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Integral Transformations of Volterra Gaussian Processes

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On thesis

This thesis consists of an introduction and of four research articles, of which I am the single author. The introduction provides an overview about the subject of Volterra Gaussian processes in general, and fractional Brownian motion in particular. The articles are:

[I] C. Jost, Transformation formulas for fractional Brownian motion. Stochastic Processes and their Applications 116, 1341-1357, 2006.

[II] C. Jost, On the connection between Molchan-Golosov and Mandelbrot-Van Ness representations of fractional Brownian motion. To appear in the Journal of Integral Equations and Applications. Available from http://www.arxiv.org/pdf/math.PR/0602356.

[III] C. Jost, Measure-preserving transformations of Volterra Gaussian processes and related bridges. Available from http://www.arxiv.org/pdf/math.PR/0701888. Earlier version as Preprint 448, Department of Mathematics and Statistics, University of Helsinki, Finland.

[IV] C. Jost, A note on ergodic transformations of self-similar Volterra Gaussian processes. Available from http://www.arxiv.org/pdf/math.PR/0702096. Earlier version as Preprint 452, Department of Mathematics and Statistics, University of Helsinki, Finland.

Contents

1 Prologue

Volterra Gaussian processes are a generalization of the simple and well-studied standard Brownian motion. More precisely, a Volterra Gaussian process is a Wiener integral process with respect to a one-sided standard Brownian motion. Thus, at every point in time, it is an infinite linear combination of i.i.d. Gaussian random variables with time-dependent coefficients. The paradigm is fractional Brownian motion (fBm).

Fractional Brownian motion is a self-similar process, meaning that its probability distribution is invariant under a suitable simultaneous scaling in time and space. Moreover, fBm has stationary increments, i.e. the probability distribution of its increments in space over given time intervals is invariant under a shift in time. Both properties combined are captured by the *Hurst index* H, which is a parameter between 0 and 1. Also, the Hurst index characterizes dependence structure of the increments and memory of the process: The increments over two non-overlapping time intervals are positively correlated for $H > \frac{1}{2}$ and negatively correlated for $H < \frac{1}{2}$. Moreover, for $H > \frac{1}{2}$, the decay of this dependence as the time intervals grow apart is slow, and referred to as long-range dependence (long memory). For $H < \frac{1}{2}$, this decay is fast, and termed accordingly by short-range dependence (short memory). For $H = \frac{1}{2}$, fBm corresponds to standard Brownian motion, where increments are independent and the process has no memory. Furthermore, the Hurst index is a measure for the roughness of the paths of fBm. More precisely, the larger the Hurst index, the smoother the paths.

The phenomena of long-range dependence and self-similarity have been observed empirically in a wide range of fields, such as in hydrodynamics, meteorology, economics and telecommunications, but also in geophysics: In fact, the index H is named after the British hydrologist H. E. Hurst, who in 1951 found evidence for the presence of long-range dependence in time series describing the level changes in reservoirs along the Nile river. These facts make fBm a fundamental modelling tool in Applied Probability.

The fractional Brownian motion was introduced by Kolmogorov in 1940 under its former name *Wiener spiral* for modelling turbulence in liquids. Kolmogorov also obtained its spectral representation. In 1962, Lamperti observed that self-similar processes are the natural limits of functional central limit theorems. In 1968, Mandelbrot and Van Ness represented fBm as a Wiener integral process with respect to a two-sided standard Brownian motion, where the integrand kernel function is a simple fractional integral. Based on this representation, the authors proposed its modern name. In 1969, Molchan and Golosov constructed fBm as a Wiener integral process with respect to a one-sided standard Brownian motion, where the integrand kernel function is a more complicated fractional integral. More information about the history of fractional Brownian motion can be found in [29].

The integral representation by Molchan and Golosov is nowadays the basis of many theoretical and practical considerations involving fBm. This is mainly due to the fact that the natural filtrations of the standard Brownian motion and of the fBm that it generates through this representation do coincide. The Volterra Gaussian process is a generalization of this representation by allowing arbitrary integrand kernel functions. FBm is the unique Gaussian process which provides a model for both self-similarity and long-range dependence. However, the general Volterra Gaussian process allows a bigger flexibility for modelling self-similarity. Clearly, Volterra Gaussian processes are not only interesting from a practical point of view: A crucial fact is that most Volterra Gaussian processes, and fBm with $H \neq \frac{1}{2}$ in particular, are not semimartingales, i.e. the usual Itô calculus is not available in order to implement many ideas. This fact makes the handling of those processes challenging and interesting, even for purely theoretical considerations.

2 The fractional Brownian motion (fBm)

Definition 2.1. The *fractional Brownian motion with Hurst index* $H \in (0,1)$, or *H-fBm*, denoted by $\left(B_t^H\right)_{t\in\mathbb{R}}$, is the centered Gaussian process with covariance function

$$R^H(s,t) \;=\; \frac{1}{2}\left(|s|^{2H} \;+\; |t|^{2H} \;-\; |t-s|^{2H}\right),\; s,t \in \mathbb{R}. \tag{2.1}$$

Clearly, it holds that

$$B^{\frac{1}{2}} \;\overset{d}{=}\; W,$$

where $\overset{d}{=}$ denotes equality of finite-dimensional distributions and W is a standard Brownian motion.

Remark 2.2. The function R^H is positive semi-definite, and thus determines a covariance function, if and only if $H \in (0,1]$ (see [44], Proposition 2.2). However, since $R^1(s,t) = st$, $s,t \in \mathbb{R}$, or equivalently, $B_t^1 = tX$, a.s., $t \in \mathbb{R}$, for some $X \sim \mathcal{N}(0,1)$, we exclude this particular and trivial case from our considerations.

2.1 Self-similarity and long-range dependence

Recall that a process $(X_t)_{t\in\mathbb{R}}$ is *β-self-similar*, where $\beta > 0$, if

$$(X_{at})_{t\in\mathbb{R}} \;\overset{d}{=}\; \left(a^\beta X_t\right)_{t\in\mathbb{R}},\; a > 0.$$

Moreover, a process $(X_t)_{t\in\mathbb{R}}$ has *stationary increments*, if

$$(X_{t+s} - X_s)_{t\in\mathbb{R}} \;\overset{d}{=}\; (X_t - X_0)_{t\in\mathbb{R}}\,,\; s \in \mathbb{R}.$$

By using (2.1), it is straightforward to check the following:

Lemma 2.3. *H-fBm is H-self-similar and has stationary increments.*

From Lemma 2.3, it follows that the sequence $\left\{B_n^H - B_{n-1}^H\right\}_{n\in\mathbb{N}}$, which is called *fractional Gaussian noise* (and *white noise* for $H = \frac{1}{2}$ in particular), is stationary. Let

$$r^H(n) \;:=\; \mathrm{Cov}_{\mathbb{P}}\left(B_{n+1}^H - B_n^H, B_1^H - B_0^H\right),\; n \in \mathbb{N},$$

denote the autocovariance function of the fractional Gaussian noise. By using the mean value theorem twice, we obtain that

$$r^H(n) \;=\; H(2H-1)(n-\theta)^{2H-2},\; n \in \mathbb{N}, \tag{2.2}$$

where $\theta = \theta(H,n) \in (-1,1)$. From this, it is straightforward to see the following:

Lemma 2.4. *For $H > \frac{1}{2}$, it holds that $r^H(n) > 0$, $n \in \mathbb{N}$, and*

$$\sum_{n\in\mathbb{N}} \left|r^H(n)\right| \;=\; \infty. \tag{2.3}$$

For $H < \frac{1}{2}$, we have that $r^H(n) < 0$, $n \in \mathbb{N}$, and

$$\sum_{n\in\mathbb{N}} \left|r^H(n)\right| \;<\; \infty. \tag{2.4}$$

(2.3) and (2.4) are called *long-range* and *short-range dependence properties* of the fractional Gaussian noise, respectively. Hence, the fractional Gaussian noise has the long-range dependence property if and only if $H > \frac{1}{2}$.

2.2 Other important properties

1. B^H has a modification $\tilde{B}^H$, which is *locally Hölder-continuous of order* γ for every $\gamma < H$. This means that for every compact set $K \subset \mathbb{R}$ and every $\gamma \in (0, H)$, there exists a finite random variable $C = C(K, \gamma)$, such that

$$\sup_{\substack{s,t \in K \\ s \neq t}} \frac{\left|\tilde{B}^H_s - \tilde{B}^H_t\right|}{|s-t|^\gamma} \;\leq\; C, \; a.s. \tag{2.5}$$

In particular, $\tilde{B}^H$ is a continuous process. Indeed, by combining stationarity of increments and H-self-similarity, we obtain that

$$\mathrm{E}_{\mathbb{P}}\left(B^H_s - B^H_t\right)^{2m} \;\leq\; \mathrm{E}_{\mathbb{P}}\left(B^H_1\right)^{2m} |t-s|^{2Hm}, \;\; s,t \in \mathbb{R}, \;\; m \in \mathbb{N}.$$

Since $\lim_{m\to\infty} \frac{2Hm-1}{2m} = H$, the claim follows from the Kolmogorov-Chentsov criterion (see [21], Theorem 2.8). We always assume that $B^H = \tilde{B}^H$.

2. For $T > 0$, let $\pi_n(T) := \left\{\frac{kT}{n} \,|\, k = 0, \ldots, n\right\}$, $n \in \mathbb{N}$, be a sequence of equidistant partitions of $[0, T]$. Then

$$L^2(\mathbb{P})\text{-}\lim_{n\to\infty} \mathrm{v}_p\left(B^H, \pi_n(T)\right) \;=\; \begin{cases} +\infty & \text{if } H < \frac{1}{p} \\ T \cdot \mathrm{E}_{\mathbb{P}}(|B^H_1|)^{\frac{1}{H}} & \text{if } H = \frac{1}{p} \\ 0 & \text{if } H > \frac{1}{p}, \end{cases} \tag{2.6}$$

where for $p > 0$,

$$\mathrm{v}_p\left(B^H, \pi(T)\right) \;:=\; \sum_{t_k \in \pi(T)} \left|B^H_{t_k} - B^H_{t_{k-1}}\right|^p$$

is the p-variation of B^H with respect to $\pi(T) := \{0 = t_0 < t_1 < \ldots < t_n = T\}$. Indeed, by using the H-self-similarity of B^H, we obtain that

$$\mathrm{v}_p\left(B^H, \pi_n(T)\right) \;\stackrel{d}{=}\; \sum_{k=1}^{n} \left(\frac{1}{n}\right)^{pH-1} T^{pH} \frac{\left|B^H_k - B^H_{k-1}\right|^p}{n}.$$

From (2.2), we have that $\lim_{n\to\infty} r^H(n) = 0$. Hence, the fractional Gaussian noise is ergodic (see [6], Theorem 2, p. 369). In particular, the sequence $\left\{\left|B^H_k - B^H_{k-1}\right|^p\right\}_{k\in\mathbb{N}}$ is ergodic. The claim follows from the ergodic theorem.

