

# ON ORLICZ-SOBOLEV CAPACITIES

JANI JOENSUU

Academic dissertation

*To be presented for public examination with the  
permission of the Faculty of Science of the University of Helsinki in  
S12 of the University Main Building on 18 December 2009 at 12 noon.*

Department of Mathematics and Statistics  
Faculty of Science  
University of Helsinki  
2009

ISBN 978-952-92-6557-2 (paperback)  
Helsinki University Print

ISBN 978-952-10-5904-9 (PDF)  
<http://ethesis.helsinki.fi/>

Helsinki 2009

## ACKNOWLEDGEMENTS

I wish to express my deepest gratitude to my advisor Ritva Hurri-Syrjänen for her guidance.

I am grateful to the pre-examiners Jana Björn and Tetsu Shimomura for suggesting many improvements to the final version of this thesis.

For the financial support I am indebted to the Academy of Finland and to the Finnish Academy of Science and Letters (Vilho, Yrjö and Kalle Väisälä foundation).

I would like to thank my parents Irma and Veikko, my sister Mira, and my brothers Mika and Toni for their support and encouragement. Finally, I would like to thank my wife Nina and my wonderful children Elmeri and Tuuli for their love and support.

Jani Joensuu  
Helsinki  
November 2009

#### LIST OF INCLUDED ARTICLES

This thesis consists of an introductory part and the following articles:

- [A] Joensuu, J.: On null sets of Sobolev-Orlicz capacities,  
to appear in Illinois J. Math.
- [B] Joensuu, J.: Orlicz-Sobolev capacities and their null sets,  
to appear in Rev. Mat. Complut., 23, (2010).  
DOI: 10.1007/s13163-009-0011-1
- [C] Joensuu, J.: Estimates for certain Orlicz-Sobolev capacities of  
an Euclidean ball, manuscript.

## 1. INTRODUCTION

In this thesis we study certain Orlicz-Sobolev capacities defined in terms of potentials on a fixed Euclidean ball. There are several versions of Orlicz-Sobolev capacity. We concentrate on two of these versions, and we assume that the defining function of the Orlicz-Sobolev space, the Young function, satisfies some conditions.

This dissertation includes three articles. In [A] and [B] we study relations of Orlicz-Sobolev capacities  $\mathcal{P}_{\alpha,\Phi}$  and  $\mathcal{B}_{\alpha,\Phi}$ . Our main interest is in the null sets of these capacities. In [C] we study metric properties of these capacities. In particular, we obtain two-sided estimates for the capacity of a ball.

The rest of the summary is organized as follows. In Section 2 we give some definitions including the definitions of Orlicz-Sobolev capacities  $\mathcal{P}_{\alpha,\Phi}$  and  $\mathcal{B}_{\alpha,\Phi}$ . Section 3 contains history of Orlicz-Sobolev capacities. In Section 4 we summarize the results of articles [A], [B] and [C].

## 2. ORLICZ-SOBOLEV CAPACITIES

In this section we give some definitions. Let  $n \geq 2$  and let  $B_R = B^n(0, R)$ , with  $R > 0$ , be a fixed ball in  $\mathbb{R}^n$ . We use the Lebesgue  $n$ -measure.

Let us first introduce Orlicz spaces.

**Definition 2.1.** A function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  is a Young function if it is continuous, strictly increasing, and convex, and it satisfies

$$\lim_{t \rightarrow 0^+} \frac{\Phi(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{\Phi(t)} = 0.$$

A Young function  $\Phi$  has a presentation  $\Phi(t) = \int_0^t \phi(s) ds$  for some non-decreasing, right-continuous function  $\phi : [0, \infty) \rightarrow [0, \infty)$ . The complementary Young function to  $\Phi$  is the function  $\Psi : [0, \infty) \rightarrow [0, \infty)$  defined by

$$\Psi(t) = \int_0^t g(s) ds,$$

where  $g(s) = \sup\{a \mid \phi(a) \leq s\}$  for  $s \geq 0$ .

Let  $\Phi$  be a Young function. Suppose that  $\Omega$  is a measurable subset of  $\mathbb{R}^n$ . The Orlicz space  $L^\Phi(\Omega)$  is defined by

$$L^\Phi(\Omega) = \left\{ f \in \mathcal{M}_0(\Omega) \mid \int_\Omega \Phi(\lambda|f(x)|) dx < \infty \text{ for some } \lambda > 0 \right\}.$$

Here  $\mathcal{M}_0(\Omega)$  is the set of those functions  $g : \Omega \rightarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$  that are measurable and almost everywhere finite. The Orlicz space

equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ b > 0 \mid \int_{\Omega} \Phi \left( \frac{|f(x)|}{b} \right) dx \leq 1 \right\}$$

is a Banach space.

