ON RANDOM PLANAR CURVES AND THEIR SCALING LIMITS

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Academic dissertation

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Antti Kemppainen
Paris
September 2009

In this thesis

This thesis includes the following articles:

[i] Kalle Kytölä and Antti Kemppainen. SLE local martingales, reversibility and duality. *Journal of Physics A: Mathematical and General*, 39(46):L657–L666, 2006.

[ii] Antti Kemppainen. Stationarity of SLE, 2009

[iii] Antti Kemppainen and Stanislav Smirnov. Describing scaling limits of random planar curves by SLEs, 2009

The paper [i] is reprinted with the permission of Institute of Physics Publishing Ltd.

Introduction

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1 Overview

In this thesis, random planar curves are studied. The study of the geometry of random curves and surfaces is a very active part of mathematical statistical physics. Let's first explain the words in the title of this thesis.

Why the *plane* $\mathbb{R}^2$? Many problems of the statistical physics are quite trivial in $\mathbb{R}$ and really hard in $\mathbb{R}^3$. Maybe surprisingly the problems are also challenging in $\mathbb{R}^2$. For this reason, the study of two dimensional systems is a benchmark. If a method works in two dimensions then there is, in principle, a good chance that it works in three dimensions. The methods that are explained in this thesis apply in two dimension; however, they are quite special. Only in two dimensions the theory of conformal mappings is so rich that it enables a full characterization of certain objects. We will come back to this in a moment.

Why *curves*? A planar curve is a continuous mapping from an interval of the real axis to $\mathbb{R}^2$. The concept of a curve includes both the locus and the order in which the locus is visited. A *simple* curve doesn't visit the same point twice. In the plane $\mathbb{R}^2$, there are naturally the left-hand side and the right-hand side of a simple curve. Both end points of the curve can be connected to infinity by a simple curve so that if we glue these three curves together they form a simple curve. And this divides the plane into exactly two connected components. These components are the left-hand side and the right-hand side of the extended curve. For this reason, simple curves are naturally *interfaces*: they divide an area into two. If we are, for example, studying a random coloring of the plane, then the interface is formed between an area with one color and an area with the other color.

What kind of *random curves*? The random curves of this thesis arise as interfaces in the statistical physics and are of the type illustrated in Figure 1. The colors red and white represent the two possible states of each individual lattice site. There might be a mechanism so that red hexagons attract or repel each other, but this effect is counterbalanced by the fluctuations caused by the finite temperature. In the case of Figure 1, each hexagon is independent of all the others and what is seen is the pure thermal fluctuations. The boundary conditions are chosen so that there are two boundary arcs and one of them is white and the other is red. This way, there are a macroscopic white cluster, a macroscopic red cluster and an interface in between them.

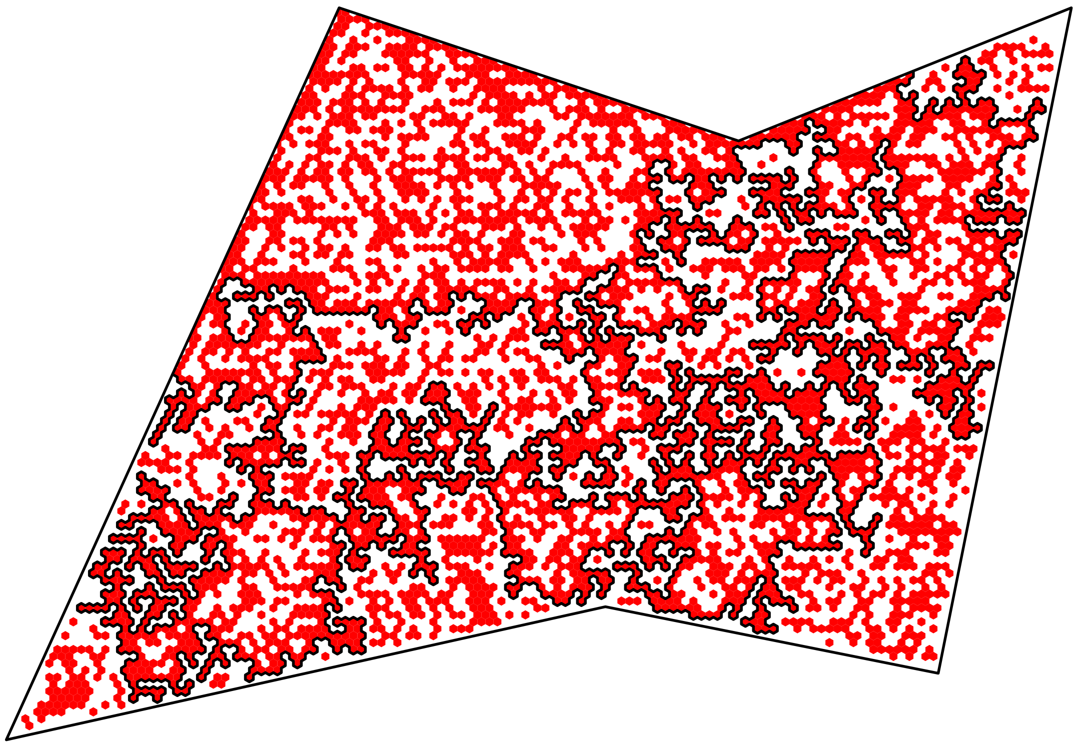


Figure 1: A sample of percolation showing the interface between a red cluster and a white cluster.

What is the *scaling limit*? Usually, a connected region of the plane is taken. Then we take a lattice like the triangular lattice formed by the centers of the hexagons in Figure 1. The scaling limit is taken when we let the mesh size of the lattice decrease to zero while keeping the region the same.

What are the *scaling limits of random planar curves*? First of all, the random curves, that this thesis is about, converge as curves. The limiting object will be a curve, not a nice polygonal curve like in Figure 1 but a rougher one. The paper [iii] of this thesis is a study how this convergence can be established from a simple estimate.

Since the origin of these random curves is in the statistical physics, we need to consider what kind of curves it can produce. Usually there are some parameters in the problems that determine the probability distribution. When these parameters are tuned in exactly right way the system is at its critical point. In Figure 1, this is a balance between red and white. Characteristic for this point is that the system is scale invariant and there is no typical size of, say, the red droplets. Also for the random curve this means scale invariance. There are details in every scale which resemble each other. The scaling limit of the interface will be a random fractal.

When the parameters are not tuned to the critical point then there is still a scaling limit of these random curves. However, the scaling limit is not that interesting: the scaling limit is often a deterministic curve that is stuck to the boundary or minimizing its length etc. This doesn't mean that the system wouldn't interesting also in this case. Almost everything we see around us is not tuned to criticality.

The scaling limits of the random curves arising from a statistical physics model at criticality are identified as Schramm–Loewner evolutions. In 1923, Charles Loewner [13] had an idea to code the information of a curve $\gamma(t)$ in the complex plane $\mathbb{C}$ into a collection of conformal mappings g_t. These mappings satisfy a differential equation

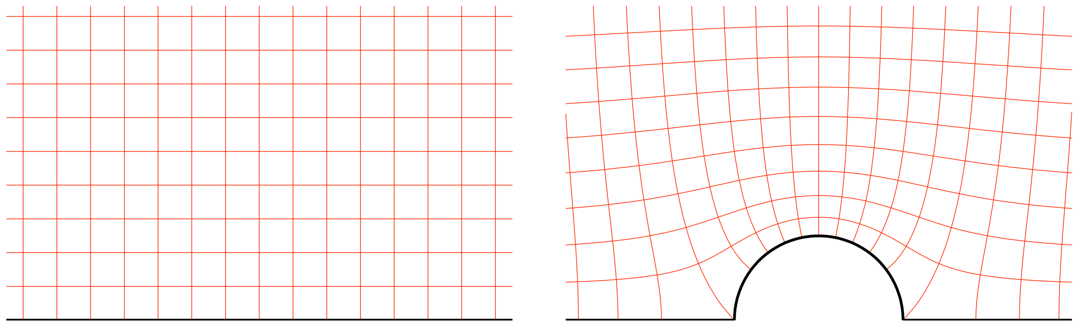
$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - W_t} \tag{1}$$

there W_t is a real valued function unique for each curve $\gamma(t)$. The function W_t acts as a steering wheel. When it increases the curve turns to the right and when it decreases the curve turns to the left. In 1999, Oded Schramm [16] used this equation to study random curves by considering a random function W_t, i.e a stochastic process.