3. If $H \neq \frac{1}{2}$, then B^H is not a semimartingale. Indeed, let $H < \frac{1}{2}$ and assume that B^H is a semimartingale. Then, for $T > 0$, it holds that

$$\mathrm{v}^0_2\left(B^H, T\right) < \infty, \; a.s., \tag{2.7}$$

where for $p > 0$,

$$\mathrm{v}^0_p\left(B^H, T\right) \;:=\; \lim_{\substack{|\pi(T)| \to 0 \\ \pi(T) \in \mathcal{P}_T}} \sum_{t_k \in \pi(T)} \left|B^H_{t_k} - B^H_{t_{k-1}}\right|^p, \; a.s.,$$

and $\mathcal{P}_T$ denotes the set of all partitions of $[0, T]$. Hence, v^0_p is the p-variation of B^H over $[0, T]$. However, (2.7) is a contradiction to (2.6), so B^H is not a semimartingale. Furthermore, if

$H > \frac{1}{2}$, then by using (2.5) with $\gamma = \frac{H}{2}$, we obtain on the one hand that

$$\begin{aligned} \mathrm{v}_2^0\left(B^H, T\right) &= \lim_{\substack{|\pi(T)| \to 0 \\ \pi(T) \in \mathcal{P}_T}} \sum_{t_k \in \pi(T)} \left| B_{t_k}^H - B_{t_{k-1}}^H \right|^2 \\ &\leq C^2 \lim_{\substack{|\pi(T)| \to 0 \\ \pi(T) \in \mathcal{P}_T}} \sum_{t_k \in \pi(T)} |t_k - t_{k-1}|^H \\ &\leq C^2 \lim_{\substack{|\pi(T)| \to 0 \\ \pi(T) \in \mathcal{P}_T}} |\pi(T)|^{\frac{1}{2}} \sum_{t_k \in \pi(T)} |t_k - t_{k-1}|^{H-\frac{1}{2}} \\ &= 0, \ a.s. \end{aligned} \tag{2.8}$$

On the other hand, it follows from (2.6) that

$$\sup_{\pi(T) \in \mathcal{P}_T} \mathrm{v}_1\left(B^H, \pi(T)\right) = \infty, \ a.s. \tag{2.9}$$

From (2.8) and (2.9), it follows that B^H is not a semimartingale.

4. If $H \neq \frac{1}{2}$, then B^H is not a Markov process. Indeed, assume that B^H is a Markov process. Since B^H is Gaussian, it follows that $R^H(s,t) = f(t)g(s)$, $s, t \in \mathbb{R}$, for some functions f, g (see [18], p. 88). This contradicts (2.1).

3 Integral representations of fBm

In this section, we show that fBm can be represented in terms of the simpler standard Brownian motion or in terms of a complex Gaussian measure. First, we review some special functions involved in these representation results.

The *gamma function* is defined by

$$\Gamma(\alpha) := \int_0^\infty \exp(-v) v^{\alpha-1} dv, \ \alpha > 0.$$

By partial integration, we obtain the recursion formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ which is used to extend Γ to all $\alpha \in \mathcal{A} := \mathbb{R} \setminus -\mathbb{N}_0$. Since $\Gamma(1) = 1$, it holds that

$$\Gamma(n) = (n-1)!, \ n \in \mathbb{N}. \tag{3.1}$$

We define

$$\frac{1}{\Gamma(\beta)} := \lim_{\substack{\alpha \to \beta \\ \alpha \in \mathcal{A}}} \frac{1}{\Gamma(\alpha)} = 0, \ \beta \in -\mathbb{N}_0. \tag{3.2}$$

The *beta function* is defined by

$$B(\alpha, \beta) := \int_0^1 (1-v)^{\alpha-1} v^{\beta-1} dv, \ \alpha, \beta > 0.$$

For $\beta > 1$, we obtain that $B(\alpha, \beta - 1) = \frac{\alpha+\beta-1}{\beta-1} B(\alpha, \beta)$. This recursion formula and the symmetry relation $B(\alpha, \beta) = B(\beta, \alpha)$ are used to extend B to all $\alpha, \beta \in \mathcal{A}$. It holds that (see [12], p. 9)

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \ \alpha, \beta \in \mathcal{A}. \tag{3.3}$$

3.1 The Molchan-Golosov representation

The one-sided fractional Brownian motion can be constructed from a one-sided standard Brownian motion:

Theorem 3.1 (Molchan & Golosov, 1969)**.** *For $H \in (0,1)$, it holds that*

$$\left(B_t^H\right)_{t\in[0,\infty)} \stackrel{d}{=} \left(\int_0^t z_H(t,s) dW_s\right)_{t\in[0,\infty)}. \tag{3.4}$$

Here, for $H > \frac{1}{2}$,

$$z_H(t,s) := \frac{C(H)}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du, \ 0 < s < t < \infty.$$

For $H \leq \frac{1}{2}$,

$$\begin{aligned} z_H(t,s) := \ & \frac{C(H)}{\Gamma\left(H + \frac{1}{2}\right)} \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\ & - \frac{C(H)}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du, \ 0 < s < t < \infty. \end{aligned}$$

Moreover,

$$C(H) := \left(\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

Proof. Note that from (3.2), it follows that $z_{\frac{1}{2}}(t,s) = 1$, $0 < s < t < \infty$, so the claim holds for $H = \frac{1}{2}$. Denote the process on the right-hand side of (3.4) by $\left(Z_t^H\right)_{t\in[0,\infty)}$. Clearly, Z^H is centered and Gaussian. Let $H > \frac{1}{2}$. Then, by substituting $v := \frac{1}{y-u}$ and $v := \frac{1}{x-u}$, and using (3.3), we obtain for $0 < s \le t < \infty$ that

$$
\begin{aligned}
&\left(\frac{\Gamma\left(H-\frac{1}{2}\right)}{C(H)}\right)^2 \int_0^s z_H(t,u) z_H(s,u)du \\
&= \int_0^s \int_u^s \int_u^t x^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}dx\, y^{H-\frac{1}{2}}(y-u)^{H-\frac{3}{2}}u^{1-2H}dydu \\
&= \int_0^s \int_0^y \int_u^t x^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}dx\, y^{H-\frac{1}{2}}(y-u)^{H-\frac{3}{2}}u^{1-2H}dudy \\
&= \int_0^s \int_0^y \int_u^y x^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}dx\, y^{H-\frac{1}{2}}(y-u)^{H-\frac{3}{2}}u^{1-2H}dudy \\
&\quad + \int_0^s \int_0^y \int_y^t x^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}dx\, y^{H-\frac{1}{2}}(y-u)^{H-\frac{3}{2}}u^{1-2H}dudy \\
&= \int_0^s \int_0^y \int_0^x (y-u)^{-2}\left(\frac{x-u}{y-u}\right)^{H-\frac{3}{2}}\left(\frac{u}{y-u}\right)^{1-2H} du\, x^{H-\frac{1}{2}}dx\, y^{H-\frac{1}{2}}dy \\
&\quad + \int_0^s \int_y^t \int_0^y \left(\frac{y-u}{x-u}\right)^{H-\frac{3}{2}}(x-u)^{-2}\left(\frac{u}{x-u}\right)^{1-2H} du\, x^{H-\frac{1}{2}}dx\, y^{H-\frac{1}{2}}dy \\
&= B\left(2-2H, H-\frac{1}{2}\right)\int_0^s \int_0^y x^{\frac{1}{2}-H}y^{\frac{1}{2}-H}(y-x)^{2H-2}x^{H-\frac{1}{2}}dx\, y^{H-\frac{1}{2}}dy \\
&\quad + B\left(2-2H, H-\frac{1}{2}\right)\int_0^s \int_y^t y^{\frac{1}{2}-H}x^{\frac{1}{2}-H}(x-y)^{2H-2}x^{H-\frac{1}{2}}dx\, y^{H-\frac{1}{2}}dy \\
&= \frac{\Gamma(2-2H)\Gamma\left(H-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}\left(\int_0^s \int_0^y (y-x)^{2H-2}dxdy + \int_0^s \int_y^t (x-y)^{2H-2}dxdy\right) \\
&= \left(\frac{\Gamma\left(H-\frac{1}{2}\right)}{C(H)}\right)^2 \frac{1}{2}\left(s^{2H}+t^{2H}-(t-s)^{2H}\right).
\end{aligned}
$$

Hence, $\mathrm{Cov}_{\mathbb{P}}\left(Z_t^H, Z_s^H\right) = R^H(s,t)$, $0 \le s \le t < \infty$. For $H < \frac{1}{2}$, this follows from a similar calculation. □

Since the integral on the right-hand side of (3.4) depends on the function $z_H(t,\cdot)$ only via its values on $(0,t)$, we can assume for later convenience that

$$z_H(t,s) \ = \ 0, \ 0 < t \le s < \infty. \tag{3.5}$$

Then the kernel z_H is called *Volterra* on $[0,\infty)^2$.

