

**DISTORTION OF DIMENSION UNDER QUASICONFORMAL
MAPPINGS**

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Academic dissertation

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This doctoral dissertation consists of this summary and the following three articles:

- [A] *A remark on quasiconformal dimension distortion on the line*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 341–352.
- [B] *A note on distortion of Hausdorff measures under quasiconformal mappings*, Preprint 457, Reports in Mathematics, University of Helsinki, 2007.
- [C] *Flatness properties of quasispheres*, Preprint 458, Reports in Mathematics, University of Helsinki, 2007.

1. INTRODUCTION

Quasiconformal mappings are generalizations of conformal mappings. They constitute a standard tool in a number of areas of complex analysis such as Teichmüller theory, Kleinian groups and complex dynamics. They also appear in various contexts in other parts of mathematics, including connections to elliptic partial differential equations, differential geometry and calculus of variations. As for their role in geometric function theory we refer to [18].

Quasiconformal maps in the plane were introduced by Grötzsch in 1928 and their importance in complex analysis was soon realized by Ahlfors and Teichmüller [1]. Higher dimensional quasiconformal mappings were already considered by Lavrentiev in the 1930's, while their systematic study began with the work of Gehring and Väisälä in the 1960's. Then in the late 1960's, Reshetnyak and the Finnish school, Martio, Rickman and Väisälä initiated the theory of quasiregular mappings, the non-injective counterpart of quasiconformal mappings. This framework offers an extension of complex analysis to \mathbb{R}^n from the viewpoint of real analysis. Recent developments include extension of quasiconformal analysis to general metric measure space setting [14] and the theory of mappings of finite distortion [17]. We refer to the survey of Gehring [12] for an overview of the topic.

Basic pointwise distortion results were established at an early stage of the theory. Much harder is to find precise bounds how quasiconformal maps distort dimension. A complete solution is known only in the plane. In this thesis we are concerned with some aspects of distortion of Hausdorff dimension under quasiconformal mappings both in the two-dimensional and higher dimensional Euclidean setting.

1.1. An example. Quasiconformal mappings constitute a class interpolating between bilipschitz maps and homeomorphisms. Most of the questions we consider

are straightforward for the bilipschitz class; bilipschitz maps preserve dimension and rectifiability. Different phenomena occur in the quasiconformal setting, since quasiconformal curves need not be rectifiable, and moreover, they can have Hausdorff dimension bigger than one. It is a classical fact that both bilipschitz and quasiconformal mappings are differentiable almost everywhere. It is the different nature of singularities at this exceptional set of measure zero that brings out the difference between quasiconformal mappings and bilipschitz mappings. The standard *von Koch snowflake* curve serves as an illustration. It has Hausdorff dimension $\log 4 / \log 3$ while being a quasiconformal image of the unit segment. For more examples of quasiconformal circles or spheres, see for instance [27].

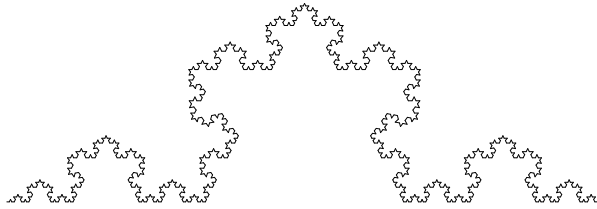


FIGURE 1. The snowflake, a quasiconformal curve

The snowflake is *wiggly* in the following sense: it oscillates around every point and at every scale. Quantitative versions of this property have been studied in [6, 24]. Wiggly or thick sets arise naturally in many parts of analysis, e.g. in connection with Kleinian groups, harmonic measure or bilipschitz extensions. Observe that if we replace the angle of 60 degrees in the snowflake construction by an angle close to 180 degrees then the oscillation becomes very small and the curve will also satisfy an opposite property, a uniform flatness condition, see Section 4.3 for details. Higher dimensional analogous “snowballs” have been constructed in [8].