The *conformal mappings* are those that preserve the angles locally. So if g is a conformal mapping and two curves make an angle of θ at a point p, then the image of these curves under g make an angle of θ at the point $g(p)$. In the plane, there are plenty of these mappings. They are the *analytic* (also called *holomorphic*) and one-to-one mappings. Figure 2 illustrates a conformal mapping in the plane.

Schramm noticed that if the random curve comes from the statistical physics and has a symmetry so that the laws of the curve in two different regions are connected through a conformal mapping, then W_t has to be a specific process, a Brownian motion. The random curves driven by such W_t are called Schramm–Loewner evolutions (SLE). The papers [i] and [ii] are studies of some SLE specific questions.

In the section 2, we present lattice models of statistical physics and define what is meant for the criticality. We also comment how the conformal invariance can be seen in them. In the section 3, we give quite complete introduction to SLE. Finally in the section 4, we present the context of the papers [i], [ii] and [iii].



(a) The upper half-plane with a square grid. (b) The image of the grid.

Figure 2: The image of a grid under a conformal mapping from the upper half-plane onto upper half-plane with a semidisc removed.

2 Lattice models of statistical physics

2.1 Statistical physics

Suppose there is given a physical system with finite number of states labelled by $1, 2, \ldots, N$. For example, there are n atoms lying on their sites on a lattice. Each atom has s different states. The total number of the states of the whole system is then $N = s^n$.

It useful to model uncertainty as randomness. Therefore, suppose that there is a probability distribution on the states of the systems, i.e. a set of real numbers p_j where $j = 1, 2, \ldots, N$ such that $0 \leq p_j \leq 1$ for each j and that

$$\sum_{j=1}^{N} p_j = 1. \tag{2}$$

If we have an observable O that takes a value O_j on jth state, then the expected value is denoted by

$$\mathbb{E}O = \sum_{j=1}^{N} p_j O_j.$$

Our starting point in statistical mechanics is the Gibbs measure. Suppose next that the energy of jth state is E_j. If we know the expected energy $\sum p_j E_j$ of the system, then the *Gibbs measure*

$$p_j = \frac{1}{Z(\beta)} e^{-\beta E_j} \tag{3}$$

is maximally random in the sense of entropy. Here $Z(\beta)$ is the partition function determined from the condition (2), i.e.

$$Z = \sum_{j=1}^{N} e^{-\beta E_j} \tag{4}$$

The parameter β can be identified as being inversely proportional to the thermodynamical variable T, the temperature. We can choose the units so that $\beta = 1/T$.

A probability measure of the form (3) is also called Boltzmann distribution and the number $\exp(-\beta E_j)$ is called a *Boltzmann weight* of the state j.

The partition function is very important object in statistical physics. Given an observable O which is positive, we can define

$$Z(O) := \sum_{j=1}^{N} O_j e^{-\beta E_j} = \sum_{j=1}^{N} e^{-\beta E_j + \log O_j}$$

which is also written in the form of partition function. Hence the expected value of O can be written as

$$\mathbb{E}O = \frac{Z(O)}{Z}$$

i.e. as a ratio of two partition functions.

2.2 Ising model

Ideally, the Gibbs measure is defined using a real physical system. However, this turns out to be a really tough job. The partition function (4) is a sum with $N = s^n$ terms, where n is typically 10^{23} and s is a large number. Actually often there is a continuum of one atom states and therefore it would be more accurate to take $s = \infty$. If also the energy variable E_j is hard to calculate, the task is hopeless. Therefore it is clear that some kind of modeling is needed.

A model of a physical system should be simple but still carry the essential features of the physical system. The hope is that the macroscopic properties, in the infinite system limit, don't depend on the details of the system. This kind of *universality* is expected at least under special circumstances, that is near the critical point of the system.

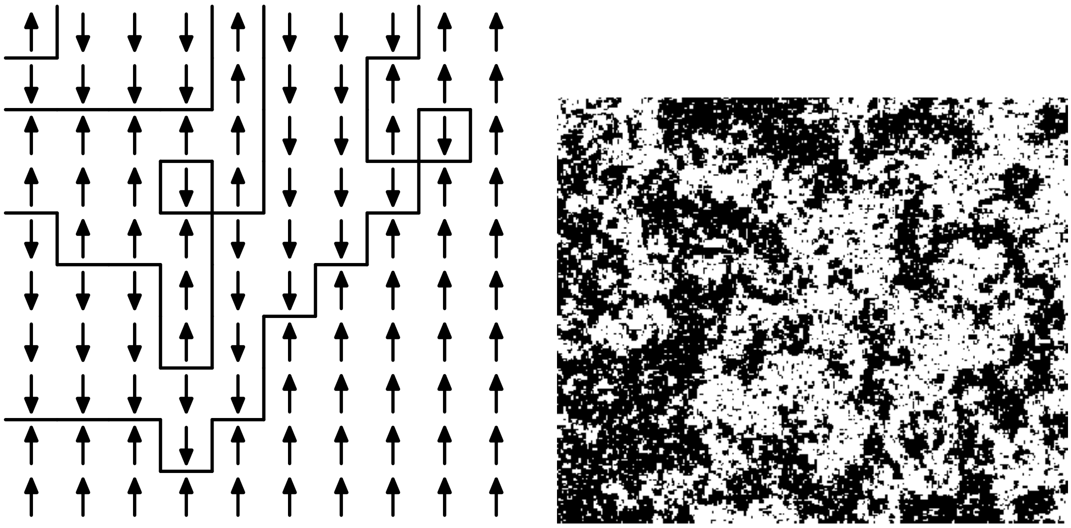
One of the most studied models of statistical physics is the *Ising model*. It is a model of ferromagnet or antiferromagnet. The system is formed of elementary magnets, *spins* σ_x. The subindex x refers to the lattice site the spin is lying on. So far the system is quite accurately physical. A big simplification is made when σ_x takes only two possible values. If the lattice is planar, then think that the spin is either pointing up, to the positive z-direction, or down, to the negative z-direction. We use both the labels $\uparrow$ and $\downarrow$ and the labels $+1$ and -1 for these two possible values. The state of the system is the collection of the all spins at different sites $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, where the different lattice sites have been named $1, 2, \ldots, n$. The setup is illustrated Figure 3(a).

The second simplification is that the interaction between the spins is really short-ranged. The Ising Hamiltonian is defined as

$$H(\underline{\sigma}) = -J \sum_{\langle xy \rangle} \sigma_x \sigma_y - B \sum_{x} \sigma_x \tag{5}$$

and it gives the energy of the configuration of $\underline{\sigma}$. The first sum is over the neighboring pairs of sites, and the second sum is over all the sites. The probability of the state is then given by the Gibbs measure (3).

If $J > 0$ then the system is ferromagnetic and the spins tend to align. In the Ising Hamiltonian (5), an aligned pair of spins has lower energy by $2J$ compared to



(a) The Ising model: spins are the arrows on the lattice sites and the lines are the interfaces between an area of ↑-spins and an area of ↓-spins.

(b) A sample of the Ising model on the square lattice, $T \approx T_c$ and $B = 0$.

Figure 3: Two-dimensional Ising model.

non-aligned. Therefore there is a price for having non-aligned spins and this effect tries to minimize the total length of interfaces, see Figure 3(a). The spins also tend to align with the the external magnetic field represented by the variable B.

For the *thermodynamical limit* of the system, in the plane take first $L \times L$ box of the lattice. Define the Ising probability distribution of the spins of the box, and then take the limit $L \to \infty$. The result is a random spin configuration on the infinite lattice. This is a different infinite system limit than the scaling limit where the region is kept fixed and the lattice mesh taken to zero, whereas for the thermodynamical limit the lattice mesh is kept constant and the size of the box is taken to infinity.

The formulation of the critical point is only possible in the infinite system. Therefore assume that the thermodynamical limit is taken. Denote the magnetization

$$M(T, B) = \mathbb{E}[\sigma_x]$$

which is a constant by translation invariance. There is a *critical point* $T = T_c$ and $B = 0$ in the following sense. For $T < T_c$, as the external magnetic field B is decreased to zero some magnetization remain, i.e. $M(T, 0_+) > 0$, and symmetrically $M(T, 0_-) < 0$. Hence there is a discontinuous phase transition between the phases $M < 0$ and $M > 0$ as B changes its sign. For $T > T_c$, there is no spontaneous magnetization, i.e. $M(T, 0) = 0$ and the $B \mapsto M(T, B)$ is continuous across $B = 0$. Therefore the system is a ferromagnet only for $T < T_c$.