The kernel z_H can be written compactly in terms of the Gauss hypergeometric function. The *Gauss hypergeometric function* of parameters a,b,c and variable $z \in \mathbb{R}$ is defined by the formal power series

$${}_2F_1(a,b,c,z) \ := \ \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

Here, $(a)_0 := 1$ and $(a)_k := a\cdot(a+1)\cdot\dots\cdot(a+k-1)$, $k\in\mathbb{N}$, is the *Pochhammer symbol.* We assume that $c \in \mathcal{A}$ for this to make sense. If $|z| < 1$ or $|z| = 1$ and $c-b-a > 0$, then the series converges absolutely. If furthermore $c > b > 0$ for $z \in [-1,1)$ and $b > 0$ for $z = 1$, then it can be represented by the Euler integral

$${}_2F_1(a,b,c,z) \ = \ \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 v^{b-1}(1-v)^{c-b-1}(1-vz)^{-a}dv \tag{3.6}$$

(see [13], p. 59). If $c > b > 0$, then the expression on the right-hand side of (3.6) is well-defined for all $z \in (-\infty, 1)$, and is therefore used as an extended definition of ${}_2F_1$. For fixed $z \in (-\infty, 1]$, ${}_2F_1$ can be extended to more general parameters by using Gauss' relations for neighbor functions. *Neighbors* of ${}_2F_1(a, b, c, z)$ are functions of type ${}_2F_1(a \pm 1, b, c, z)$, ${}_2F_1(a, b \pm 1, c, z)$ or ${}_2F_1(a, b, c \pm 1, z)$. For any two neighbors $F_1(z)$, $F_2(z)$ of ${}_2F_1(a, b, c, z)$, one has a linear relation of type

$$A(z){}_2F_1(a, b, c, z) \;+\; A_1(z)F_1(z) \;+\; A_2(z)F_2(z) \;=\; 0,$$

where A, A_1 and A_2 are first-degree polynomials. See [1], p. 558 for all 15 neighbor relations. Based on these relations, we extend ${}_2F_1$ for $z \in (-\infty, 1)$ to all $a, b, c \in \mathbb{R}$ such that $c \in \mathcal{A}$, and for $z = 1$ to all parameters that satisfy $c, c - b - a \in \mathcal{A}$. Important properties of ${}_2F_1$ are the symmetry relation

$${}_2F_1(a, b, c, z) \;=\; {}_2F_1(b, a, c, z)$$

and the reduction formula

$${}_2F_1(0, b, c, z) \;=\; 1. \tag{3.7}$$

Moreover, it holds that (see [1], p. 559)

$${}_2F_1(a, b, c, z) \;=\; (1 - z)^{-a}{}_2F_1\left(a, c - b, c, \frac{z}{z - 1}\right),\; z < 1. \tag{3.8}$$

By combining (3.8) and (3.7), we obtain that

$${}_2F_1(a, b, b, z) \;=\; (1 - z)^{-a},\; z < 1. \tag{3.9}$$

Furthermore, by combining the neighbor relations (15.2.17) and (15.2.25) in [1], we obtain that

$$c \cdot {}_2F_1(a, b, c, z) \;-\; c \cdot {}_2F_1(a, b + 1, c, z) \;+\; az \cdot {}_2F_1(a + 1, b + 1, c + 1, z) \;=\; 0. \tag{3.10}$$

Based on these facts, we can show the following:

Lemma 3.2 (Decreusefond & Üstünel, 1999). *For $H \in (0, 1)$ and $0 < s < t < \infty$, it holds that*

$$z_H(t, s) \;=\; \frac{C(H)}{\Gamma\left(H + \frac{1}{2}\right)}(t - s)^{H - \frac{1}{2}} \cdot {}_2F_1\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, \frac{s - t}{s}\right).$$

Proof. For $H > \frac{1}{2}$, this follows from (3.6) by substitution. For $H = \frac{1}{2}$, this is clear by using (3.7). For $H < \frac{1}{2}$, it follows by using (3.10) and then combining (3.9) and (3.6). □

Remark 3.3. 1. From Lemma 3.2 and (3.8), we obtain for $0 < s < t < \infty$ that

$$z_H(t, s) \;=\; \frac{C(H)}{\Gamma\left(H + \frac{1}{2}\right)}(t - s)^{H - \frac{1}{2}}\left(\frac{s}{t}\right)^{\frac{1}{2} - H} \cdot {}_2F_1\left(\frac{1}{2} - H, 1, H + \frac{1}{2}, \frac{t - s}{t}\right). \tag{3.11}$$

Since the Gauss hypergeometric function is continuous in z over $(-\infty, 1]$, hence bounded in z over $[0, 1]$, we see from (3.11) that $z_H(t, \cdot)$ behaves like $\cdot^{\frac{1}{2} - H}$ close to 0 and like $(t - \cdot)^{H - \frac{1}{2}}$ close to t. In particular, from (3.11) we easily see that $z_H(t, \cdot)$ is indeed square-integrable for every $t \in (0, \infty)$.
2. After its discoverers, (3.4) is called the *Molchan-Golosov representation* or *integral transform of fBm*, respectively. Alternatively, it is also called the *time domain representation of fBm over* $[0, \infty)$. It has the useful feature that the natural filtrations of $(W_t)_{t \in [0, \infty)}$ and of the fractional Brownian motion that it generates do coincide. Many results for fBm, such as a Girsanov formula (see [31]), a Donsker theorem (see [43], item [a]), a prediction formula (see [37]), a Lévy characterization (see [26]), and results on deterministic and stochastic integration with respect to fBm (see [35], and [2] and [7], respectively) are based on the Molchan-Golosov representation.

3.2 The Mandelbrot-Van Ness representation

The two-sided fractional Brownian motion can be written in terms of a two-sided standard Brownian motion:

Theorem 3.4 (Mandelbrot & Van Ness, 1968)**.** *For $H \in (0,1)$, it holds that*

$$\left(B_t^H\right)_{t\in\mathbb{R}} \overset{d}{=} \left(\int_{\mathbb{R}} z'_H(t,s)dW_s\right)_{t\in\mathbb{R}}, \tag{3.12}$$

where

$$z'_H(t,s) := \frac{1}{C'(H)}\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right), \ s,t\in\mathbb{R},$$
$$x_+^a := x^a\cdot 1_{(0,\infty)}(x), \ x,a\in\mathbb{R},$$

and

$$C'(H) := \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}}.$$

Proof. We denote the process on the right-hand side of (3.12) by $\left(Z_t^H\right)_{t\in\mathbb{R}}$. Clearly, Z^H is centered and Gaussian. Moreover, since W has stationary increments, we obtain for $s \in \mathbb{R}$ that

$$\begin{aligned}
\left(Z_t^H - Z_s^H\right)_{t\in\mathbb{R}} &= \left(\frac{1}{C'(H)}\int_{\mathbb{R}}\left((t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}}\right)dW_u\right)_{t\in\mathbb{R}}\\
&= \left(\frac{1}{C'(H)}\int_{\mathbb{R}}\left((t-s-v)_+^{H-\frac{1}{2}} - (-v)_+^{H-\frac{1}{2}}\right)dW_{v+s}\right)_{t\in\mathbb{R}}\\
&\overset{d}{=} \left(\frac{1}{C'(H)}\int_{\mathbb{R}}\left((t-s-v)_+^{H-\frac{1}{2}} - (-v)_+^{H-\frac{1}{2}}\right)dW_v\right)_{t\in\mathbb{R}}\\
&= \left(Z_{t-s}^H\right)_{t\in\mathbb{R}}.
\end{aligned}$$

Hence, Z^H has stationary increments. Furthermore, for $t \in \mathbb{R}\setminus\{0\}$, we have by substituting $y := \frac{s}{t}$ for $t > 0$ and $y := 1 - \frac{s}{t}$ for $t < 0$ that

$$\begin{aligned}
\mathrm{E}_{\mathbb{P}}\left(Z_t^H\right)^2 &= \left(\frac{1}{C'(H)}\right)^2\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right)^2 ds\\
&= \left(\frac{1}{C'(H)}\right)^2 |t|^{2H}\int_{\mathbb{R}}\left((1-y)_+^{H-\frac{1}{2}} - (-y)_+^{H-\frac{1}{2}}\right)^2 dy\\
&= \left(\frac{1}{C'(H)}\right)^2 |t|^{2H}\left(\int_{-\infty}^0\left((1-y)^{H-\frac{1}{2}} - (-y)^{H-\frac{1}{2}}\right)^2 dy + \int_0^1 (1-y)^{2H-1}dy\right)\\
&= |t|^{2H}.
\end{aligned}$$

By combining these facts, we obtain that

$$\begin{aligned}
\mathrm{Cov}_{\mathbb{P}}\left(Z_s^H, Z_t^H\right) &= \frac{1}{2}\left(\mathrm{E}_{\mathbb{P}}\left(Z_s^H\right)^2 + \mathrm{E}_{\mathbb{P}}\left(Z_t^H\right)^2 - \mathrm{E}_{\mathbb{P}}\left(Z_s^H - Z_t^H\right)^2\right)\\
&= \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \ s,t\in\mathbb{R}. \quad \square
\end{aligned}$$

Remark 3.5. 1. For every $t \in \mathbb{R}$, it follows from the mean value theorem that $z'_H(t,s)$ behaves like $(-s)^{H-\frac{3}{2}}$ as $s \to -\infty$.
2. (3.12) is called the *Mandelbrot-Van Ness representation of fBm*, the *Mandelbrot-Van Ness integral transform of fBm* or the *time domain representation of fBm over* $\mathbb{R}$. Compared to z_H, the kernel z'_H has a simple structure. However, the natural filtrations of $(W_t)_{t\in\mathbb{R}}$ and of the fractional Brownian motion that it generates via the Mandelbrot-Van Ness representation do not coincide. This feature makes the Mandelbrot-Van Ness representation less interesting than the Molchan-Golosov representation for potential applications.

3.3 The spectral representation

Alternatively, based on the stationarity of its increments, a two-sided fractional Brownian motion can be generated by a suitable complex Gaussian random measure. In the following, let $\mathcal{W}$ be a complex Gaussian random measure, meaning that $\mathcal{W} := \mathcal{W}_1 + i\mathcal{W}_2$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are independent real-valued Gaussian random measures. We assume that $\mathcal{W}_1$ and $\mathcal{W}_2$ are independently scattered on $[0,\infty)$, and that they satisfy $\mathcal{W}_1(-A) = \mathcal{W}_1(A)$, $\mathcal{W}_2(-A) = -\mathcal{W}_2(A)$ and $\mathrm{E}_{\mathbb{P}}(\mathcal{W}_i(A))^2 = \frac{1}{2}\lambda(A)$, $i \in \{1,2\}$, for every Borel set A with $\lambda(A) < \infty$, where λ denotes Lebesgue measure. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f(s) = \overline{f(-s)}$, $s \in \mathbb{R}$, and $|f| \in L^2(\mathbb{R})$. Here, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. The integral of f with respect to $\mathcal{W}$ is defined by

$$\int_{\mathbb{R}} f(s)\mathcal{W}(ds) \;:=\; \int_{\mathbb{R}} \mathrm{Re}(f)(s)\mathcal{W}_1(ds) \;-\; \int_{\mathbb{R}} \mathrm{Im}(f)(s)\mathcal{W}_2(ds).$$

For details on this type of integral, see [40], p. 325. Based on this definition, we have the following:

Theorem 3.6 (Kolmogorov, 1940)**.** *For $H \in (0,1)$, it holds that*

$$\left(B_t^H\right)_{t\in\mathbb{R}} \overset{d}{=} \left(\int_{\mathbb{R}} z_H''(t,s)\mathcal{W}(ds)\right)_{t\in\mathbb{R}}, \tag{3.13}$$

where

$$z_H''(t,s) \;:=\; \frac{1}{C''(H)}\frac{\exp(ist)-1}{is}|s|^{\frac{1}{2}-H}, \;\; s,t \in \mathbb{R}, \;\; s \neq 0,$$

and

$$\begin{aligned} C''(H) \;&:=\; \left(-4\Gamma(-2H)\cos(-H\pi)\right)^{\frac{1}{2}} \\ &=\; \left(\frac{\pi}{H\Gamma(2H)\sin(H\pi)}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.14}$$

