verlag, Lecture Notes in Physics 344 (1989), 207-215.

[BJ] Ball, J.M; James, R.J. Fine phase mixtures as minimizers of energy, Arch,Rat. Mech, Anal.100 (1987), 13-52.

[BJ1] Ball, J.M; James, R.J. Proposed experimental test of a theory of fine microstructure and the two well problem, Phil. Trans. R. Proc. London A, 338 (1992), 389-450.

[BP] Bellido, J.C; Pedregal, P. Explicit quasiconvexification for some cost functionals depending on derivatives of the state in optimal design. Preprint 2002.

[D] Dacorogna, B. Direct methods in the calculus of variations. Applied Mathematical Sciences, 78. Springer-Verlag, Berlin, 1989.

[DD] DeSimone, A; Dolzmann, G. Material instabilities in nematic elastomers. Phys. D 136 (2000), no. 1-2, 175–191.

[F] Faraco, D. Quasiconvex hulls and the homogenization of differential operators. Licenciate thesis. University of Jyväskylä. (2000)

[KP] Kinderlehrer, D; Pedregal, P. Characterizations of Young measures generated by gradients. Arch. Rational Mech. Anal. 115 (1991), no. 4, 329–365.

[K] Kirchheim, B. Geometry and rigidity of microstructures. Habilitation Thesis, Leipzig 2001.

[LN] Leonetti, F; Nesi, V. Quasiconformal solutions to certain first order systems and the proof of a conjecture of G. W. Milton, J. Math. Pures. Appl (9) 76 (1997), 109-124.

[Mi] Milton, G. Modelling the properties of composites by laminates, in *Homogenization and effective moduli of materials and media*, Ericksen, Kinderleher, Kohn, Lions, eds., IMA volumes in mathematics and its applications 1, Springer-Verlag, New York, 1986

[Mü] Müller, S. Variational models for microstructure and phase transitions. Calculus of variations and geometric evolution problems (Cetraro, 1996), 85–210, Lecture Notes in Math., 1713, Springer, Berlin, 1999.

[Mü1] Müller, S. A sharp version of Zhang's theorem on truncating sequences of gradients. Trans. Amer. Math. Soc. 351 (1999), no. 11, 4585–4597.

[N] Nesi, V. Quasiconformal mappings as a tool to study certain two-dimensional G-closure problems. Arch. Rational Mech. Anal. 134 (1996), no. 1, 17–51.

[P] Pedregal, P. Laminates and microstructure, Europ. J. Appl. Math. 4 (1993), 121-149.

[P1] Pedregal, P. Parametrized measures and variational principles. Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser Verlag, Basel, 1997.

[P2] Pedregal, P. Fully explicit quasiconvexification of the mean-square deviation of the gradient of the state in optimal design. Electron. Res. Announc. Amer. Math. Soc. 7 (2001), 72–78.

[PS] Piccinini, L.C; Spagnolo, S. On the Hölder continuity of solutions of second order elliptic equations in two variables. Ann. Scuola Norm. Sup. Pisa (3) 26 (1972), 391–402.

[S] Šverák, V. On regularity for the Monge-Ampere Equation without convexity assumptions. Preprint (1991).

[S1] Šverák, V. New examples of quasiconvex functions. Arch. Rat. Mech. Anal., 19 (1992), 293-300.

[S2] Šverák, V. On Tartar's conjecture. Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993), no. 4, 405–412.

[Ta1] Tartar, L. Compensated compactness and applications to partial differential equations. Heriot-Watt Symposium,Vol IV, Research Notes in Mathematics 39 (1979), 136-212.

[Z] Zhang, K. A construction of quasiconvex functions with linear growth at infinity. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), no. 3, 313–326.

[Z1] Zhang, K. On connected subsets of $M^{2\times 2}$ without rank-one connections. Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), no. 1, 207–216.

[Z2] Zhang, K. On non-negative quasiconvex functions with unbounded zero sets. Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), no. 2, 411–422.

[Z3] Zhang, K. On some quasiconvex functions with linear growth. J. Convex Anal. 5 (1998), no. 1, 133–146.

[Z4] Zhang, K. Rank-one connections at infinity and quasiconvex hulls. J. Convex Anal. 7 (2000), no. 1, 19–45.