The region $T > T_c$ is called the disordered phase. Even there the spins are correlated, the correlations decay as

$$G(x, y) := \mathbb{E}[\sigma_x \sigma_y] - \mathbb{E}[\sigma_x]\,\mathbb{E}[\sigma_y] \sim e^{-\frac{|x-y|}{\xi(T,B)}} \tag{6}$$

where $\xi(T, B)$ is the correlation length. The correlation length $\xi(T, B)$ also tells the typical size of the connected component of aligned spins.

The correlation length diverges $\xi(T, B) \to \infty$ as we approach the critical point. This implies that there is no typical length scale and there should be some similarity between different scales. In Figure 3(b) a sample of Ising model $T \approx T_c$ is presented.

2.3 Conformal invariance at criticality

Since 80's it has been conjectured that statistical physics models at criticality are conformally invariant in some sense. In Chapter 11 of [3] there is an excellent explanation why conformal invariance should hold. We will review this in this section. This argument is very much heuristic and any implication should be considered as a conjecture that needs a proof. Think that the system is the Ising model; although, the argument isn't restricted to it.

The renormalization group (RG) is a widely used method in statistical physics worth of a Nobel prize. The reader may want to check prize winner's review paper [27]. To illustrate the method, let's consider Kadanoff's block spin transformation. Let B_x be a cube centered at x and of linear size b. Cover the lattice with separate cubes B_x. The centers form a new, sparser lattice V'. Define the block spin

$$\sigma'_x = \sum_{y \in B_x} \sigma_y$$

for each $x \in V'$. Hence for the Ising model in the square lattice $\sigma'_x \in \{-b^n, -b^n + 2, \ldots, b^n - 2, b^n\}$

Redefine the Hamiltonian so that βH is repleaced by H, i.e. the parameter β is absorbed to the constants J and B of the Ising model, and hence the partition function is

$$Z = \sum_{\underline{\sigma}} e^{-H(\underline{\sigma})}$$

Sum in the partition function first in each cube over spins keeping the value of the block spin fixed. And only after this sum over the block spins. One can write the partition function in the form

$$Z = \sum_{\underline{\sigma}'} e^{-H'(\underline{\sigma}')}$$

which defines the renormalized Hamiltonian H'. Continue in the same manner and define V'' and H''. For an infinite system you could repeat this infinitely many times, but not for a finite system.

Denote

$$L_1 : H \mapsto H'$$

and $L_n = (L_1)^n = L_1 \circ L_1 \circ \ldots \circ L_1$, where n maps are composed. The mappings $(L_n)_{n \in \mathbb{N}}$ form a semigroup which is called the *renormalization group*.

The *critical points* of stastitical physics systems are the fixed points of the renormalization group. To see why this is the case and to see some implications of this consider the following. Since b is the linear size of B_x, a distance d in the lattice V is mapped to distance db^{-1} in the lattice V'. Hence the fixed point has to be scale invariant. This is the lack of typical length scale, i.e. $\xi = \infty$, that was characteristic for a critical point.

Also if the Hamiltonian of the system is symmetric under a large enough subgroup of the rotation group, then it is not hard to believe that in the fixed point Hamiltonian has full rotational invariance. For example, the Ising Hamiltonian is invariant under 90 degrees rotations. In the same manner, invariance under the lattice translations develops to full translation invariance.

So the fixed point should be at least invariant under *global* scaling, rotation and translation. If the fixed point Hamiltonian has only short range interactions then different scaling and rotation can be used in the different parts of the space. Hence the Hamiltonian is invariant under transformations that are *locally* scaling and rotation, or briefly it is invariant under *conformal transformations*.

This explanation is satisfactory in a heuristic level, but a question arises: what are the limiting Hamiltonian and the limiting spin field? This kind of constructive approach is not completely available. There is the method of conformal field theory (CFT) which gives at least a partial, but non-constructive, answer in the planar case. Using CFT it is possible to write explicit formulas for correlation functions such as one in the equation (6). This is, in some extent, complementary to the mathematical method of Schramm–Loewner evolutions which gives more easily access to, for example, global connectivity properties.

2.4 Percolation

One of the simplest models of statistical physics to formulate is the percolation. We will here introduce the site percolation on the triangular lattice.

Each vertex or site of the triangular lattice is either open or closed. Pick a value for the parameter $p \in [0,1]$. We independently toss a coin and with the probability p the site is open and with the probability $1-p$ it is closed.

There is a *critical* value p_c for the percolation parameter in the following sense. For $p < p_c$, almost surely there is no infinite, connected cluster of open sites and the probability that two sites x and y can be connected by a open path decays as $\exp(-|x-y|/\xi(p))$ where the constant $\xi(p)$ depends only on p and is called the *correlation length*. For $p > p_c$, almost surely there is an infinite, connected cluster of open sites. For the triangular lattice, the critical value is $p_c = 1/2$.

Let V be the set of vertices of a finite piece of the triangular lattice. Consider the percolation on these vertices. The Figure 1, in the section 1, illustrates this configuration. The centers of the hexagons form the triangular lattice. The closed sites are red hexagons and the open sites are white hexagons. The boundary conditions are such that there are two boundary arc and the other has only red hexagons and the other only white ones. An interface is formed between the cluster of red hexagons attached to the red boundary arc and the white cluster attached to the white arc.

The model is named percolation, since we can think we have a piece V of porous material, say porous rock: the red hexagons are the actual solid rock. The white (open) hexagons are the free space and they form channels inside the piece of rock and they cause the porousness. When you put the piece of rock into water it will wet. The interface in Figure 1 can be interpreted to be frontier where the water reaches. Here we assume that the elementary pieces of the rock (hexagons) are big compared to the molecule size of the water etc.

The conformal invariance of the scaling limit of the percolation at criticality is seen from the Cardy–Smirnov formula. It is an example of an *conformally invariant observable.* It was proposed in [4] and proven for the site percolation on the triangular lattice in [21].

Theorem 2.1 (Cardy–Smirnov formula)**.** *Consider the critical site percolation on the triangular lattice, $p = p_c = 1/2$, in a simply connected domain $U \subset \mathbb{C}$. Let a, b, c, x be four boundary points of U in a counterclockwise order. The domain U can be conformally mapped to an equilateral triangle ABC so that a, b, c are mapped to A, B, C, respectively. Let X be the image of x. Then the probability that there is an open path from the boundary arc xa to the boundary arc bc converges to the ratio*

$$\frac{AX}{AC} \tag{7}$$

as the lattice mesh goes to zero.

2.5 Fortuin-Kasteleyn model

In this section we present a model that is closely related to the Ising model. It is also a weighted version of the edge percolation (instead of the sites the percolation is done on the edges of the lattice).

Consider the Ising model with vanishing external magnetic field, $B = 0$, on a finite piece of the square lattice $\mathbb{Z}^2$. Let V be the set of vertices, i.e. lattice sites, and let E be the set of edges, i.e. nearest neighbor pairs. Assume that the constant β is absorbed in J, as before. Using the fact that $\sigma_x\sigma_y = 2\mathbb{1}_{\{\sigma_x=\sigma_y\}} - 1$, the partition function of the Ising model can be written as

$$Z = \sum_{\underline{\sigma}} \prod_{\langle xy \rangle} \exp(J\sigma_x\sigma_y) = \exp(-Jn) \sum_{\underline{\sigma}} \prod_{\langle xy \rangle} (1 + \mathbb{1}_{\{\sigma_x=\sigma_y\}} v)$$

where $v = \exp(2J) - 1$ and n is the number of lattice sites. The constant in the front can be discarded. Upto a constant this can now be written as

$$Z = \sum_{E' \subset E} 2^{\text{number of components}} \, v^{\text{number of edges}}.$$

where sum is over all the subsets of the set edges E.

The above motivates the definition of the following probability measure on subgraphs $\omega \subset E$: Denote

$$|\omega| = \text{number of edges in } \omega$$
$$C(\omega) = \text{number of components in the graph } (V, \omega).$$

For each $0 < p < 1$, $q > 0$, define a probability measure by

$$\mathbb{P}(\omega) = \frac{1}{Z} \left(\frac{p}{1-p} \right)^{|\omega|} q^{C(\omega)} \tag{8}$$

where Z is a normalizing constant that makes this a probability measure. This is the *Fortuin–Kasteleyn model* (FK model) with weight p per open edge (the edge is present in ω) and $1-p$ per closed edge (the edge is not present in ω) and weight q per cluster. The construction can be modified so that ω is restricted to satisfy boundary conditions: given a set $E_W \subset E$ the above probability measure is conditioned on $E_W \subset \omega$.