Proof. Note first that (3.14) is obtained by combining identities $\frac{\pi}{\cos(\pi\alpha)} = \Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1}{2}-\alpha\right)$, $\Gamma(2\alpha) = \frac{2^{2\alpha-1}}{\sqrt{\pi}}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)$ and $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$ (see [12], p. 3 and p. 5). Let $\left(Z_t^H\right)_{t\in\mathbb{R}}$ be the process on the right-hand side of (3.13). Clearly, Z^H is centered and Gaussian. Furthermore, by using the identities $\cos(x) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})$, $x \in \mathbb{R}$, and

$$\int_0^\infty \sin^2(au)\,u^{-2H-1}du \;=\; \frac{-\Gamma(-2H)\cos(-H\pi)a^{2H}}{2^{-2H+1}}, \;\; a \geq 0,$$

(see [15], p. 447), we obtain that

$$\begin{aligned} \mathrm{Cov}_{\mathbb{P}}\left(Z_t^H, Z_s^H\right) \;&=\; \int_{\mathbb{R}} \overline{z_H''(t,u)}z_H''(s,u)du \\ =\;& \frac{1}{(C''(H))^2}\int_{\mathbb{R}} \overline{\left(\frac{\exp(iut)-1}{iu}\right)}\left(\frac{\exp(ius)-1}{iu}\right)|u|^{1-2H}du \\ =\;& \frac{1}{(C''(H))^2}\int_{\mathbb{R}} \Big(\exp(-iu(t-s)) - \exp(-iut) - \exp(ius) + 1\Big)\frac{|u|^{1-2H}}{u^2}du \\ =\;& \frac{2}{(C''(H))^2}\int_0^\infty \Big(\big(\cos(u|t-s|)-1\big) - \big(\cos(u|t|)-1\big) - \big(\cos(u|s|)-1\big)\Big)u^{-1-2H}du \\ =\;& \frac{-4}{(C''(H))^2}\int_0^\infty \left(\sin^2\left(\frac{u|t-s|}{2}\right) - \sin^2\left(\frac{u|t|}{2}\right) - \sin^2\left(\frac{u|s|}{2}\right)\right)u^{-1-2H}du \\ =\;& \frac{-4}{(C''(H))^2}\frac{-\Gamma(-2H)\cos(-H\pi)}{2^{-2H+1}}\left(\left(\frac{|t-s|}{2}\right)^{2H} - \left(\frac{|t|}{2}\right)^{2H} - \left(\frac{|s|}{2}\right)^{2H}\right) \\ =\;& \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \;\; s,t \in \mathbb{R}. \quad \square \end{aligned}$$

Remark 3.7. The spectral representation was the starting point to obtain a series expansion for fBm (see [9]).

3.4 Connection between Mandelbrot-Van Ness and spectral representations

The Mandelbrot-Van Ness representation and the spectral representation of fBm are inherently related via Parseval's equality (see [38], p. 172), as the following lemma shows:

Lemma 3.8. *Let $H \in (0,1)$. Then it holds that*

$$\widehat{z'_H(t,\cdot)}(s) \;=\; -i\cdot \operatorname{sgn}(s)\exp\left(\operatorname{sgn}(s)\Big(H+\frac{1}{2}\Big)\frac{i\pi}{2}\right)\overline{z''_H(t,s)},\;\; s,t\in\mathbb{R},\;\; s\neq 0, \tag{3.15}$$

where $\widehat{f}(s) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(-isu)f(u)du$, $s\in\mathbb{R}$, is the Fourier transform of f. Furthermore,

$$C'(H) \;=\; C''(H)\frac{\Gamma\big(H+\frac{1}{2}\big)}{\sqrt{2\pi}} \;=\; \frac{\Gamma\big(H+\frac{1}{2}\big)}{\big(\Gamma\left(2H+1\right)\sin(\pi H)\big)^{\frac{1}{2}}}. \tag{3.16}$$

Proof. First, note that (see [12], combine (37) and (38) on p. 13)

$$\int_0^\infty \exp(isu)u^{\alpha-1}du \;=\; \Gamma(\alpha)\exp\Big(\operatorname{sgn}(s)\alpha\frac{i\pi}{2}\Big)|s|^{-\alpha}, \quad \alpha\in(0,1),\;\; s\in\mathbb{R}\setminus\{0\}.$$

Let $H<\frac{1}{2}$. Then for $t\in\mathbb{R}$ and $s\in\mathbb{R}\setminus\{0\}$, we obtain that

$$\begin{aligned}
&\widehat{z'_H(t,\cdot)}(s)\\
=\;& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(-isu)z'_H(t,u)du\\
=\;& \frac{\exp(-ist)-1}{C'(H)\sqrt{2\pi}}\int_0^\infty \exp(isu)u^{H-\frac{1}{2}}du\\
=\;& \frac{\exp(-ist)-1}{C'(H)\sqrt{2\pi}}\Gamma\Big(H+\frac{1}{2}\Big)\exp\left(\operatorname{sgn}(s)\Big(H+\frac{1}{2}\Big)\frac{i\pi}{2}\right)|s|^{-H-\frac{1}{2}}\\
=\;& \frac{C''(H)\Gamma\big(H+\frac{1}{2}\big)}{C'(H)\sqrt{2\pi}}(-i)\cdot\operatorname{sgn}(s)\exp\left(\operatorname{sgn}(s)\Big(H+\frac{1}{2}\Big)\frac{i\pi}{2}\right)\overline{z''_H(t,s)}. \qquad (3.17)
\end{aligned}$$

By using an analyticity argument, we obtain that identity (3.17) also holds for $H\geq\frac{1}{2}$. Furthermore, by using Theorem 3.6, Theorem 3.4, Parseval's equality and (3.17), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}}\Big|z''_H(t,s)\Big|^2 ds \;&=\; R^H(t,t)\\
&=\; \int_{\mathbb{R}}\Big(z'_H(t,s)\Big)^2 ds\\
&=\; \int_{\mathbb{R}}\Big|\widehat{z'_H(t,\cdot)}(s)\Big|^2 ds\\
&=\; \left(\frac{C''(H)\Gamma\big(H+\frac{1}{2}\big)}{C'(H)\sqrt{2\pi}}\right)^2\int_{\mathbb{R}}\Big|z''_H(t,s)\Big|^2 ds,\;\; t\in\mathbb{R}.
\end{aligned}$$

This proves (3.16). Moreover, by combining (3.17) and (3.16), we obtain (3.15). □

4 Fractional calculus for fBm

Next, we show that the kernels z_H and z'_H defined in Theorem 3.1 and Theorem 3.4, respectively, can be expressed in terms of fractional integrals and derivatives. Fractional calculus in a generalization of usual calculus. An extensive source on this subject is [39]. Moreover, see [37] for details on fractional calculus in connection with fBm.

4.1 Fractional calculus over $[0, T]$

Usual n-fold integrals, where $n \in \mathbb{N}$, can be generalized to n-fold integrals, or *fractional integrals of order* n, where $n > 0$. In fact, let $n \in \mathbb{N}$, $T > 0$ and f be suitably integrable. Then, by using Fubini's theorem, iterating and using (3.1), we obtain that

$$\int_s^T \int_{s_{n-1}}^T \dots \int_{s_1}^T f(u) du ds_1 \dots ds_{n-1} \;=\; \frac{1}{\Gamma(n)} \int_s^T f(u)(u-s)^{n-1} du. \tag{4.1}$$

The expression on the right-hand side of (4.1) is well-defined not only for $n \in \mathbb{N}$, but for all $n > 0$. This motivates the following definition:

Definition 4.1. Let $T > 0$. The *right-sided Riemann-Liouville fractional integral operator of order* α *over* $[0, T]$ is defined by

$$\left(\mathcal{I}^{\alpha}_{T-} f\right)(s) \;:=\; \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^T f(u)(u-s)^{\alpha-1} du, \; s \in [0,T], & \text{if } \alpha > 0 \\ \\ f(s), \; s \in [0,T], & \text{if } \alpha = 0. \end{cases}$$

The *right-sided Riemann-Liouville fractional derivative operator of order* α *over* $[0, T]$ is defined by

$$\left(\mathcal{D}^{\alpha}_{T-} f\right)(s) \;:=\; \begin{cases} \frac{-d}{ds}\left(\mathcal{I}^{1-\alpha}_{T-} f\right)(s), \; s \in (0,T), & \text{if } \alpha \in (0,1) \\ \\ \frac{-d}{ds} f(s), \; s \in (0,T), & \text{if } \alpha = 1 \\ \\ f(s), \; s \in (0,T), & \text{if } \alpha = 0. \end{cases}$$

For convenience, we set

$$\mathcal{I}^{-\alpha}_{T-} f \;:=\; \mathcal{D}^{\alpha}_{T-} f, \; \alpha \in (0,1].$$

If $\alpha \geq 0$, then $\mathcal{I}^{\alpha}_{T-}$ is a bounded endomorphism on $L^p([0,T])$ for every $p \geq 1$ (see [39], Theorem 2.6). By combining Fubini's theorem and (3.3), we obtain the semigroup property

$$\mathcal{I}^{\alpha}_{T-} \mathcal{I}^{\beta}_{T-} f \;=\; \mathcal{I}^{\alpha+\beta}_{T-} f, \; f \in L^1([0,T]), \; \alpha, \beta \geq 0. \tag{4.2}$$

As for the ordinary derivative, the fractional derivative is defined only if f is suitably smooth. More precisely, $\mathcal{D}^{\alpha}_{T-} f$ is well-defined if $f = \mathcal{I}^{\beta}_{T-} g$ for some $g \in L^1([0,T])$ and some $\beta \geq \alpha$ (see [39], Theorem 2.4). By combining (4.2) and the identity $\mathcal{D}^1_{T-} \mathcal{I}^1_{T-} f = f$, we obtain that

$$\mathcal{D}^{\alpha}_{T-} \mathcal{I}^{\beta}_{T-} f \;=\; \mathcal{I}^{\beta-\alpha}_{T-} f, \quad f \in L^1([0,T]), \; 0 \leq \alpha \leq \beta < \infty, \; \alpha \leq 1. \tag{4.3}$$

Similarly as for ordinary calculus, we have that $\mathcal{I}^{\alpha}_{T-} \mathcal{D}^{\alpha}_{T-} f \neq f$ in general.