2. QUASICONFORMAL MAPS AND HAUSDORFF DIMENSION

2.1. Quasiconformal mappings. According to the *analytic definition* a (sense preserving) homeomorphism $f: \Omega \rightarrow \Omega'$ between domains in \mathbb{R}^n , $n \geq 2$, is called *quasiconformal* if $f \in W_{\text{loc}}^{1,n}(\Omega)$ and there exists $1 \leq K < \infty$ such that

$$(2.2) \quad \max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi| \quad \text{a.e. } x \in \Omega.$$

Quantifying this we speak of K -quasiconformal mappings if (2.2) holds. If $K = 1$ we recover conformal maps. According to Liouville’s rigidity theorem it is crucial to allow the dilatation $K > 1$ in order to get an interesting theory in higher dimensions. We refer to [23] for other equivalent definitions and for foundations of quasiconformal mappings. See also [17, 25] for different approaches.

Condition (2.2) expresses that balls are distorted in a uniform manner on the infinitesimal scale. Eventually, this property also leads to global distortion estimates. The following definition from [22] captures a similar phenomenon globally.

2.3. Quasisymmetric maps. Let $\eta: [0, \infty) \rightarrow [0, \infty)$ be an increasing homeomorphism. A homeomorphism $f: X \rightarrow Y$ between metric spaces is η -*quasisymmetric* if

$$(2.4) \quad \frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \eta \left(\frac{|a - x|}{|b - x|} \right),$$

for all $a, b, x \in X$ ($b \neq x$). The mapping f is called *quasisymmetric* if it is η -quasisymmetric with some function η .

Quasisymmetric maps (between domains in \mathbb{R}^n) are always quasiconformal. In the other direction, quasiconformal maps satisfy the quasisymmetry condition semi-globally, in particular, a K -quasiconformal map of the whole space $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, is $\eta_{K,n}$ -quasisymmetric.

In many ways quasiconformal maps interpolate between bilipschitz maps and homeomorphisms. We will see how this is reflected in the way these maps distort Hausdorff dimension.

2.5. Hausdorff dimension. Let $\delta: [0, \infty) \rightarrow [0, \infty)$ be a continuous non-decreasing function with $\delta(0) = 0$. We call δ a *measure function* and define the *Hausdorff δ -measure* for a set E as

$$\mathcal{H}^\delta(E) = \liminf_{\varepsilon \rightarrow 0} \sum \delta(\text{diam}(E_i)),$$

where the infimum is taken over all countable coverings of E by sets E_i with $\text{diam}(E_i) < \varepsilon$. If we set $\delta(r) = r^t$ for some $t \in (0, \infty)$, then we obtain the t -dimensional Hausdorff measure and denote it simply by \mathcal{H}^t . The *Hausdorff dimension* of E is given by

$$\dim E = \inf \{t: \mathcal{H}^t(E) = 0\}.$$

Hausdorff measures and dimension provide a general way to measure metric size; for further details see [19]. The term *dimension* always refers to Hausdorff dimension in this thesis.

2.6. Higher integrability. It is well known that K -quasiconformal maps are locally Hölder continuous with exponent $1/K$, see [9]. The sharpness of the exponent is seen by considering the radial stretching of the form $f(x) = x|x|^{\frac{1}{K}-1}$. In fact, this example is believed to be extremal for many problems, providing maximal expansion at a point. A remarkable result of Bojarski [7] ($n = 2$) and Gehring [10] ($n \geq 3$) is the higher integrability phenomenon: a K -quasiconformal map f has higher Sobolev regularity than the natural exponent n , that is $f \in W_{\text{loc}}^{1,p}$ for every $p < p_0$ where $p_0 = p_0(K, n) > n$. It is an important problem to identify the precise exponent $p_0(K, n)$.

2.7. Conjecture (Higher integrability conjecture (Gehring)). We may take

$$p_0(K, n) = \frac{nK}{K-1}.$$

Note that the above value of p_0 and the Hölder exponent $1/K$ are related via the Sobolev embedding theorem. This conjecture has been proved in the case $n = 2$ by Astala; for further details see the next section.

Hölder continuity implies that sets of *zero* dimension are preserved, while sets of dimension n are preserved because of the higher integrability phenomenon. However, in general, quasiconformal maps can change the Hausdorff dimension, see [13]. Bishop [5] showed that the dimension of any compact set of positive dimension can, in fact, be raised arbitrarily close to n by a quasiconformal homeomorphism of \mathbb{R}^n .