The dual lattice of $\mathbb{Z}^2$ is the square lattice $(\mathbb{Z}+1/2)^2$. The vertices of the dual lattice are the centers of the faces of the original lattice, and for each edge of the lattice there is one dual edge crossing it, in this case, perpendicularly. The dual graph ω' of ω is formed if for each edge e in E that is not present in ω, the edge e' in the dual lattice crossing e is in ω'.

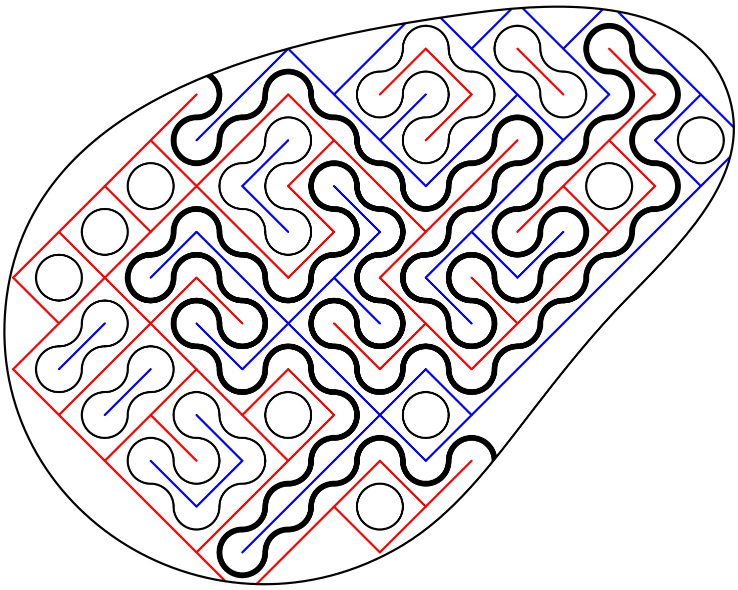


Figure 4: FK model: red lines form the subgraph ω of the lattice $\mathbb{Z}^2$ and blue lines form the dual graph ω' which is a subgraph of $(\mathbb{Z}+1/2)^2$. In the picture these lattices are rotated by 45°.

The setup is illustrated in Figure 4. Similarly as for the percolation, E_W is chosen to be approximation of a boundary arc. There are interfaces between ω and its dual ω', and all but one of these interfaces are closed loops. One of the interfaces is a curve running from near one end point of E_W to the other. See Figure 4. Denote this curve by $\gamma(\omega)$.

The probability measure $\mathbb{P}$ can be written using the loop configuration as

$$\mathbb{P}(\omega) = \frac{1}{Z'} \left(\frac{p}{(1-p)\sqrt{q}} \right)^{|\omega|} (\sqrt{q})^{\text{number of loops}}.$$

When the expression inside the first brackets equals 1, the sets ω and ω' are in a symmetric position. Hence, for a given q, the value $p = p_c(q)$ where

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$$

is self-dual for the square lattice. It turns out that the self-dual value $p = p_c(q)$ is also the critical value (at least for $q = 1$ and $q = 2$ [7]) in the same sense as for the percolation: in the thermodynamical limit, for $p > p_c(q)$, a lattice site is in an infinite cluster with positive probability, and for $p < p_c(q)$, the cluster of a lattice site is finite almost surely.

When $q = 2$ the FK model is related to the Ising model by the above calculation. In fact, the models can be coupled in a useful way and the two point correlation of the Ising model can be given in terms of a two point connectivity probability of the FK model with $q = 2$. Hence we will call the FK model $q = 2$ as *FK Ising model.* The critical value $p = p_c(2)$ on the square lattice corresponds to the critical value of J of the Ising model on the square lattice; namely,

$$J_c = \frac{1}{2}[\log(1 + \sqrt{2})].$$

Conformal invariance of the scaling limit of the FK Ising model at criticality is manifested in a conformally covariant observable. This means that there is a function $F_{U,a,b}(z)$ for each domain U and for any boundary points a and b, which is an expected value, and it transforms as

$$F_{U,a,b}(z) = (\phi'(z))^{\alpha} F_{\phi(U),\phi(a),\phi(b)}(\phi(z))$$

for some α. In this case $\alpha = 1/2$. The following result is also by Smirnov [24].

Theorem 2.2. *Consider the FK Ising model at criticality, $q = 2$ and $p = p_c(2)$. Denote the event that γ passes through z as $\gamma \to z$ and let $w(\gamma, z)$ be the winding of the curve γ from a to z measured in radians. The weighted probability*

$$F_h(z) = \mathbb{E}[\mathbb{1}_{\gamma \to z}\, e^{i\frac{1}{2}w(\gamma,z)}] \tag{9}$$

satisfies a discrete version of the Cauchy–Riemann equations and is hence discrete holomorphic and the function $h^{-\frac{1}{2}}F_h$ converges to $(\Phi'(z))^{\frac{1}{2}}$ where Φ is the conformal mapping from U onto the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < c\}$ so that a and b are mapped to the end points $-\infty$ and $+\infty$, respectively, and $c > 0$ is an universal constant.

3 Schramm–Loewner evolution

In this section we introduce the Loewner equation, Loewner chains and Schramm–Loewner evolution.

3.1 Conformal mappings

Remember that the complex plain is denoted by $\mathbb{C}$. The standard choices for a reference domain with a boundary are the upper half-plane and the unit disc, which are denoted as

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\} \quad \text{and} \quad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

respectively.

A complex valued function of a complex variable is a conformal mapping if it is analytic and one-to-one. A function f that is analytic near z_0 can be expanded as

$$f(z) = f(z_0) + f'(z_0)\,(z - z_0) + \frac{1}{2} f''(z_0)\,(z - z_0)^2 + \ldots$$

Locally near z_0, f is conformal if it is analytic and $f'(z_0) \neq 0$. Then the modulus $\lambda = |f'(z_0)|$ is positive and acts as the scaling and $R = f'(z_0)/|f'(z_0)|$ has unit modulus and acts as the rotation, i.e.

$$f(z) \approx f(z_0) + \lambda R\,(z - z_0)$$

near z_0. Hence, this is equivalent for the two other definitions of the conformal mapping.

The conformal mappings, that are defined on the whole plane or rather in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, are always Möbius transformations, i.e. of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Especially, the set of the conformal mappings of $\hat{\mathbb{C}}$ are parameterized by three complex parameters and therefore it is finite (dimensional) in a natural sense.

The special property of the plane $\mathbb{C}$ compared to $\mathbb{R}^n$, $n > 2$, is that there is a richness of the conformal mappings. To see this, the assumption, that the maps are defined on the whole plane, has to be discarded. Hence it is essential that the map is defined on a domain with a boundary. A simply connected domain is an open subset of $\mathbb{C}$ so that the set and its complement are both connected. For such domains we have the following theorem.

Theorem 3.1 (The Riemann mapping thorem)**.** *Let U be a simply connected domain in $\mathbb{C}$ not equal to the whole plane $\mathbb{C}$. Let $z_0 \in U$. Then there is a unique conformal map f from $\mathbb{D}$ onto U so that $f(0) = z_0$ and the complex number $f'(0)$ is a positive real number.*

3.2 Domain Markov property and conformal invariance

As a motivation for the Loewner equation and Schramm–Loewner evolutions, let's introduce so called *Schramm's principle* (originally in [16], a clear presentation in [23]).

Given a model of statistical physics, the law of an interface in the model determines a collection of probability measures $(\mathbb{P}^{U,a,b})$, where each measure defined on the corresponding set of curves γ in $\overline{U}$ connecting two boundary points a and b of a simply connected domain U. Choose some consistent parameterization for such curves so that they are parametrized by $t \in [0, \infty)$. Suppose that $(\mathbb{P}^{U,a,b})$ satisfies the following two requirements:

(CI) Conformal invariance: For any triplet (U, a, b) and any conformal mapping $\phi : U \to \mathbb{C}$, it holds that $\phi\mathbb{P}^{U,a,b} = \mathbb{P}^{\phi(U),\phi(a),\phi(b)}$.