Let  $k$  be a non-negative integer. A function  $u \in L^\Phi(\mathbb{R}^n)$  is in the Orlicz-Sobolev space  $W^{k,\Phi}(\mathbb{R}^n)$  if its weak derivatives up to the order  $k$  belong to  $L^\Phi(\mathbb{R}^n)$ . If  $\Phi$  and  $\Psi$  satisfy the following growth condition, the  $\Delta_2$ -condition, then the Orlicz-Sobolev space  $W^{k,\Phi}(\mathbb{R}^n)$  has properties similar to Sobolev spaces for  $p \in (1, \infty)$ . For example, then  $W^{k,\Phi}(\mathbb{R}^n)$  is separable and reflexive. A Young function  $\Phi$  satisfies the  $\Delta_2$ -condition, if there exists a positive constant  $C$  such that

$$\Phi(2t) \leq C\Phi(t), \quad \text{for all } t \geq 0.$$

Standard references of Orlicz spaces and Orlicz-Sobolev spaces are [13] and [16].

The Orlicz potential space  $L^{\alpha,\Phi}(\mathbb{R}^n)$  is defined for  $\alpha > 0$  by

$$L^{\alpha,\Phi}(\mathbb{R}^n) = \{u \mid u = G_\alpha * f, f \in L^\Phi(\mathbb{R}^n)\}.$$

Here  $G_\alpha$  is the Bessel kernel, and  $G_\alpha * f$  is the convolution of  $G_\alpha$  and  $f$ . For the definition and properties of the Bessel kernel we refer to [1]. Here we mention that  $G_\alpha$  is equivalent to the Riesz kernel on  $B_R$ . The norm in  $L^{\alpha,\Phi}(\mathbb{R}^n)$  is  $\|u\|_{L^{\alpha,\Phi}(\mathbb{R}^n)} := \|f\|_{L^\Phi(\mathbb{R}^n)}$ . If a Young function  $\Phi$  and its complementary Young function satisfy the  $\Delta_2$ -condition, and  $k$  is a positive integer, then the Orlicz-Sobolev space  $W^{k,\Phi}(\mathbb{R}^n)$  coincides with the Orlicz potential space  $L^{k,\Phi}(\mathbb{R}^n)$ .

We are interested in the following two versions of Orlicz-Sobolev capacity.

**Definition 2.2.** Let  $E$  be a subset of  $B_R$  and let  $\alpha > 0$ . Suppose that a Young function  $\Phi$  satisfies the  $\Delta_2$ -condition. Then  $\mathcal{P}_{\alpha,\Phi}$ -capacity of  $E$  is

$$\mathcal{P}_{\alpha,\Phi}(E) = \inf \left\{ \|f\|_{L^\Phi(B_R)} \mid f \geq 0, \text{ spt } f \subset B_R, G_\alpha * f \geq 1 \text{ on } E \right\}$$

and  $\mathcal{B}_{\alpha,\Phi}$ -capacity of  $E$  is

$$\mathcal{B}_{\alpha,\Phi}(E) = \inf \left\{ \int_{B_R} \Phi(f(x)) dx \mid f \geq 0, \text{ spt } f \subset B_R, G_\alpha * f \geq 1 \text{ on } E \right\},$$

here  $\text{spt } f$  is the support of  $f$ ,

$$\text{spt } f = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}.$$

These capacities have been studied in [2]-[7], [11], [12], [14], [15]. If  $\Phi(t) = t^p$  with  $p \in (1, \infty)$ , then the theory of these capacities reduces to the theory of  $L^p$ -capacities which has been extensively studied in [1] and [17].