Based on these facts, we can show the following:

Lemma 4.2 (Molchan & Golosov, 1969)**.** *Let* $H \in (0,1)$ *and* $T > 0$. *Then*

$$z_H(t,s) \;=\; C(H) s^{\frac{1}{2}-H} \left(\mathcal{I}^{H-\frac{1}{2}}_{T-} \cdot^{H-\frac{1}{2}} 1_{[0,t)}\right)(s), \; s,t \in (0,T].$$

Proof. Clearly, for $s \geq t$, the claim follows from assumption (3.5). Let $s < t$. For $H \geq \frac{1}{2}$, the claim is clear by definition. For $H < \frac{1}{2}$, it follows from partial integration that

$$\left(\mathcal{I}_{T-}^{H+\frac{1}{2}} \cdot^{H-\frac{1}{2}} 1_{[0,t)}\right)(s) = \frac{1}{\Gamma\left(H+\frac{3}{2}\right)} t^{H-\frac{1}{2}}(t-s)^{H+\frac{1}{2}} - \left(H-\frac{1}{2}\right)\left(\mathcal{I}_{T-}^{H+\frac{3}{2}} \cdot^{H-\frac{3}{2}} 1_{[0,t)}\right)(s).$$

By using this and (4.3), we obtain that

$$\begin{aligned}
& C(H)s^{\frac{1}{2}-H}\left(\mathcal{D}_{T-}^{\frac{1}{2}-H} \cdot^{H-\frac{1}{2}} 1_{[0,t)}\right)(s) \\
= \; & C(H)s^{\frac{1}{2}-H}\left(\mathcal{D}_{T-}^{1}\mathcal{I}_{T-}^{H+\frac{1}{2}} \cdot^{H-\frac{1}{2}} 1_{[0,t)}\right)(s) \\
= \; & \frac{C(H)}{\Gamma(H+\frac{1}{2})}\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - C(H)\left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\left(\mathcal{I}_{T-}^{H+\frac{1}{2}} \cdot^{H-\frac{3}{2}} 1_{[0,t)}\right)(s) \\
= \; & z_H(t,s). \quad \square
\end{aligned}$$

Remark 4.3. The formal derivative of B^H, denoted by $\dot{B}^H$ (and, with some abuse of terminology, called the *fractional Gaussian noise*), can be interpreted as a weighted *left-sided* fractional integral of order $H - \frac{1}{2}$ of $\dot{W}$. The *left-sided Riemann-Liouville fractional integral operator of order* $\alpha > 0$ *over* $[0,T]$ is defined by

$$\left(\mathcal{I}_{0+}^{\alpha} f\right)(s) := \frac{1}{\Gamma(\alpha)} \int_0^s f(u)(s-u)^{\alpha-1} du, \; s \in [0,T].$$

The operators $\mathcal{I}_{0+}^{-\alpha}$, $\alpha \in [0,1]$, are defined accordingly, similarly as for the right-sided fractional integral operators. Indeed, by combining Lemma 4.2 and the formula for fractional integration by parts (see [39], p. 34), we formally obtain that

$$B_t^H = C(H) \int_0^T 1_{[0,t)}(s) s^{H-\frac{1}{2}} \left(\mathcal{I}_{0+}^{H-\frac{1}{2}} \cdot^{\frac{1}{2}-H} \dot{W}_\cdot\right)(s)\, ds, \; t \in [0,T].$$

Hence, formally,

$$\dot{B}_t^H = C(H) t^{H-\frac{1}{2}} \left(\mathcal{I}_{0+}^{H-\frac{1}{2}} \cdot^{\frac{1}{2}-H} \dot{W}_\cdot\right)(t), \; t \in [0,T].$$

4.2 Fractional calculus over $\mathbb{R}$

Definition 4.4. The *right-sided Riemann-Liouville fractional integral operator of order* α *over* $\mathbb{R}$ is defined by

$$\left(\mathcal{I}_{-}^{\alpha} f\right)(s) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^\infty f(u)(u-s)^{\alpha-1} du, \; s \in \mathbb{R}, & \text{if } \alpha > 0 \\ f(s), \; s \in \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

The *right-sided Marchaud fractional derivative operator of order* $\alpha \in (0,1)$ *over* $\mathbb{R}$ is defined by

$$\left(\mathbf{D}_{-}^{\alpha} f\right)(s) := \lim_{\epsilon \searrow 0} \left(\mathbf{D}_{-,\epsilon}^{\alpha} f\right)(s), \; s \in \mathbb{R},$$

where

$$\left(\mathbf{D}_{-,\epsilon}^{\alpha} f\right)(s) := \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty \left(f(s) - f(u+s)\right) u^{-\alpha-1} du.$$

Moreover,

$$\mathbf{D}_{-}^{0} f := f.$$

For convenience, we set

$$\mathcal{I}_{-}^{-\alpha} \;:=\; \mathbf{D}_{-}^{\alpha},\ \alpha \in (0,1).$$

If $\alpha \in (0,1)$ and $f \in L^p(\mathbb{R})$, where $p \in \left[1, \frac{1}{\alpha}\right)$, then $\mathcal{I}_{-}^{\alpha} f$ is well-defined (see [39], p. 94-95). Moreover, if $\alpha \in (0,1)$, $p \in \left(1, \frac{1}{\alpha}\right)$ and $q = \frac{p}{1-\alpha p}$, then $\mathcal{I}_{-}^{\alpha}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ (see [39], Theorem 5.3). If f is suitably integrable, then it holds that

$$\mathcal{I}_{-}^{\alpha}\mathcal{I}_{-}^{\beta} f \;=\; \mathcal{I}_{-}^{\alpha+\beta} f,\ \alpha,\beta \geq 0.$$

Furthermore, we have that (see [39], Theorem 6.1)

$$\mathbf{D}_{-}^{\alpha}\mathcal{I}_{-}^{\alpha} f \;=\; f,\ \alpha \in \left(0, \frac{1}{2}\right),\ f \in L^2(\mathbb{R}). \tag{4.4}$$

By straightforward calculation, we can show the following:

Lemma 4.5 (Mandelbrot & Van Ness, 1968)**.** *Let $H \in (0,1)$. For $t < 0$, denote*

$$1_{[0,t)} \;:=\; -1_{[t,0)}.$$

Then

$$z'_H(t,s) \;=\; \frac{\Gamma\left(H + \frac{1}{2}\right)}{C'(H)} \left(\mathcal{I}_{-}^{H-\frac{1}{2}} 1_{[0,t)}\right)(s),\ s,t \in \mathbb{R}.$$

Remark 4.6. The operator $\mathbf{D}_{-}^{\alpha}$ can be formally derived from the corresponding right-sided Riemann-Liouville fractional derivative operator $\mathcal{D}_{-}^{\alpha}$ by suitable transformations (see [39], p. 109). The reason why we present $\mathbf{D}_{-}^{\alpha}$ instead of $\mathcal{D}_{-}^{\alpha}$ is that $\mathcal{D}_{-}^{\alpha}\mathcal{I}_{-}^{\alpha} f$ is not well-defined for $f \in L^2(\mathbb{R})$, unless also $f \in L^1(\mathbb{R})$. Hence, $\mathbf{D}_{-}^{\alpha}$ can not be replaced by $\mathcal{D}_{-}^{\alpha}$ in (4.4). The reason for this is that contrary to $\mathbf{D}_{-}^{\alpha}$, the operator $\mathcal{D}_{-}^{\alpha}$ requires integrability of the function at infinity: for example, we have that $\mathbf{D}_{-}^{\alpha} 1 \equiv 0$, whereas $\mathcal{D}_{-}^{\alpha} 1$ does not exist.

5 Wiener integrals with respect to fBm

The Molchan-Golosov and Mandelbrot-Van Ness integral transforms are useful in order to construct integrals with respect to fBm:

5.1 Wiener integrals over $[0, T]$

For $H \in (0,1)$ and $T > 0$, define power-weighted fractional integral operators by

$$\left(\mathbf{K}^H f\right)(s) \ := \ \left(\mathbf{K}_T^H f\right)(s) \ := \ C(H) s^{\frac{1}{2}-H} \left(\mathcal{I}_{T-}^{H-\frac{1}{2}} \cdot^{H-\frac{1}{2}} f\right)(s), \ s \in (0,T),$$

and

$$\left(\mathbf{K}^{H,*} f\right)(s) \ := \ \left(\mathbf{K}_T^{H,*} f\right)(s) \ := \ C(H)^{-1} s^{\frac{1}{2}-H} \left(\mathcal{I}_{T-}^{\frac{1}{2}-H} \cdot^{H-\frac{1}{2}} f\right)(s), \ s \in (0,T).$$

The operators $\mathbf{K}^H$ and $\mathbf{K}^{H,*}$ are mutually inverse in the following sense:

Lemma 5.1. *Let $H > \frac{1}{2}$. Then*

$$\mathbf{K}^{H,*}\mathbf{K}^H f \ = \ f, \ f \in L^2([0,T]), \tag{5.1}$$

and

$$\mathbf{K}^H \mathbf{K}^{H,*} 1_{[0,t)} \ = \ 1_{[0,t)}, \ t \in [0,T]. \tag{5.2}$$

Moreover, let $H < \frac{1}{2}$. Then

$$\mathbf{K}^H \mathbf{K}^{H,*} f \ = \ f, \ f \in L^2([0,T]), \tag{5.3}$$

and

$$\mathbf{K}^{H,*}\mathbf{K}^H 1_{[0,t)} \ = \ 1_{[0,t)}, \ t \in [0,T]. \tag{5.4}$$

Proof. Let $H > \frac{1}{2}$. If $f \in L^2([0,T])$, then $\cdot^{H-\frac{1}{2}} f(\cdot) \in L^1([0,T])$. Hence by using (4.3), we obtain (5.1). By using the fact that $\mathcal{I}_{T-}^{H-\frac{1}{2}} = \mathcal{D}_{T-}^{\frac{3}{2}-H}\mathcal{I}_{T-}^{1}$ and a straightforward calculation, we obtain (5.2). The identities (5.3) and (5.4) are obtained similarly. □

From Lemma 4.2, it follows that the Molchan-Golosov representation restricted to the interval $[0,T]$ can be written as

$$\left(B_t^H\right)_{t\in[0,T]} \ \stackrel{d}{=} \ \left(\int_0^T \left(\mathbf{K}_T^H 1_{[0,t)}\right)(s) dW_s\right)_{t\in[0,T]}. \tag{5.5}$$

Let $\mathcal{E}_T$ denote the space of elementary functions on $[0,T]$, i.e. functions of type

$$f(s) \ := \ \sum_{i=1}^m a_i 1_{[0,t_i)}(s), \ s \in [0,T],$$

where $a_i \in \mathbb{R}$, $t_i \in [0,T]$, $i = 1, \ldots, m$, and $m \in \mathbb{N}$. For $f \in \mathcal{E}_T$, the *(fractional) Wiener integral with respect to B^H* is defined by

$$I_T^H(f) \ := \ \int_0^T f(s) dB_s^H \ := \ \sum_{i=1}^m a_i B_{t_i}^H.$$

From (5.5), we obtain that

$$\int_0^T f(s) dB_s^H \ \stackrel{d}{=} \ \int_0^T \left(\mathbf{K}^H f\right)(s) dW_s. \tag{5.6}$$