(DMP) Domain Markov property: Suppose we are given $\gamma[0, t]$, $t > 0$. The conditional law of $\gamma(t + s)$ given $\gamma[0, t]$ is the same as the law of $\gamma(s)$ in the slit domain $(U \setminus \gamma[0, t], \gamma(t), b)$. That is

$$\mathbb{P}^{U,a,b}(\,\cdot \mid \gamma[0, t]) = \mathbb{P}^{U\setminus\gamma[0,t],\gamma(t),b}$$

Then Schramm's principle states that such $(\mathbb{P}^{U,a,b})$ can only be one of the chordal SLE_κ-processes. We will comment this in the end of the section 3.6.

CI is the property that the law of the interface in U and the law of the interface in $\phi(U)$ are connected through the conformal mapping ϕ. This property holds at criticality.

The DMP holds basically for all interfaces of statistical physics, also outside criticality. The interface is parameterized as a curve. If one moves along the curve, one explores the configuration of the system next to the curve. In Figure 1, during this exploration the curve meets red and white hexagons. The red hexagons are on one side of the curve, let's say on the left-hand side, and the white hexagons are on the other side, on the right-hand side. If this process is stopped at any time, then the left-hand side of the curve up to this moment consists of red hexagons and the right-hand side consists of white hexagons. From the left-hand boundary, i.e. from the tip $\gamma(t)$ of the curve to the target point b, of the slit domain $U \setminus \gamma[0,t]$ consists of red hexagons. The boundary conditions in the slit domain are the same as in the original domain: there is one red arc and one white arc. The DMP holds if the model has a Markov property in the sense, that if the system is conditioned to have a certain configuration in a part of the system, then the conditioned model is the same model in the smaller region with the conditioning acting as a boundary condition.

3.3 Capacity of a hull

Suppose we are given a simple curve γ in the upper half plane starting from the boundary, i.e. γ is a continuous, one-to-one mapping $[0,\infty)$ to $\mathbb{C}$ so that $\gamma(0) \in \mathbb{R}$ and $\gamma\big((0,\infty)\big) \subset \mathbb{H}$. For each $t > 0$, the set $\gamma\big((0,t]\big) \subset \mathbb{H}$ is closed in $\mathbb{H}$ and its complement is simply connected. We want measure the size of such a set.

Definition 3.2. A subset $K \subset \mathbb{H}$ is said to be a *hull*, if K is bounded, K is closed in $\mathbb{H}$ (i.e. $K = \mathbb{H} \cap \overline{K}$) and $\mathbb{H} \setminus K$ is simply connected.

If K is a hull then the complement $H = \mathbb{H} \setminus K$ is open and simply connected. By the Riemann mapping theorem there are conformal mappings from H onto $\mathbb{H}$. Let g_K be such a mapping. We can choose g_K so that $g_K(\infty) = \infty$.

Let's state some consequences of choosing g_K this way. Since K is bounded,

$$\mathrm{rad}(K) = \inf\{r > 0 : K \subset B(0,r)\}$$

is finite. The mapping g_K extends continuously to the part of the real axis away from K. Since $\mathbb{R} \setminus [-\mathrm{rad}(K), \mathrm{rad}(K)]$ is mapped in $\mathbb{R}$, the imaginary part of g_K vanishes on this part of the boundary and therefore the mapping g_K can be extended to $\mathbb{C} \setminus \overline{B}(0, \mathrm{rad}(K))$ by the Schwarz reflection principle. This extended map is analytic also at infinity in the sense that $1/g_K(1/z)$ is analytic at 0. From this it follows that the expansion

$$g_K(z) = bz + a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots \qquad (10)$$

holds near the infinity, infact for $z \in \mathbb{H}$ and $|z| > \mathrm{rad}(K)$. Again since $\mathbb{R} \setminus [-\mathrm{rad}(K), \mathrm{rad}(K)]$ is mapped in $\mathbb{R}$, the coefficients $b, a_0, a_1, \dots$ are real. Furthermore, since the image of g_K is $\mathbb{H}$, $b > 0$.

We want to choose g_K uniquely. Any a mapping of the form $\beta g_K + \alpha$ where $\beta > 0$ and $\alpha \in \mathbb{R}$ is a conformal map from H onto $\mathbb{H}$ so that infinity is mapped to itself. Now it is clear that by using scaling and translation we can choose g_K to satisfy the *hydrodynamical normalization*

$$\lim_{z\to\infty} \big[g_K(z) - z\big] = 0.$$

With this choice the mapping is unique. This choice is equivalent for choosing a mapping that has the expansion (10) with $b = 1$ and $a_0 = 0$. Hence with this choice

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \dots \tag{11}$$

near infinity. The coefficient $a_1 = a_1(g_K) = a_1(K)$ is called the *upper half-plane capacity* of K or simply the capacity of K. Note that it is notationally useful to think the capacity both a property of the hull and of the mapping.

A straightforward application of the expansion (11) can be used to check the capacity satisfies the following properties:

- The *additivity* property: If g and h are hydrodynamically normalized, i.e.

  $$g(z) = z + \frac{a_1(g)}{z} + \dots \qquad \text{and} \qquad h(z) = z + \frac{a_1(h)}{z} + \dots$$

  then $g \circ h$ is hydrodynamically normalized and

  $$a_1(g \circ h) = a_1(g) + a_1(h).$$

- The *scaling* property: For any $\lambda > 0$, define $g_\lambda(z) = \lambda g(z/\lambda)$ whenever this makes sense. If g is hydrodynamically normalized, then g_λ is hydrodynamically normalized and

  $$a_1(g_\lambda) = \lambda^2 a_1(g).$$

  This implies that if the hull K scaled by λ is denoted by λK then $a_1(\lambda K) = \lambda^2 a_1(K)$.

A collection of hulls $(K_t)_{t\geq 0}$ is *growing*, if for each $t < s$, $K_t \subset K_s$. If γ is a simple curve in $\mathbb{H}$ parameterized by $[0,\infty)$ and $\gamma(0) \in \mathbb{R}$, then

$$K_t = \gamma\big((0,t)\big)$$

defines a hull and the collection $(K_t)_{t\geq 0}$ is growing. In this case we identify $(K_t)_{t\geq 0}$ with γ and say that $(K_t)_{t\geq 0}$ is a simple curve γ.

It is possible to show that the capacity is strictly increasing in the sense of the following lemma.

Lemma 3.3. *(i) Let K and L be two hulls. If $K \subset L$, then*

$$0 \leq a_1(K) \leq a_1(L).$$

The equality holds in the first inequality only if $K = \emptyset$ and in the second inequality only if $K = L$.

(ii) Let $(K_t)_{t\geq 0}$ be a growing collection of hulls. Then $t \mapsto a_1(K_t)$ is a non-decreasing map from $[0,\infty)$ to $[0,\infty)$.

In short, the proof of the lemma is based on that $h_K(z) = \mathrm{Im}(z - g_K(z))$ is a bounded harmonic function with non-negative boundary values, and not identically zero. By the minimum it is positive in the interior points. The capacity can be written as $a_1(K) = \frac{2R}{\pi} \int_0^\pi h_K\left(Re^{i\theta}\right) \sin(\theta) \mathrm{d}\theta$ for any $R > \mathrm{rad}(K)$. The rest follows from the additivity.

The capacity is continuous in the sense of the following lemma.

Lemma 3.4. *(i) Denote* $K^\varepsilon = \mathbb{H} \setminus H^\varepsilon$*, where* H^ε *the unbounded component of* $\mathbb{H} \setminus \{z \in \mathbb{H} : \mathrm{d}(z, K) \leq \varepsilon\}$*. Let* $M > 0$*. Then uniformly for any hull* K *so that* $\mathrm{rad}(K) \leq M$ *the following holds: for each* $\delta > 0$ *there is* $\varepsilon > 0$ *so that*

$$a_1(K^\varepsilon) < a_1(K) + \delta$$

(ii) Let M, δ, ε *be as above. If* K *and* L *are hulls so that* $\mathrm{rad}(K) \leq M$, $\mathrm{rad}(L) \leq M$ *and* K *and* L *are closer then* ε *to each other in the sense that* $K \subset L^\varepsilon$ *and* $L \subset K^\varepsilon$ *then*

$$|a_1(K) - a_1(L)| < \delta$$

This lemma can be proven almost the same way as Lemma 3.3. The function $\hat{h}_{K,\varepsilon} = \mathrm{Im}(g_K(z) - g_{K^\varepsilon}(z))$ is harmonic, and the boundary values can be proven to be small.