### 3. HISTORY

The foundations for the study of Orlicz-Sobolev capacity were laid by N. Aïssaoui in [5] and Aïssaoui and A. Benkirane in [4] and [7]. It is shown in [5] that if  $\Phi$  and its complementary Young function satisfy the  $\Delta_2$ -condition, then  $\mathcal{P}_{\alpha, \Phi}$ -capacity has the important Choquet-property, that is,  $\mathcal{P}_{\alpha, \Phi}$ -capacity of a Suslin set  $E \subset B_R$  can be estimated by open sets and compact sets,

$$\mathcal{P}_{\alpha, \Phi}(E) = \inf\{\mathcal{P}_{\alpha, \Phi}(U) \mid U \text{ is open, } E \subset U\}$$

and

$$\mathcal{P}_{\alpha, \Phi}(E) = \sup\{\mathcal{P}_{\alpha, \Phi}(K) \mid K \text{ is compact, } K \subset E\}.$$

A. Cianchi and B. Stroffolini [11], and D.R. Adams and R. Hurri-Syrjänen [2],[3], and Y. Mizuta and T. Shimomura [15] used Orlicz-Sobolev capacity for the study of extensions of Trudinger-type inequalities and Lebesgue point theory. In these articles the sets of zero capacity, the null sets, play an important role. As it is natural to work outside the set of measure zero in the Orlicz space setting, it is natural to work outside the set of capacity zero in the Orlicz-Sobolev space setting. If  $\mathcal{P}_{\alpha, \Phi}$  and  $\mathcal{B}_{\alpha, \Phi}$  have the same null sets, then in many situations it does not matter which one of these capacities is used. Hence, it is important to know whether  $\mathcal{P}_{\alpha, \Phi}$  and  $\mathcal{B}_{\alpha, \Phi}$  have the same null sets.

Adams and Hurri-Syrjänen [2] proved the following theorem concerning the null sets of capacity.

**Theorem 3.1.** *Let  $E$  be a subset of  $B_R$ . If the Young function  $\Phi$  is defined by  $\Phi(t) = t^p(\log(e+t))^\theta$  with  $p \in (1, \infty)$  and  $\alpha p = n$ , then  $\mathcal{P}_{\alpha, \Phi}(E) = 0$  if and only if  $\mathcal{B}_{\alpha, \Phi}(E) = 0$ .*

This is the most general result concerning the null sets of Orlicz-Sobolev capacity as far as we know excluding [A] and [B].

The method the authors used in [3] required estimates for the capacity of a ball. Therefore Adams and Hurri-Syrjänen proved that if  $r$  is sufficiently small and  $B_R$  is a fixed ball, then there is a positive constant  $C$ , depending on  $n$ ,  $p$  and  $R$  only, such that

$$(3.1) \quad C^{-1} \left( \log \log \frac{1}{r} \right)^{\frac{1-p}{p}} \leq \mathcal{P}_{\alpha, \Phi}(B^n(0, r)) \leq C \left( \log \log \frac{1}{r} \right)^{\frac{1-p}{p}},$$

when  $\theta = p - 1$ , and there is a positive constant  $C$ , depending on  $n$ ,  $p$ ,  $R$  and  $\theta$  only, such that

$$(3.2) \quad C^{-1} \left( \log \frac{1}{r} \right)^{\frac{1-p+\theta}{p}} \leq \mathcal{P}_{\alpha, \Phi}(B^n(0, r)) \leq C \left( \log \frac{1}{r} \right)^{\frac{1-p+\theta}{p}},$$

when  $\theta \in [0, p - 1)$ , we refer to [3]. Article [C] contains these estimates as a special case.

#### 4. SUMMARY OF ARTICLES

The class of Young functions we study in this thesis is the following.

**$\Delta_2^+$ -condition 4.1.** Suppose that  $p \in (1, \infty)$ . Let  $\varphi : [0, \infty) \rightarrow [1, \infty)$  be a differentiable and increasing function such that  $t \mapsto t^p \varphi(t)$  is a Young function. Further, suppose that

$$(4.1) \quad \varphi(t^2) \sim \varphi(t) \quad \text{on } (0, \infty),$$

and there is a positive number  $M_1$  such that for all  $t \in (0, \infty)$

$$(4.2) \quad \frac{t\varphi'(t)}{\varphi(t)} \leq M_1 < p,$$

and there is a positive number  $M_2$  such that for all  $t \in (0, \infty)$

$$(4.3) \quad \varphi'(t) \leq M_2,$$

and

$$(4.4) \quad \lim_{t \rightarrow \infty} \frac{t\varphi'(t)}{\varphi(t)} = 0.$$

Then we say that the function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(t) = t^p \varphi(t)$  satisfies the  $\Delta_2^+$ -condition.