We are interested in bounds in terms of the dilatation K . Let us note that the Higher integrability conjecture would imply the following (see [13, 15]).

2.8. Conjecture. Let $f: \Omega \rightarrow \Omega'$ be K -quasiconformal in \mathbb{R}^n and suppose $E \subset \Omega$ is compact. Then

$$(2.9) \quad \frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{n} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{n} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{n} \right).$$

Examples built on Cantor sets via iterations of radial stretchings show that we can have equality on either side.

3. AREA DISTORTION

In this section we confine ourselves to the theory of planar quasiconformal mappings in which case one has an essentially complete understanding of the regularity issues discussed above, due to the work of Astala [2].

In the two dimensional situation there is a strong interaction with elliptic PDE's because of the connection to the *Beltrami equation*

$$(3.1) \quad \bar{\partial}f(z) = \mu(z)\partial f(z) \quad \text{a.e. } z \in \Omega,$$

which is equivalent to (2.2) if we require $\|\mu\|_\infty \leq (K-1)/(K+1) < 1$. One of the cornerstones of the theory is the measurable Riemann mapping theorem which asserts that (3.1) has always (an essentially unique) homeomorphic solution when $\|\mu\|_\infty < 1$.

As we remarked earlier the Higher integrability conjecture has been solved in the plane by Astala. Higher integrability is closely connected with metric distortion properties of quasiconformal maps, and in fact Astala proved the optimal regularity via establishing the Gehring-Reich conjecture on area distortion of quasiconformal maps. Let us record these results.

3.2. Theorem (Area distortion [2]). *Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a K -quasiconformal mapping in the unit disk $\mathbb{D} \subset \mathbb{C}$ with $f(0) = 0$. Then we have*

$$|fE| \leq C(K)|E|^{1/K},$$

for all Borel measurable sets $E \subset \mathbb{D}$.

3.3. Theorem (Higher integrability [2]). *Let $f: \Omega \rightarrow \Omega'$ be K -quasiconformal in \mathbb{C} . Then*

$$f \in W_{\text{loc}}^{1,p}(\Omega) \quad \text{for all } p < \frac{2K}{K-1}.$$

Higher integrability also controls the change of Hausdorff dimension, thus confirming Conjecture 2.8 for $n = 2$.

3.4. Theorem (Dimension distortion [2]). *Let $f: \Omega \rightarrow \Omega'$ be K -quasiconformal in \mathbb{C} and suppose $E \subset \Omega$ is compact. Then*

$$(3.5) \quad \frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{2} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{2} \right).$$

This inequality is best possible.

The previous theorem gives a complete description of dimension distortion under planar quasiconformal mappings. We shall be concerned with two related issues which remain unsettled: (A) *improved distortion* on the line, and (B) *distortion of Hausdorff measures*.

Let us first discuss (B). It is natural to ask, see [2, 3], whether the estimates of (3.5) hold on the level of Hausdorff measures \mathcal{H}^t . That is, if f is a planar K -quasiconformal mapping, $0 < t < 2$ and $d = \frac{2Kt}{2+(K-1)t}$, is it true that

$$(3.6) \quad \mathcal{H}^t(E) = 0 \Rightarrow \mathcal{H}^d(f(E)) = 0?$$

In other words, do we have absolute continuity $f^*\mathcal{H}^d \ll \mathcal{H}^t$? It is classical that quasiconformal mappings are absolutely continuous with respect to the Lebesgue measure, and the Area distortion theorem proves this in a quantitatively optimal form. Very recently, the authors of [3] confirmed (3.6) in the case $d = 1$ and obtained partial results when $d > 1$.

3.7. Dimension of quasicircles. In this paragraph we discuss phenomenon (A). We call a Jordan curve a K -*quasicircle* if it is the image of the unit circle under a global K -quasiconformal map of the plane \mathbb{C} . Quasicircles and domains they bound (quasidisks) have been proved to possess many important function theoretic properties [11]. Here we concentrate on the question on their Hausdorff dimension, and for convenience we fix the notation $k = (K - 1)/(K + 1)$.