This section can be summarized in the following way: the upper half-plane capacity measures the size of a hull, since the capacity is non-negative and strictly increasing. By the additivity, the most natural way to use it as the parameterization of a collection of hulls $(K_t)_{t \geq 0}$ is linear in t so that

$$a_1(K_t) = ct$$

for some constant $c > 0$. For historical reason, the standard choice is $c = 2$.

3.4 Loewner equation

In this section we present the Loewner equation which is an ordinary differential equation (ODE) in t satisfied by $g_t(z)$ for each z as long as the collection of hulls $(K_t)_{t \geq 0}$ is parameterized so that $a_1(K_t) = 2t$ and the growth is *local* in a suitable sense. Especially this applies to $(K_t)_{t \geq 0}$ corresponding to a simple curve.

We will first motivate the form of the Loewner equation by considering an iteration of conformal mappings. Let $x_0 \in \mathbb{R}$ and $\delta > 0$. The mapping defined by

$$\phi_{x_0,\delta}(z) = z + \frac{2\delta}{z - x_0} \tag{12}$$

is a conformal mapping from the upper half-plane with a semi-disc removed $\mathbb{H} \setminus \overline{B}(x_0, \sqrt{2\delta})$ onto the upper half-plane $\mathbb{H}$. The mapping is hydrodynamically normalized and $a_1(\phi_{x_0,\delta}) = 2\delta$.

The inverse map $\phi_{x_0,\delta}^{-1}$ maps the upper half-plane to the upper half-plane with a semi-disc removed. Add these semi-discs on top of each other by defining a map

$$f_{\delta n} = \phi_{x_1,\delta}^{-1} \circ \phi_{x_2,\delta}^{-1} \circ \ldots \circ \phi_{x_n,\delta}^{-1}$$

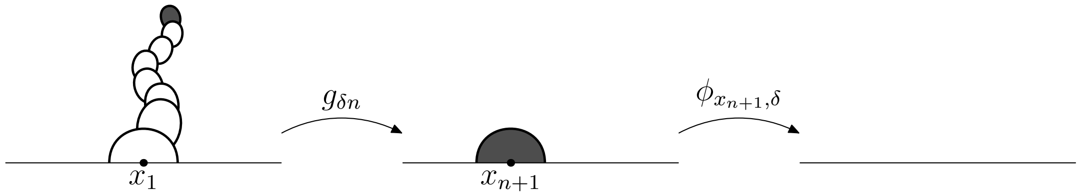


Figure 5: An iteration of conformal mappings. The mapping $g_{\delta(n+1)}$ can be decomposed as $\phi_{x_{n+1},\delta} \circ g_{\delta n}$.

for each $n \in \mathbb{N}$. Here x_n is a sequence of real numbers. Let the image of $f_{\delta n}$ be $H_{\delta n}$. And let $K_{\delta n}$ be the complement of $H_{\delta n}$. The shape of $K_{\delta(n+1)}$ is schematically illustrated in the leftmost part of Figure 5.

The inverse of this map of $f_{\delta n}$ and the hydrodynamically normalized map of the hull $K_{\delta n}$ is

$$g_{\delta n} = \phi_{x_n,\delta} \circ \phi_{x_{n-1},\delta} \circ \ldots \circ \phi_{x_1,\delta}.$$

We can write the difference of two consecutive mappings by

$$\begin{aligned} g_{\delta(n+1)}(z) - g_{\delta n}(z) &= (\phi_{x_{n+1},\delta} \circ g_{\delta n})(z) - g_{\delta n}(z) \\ &= \frac{2\delta}{g_{\delta n}(z) - x_{n+1}}. \end{aligned} \tag{13}$$

Suppose that we can approximate a simple curve γ with the above sets. As we take $\delta \searrow 0$ we have to take also the increments $x_{n+1} - x_n$ smaller. Hence it is natural that if this works, in the limit x_n is replaced by a continuous function. The continuum version of the difference equation (13) as $\delta \searrow 0$ will be the ordinary differential equation of the following theorem.

Theorem 3.5. *Let $\gamma : [0,T] \to \mathbb{C}$ be a simple curve and let $K_t = \gamma(0,t]$. Assume that $\gamma(0) \in \mathbb{R}$, $\gamma(0,T] \subset \mathbb{H}$ and $a_1(K_t) = 2t$ for each $t \in [0,T]$. Then $g_t = g_{K_t}$ satisfies for each $z \in \mathbb{H} \setminus K_T$ the ordinary differential equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}. \tag{14}$$

Here $W_t = g_t(\gamma(t))$ which is well-defined.

The equation (14) is called the *upper half-plane Loewner equation* or simply the Loewner equation. Note that the Loewner equation applies upto time t when z becomes a part of the hull: if $z = \gamma(s)$ for some $s > 0$ then choose $0 < T < s$. Then $z \in \mathbb{H} \setminus K_T$ and $g_t(z)$ satisfies the Loewner equation by the above theorem.

The proof of Theorem 3.5 uses a uniform estimate such as the following lemma from [11].

Lemma 3.6. *Let $f_K = g_K^{-1}$. There is an universal constant $C > 0$ so that the following holds: If K is a hull and $x_0 \in \mathbb{R}$ and $R > 0$ are such that $K \subset B(x_0, R)$, then*

$$\left| f_K(z) - z + \frac{a_1(K)}{z - x_0} \right| \leq \frac{C\, R\, a_1(K)}{|z - x_0|^2}$$

for any $z \in \mathbb{H}$ so that $|z - x_0| \geq CR$.

The proof of the lemma shows more fundamental origin of the form of the right-hand side of the Loewner equation. It comes from the Poisson kernel of the upper half-plane.

Let $t > 0$ and $\delta > 0$. Denote $\hat{K}_{t,\delta} = g_t(K_{t+\delta} \setminus K_t)$ and denote the corresponding mappings by $\hat{g}_{t,\delta}$ and $\hat{f}_{t,\delta}$. The capacity of $\hat{K}_{t,\delta}$ is δ. Then Lemma 3.6 applies to the mapping $\hat{f}_{t,\delta}$ and using $g_t = \hat{f}_{t,\delta} \circ g_{t+\delta}$ we have that

$$\begin{aligned} \left| \frac{g_{t+\delta}(z) - g_t(z)}{\delta} - \frac{2}{g_{t+\delta}(z) - W_t} \right| &= \left| \frac{g_{t+\delta}(z) - \hat{f}_{t,\delta}(g_{t+\delta}(z))}{\delta} - \frac{2\delta}{g_{t+\delta}(z) - W_t} \right| \\ &= \frac{C\, R_{t,\delta}}{|g_{t+\delta}(z) - W_t|^2}. \end{aligned}$$

Here $R_{t,\delta}$ is the smallest radius $r > 0$ so that $\hat{K}_{t,\delta} \subset B(W_t, r)$. It can be shown that for a given curve, uniformly in $t \in [0, T)$ the radius $R_{t,\delta} \searrow 0$ as $\delta \searrow 0$. Hence Theorem 3.5 is proven.

3.5 Loewner chains

A nice feature of the Loewner equation is that any continuous function $t \mapsto W_t$ corresponds to a growing family of hulls. Given a function that drives the Loewner equation we construct the hulls $(K_t)_{t\geq 0}$ as follows.

For each $z \in \mathbb{H}$ define $g_t(z) = z_t$ as the solution of the Loewner equation

$$\frac{\mathrm{d}z_t}{\mathrm{d}t} = \frac{2}{z_t - W_t}, \quad z_0 = z. \tag{15}$$

The equation is the same as the equation (14), but written for just single point z. Then $g_t(z)$ is well defined for $0 < t < \hat{T}(z)$ where

$$\hat{T}(z) = \sup \{t \geq 0 \,:\, g_s(z) \neq W_s \text{ for any } s \in [0, t]\}. \tag{16}$$

Let's define

$$K_t = \{z \in \mathbb{H} \,:\, \hat{T}(z) \leq t\} \text{ and } H_t = \mathbb{H} \setminus K_t = \{z \in \mathbb{H} \,:\, \hat{T}(z) > t\}. \tag{17}$$

Indeed the following theorem shows K_t is a hull and g_t is the conformal map g_{K_t}.

Theorem 3.7. *Let W_t, g_t, K_t and H_t be as above. Then for each $t > 0$, K_t a hull and g_t is the conformal map from H_t onto $\mathbb{H}$ so that g_t is hydrodynamically normalized, i.e.*

$$g_t(z) = z + 2tz^{-1} + \ldots \tag{18}$$

near infinity.