An example of a function, which satisfies the  $\Delta_2^+$ -condition is the function  $\Phi$ ,

$$\Phi(t) = t^p (\log(C + t))^\theta \exp([\log \log(C + t)]^\gamma),$$

when  $p \in (1, \infty)$ ,  $\theta \in [0, p - 1]$ ,  $\gamma \in [0, 1)$ , and  $C \geq e^e$  is a positive constant depending on  $p$ ,  $\theta$  and  $\gamma$ , see [C, Example 6.4].

##### 4.1 Article [A]

Our main theorem [A, Theorem 8.1] gives a relation for capacities  $\mathcal{P}_{\alpha, \Phi}$  and  $\mathcal{B}_{\alpha, \Phi}$  when the Young function  $\Phi$  satisfies the  $\Delta_2^+$ -condition.



**Theorem 4.2.** *Let  $E$  be a subset of  $B_R$ . Suppose that  $\Phi$  satisfies the  $\Delta_2^+$ -condition. Let  $\alpha$  be a positive real number such that  $\alpha p = n$ . Then there is a positive constant  $C$ , depending on  $n$ ,  $p$ ,  $R$  and  $\varphi$  only, such that*

$$(4.5) \quad \mathcal{B}_{\alpha, \Phi}(E) \leq C \mathcal{P}_{\alpha, \Phi}(E)^p \varphi(\mathcal{P}_{\alpha, \Phi}(E)).$$

Further, if  $\mathcal{P}_{\alpha, \Phi}(E) > 0$ , then

$$(4.6) \quad C^{-1} \mathcal{P}_{\alpha, \Phi}(E)^p \left[ \varphi \left( \frac{1}{\mathcal{P}_{\alpha, \Phi}(E)} \right) \right]^{-1} \leq \mathcal{B}_{\alpha, \Phi}(E).$$

We set  $\mathcal{P}_{\alpha, \Phi}(E)^{-1} = 0$  in (4.6), if  $\mathcal{P}_{\alpha, \Phi}(E) = \infty$ .

The proof of (4.5) applies a method from [10]. We show that there is a quasi-norm which is equivalent to the Luxemburg norm, when  $\Phi$  satisfies the  $\Delta_2^+$ -condition, see [A, Theorem 7.6]. In order to obtain this quasi-norm, we prove a Hardy-type inequality [A, Lemma 7.2]. The other inequality in Theorem 4.2 is a straight-forward calculation based on the properties of  $\Phi$  and the fact that

$$\int_{B_R} \Phi \left( \frac{f(x)}{\|f\|_{L^\Phi(B_R)}} \right) dx = 1.$$

Theorem 4.2 implies that if the capacity of a set is zero for one of the capacities  $\mathcal{P}_{\alpha, \Phi}$  and  $\mathcal{B}_{\alpha, \Phi}$ , then the other capacity of this set is zero as well.

**Corollary 4.3.** *Suppose that  $\Phi$  satisfies the  $\Delta_2^+$ -condition, and  $\alpha > 0$ . Then capacities  $\mathcal{B}_{\alpha, \Phi}$  and  $\mathcal{P}_{\alpha, \Phi}$  have the same null sets.*

## 4.2 Article [B]

Article [A] raised a question whether it is possible to obtain a result corresponding to Corollary 4.3 for capacities  $\mathcal{B}_{\alpha, \Psi}$  and  $\mathcal{P}_{\alpha, \Psi}$ , when  $\Psi$  is the complementary Young function to  $\Phi$  which satisfies the  $\Delta_2^+$ -condition. It turned out that it is possible since [B, Corollary 4.4] and [B, Example 4.5] yield the following.

**Theorem 4.4.** *Assume that a Young function  $\Phi$  satisfies the  $\Delta_2^+$ -condition. Let  $\Psi$  be the complementary Young function to  $\Phi$ . Suppose that  $\alpha$  is a positive real number. Then capacities  $\mathcal{B}_{\alpha, \Psi}$  and  $\mathcal{P}_{\alpha, \Psi}$  have the same null sets.*

It is shown in [C, Lemma 5.2] that the function  $\Psi$  is equivalent to the Young function

$$t \mapsto t^{\frac{p}{p-1}} \varphi(t)^{-\frac{1}{p-1}}.$$

Partly this article was motivated by the fact that we were simultaneously writing article [C] where we needed an upper estimate for

$$(4.7) \quad \left\| G_\alpha * \chi_{B^n(x_0, r)} \right\|_{L^\Psi(B_R)},$$

whenever  $r \in (0, \frac{R}{2})$  and  $B^n(x_0, r) \subset B_R$ . Here  $\chi_{B^n(x_0, r)}$  is the characteristic function of  $B^n(x_0, r)$ . It turned out that the next theorem, see [B, Theorem 3.7], was useful in obtaining (4.7).