From the inequalities (3.5) we see that one can map a 1-dimensional set to a set of $1 + k$ dimension (or $1 - k$ resp.) under a K -quasiconformal map and these bounds are optimal. However, the extremal distortion is achieved for sets of highly irregular character and one can expect better estimates to hold for subsets of rectifiable curves, or more concretely for subsets of the real line. In fact, Becker and Pommerenke showed that the correct asymptotic behavior of the dimension for quasicircles is quadratic in k as $K \rightarrow 1$.

3.8. Theorem ([4]). *For every K -quasicircle Γ , we have*

$$\dim \Gamma \leq 1 + 37k^2.$$

Conversely, for every $K \geq 1$, there exists a K -quasicircle with dimension at least $1 + 0.09k^2$.

Later S. Smirnov improved this to the following.

3.9. Theorem (Smirnov (2000, unpublished)). *For every K -quasicircle Γ , we have*

$$\dim \Gamma \leq 1 + k^2.$$

It would be of particular interest to know whether this estimate is sharp. To date, lower bounds are relatively far from the conjectured value of $1 + k^2$.

3.10. Higher dimensions. Conjectures 2.7 and 2.8 remain widely open in higher dimensions, $n \geq 3$. The solution in the planar case by Astala is largely based on the theory of holomorphic motions. As these planar methods do not carry over to higher dimensions one inevitably needs to find other approaches. See [16, 17] for developments in this direction.

Somewhat similar remarks apply to the arguments in Theorems 3.8 and 3.9, they are analytical and not applicable in higher dimensions. Mattila and Vuorinen in [20] studied related problems from a more geometric point of view and obtained qualitatively the same estimates as in Theorem 3.8. Their idea is to show that quasicircles are flat in a weak sense and this in turn implies a bound on their dimension. For precise definitions, see Subsection 4.3. The advantage is that this approach generalizes to higher dimensions, that is, we can e.g. study the dimension of quasispheres (quasiconformal images of a sphere). The drawback is that one cannot obtain sharp results this way, but nevertheless, it suffices to analyze the asymptotics as $K \rightarrow 1$. *Flatness properties* of quasispheres constitute question (C) of our study.

4. MAIN RESULTS

The papers [A], [B] and [C] contribute to the issues (A), (B) and (C) mentioned above, respectively. We describe the main results in the next three subsections.

4.1. Improved distortion

As we discussed above one expects improved dimension distortion bounds to hold for subsets of the line. The next theorem expresses this in a special case. Recall that $k = (K - 1)/(K + 1)$.

4.1. Theorem ([A, 1.6]). *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal map with $0 < k < 1/\sqrt{8}$ and $E \subset \mathbb{R}$. Then $\dim fE < 1$ provided that $\dim E \leq 1 - 8k^2$. Conversely, if $\dim E = 1$ then $\dim fE > 1 - 8k^2$.*

In view of Stoilow factorization, quasiconformal distortion results have immediate applications to quasiregular removability questions. In fact, the result above (with unspecified constants in place of 8) is due to [3], where the authors studied this problem in connection with their improved version of Painlevé removability for quasiregular mappings. Our approach relies on the area distortion argument from [2] and the quasicircle dimension estimate of Theorem 3.9.

Under the additional assumption that f fixes the real line, we obtain a refined estimate, a dual result to Theorem 3.9. The relevance of the refinement is that it could very well be sharp.

4.2. Theorem ([A, 3.1], Smirnov (unpublished)). *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal map for which $f(\mathbb{R}) = \mathbb{R}$. Then for a 1-dimensional set $E \subset \mathbb{R}$,*

$$\dim fE \geq 1 - k^2.$$

4.2. Distortion of Hausdorff measures

The objective of [B] is to point out that the methods of [2] allow to establish Theorem 3.4 in a slightly stronger form, that is, to show absolute continuity as in (3.6) with respect to some weaker Hausdorff measures.