The proof is done by analyzing the ODE. For example, to prove that K_t is bounded we need to separately analyze the real part and the imaginary part of the equation (15). On the interval $s \in [0, t]$, the real part $x_s = \mathrm{Re}(z_s)$ flows away from U_s and hence the point z with large $|\mathrm{Re}(z)|$ cannot be reached. The imaginary part $y_s = \mathrm{Im}(z_s)$ is monotonically decreasing but we can control the speed so that $y_t \geq \sqrt{y_0^2 - 4t}$. So the points z such that $\mathrm{Im}(z) > \sqrt{4t}$ cannot be reached during the interval $[0, t]$. The proofs of the other claims can be found in [26].

The growing family of hulls $(K_t)_{t\geq 0}$ constructed this way is called the *Loewner chain* associated to driving function $(W_t)_{t\geq 0}$. We say that $(K_t)_{t\geq 0}$ is

- a *simple curve*, if there exists a simple curve γ such that $K_t = \gamma((0,t])$,
- *generated by a curve*, if there exists a curve γ such that $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma((0,t])$,

In general, a Loewner chain is neither a simple curve nor generated by a curve. Simplest example of such a pathology is an infinite spiral that first winds around, say, a disc infinitely many times and then unwinds. Following theorem gives necessary and sufficient condition for a collection of growing hulls to have a continuous driving function.

Theorem 3.8. *Let $(K_t)_{t\geq 0}$ be an increasing family of hulls such that $a_1(K_t) = 2t$, for any $t \geq 0$. Then the following are equivalent:*

- *$(K_t)_{t\geq 0}$ is a Loewner chain associated to a continuous driving function $(W_t)_{t\geq 0}$.*
- *For all $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \leq T$ there exist a bounded connected set $S \subset \mathbb{H} \setminus K_t$ with diameter $\leq \varepsilon$ disconnecting $K_{t+\delta} \setminus K_t$ from infinity in $\mathbb{H} \setminus K_t$.*

For the proof see [11]. The proof the direct implication if proved by the same methods as Theorem 3.7 and the inverse implication by Lemma 3.6 as before.

3.6 Schramm–Loewner evolution

If the methods related to the Loewner equation are applied to a random curve, then the driving function $(W_t)_{t\geq 0}$ is random. The next definition gives a very important example of this.

Definition 3.9. Let $\kappa \geq 0$. The chordal *Schramm–Loewner evolution* SLE_κ is the Loewner chain associated with $W_t = \sqrt{\kappa} B_t$, where B_t is a standard, one-dimensional Brownian motion with $B_0 = 0$.

A *standard, one-dimensional Brownian motion* is a continuous stochastic process $(B_t)_{t\geq 0}$ so that

- $B_0 = 0$
- For each n and for any $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n$, the increments $B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}, \ldots, B_{t_n} - B_{s_n}$ are independent.
- For each $s > 0$, the distribution of $B_{t+s} - B_t$ is the normal distribution $N(0,s)$, the same of all $t \geq 0$.

By the second and the third property, a Brownian motion is said to have independent and stationary increments.

SLE_κ can be seen as a limit $\delta \to 0$ of the iteration of the semi-disc maps as in Figure 5. The fact that a Brownian motion has independent and stationary increments corresponds to $x_1, x_2 - x_1, x_3 - x_2, \ldots$ being independent and identically distributed.

If we don't want to emphasize the value of κ we often write just SLE. Consider a chordal SLE_κ in $\mathbb{H}$, and let g_t be the conformal maps associated with K_t. Let's list few properties of SLE.

- SLE is scale-invariant: $(\lambda K_{t\lambda^{-2}})_{t\geq 0}$ has the same law as $(K_t)_{t\geq 0}$.
- The law of SLE is symmetric respect to the imaginary axis.
- SLE has the conformal Markov property: Let $t \geq 0$. For any $s \geq 0$ let

  $$\tilde{K}_s = g_t(K_{t+s} \setminus K_t) - W_t.$$

  Then $(\tilde{K}_t)_{t\geq 0}$ has the same law as $(K_t)_{t\geq 0}$ and $(\tilde{K}_t)_{t\geq 0}$ is independent of $(K_t)_{0\leq t\leq s}$.

The last property correspond to the DMP of the section 3.2. The fact that $(\tilde{K}_t)_{t\geq 0}$ has the same law as $(K_t)_{t\geq 0}$ is called the *stationarity* of SLE and it is the main concept in [ii].

Let U be a simply connected domain in $\mathbb{C}$ and let a and b be two boundary points. Let Φ be a conformal map from $\mathbb{H}$ onto U such that $\Phi(0) = a$ and $\Phi(\infty) = b$. This doesn't determine Φ fully. It is determined up to a multiplicative factor: for each $\lambda > 0$, Φ_λ defined by $\Phi_\lambda(z) = \Phi(\lambda z)$ would satisfy these conditions.

Definition 3.10. A collection $(K_t)_{t\geq 0}$ of subsets of U is said to be the *chordal SLE$_\kappa$ in U from a to b* if $(\Phi^{-1}(K_t))_{t\geq 0}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ and Φ is as above.

Although Φ is only defined up to a multiplicative factor, SLE$_\kappa$ in U is unique because SLE$_\kappa$ in $\mathbb{H}$ is scale-invariant. This definition makes conformal invariance immediate. The DMP can be proven using CI and the conformal Markov property of SLE.

The following result tells that chordal SLEs are curves.

Theorem 3.11. *If $\kappa \in [0, 4]$, SLE$_\kappa$ is a simple curve. If $\kappa > 4$, SLE$_\kappa$ is generated by a curve that is not simple. If $\kappa \geq 8$, it is a space filling curve.*

For the proof see [15].

3.7 SLE martingales

In this section, a couple of example calculations are made using SLE. The goal is to derive quantities which are related to the percolation and to the FK Ising model.

For a basic introduction to Itô calculus the reader is referred some introductory text such as [14]. Our starting point is the following version of Itô's lemma for complex valued processes and analytic functions.

Lemma 3.12. *Let f is analytic and Z_t is a complex valued semimartingale, i.e. $Z_t = X_t + iY_t$ and X_t and Y_t are (real) continuous semimartingales. Then*

$$\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t + \frac{1}{2}f''(Z_t)(\mathrm{d}Z_t)^2 \tag{19}$$

where $\mathrm{d}Z_t = \mathrm{d}X_t + i\mathrm{d}Y_t$, $(\mathrm{d}Z_t)^2 = (\mathrm{d}X_t)^2 + 2i\mathrm{d}X_t\mathrm{d}Y_t - (\mathrm{d}Y_t)^2$ and the notation $(\mathrm{d}X_t)^2$ really means the differential of the quadratic variation (for example $(\mathrm{d}B_t)^2 = \mathrm{d}t$).

The differential of the quadratic variation $(\mathrm{d}Z_t)^2$ is is quite naturally interpreted as $(\mathrm{d}(X_t + iY_t))^2$ expanded in using the usual arithmetics.

The first example we calculate is a local martingale depending on one point $z \in \mathbb{H}$. For a fixed $z \in \mathbb{H}$, denote

$$\begin{aligned} Z_t &= g_t(z) - W_t \\ A_t &= g_t'(z) \end{aligned}$$

Calculate the Itô differential

$$\mathrm{d}\left(Z_t^\alpha A_t^\beta\right) = Z_t^{\alpha-2} A_t^\beta \left[2\alpha + \frac{\kappa}{2}\alpha(\alpha-1) - 2\beta\right] \mathrm{d}t - Z_t^{\alpha-1} A_t^\beta \sqrt{\kappa}\mathrm{d}B_t \tag{20}$$

If we set $\alpha = -\beta$ in the equation (20), then $(A_t/Z_t)^\beta$ is a local martingale if and only if the drift vanishes, i.e. $\beta = 0$ or

$$\beta = \frac{8}{\kappa} - 1$$

For $\kappa = 16/3$, the non-trivial local martingale is $(A_t/Z_t)^{1/2}$ which corresponds to the observable of Theorem 2.2. If it is stopped before Z_t hits zero, then it is a martingale.