**Theorem 4.5.** *Suppose that a Young function  $\Phi$  satisfies the  $\Delta_2^+$ -condition. Let  $\Omega$  be a measurable subset of  $\mathbb{R}^n$  with  $m(\Omega) = 1$ . If a function  $f$  is measurable and almost everywhere finite on  $\Omega$ , then  $\|f\|_{L^\Phi(\Omega)}$  is equivalent to*

$$\left( \int_0^1 f^*(t)^p \varphi \left( \frac{1}{t} \right)^{\theta p} dt \right)^{\frac{1}{p}}.$$

Here  $f^*$  is the decreasing rearrangement of  $f$ , we refer to [B, Section 2]. One essential part of proving Theorem 4.5 was a Hardy-type inequality, see [B, Proposition 3.5], which we proved by using a method from [9].

### 4.3 Article [C]

It is usually difficult or impossible to calculate the capacity of a given set. In applications it is often useful to have some estimates for the capacity of a ball. In [C] we obtain a lower bound and an upper bound for  $\mathcal{P}_{\alpha, \Phi}$ -capacity of a ball, when  $\Phi$  satisfies the  $\Delta_2^+$ -condition.

**Theorem 4.6.** *Assume that  $B^n(0, R)$  is a fixed ball. Suppose that a Young function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  defined by  $\Phi(t) = t^p \varphi(t)$  satisfies the  $\Delta_2^+$ -condition. Let  $\alpha = n/p$ . Suppose that  $r \in (0, \frac{R}{2})$ , and*

$$F(r) = \int_r^R s^{-1} \varphi \left( \frac{1}{s} \right)^{-\frac{1}{p-1}} ds.$$

*Then there is a positive constant  $C$ , depending on  $n, p, R$  and  $\varphi$  only, such that*

$$C^{-1} F(r)^{\frac{1-p}{p}} \leq \mathcal{P}_{\alpha, \Phi}(B^n(0, r)) \leq C F(r)^{\frac{1-p}{p}},$$

For the upper bound we construct a non-negative function  $f$  such that  $G_\alpha * f \geq 1$  on  $B^n(0, r)$  and  $\text{spt } f \subset B^n(0, R)$  and

$$\|f\|_{L^\Phi(B^n(0, R))} \leq C F(r)^{\frac{1-p}{p}}.$$

Then the upper bound for  $\mathcal{P}_{\alpha, \Phi}(B^n(0, r))$  follows from the definition of  $\mathcal{P}_{\alpha, \Phi}$ .

In order to obtain the lower bound for  $\mathcal{P}_{\alpha,\Phi}(B^n(0,r))$  it is convenient to use [5, Théorème 4, p. 112] which states that for all Borel sets  $E$  in  $B_R$

$$(4.8) \quad \mathcal{P}_{\alpha,\Phi}(E) = \sup\{\mu(E) \mid \mu \in \mathcal{M}_E, \|G_\alpha * \mu\|_{L^\Psi(B_R)} \leq 1\}.$$

Here  $\mathcal{M}_E$  is the set of those Radon measures  $\mu : \mathbb{R}^n \rightarrow [0, \infty]$  that are supported in  $E$ , and

$$G_\alpha * \mu(x) = \int_{\mathbb{R}^n} G_\alpha(x-y) d\mu(y).$$

Combining Theorem 4.6 with estimates for the capacity  $B_{\alpha,\Phi}$  of a ball by Y. Mizuta [14, Lemma 7.3, p. 104] yields an interesting relationship of capacities  $\mathcal{P}_{\alpha,\Phi}$  and  $B_{\alpha,\Phi}$ , when  $\Phi$  satisfies the  $\Delta_2^+$ -condition. There is a constant  $C$ , depending on  $n, p, R$  and  $\varphi$  only, such that

$$(4.9) \quad C^{-1}\mathcal{B}_{\alpha,\Phi}(B^n(0,r)) \leq (\mathcal{P}_{\alpha,\Phi}(B^n(0,r)))^p \leq C\mathcal{B}_{\alpha,\Phi}(B^n(0,r)),$$

when  $\alpha = n/p$ , and  $r \in (0, R/2)$ , see [C, Theorem 7.2].