By considering (5.6) and the standard Wiener isometry, for $H > \frac{1}{2}$, it is natural to define a space of *time domain Wiener integrands* by

$$\Lambda_T(H) \;:=\; \left\{ f : [0,T] \to \mathbb{R} \;\middle|\; \cdot^{H-\frac{1}{2}} f(\cdot) \in L^1([0,T]) \text{ and } \int_0^T \left(\mathbf{K}^H f\right)(s)^2 ds \;<\; \infty \right\}.$$

For $H < \frac{1}{2}$, $\mathbf{K}^H$ is a weighted fractional derivative operator, so its arguments must be sufficiently smooth. Hence, based on Lemma 5.1, we define

$$\Lambda_T(H) \;:=\; \left\{ f : [0,T] \to \mathbb{R} \mid \exists\, \phi_f \in L^2([0,T]) \text{ such that } f = \mathbf{K}^{H,*}\phi_f \right\}.$$

$\mathcal{E}_T$ is dense in $\Lambda_T(H)$ with respect to the scalar product

$$(f,g)_{\Lambda_T(H)} \;:=\; \left(\mathbf{K}^H f, \mathbf{K}^H g\right)_{L^2([0,T])},\;\; f,g \in \Lambda_T(H).$$

Let

$$\mathcal{H}_T\big(B^H\big) \;:=\; \overline{\operatorname{span}\left\{B_t^H \,|\, t \in [0,T]\right\}} \;\subseteq\; L^2(\mathbb{P})$$

denote the first Wiener chaos of B^H over $[0,T]$. The *time domain (fractional) Wiener integral with respect to* $(B_t^H)_{t\in[0,T]}$ is defined by

$$\begin{aligned} I_T^H \;:\; \Lambda_T(H) \;&\to\; \mathcal{H}_T\big(B^H\big) \\ f \;&\mapsto\; I_T^H(f) \;:=\; \int_0^T f(s)dB_s^H \;:=\; L^2(\mathbb{P})\text{-}\lim_{n\to\infty} \int_0^T f_n(s)dB_s^H, \end{aligned}$$

where $(f_n)_{n\in\mathbb{N}} \in \mathcal{E}_T^{\mathbb{N}}$ converges to f in the norm induced by $(\cdot,\cdot)_{\Lambda_T(H)}$. By construction, (5.6) holds for all $f \in \Lambda_T(H)$.

Remark 5.2. For $H > \frac{1}{2}$, it holds that $\Lambda_T(H) \subsetneq \overline{\mathcal{E}_T}$, i.e. $\Lambda_T(H)$ is not complete. This is due to the fact that $\mathbf{K}^H f$ is a weighted fractional integral of positive order, i.e. more smooth than a general square-integrable function. With other words, $\mathbf{K}^H\big(\Lambda_T(H)\big) \subsetneq L^2([0,T])$. Contrary, for $H < \frac{1}{2}$, $\Lambda_T(H)$ is complete.

By using Lemma 5.1, we can derive the following reciprocal of the Molchan-Golosov integral transform:

Lemma 5.3. *Let $H \in (0,1)$. Then*

$$(W_t)_{t\in[0,\infty)} \;\stackrel{d}{=}\; \left(\int_0^t z_H^*(t,s)dB_s^H\right)_{t\in[0,\infty)},$$

where for $T > 0$ and $s,t \in [0,T]$, it holds that

$$\begin{aligned} z_H^*(t,s) \;&=\; \left(\mathbf{K}_T^{H,*} 1_{[0,t)}\right)(s) \\ &=\; \frac{C(H)^{-1}}{\Gamma\big(\frac{3}{2}-H\big)}(t-s)^{\frac{1}{2}-H} \cdot {}_2F_1\left(\frac{1}{2}-H, \frac{1}{2}-H, \frac{3}{2}-H, \frac{s-t}{s}\right) 1_{[0,t)}(s). \end{aligned}$$

5.2 Wiener integrals over $\mathbb{R}$

From Lemma 4.5, we obtain that the Mandelbrot-Van Ness representation is equivalent to

$$\big(B_t^H\big)_{t\in\mathbb{R}} \;\stackrel{d}{=}\; \left(\frac{\Gamma\big(H+\frac{1}{2}\big)}{C'(H)} \int_{\mathbb{R}} \left(\mathcal{I}_-^{H-\frac{1}{2}} 1_{[0,t)}\right)(s)dW_s\right)_{t\in\mathbb{R}}. \tag{5.7}$$

Let $\mathcal{E}$ be the space of elementary functions on $\mathbb{R}$, i.e. functions of type

$$f(s) \;:=\; \sum_{i=1}^m a_i(s) 1_{[0,t_i)}(s),\;\; s \in \mathbb{R},$$

where $a_i, t_i \in \mathbb{R}$, $i = 1, \ldots, m$, and $m \in \mathbb{N}$. For $f \in \mathcal{E}$, the *(fractional) Wiener integral with respect to* B^H is defined by

$$I^H(f) \;:=\; \int_{\mathbb{R}} f(s) dB_s^H \;:=\; \sum_{i=1}^{m} a_i B_{t_i}^H.$$

Then from (5.7), it follows that

$$\int_{\mathbb{R}} f(s) dB_s^H \;\stackrel{d}{=}\; \frac{\Gamma(H+\frac{1}{2})}{C'(H)} \int_{\mathbb{R}} \left(\mathcal{I}_{-}^{H-\frac{1}{2}} f\right)(s) dW_s. \tag{5.8}$$

For $H > \frac{1}{2}$, we define the space of *time domain Wiener integrands* by

$$\Lambda(H) \;:=\; \left\{ f \in L^1(\mathbb{R}) \;\Big|\; \int_{\mathbb{R}} \left(\mathcal{I}_{-}^{H-\frac{1}{2}} f\right)(s)^2 ds \;<\; \infty \right\}.$$

If $H < \frac{1}{2}$, then in view of (4.4), we set

$$\Lambda(H) \;:=\; \left\{ f : \mathbb{R} \to \mathbb{R} \;\Big|\; \exists\, \phi_f \in L^2(\mathbb{R}) \text{ such that } f \;=\; \mathcal{I}_{-}^{\frac{1}{2}-H} \phi_f \right\}.$$

$\mathcal{E}$ is dense in $\Lambda(H)$ with respect to the scalar product

$$(f,g)_{\Lambda(H)} \;:=\; \left(\frac{\Gamma(H+\frac{1}{2})}{C'(H)}\right)^2 \left(\mathcal{I}_{-}^{H-\frac{1}{2}} f, \mathcal{I}_{-}^{H-\frac{1}{2}} g\right)_{L^2(\mathbb{R})}, \;\; f, g \in \Lambda(H).$$

Let

$$\mathcal{H}(B^H) \;:=\; \overline{\operatorname{span}\{B_t^H \,|\, t \in \mathbb{R}\}} \;\subseteq\; L^2(\mathbb{P})$$

be the first Wiener chaos of B^H over $\mathbb{R}$. The *time domain (fractional) Wiener integral with respect to* $(B_t^H)_{t\in\mathbb{R}}$ is defined by

$$\begin{aligned} I^H \;:\; \Lambda(H) \;&\to\; \mathcal{H}(B^H) \\ f \;&\mapsto\; I^H(f) \;:=\; \int_{\mathbb{R}} f(s) dB_s^H \;:=\; L^2(\mathbb{P})\text{-}\lim_{n\to\infty} \int_{\mathbb{R}} f_n(s) dB_s^H, \end{aligned}$$

where $(f_n)_{n\in\mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$ approximates f in the norm induced by $(\cdot,\cdot)_{\Lambda(H)}$. By construction, (5.8) holds for all $f \in \Lambda(H)$.

From a straightforward calculation, we obtain the following reciprocal of the Mandelbrot-Van Ness integral transform:

Lemma 5.4. *For $H \in (0,1)$, it holds that*

$$(W_t)_{t\in\mathbb{R}} \;\stackrel{d}{=}\; \left(\frac{C'(H)}{\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)} \int_{\mathbb{R}} \left((t-s)_{+}^{\frac{1}{2}-H} - (-s)_{+}^{\frac{1}{2}-H}\right) dB_s^H\right)_{t\in\mathbb{R}}.$$

6 The Volterra Gaussian process

Some considerations for fBm which are based on its Molchan-Golosov representation do not rely on the exact structure of the kernel z_H, but merely on the fact that there exists a standard Brownian motion, or a convenient Gaussian martingale, having the same filtration (and the same first Wiener chaos) as fBm. This is one fact motivating the following definition:

Definition 6.1. A Gaussian process $(X_t)_{t\in[0,\infty)}$ is called *Volterra*, if there exists a standard Brownian motion $(W_t)_{t\in[0,\infty)}$ and a Volterra kernel $z_X \in L^2_{\text{loc}}\left([0,\infty)^2\right)$, such that

$$X_t \;=\; \int_0^t z_X(t,s)dW_s, \;\; a.s., \;\; t\in[0,\infty). \tag{6.1}$$

Recall from before that z_X is Volterra if and only if $z_X(t,s)=0$, $0<t\le s<\infty$. The Volterra Gaussian process is centered and has covariance function

$$R^X(s,t) \;=\; \int_0^{s\wedge t} z_X(t,u)z_X(s,u)du, \;\; s,t\in[0,\infty). \tag{6.2}$$

Remark 6.2. The label *Volterra* originates from the *Volterra integral equation (of the first kind)*, which is of type

$$x(t) \;=\; \int_0^t z(t,s)y(s)ds, \;\; t\in[0,\infty),$$

where the function x and the kernel z are known, and the function y is unknown. Indeed, we can consider (6.1) as a generalized, stochastic Volterra integral equation with solution $\dot{W}$. Correspondingly, X can be considered as a generalized, stochastic Volterra integral transform of the white noise.

We assume that z_X is *non-degenerate*, meaning that the family $\{z_X(t,\cdot)\,|\,t\in(0,\infty)\}$ is linearly independent and complete in $L^2([0,\infty))$. From the linear independence, it follows by using (6.2) that R^X is positive definite on $(0,\infty)$. The completeness ensures that $\Gamma_t(X)=\Gamma_t(W)$, $t\in(0,\infty)$. Here,

$$\Gamma_t(Y) \;:=\; \overline{\text{span}\{Y_s\,|\,s\in[0,t]\}} \;\subseteq\; L^2(\mathbb{P})$$

is the first Wiener chaos of the Gaussian process Y with $Y_0=0$, a.s., over $[0,t]$.