We consider measure functions $\delta(r) = r^d \varepsilon(r)$ satisfying

$$(4.3) \quad \int_0^\infty \varepsilon(r)^{\frac{Kt}{Kt-d}} \frac{dr}{r} < \infty.$$

We also make the technical assumption that the integrand is decreasing and $\varepsilon(r)$ is increasing in $(0, r_0)$ for some $r_0 > 0$. For instance, we can take $\varepsilon(r) = |\log r|^{-s}$ with $s > 1 - \frac{d}{Kt}$, so that \mathcal{H}^δ has the right dimension d .

4.4. Theorem ([B, 1.9]). *Let $E \subset \mathbb{D}$ be a compact set and let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal mapping conformal outside \mathbb{D} , normalized by $f(z) = z + O(1/|z|)$ as $z \rightarrow \infty$. Let $t \in (0, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then we have*

$$\mathcal{H}^\delta(f(E)) \leq C (\mathcal{H}^t(E))^{\frac{d}{Kt}},$$

where the measure function δ satisfies (4.3). The constant C depends only on δ and K .

This is a complementary result to [3, Corollary 2.12] which proves the same result under the assumption that $d > 1$ and $\delta(r) = r^d \varepsilon(r)$ is such that

$$(4.5) \quad \int_0^\infty \varepsilon(r)^{\frac{1}{d-1}} \frac{dr}{r} < \infty.$$

The two results complement each other in the following way: [3, Corollary 2.12] gets sharper as $d \rightarrow 1$, while Theorem 4.4 improves as $K \rightarrow 1$.

4.3. Flatness properties of quasispheres

Although quasispheres need not be rectifiable, they become more and more flat as $K \rightarrow 1$. This flatness property appears uniformly at all scales and locations. We shall work with the following definition due to Mattila and Vuorinen [20].

4.6. LAP property. Let $0 \leq \delta < 1$. We say that a closed set E in \mathbb{R}^n satisfies the d -dimensional δ -linear approximation property (δ -LAP) if there is an $r_0 > 0$ such that for each $x \in E$ and for each $0 < r < r_0$ there exists a d -dimensional affine subspace V through x such that

$$E \cap B^n(x, r) \subset V(\delta r).$$

Here $V(r)$ denotes the r -neighborhood of V ; $V(r) = \{x : d(x, V) < r\}$.

The authors of [20] showed that K -quasispheres satisfy the $(n - 1)$ -dimensional δ -LAP property with $\delta = \delta(K) \rightarrow 0$ as K tends to 1. In [C] we study this and related properties, and in particular, we show the following sharp estimate in terms of the quasimetry function η in (2.4).

4.7. Theorem ([C, 5.1]). *Let $1 < K < K_0$ and let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasi-conformal homeomorphism of \mathbb{R}^n . Then the image of a hyperplane H satisfies the $(n - 1)$ -dimensional δ -LAP property with $\delta = \delta(K) = O(\eta_{K,n}(1) - 1)$.*

LAP property implies the following bound on the dimension.

4.8. Theorem ([20]). *There is a positive number δ_0 depending only on d and n such that if a set $E \subset \mathbb{R}^n$ has the d -dimensional δ -LAP property and $0 < \delta < \delta_0$, then*

$$\dim E \leq d + c(d)\delta^2.$$

Combining the two previous theorems and the best-known bounds for $\eta_{K,n}$ [26, 21], we obtain the following.

4.9. Corollary ([C, 5.4]). *For a K -quasisphere E in \mathbb{R}^n with $1 < K < K_0$ we have*

$$\dim E = n - 1 + O((\eta_{K,n}(1) - 1)^2) = n - 1 + O\left((K - 1)^2 \log^2 \frac{1}{K - 1}\right).$$

This result can be considered satisfactory except for the logarithmic term involved, see Questions [27, 1.41 and 1.42]. Nevertheless, it reveals that we have a phenomenon in higher dimensions similar to that of the plane: K -quasispheres have much smaller dimension than K -quasiconformal images of general $(n - 1)$ -dimensional sets (in the case $K \rightarrow 1$). Similar results hold for quasiconformal images of lower dimensional subspaces [C, 5.6 and 5.7].

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