#### REFERENCES

- [A] JOENSUU, J.: On null sets of Sobolev-Orlicz capacities, to appear in Illinois J. Math.
- [B] JOENSUU, J.: Orlicz-Sobolev capacities and their null sets, to appear in Rev. Mat. Complut., 23, (2010). DOI: 10.1007/s13163-009-0011-1
- [C] JOENSUU, J.: Estimates for certain Orlicz-Sobolev capacities of an Euclidean ball, manuscript.
- [1] ADAMS, D.R., HEDBERG, L.I.: Function Spaces and Potential Theory. Springer-Verlag, Berlin, 1996.
- [2] ADAMS, D.R., HURRI-SYRJÄNEN, R.: Capacity estimates, Proc. Amer. Math. Soc., 131, (2003), 1159-1167.
- [3] ADAMS, D.R., HURRI-SYRJÄNEN, R.: Vanishing exponential integrability for functions whose gradients belong to  $L^n(\log(e+L))^\alpha$ , J. Funct. Anal., 197, (2003), 162-178.
- [4] AÏSSAOUI, N., BENKIRANE, A.: Potentiel non linéaire dans les espaces d'Orlicz, Ann. Sci. Math. Québec, 18, (1994), 105-118.
- [5] AÏSSAOUI, N.: Note sur la capacitabilité dans les espaces d'Orlicz, Ann. Sci. Math. Québec, 19, (1995), 107-113.
- [6] AÏSSAOUI, N.: Bessel potentials in Orlicz spaces, Rev. Mat. Univ. Complut. Madrid, 10, (1997), 55-79.
- [7] AÏSSAOUI, N., BENKIRANE, A.: Capacités dans les espaces d'Orlicz, Ann. Sci. Math. Québec, 18, (1994), 1-23.
- [8] BENKIRANE, A., GOSSEZ, J.-P.: An approximation theorem in higher order Orlicz-Sobolev spaces and applications, Studia Math., 92, (1989), 231-255.
- [9] BENNETT, C., RUDNICK, K.: On Lorentz-Zygmund spaces, Dissertationes Math., 175, (1980), 1-67.
- [10] BENNETT, C., SHARPLEY, R.: Interpolation of Operators, Academic Press, Inc., New York, 1988.

- [11] CIANCHI, A., STROFFOLINI, B.: An extension of Hedberg's convolution inequality and applications, *J. Math. Anal. Appl.*, 227, (1998), 166-186.
- [12] JOENSUU, J.: Capacity estimates, Licentiate thesis, Department of Mathematics and Statistics, University of Helsinki, 2005.
- [13] KUFNER, A., JOHN, O., FUČÍK, S.: *Function Spaces*, Noordhoff International Publishing, Leyden, 1977.
- [14] MIZUTA, Y.: Continuity properties of potentials and Beppo-Levi-Deny functions, *Hiroshima Math. J.*, 23, (1993), 79-153.
- [15] MIZUTA, Y., SHIMOMURA, T.: Vanishing exponential integrability for Riesz potentials of functions in Orlicz classes, *Illinois J. Math.*, 51, (2007), 1039-1060.
- [16] RAO, M.M. and REN, Z.D.: *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, 1991.
- [17] ZIEMER, W.P.: *Weakly Differentiable functions*, Springer-Verlag, New York, 1989.

## Errata

On p.6 the correct version of Theorem 4.5 is the following:

It turned out that the next theorem, which follows from [B, Theorem 3.7], was useful in obtaining (4.7).

**Theorem 4.5** *Suppose that a Young function  $\Phi$  satisfies the  $\Delta_2^+$ -condition. Let  $\Psi$  be the complementary Young function to  $\Phi$ . Let  $\Omega$  be a measurable subset of  $\mathbb{R}^n$  with  $m(\Omega) = 1$ . If a function  $f$  is measurable and almost everywhere finite on  $\Omega$ , then  $\|f\|_{L^\Psi(\Omega)}$  is equivalent to*

$$\left( \int_0^1 f^*(t)^{\frac{p}{p-1}} \varphi\left(\frac{1}{t}\right)^{-\frac{1}{p-1}} dt \right)^{\frac{p-1}{p}}.$$