6.1 Self-similarity

It is easy to identify the family of those Volterra kernels that generate β-self-similar Volterra Gaussian processes for $\beta>0$. Indeed, X is β-self-similar if and only if there exists a function $F_X \in L^2\left((0,1),(1-x)^{2\beta-1}dx\right)$ such that

$$z_X(t,s) \;=\; (t-s)^{\beta-\frac{1}{2}}F_X\left(\frac{s}{t}\right), \;\; 0<s<t<\infty,$$

(see [IV], Lemma 2.4). For example, for fBm, it follows from Lemma 3.2 that

$$F_{B^H}(x) \;=\; \frac{C(H)}{\Gamma\left(H+\frac{1}{2}\right)}\cdot {}_2F_1\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, \frac{x-1}{x}\right), \;\; x\in(0,1).$$

A second example is the *Riemann-Liouville process with index* $H>0$ defined by

$$U_t^H \;:=\; \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dW_s, \;\; t\in[0,\infty).$$

Hence, U^H is H-self-similar and $F_{U^H}\equiv\sqrt{2H}$. The Riemann-Liouville process was introduced by Lévy (see [24]) and is therefore sometimes referred to as *Lévy fBm*. The increments of U^H

are not stationary. Indeed, it is easy to show that B^H is the unique centered H-self-similar Gaussian process with stationary increments. However, contrary to fBm, the Riemann-Liouville process allows to model H-self-similarity for any index $H > 0$. We conclude that, in addition to fBm, also the general Volterra Gaussian process is a suitable instrument for modelling self-similarity. In contrast to this, it is difficult to identify the family of those Volterra kernels that produce Volterra Gaussian processes with stationary increments.

Remark 6.3. For a corresponding generalization of the Mandelbrot-Van Ness representation of fBm, i.e. for a process of type

$$X_t \;=\; \int_{\mathbb{R}} z'_X(t,s)dW_s,\; a.s.,\; t \in \mathbb{R},$$

where $X_0 = 0$, a.s., and $z'_X(t,\cdot) \in L^2(\mathbb{R})$, $t \in \mathbb{R}$, both families of kernels are easy to identify: First, assume that $\overline{\operatorname{span}\{X_s \,|\, s \in (-\infty,t]\}} = \overline{\operatorname{span}\{W_s \,|\, s \in (-\infty,t]\}}$, $t \in [0,\infty)$. Then $(X_t)_{t\in[0,\infty)}$ is β-self-similar if and only if there exists $F'_X \in L^2\left((-\infty,1),(1-x)^{2\beta-1}dx\right)$ such that $z'_X(t,s) = (t-s)^{\beta-\frac{1}{2}} F'_X\left(\frac{s}{t}\right)$, $s < t$.
Second, X has stationary increments if and only if there exists some $G'_X : \mathbb{R} \to \mathbb{R}$ such that $z'_X(t,s) = G'_X(t-s) - G'_X(-s)$, $s,t \in \mathbb{R}$.

6.2 Abstract Wiener integrals over $[0,T]$

Let $T > 0$. For $f := \sum_{i=1}^m a_i 1_{[0,t_i)} \in \mathcal{E}_T$, the *Wiener integral with respect to* $(X_t)_{t\in[0,T]}$ is defined by

$$I_T^X(f) \;:=\; \int_0^T f(s)dX_s \;:=\; \sum_{i=1}^m a_i X_{t_i}.$$

Let $\Lambda_T(X)$ be the completion of $\mathcal{E}_T$ with respect to the scalar product

$$(1_{[0,s)}, 1_{[0,t)})_X \;:=\; R^X(s,t),\;\; s,t \in [0,T].$$

Hence, $f \in \Lambda_T(X)$ is an equivalence class of Cauchy sequences $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{E}_T^{\mathbb{N}}$, where $\{f_n\}_{n\in\mathbb{N}} \sim \{g_n\}_{n\in\mathbb{N}} :\Leftrightarrow (f_n - g_n, f_n - g_n)_X \to 0$, $n \to \infty$. The scalar product on $\Lambda_T(X)$ is given by

$$(f,g)_{\Lambda_T(X)} := \lim_{n\to\infty} (f_n, g_n)_X,\;\; f,g \in \Lambda_T(X),\;\; \{f_n\}_{n\in\mathbb{N}} \in f,\;\; \{g_n\}_{n\in\mathbb{N}} \in g.$$

The *abstract Wiener integral with respect to* $(X_t)_{t\in[0,T]}$ is defined by

$$\begin{aligned} I_T^X \;:\; \Lambda_T(X) \;&\to\; \Gamma_T(X) \\ f \;&\mapsto\; I_T^X(f) \;:=\; \int_0^T f(s)dX_s \;:=\; L^2(\mathbb{P})\text{-}\lim_{n\to\infty} \int_0^T f_n(s)dX_s, \end{aligned}$$

where $\{f_n\}_{n\in\mathbb{N}} \in f$. The random variable $\int_0^T f(s)dX_s$ is centered, Gaussian and satisfies $\mathrm{E}_{\mathbb{P}}\left(\int_0^T f(s)dX_s\right)^2 = |f|^2_{\Lambda_T(X)}$, where $|\cdot|_{\Lambda_T(X)}$ is the norm induced by $(\cdot,\cdot)_{\Lambda_T(X)}$.

Remark 6.4. 1. The construction of the abstract Wiener integral does not rely on the fact that the Gaussian process X is Volterra. Similarly as for fBm, one can define time domain Wiener integrals with respect to $(X_t)_{t\in[0,T]}$ by defining a linear operator on $\mathcal{E}_T$ by $\mathbf{K}_T^X 1_{[0,t)} := z_X(t,\cdot)$, $t \in [0,T]$, and then considering a space of time domain Wiener integrands. However, in general, as it is the case for H-fBm with $H > \frac{1}{2}$, the obtained time domain Wiener integrand space is incomplete.
2. The same notation as in articles [I - IV] is also used in this introduction. Therefore, there appears a possible source of confusion: Indeed, we have that $R^{B^H} = R^H$ and $z_{B^H} = z_H$. However, for $H > \frac{1}{2}$, the space $\Lambda_T(B^H)$ and the map $I_T^{B^H}$ in this section are different from the space $\Lambda_T(H)$ and the map I_T^H in Section 5, respectively. Also, note that $\Gamma_T(B^H) = \mathcal{H}_T(B^H)$, where the latter notation was used in Section 5.

7 Summaries of the articles

I. Transformation formulas for fractional Brownian motion.
First, for given H-fBm $\left(B_t^H\right)_{t\in[0,\infty)}$ and Hurst index $K \in (0,1)$, we show that there exists a K-fBm $\left(B_t^K\right)_{t\in[0,\infty)}$, such that for all $t \in [0,\infty)$ we have that

$$B_t^H \;=\; C(K,H)\int_0^t (t-u)^{H-K}\cdot {}_2F_1\left(1-K-H, H-K, 1+H-K, \frac{u-t}{u}\right) dB_u^K, \; a.s. \quad (7.1)$$

Here, $C(K,H)$ is a normalizing constant and the integral is a time domain fractional Wiener integral. This result generalizes both the Molchan-Golosov integral transform (see Theorem 3.1 and Lemma 3.2) which corresponds to the case $K=\frac{1}{2}$, and its reciprocal (see Lemma 5.3) corresponding to the case $H=\frac{1}{2}$ and $K=H$. In order to prove this, we consider the K-fBm defined by $B_t^K := \int_0^t z_K(t,s)dW_s$, $t\in[0,\infty)$, where $W_t := \int_0^t z_H^*(t,s)dB_s^H$, $t\in[0,\infty)$. For $T>0$, the kernels z_H and z_K^* can be written in terms of power weighted fractional integral operators of order $H-\frac{1}{2}$ and $\frac{1}{2}-K$, respectively, over $[0,T]$ (see Lemma 4.2 and Lemma 5.3). Due to suitable powers, their composition reduces to a weighted fractional integral operator of order $H-K$, from which we deduce the kernel in (7.1). Then, for $t\in[0,T]$, the result is derived directly from the definition of the time domain fractional Wiener integral. By letting $T\to\infty$, we obtain the result for $t\in[0,\infty)$. Second, we demonstrate heuristically, how the corresponding generalized Mandelbrot-Van Ness integral transform, which states that the process

$$Z_t^{H,\infty} \;:=\; C(K,H)\int_{\mathbb{R}}\left((t-u)_+^{H-K} \;-\; (-u)_+^{H-K}\right)dB_u^K, \; t\in\mathbb{R},$$

(where B^K is continued to a two-sided K-fBm) is an H-fBm (see [36]), follows from this. Indeed, by combining representation (7.1) and the stationarity of increments of both B^H and B^K, we obtain that for every $s>0$, the process

$$\begin{aligned} Z_t^{H,s} \;:=\;& C(K,H) \\ \times\;& \left(\int_{-s}^t (t-u)^{H-K}\cdot {}_2F_1\left(1-K-H, H-K, 1+H-K, \frac{u-t}{u+s}\right) dB_u^K\right. \\ &\left. -\int_{-s}^0 (-u)^{H-K}\cdot {}_2F_1\left(1-K-H, H-K, 1+H-K, \frac{u}{u+s}\right) dB_u^K\right), \end{aligned}$$

$t\in[-s,\infty)$, is an H-fBm. For $t\in[-s,\infty)$, the kernel of $Z_t^{H,s}$ converges pointwise to the kernel of $Z_t^{H,\infty}$. Hence, formally, we obtain that $Z_t^{H,s}\to Z_t^{H,\infty}$, as $s\to\infty$.

II. On the connection between Molchan-Golosov and Mandelbrot-Van Ness representations of fractional Brownian motion.
We prove the second, only formally obtained result in [I] rigorously. More precisely, we show that for every $K\geq\frac{1}{2}$ and $t\in\mathbb{R}$, there exist constants $C_1(K,H,t)$ and $s_1(t)>0$, such that

$$\mathrm{E}_{\mathbb{P}}\left(Z_t^{H,s} \;-\; Z_t^{H,\infty}\right)^2 \;\leq\; C_1(K,H,t)s^{2H-2}, \; s>s_1(t).$$

Also, we show that for every $K<\frac{1}{2}$ and $t\in\mathbb{R}$, there exist constants $C_2(K,H,t)$, $C_3(K,H,t)$ and $s_2(t)>0$, such that

$$\mathrm{E}_{\mathbb{P}}\left(Z_t^{H,s} \;-\; Z_t^{H,\infty}\right)^2 \;\leq\; C_2(K,H,t)s^{2H-2} \;+\; C_3(K,H,t)s^{2K-2}, \; s>s_2(t).$$

The constants are specified exactly. The proof for these estimates is extensive and technical, and therefore, we only present it for $t>0$. We consider $Z_t^{H,s}-Z_t^{H,\infty}$ as the sum of two fractional Wiener integrals, one over $(-\infty,-s)$ and the other over $(-s,t)$. The second moments of these summands can be estimated by transforming them into standard Wiener integrals over $\mathbb{R}$, and then using the Wiener isometry. Then the original problem is reduced to estimating the $L^2(\mathbb{R})$-norm of two functions. The first one of them is simple, but the second one is simple only for $K = \frac{1}{2}$, but complicated for $K > \frac{1}{2}$, and even more complicated for $K < \frac{1}{2}$. By splitting functions, using well-known calculation formulas for the Gauss hypergeometric function and recombining results, we obtain the estimates.

III. Measure-preserving transformations of Volterra Gaussian processes and related bridges.
First, we consider a continuous Volterra Gaussian process $(X_t)_{t\in[0,T]}$, where $T > 0$ is a fixed time horizon. A measurable map $\mathcal{T}$ from the coordinate space of X to itself is a *measure-preserving transformation* if $\mathcal{T}(X) \overset{d}{=} X$. Recall that a process $\left(X_t^T\right)_{t\in[0,T]}$ is a *bridge of* X if $\mathrm{Law}_{\mathbb{P}}\left(X^T\right) = \mathrm{Law}_{\mathbb{P}}(X \,|\, X_T = 0)$. We call a measurable map $\mathcal{B}$ from the coordinate space of X to itself a *bridge transformation* if $\mathcal{B}(X) \overset{d}{=} X^T$. It is well-known that the process

$$\widehat{X}_t^T \;:=\; X_t \;-\; \frac{R^X(t,T)}{R^X(T,T)} X_T,\; t \in [0,T],$$

is a bridge of X that satisfies

$$\Gamma_T\left(\widehat{X}^T\right) \;\perp\; \mathrm{span}\{X_T\} \;=\; \Gamma_T(X),$$

where $\perp$ denotes the orthogonal direct sum (see [14]). Based on this fact, we derive two measure-preserving transformations $\mathcal{T}^{(i)}$, $i \in \{1,2\}$, satisfying

$$\Gamma_T\big(\mathcal{T}^{(i)}(X)\big) \;=\; \Gamma_T\Big(\widehat{X}^T\Big),\; i \in \{1,2\}.$$

Furthermore, as an inherently, inversely related problem, we derive two bridge transformations $\mathcal{B}^{(i)}$, $i \in \{1,2\}$, such that

$$\Gamma_T\big(\mathcal{B}^{(i)}(X)\big) \;=\; \Gamma_T(X),\; i \in \{1,2\}.$$

The transformations $\mathcal{T}^{(1)}$ and $\mathcal{B}^{(1)}$ are connected to $\mathcal{T}^{(2)}$ and $\mathcal{B}^{(2)}$, respectively, by suitable time transformations. In order to do this, we follow ideas of Jeulin and Yor, and Peccati, which considered the case $X = W$ (see [20] and [33], respectively). First, we note that

$$\widehat{X}_t^T \;=\; \int_0^T \left(\eta^X 1_{[0,t)}\right)(s)dX_s,\; a.s.,\; t \in [0,T],$$

where η^X is the linear orthoprojection from $\Lambda_T(X)$ onto the subspace

$$\Lambda_{T,0}(X) \;:=\; \left\{f \in \Lambda_T(X) \,\middle|\, \left(f, 1_{[0,T)}\right)_{\Lambda_T(X)} = 0\right\}.$$

Next, we define endomorphisms on $\Lambda_T(X)$, denoted by $\alpha^{X,i}$ and $\beta^{X,i}$, such that $\alpha^{X,i} : \Lambda_{T,0}(X) \to \Lambda_T(X)$ and $\beta^{X,i} : \Lambda_T(X) \to \Lambda_{T,0}(X)$ are mutually inverse isometries and $\alpha^{X,i}\eta^X = \alpha^{X,i}$, $i \in \{1,2\}$. If X is a martingale, then the space $\Lambda_T(X)$ is a simple L^2-space, and the structure of these operators is simple and a straightforward generalization of the corresponding operators for W. For the general case, the construction of the operators is more involved. It is based on the *prediction martingale of* X_T *with respect to* $\left(\mathcal{F}_t^X\right)_{t\in[0,T]}$, the natural filtration of X, which is defined by

$$M_t \;:=\; \mathrm{E}_{\mathbb{P}}\left(X_T \,|\, \mathcal{F}_t^X\right),\; t \in [0,T].$$

This martingale has key features $M_T = X_T$ and $\Gamma_T\left(\widehat{M^T}\right) = \Gamma_T\left(\widehat{X}^T\right)$, respectively, yielding $\eta^X = \kappa^{-1}\eta^M\kappa$, where κ is the Wiener isometry between $\Lambda_T(X)$ and $\Lambda_T(M)$. By setting

$\alpha^{X,i} := \kappa^{-1}\alpha^{M,i}\kappa$ and $\beta^{X,i} := \kappa^{-1}\beta^{M,i}\kappa$, $i \in \{1,2\}$, we obtain operators with the suitable features. The desired transformation are obtained by setting

$$\mathcal{T}_t^{(i)}(X) \;:=\; \int_0^T \left(\beta^{X,i}1_{[0,t)}\right)(s)dX_s,\; t \in [0,T],\; i \in \{1,2\},$$

and

$$\mathcal{B}_t^{(i)}(X) \;:=\; \int_0^T \left(\alpha^{X,i}1_{[0,t)}\right)(s)dX_s,\; t \in [0,T],\; i \in \{1,2\},$$

respectively, where the integrals are abstract Wiener integrals. Second, we derive a two-sided Fourier-Laguerre series expansion for the first Wiener chaos of a Gaussian martingale $(M_t)_{t\in[0,\infty)}$. This generalizes a result by Jeulin and Yor (see [20]), where a one-sided series expansion for W was obtained. For a Gaussian martingale $(M_t)_{t\in[0,\infty)}$, the transformation $\mathcal{T}^{(1)}$ is independent of T, and can hence be considered on the coordinate space of $(M_t)_{t\in[0,\infty)}$, where it is invertible. The inverse $\mathcal{T}^{(1),-1}$ satisfies

$$\Gamma_{[T,\infty)}(M) \;=\; \Gamma_{[T,\infty)}\left(\mathcal{T}^{(1),-1}(M)\right) \;\perp\; \text{span}\left\{\mathcal{T}_T^{(1),-1}(M)\right\},$$

where $\Gamma_{[T,\infty)}(M)$ is the orthogonal of $\Gamma_T(M)$ in $\Gamma_\infty(M) := \overline{\text{span}\{M_t \,|\, t \in [0,\infty)\}}$. By iterating this, together with the fact that $\Gamma_T(M) = \Gamma_T\big(\mathcal{T}^{(1)}(M)\big) \perp \text{span}\{M_T\}$, we obtain that the sequence $\left\{\mathcal{T}_T^{(1),n}(M)\right\}_{n\in\mathbb{Z}}$ is a complete orthogonal system in $\Gamma_\infty(M)$. Here, $\mathcal{T}^{(1),n}$ denotes the n-th iterate of $\mathcal{T}^{(1)}$. Based on this, we obtain the series expansion.

IV. A note on ergodic transformations of self-similar Volterra Gaussian processes. Given a continuous β-self-similar Volterra Gaussian process $X_t := \int_0^t z_X(t,s)dW_s$, $t \in [0,\infty)$, where $\beta > 0$, we show that, for every $\alpha > \frac{-1}{2}$, the map

$$\mathcal{Z}_t^\alpha(X) \;:=\; X_t \;-(2\alpha+1)t^{\beta-\alpha-\frac{1}{2}}\int_0^t s^{\alpha-\beta-\frac{1}{2}}X_s ds,\; t \in [0,\infty),$$

is an ergodic (measure-preserving) transformation on the coordinate space of X. This result generalizes and refines a result by Molchan, where $X = B^H$ and $\alpha = H - \frac{1}{2}$ (see [28]). In order to prove this, we first express the Lamperti transform of the β-self-similar process $\mathcal{Z}^\alpha(X)$ in terms of the Lamperti transform of X. Recall that the *Lamperti transform of* X is the map $X_t \mapsto \exp(-\beta t)X_{\exp(t)}$, $t \in [0,\infty)$, which transforms a β-self-similar process into a stationary process. By using a well-known result from the theory of linear transformations of stationary processes, we obtain that $\mathcal{Z}^\alpha(X) \overset{d}{=} X$, i.e. $\mathcal{Z}^\alpha$ is measure-preserving. Then, based on the special structure of z_X due to the β-self-similarity of X, we show that

$$\mathcal{Z}_t^\alpha(X) \;=\; \int_0^t z_X(t,s)d\mathcal{Z}_s^\alpha(W),\; a.s.,\; t \in [0,\infty). \tag{7.2}$$

Next, we consider the $\left(\alpha+\frac{1}{2}\right)$-self-similar martingale $N_t^\alpha := \int_0^t s^\alpha dW_s$, $t \in [0,\infty)$. If the underlying space is the coordinate space of $\left(N_t^\alpha\right)_{t\in[0,\infty)}$, then $\mathcal{Z}^\alpha$ coincides with the transformation $\mathcal{T}^{(1)}$ obtained in [III], i.e. $\mathcal{Z}^\alpha\big(N^\alpha\big) \equiv \mathcal{T}^{(1)}\big(N^\alpha\big)$. Hence, for fixed $T > 0$, we obtain from [III] that $\left\{\mathcal{Z}_T^{\alpha,n}\left(N^\alpha\right)\right\}_{n\in\mathbb{Z}}$, where $\mathcal{Z}^{\alpha,n}$ is the n-th iterate of $\mathcal{Z}^\alpha$, is a complete orthogonal system in $\Gamma_\infty\big(N^\alpha\big) = \Gamma_\infty(X) := \overline{\text{span}\{X_t \,|\, t \in [0,\infty)\}}$. By combining this fact with (7.2), we derive that $\left\{\mathcal{Z}_T^{\alpha,n}(X)\right\}_{n\in\mathbb{Z}}$ is a complete and free system in $\Gamma_\infty(X)$. Hence $\mathcal{F} = \vee_{n\in\mathbb{Z}}\sigma\big(\mathcal{Z}_T^{\alpha,n}(X)\big)$, where $\mathcal{F}$ is the σ-algebra of the coordinate space. Then, by using a well-known result from ergodic theory, we obtain that $\mathcal{Z}^\alpha$ is ergodic.

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