The second example is a martingale depending on two points. Let

$$\hat{Z}_t = \frac{g_t(z) - W_t}{g_t(1) - W_t}.$$

The Itô differential is

$$\mathrm{d}\hat{Z}_t = \frac{1}{(g_t(1) - W_t)^2}\left[\frac{2}{\hat{Z}_t} - 2\hat{Z}_t + \kappa(\hat{Z}_t - 1)\right]\mathrm{d}t - \frac{\hat{Z}_t - 1}{g_t(1) - W_t}\sqrt{\kappa}\mathrm{d}B_t \tag{21}$$

One possible simplification to this formula is to define a time change

$$\phi(t) = \int_0^t \frac{\mathrm{d}u}{(g_u(1) - W_u)^2}$$

and $s = \phi(t)$. $\tilde{Z}_s = \hat{Z}_{\phi^{-1}(s)}$ Then

$$\mathrm{d}\tilde{Z}_s = \left[\frac{2}{\tilde{Z}_s} - 2\tilde{Z}_s + \kappa(\tilde{Z}_s - 1)\right]\mathrm{d}s - (\tilde{Z}_s - 1)\sqrt{\kappa}\mathrm{d}\tilde{B}_s. \tag{22}$$

For a smooth function F we have

$$\begin{aligned} \mathrm{d}F(\tilde{Z}_s) = &\left[\left(2\frac{1+\tilde{Z}_s^2}{\tilde{Z}_s} + \kappa(\tilde{Z}_s - 1)\right)F'\left(\tilde{Z}_s\right) + \frac{\kappa}{2}(\tilde{Z}_s - 1)^2 F''\left(\tilde{Z}_s\right)\right]\mathrm{d}s \\ &- (\tilde{Z}_s - 1)F'\left(\tilde{Z}_s\right)\sqrt{\kappa}\mathrm{d}\tilde{B}_s \end{aligned} \tag{23}$$

which is a local martingale if

$$\begin{aligned} 0 &= F''(z) + \left(-\frac{4}{\kappa}\,\frac{z+1}{z(z-1)} + 2\frac{1}{z-1}\right)F'(z) \\ &= F''(z) + \left(\frac{4}{\kappa}\,\frac{1}{z} + \left(2 - \frac{8}{\kappa}\right)\frac{1}{z-1}\right)F'(z). \end{aligned}$$

For any solution F of this equation

$$F'(z) = C\, z^{\alpha_0 - 1}(z-1)^{\alpha_1 - 1}$$

where

$$\alpha_0 = 1 - \frac{4}{\kappa} \quad \text{and} \quad \alpha_1 = \frac{8}{\kappa} - 1.$$

For $4 < \kappa < 8$, this defines a conformal mapping F from $\mathbb{H}$ onto a triangle such that the points 0, 1 and ∞ are mapped to the vertices of the triangle, and the triangle has angles $\pi\alpha_0$, $\pi\alpha_1$ and $\pi\alpha_\infty$, where

$$\alpha_\infty = 1 - \alpha_0 - \alpha_1 = 1 - \frac{4}{\kappa}.$$

It is an example of Schwarz–Christoffel mapping. This is useful since F is then bounded and both $\mathrm{Re} F(\tilde{Z}_s)$ and $\mathrm{Im} F(\tilde{Z}_s)$ are martingales.

For $\kappa = 6$, $\alpha_0 = \alpha_1 = \alpha_\infty = 1/3$ and F is a conformal mapping from $\mathbb{H}$ onto an equilateral triangle. This corresponds to the Cardy–Smirnov formula (Theorem 2.1).

4 On the results of this thesis

4.1 Convergence of an interface of a lattice model to SLE

Consider a model such as the percolation or the FK Ising model. The full conformal invariance in the sense of random geometry is the conformal invariance of the law of all the interfaces. The interfaces can be of two different types. There are curves starting and ending to the boundary and curves that are closed loops. The starting point of establishing the full conformal invariance is to prove it for a single interface. The paper [iii] is proving large part methods needed for the convergence of a single interface to SLE. The methods should be applicable for the full conformal invariance since the collection of all the interfaces can also be described by SLE–type process, called the *exploration tree* [20]. The authors of [iii] plan to report on this.

Let's try to understand the general features of the proof that an interface of a given model converges to SLE_κ, for some $\kappa > 0$. As before take a bounded, simply connected domain U and its boundary points a and b. Define a graph U_h which is a piece of the lattice that the model is defined on. Here $h > 0$ is the lattice mesh. Take vertices a_h and b_h of U_h that lying on the outer face of the graph. Assume that U_h approximates the domain U in a suitable sense. This could be for example, so that discrete harmonic functions U_h converge to the harmonic functions on U with same boundary data.

The boundary conditions of the model are chosen so that an interface is formed connecting a_h and b_h. Let the curve be denoted by γ and its law by $\mathbb{P}_h$ for fixed U, a and b.

In order to make a mathematical theory on the convergence of random curves, a metric is needed to measure distances between two curves in the space of curves. A choice is made so that the curves are parametrized by the interval $[0,1]$. Define a metric as the infimum over all reparameterizations of the supremum metric:

$$\mathrm{d}_X(\gamma_1, \gamma_2) = \inf \left\{ \|\gamma_1 \circ \phi_1 - \gamma_2 \circ \phi_2\|_\infty \ : \ \begin{array}{c} \text{for } j = 1,2,\ \phi_j : [0,1] \to [0,1] \\ \text{stricly increasing and onto} \end{array} \right\}.$$

It is then natural to identify γ_1 and γ_2 if $\mathrm{d}_X(\gamma_1, \gamma_2) = 0$. Consequently, the space of curves X is defined as the set of continuous mappings $\gamma : [0, 1] \to \mathbb{C}$ modulo the strictly increasing and onto reparameterizations of the interval $[0, 1]$.

For a given model of statistical physics, the law of the interface defines a probability measure $\mathbb{P}_h$ on X such that the end points of the curve are a_h and b_h and the curve is polygonal path in a lattice with lattice mesh $h > 0$. The interface has a scaling limit as a curve if the sequence of probability measures $(\mathbb{P}_h)_{h>0}$ converges to a probability measure $\mathbb{P}$. There are many notions of the convergence of measures, but the one used here is the weak convergence, which is natural in many ways. The sequence $(\mathbb{P}_h)_{h>0}$ converges to $\mathbb{P}$, if the expected values $\mathbb{E}_h f$ converge to $\mathbb{E} f$ for each continuous function f on X.

A general structure of the proof of the convergence is the following

- **(1a)** Establish relative compactness of the sequence $(\mathbb{P}_h)_{h>0}$; consequently, each subsequence has a converging subsequence. Hence, a sequence $h_n \searrow 0$ can be chosen so that $\mathbb{P}_{h_n}$ converges to a limit $\mathbb{P}$.

- **(1b)** The curves of the domain U can be transformed conformally to the upper-half plane. Check that the limiting measure $\mathbb{P}$ is supported on curves such that when transformed to the upper half-plane $\mathbb{H}$, the curve can be parametrized by the capacity and the corresponding hulls satisfy the assumption of Theorem 3.8 characterizing Loewner chains.

- **(2)** Identify the limit as SLE_κ, for a unique $\kappa > 0$.

First two aims of the theory, **(1a)** and **(1b)**, are collectively called *a priori bounds*. The question **(1a)** was already studied in [1] prior to the invention of SLE. The study therein is not limited to $\mathbb{C}$, but can be carried out for any $\mathbb{R}^n$. In this thesis, paper [iii] is intended to answer the questions **(1a)** and **(1b)** for a large range of models. Namely, the paper gives a sufficient condition for **(1a)** and **(1b)**; see the section 2.5 therein and the equivalent conditions, Condition A and Condition B, and the main result, Theorem 2.5. The condition is checked to hold for the FK Ising model. Also paper [iii] deals the regularity of the random curve near the end points. It has to be stressed that this is the more generic part of the above principle, whereas **(2)** contains parts that are more specific for the model.

The common part of **(2)** for different models is the existence of an martingale observable that has nice properties in conformal transformations. Consider a filtration $(\mathcal{F}_t)$ where $\mathcal{F}_t$ is generated by $\gamma(s)$, $0 \le s \le t$, and γ is parametrized, say, by the length of the path. Remember that $\mathbb{P}_h$ is supported on quite regular curves such as broken lines with line segments of length h. The quantities of the form

$$M_t^h = \mathbb{E}_h[\, X \,|\, \mathcal{F}_t \,]$$

are martingales, where X is an integrable random variable. Note that this kind of martingale is a martingale in any parameterization: if $(\hat{\mathcal{F}}_s)$ is another filtration then

$$\hat{M}_s^h = \mathbb{E}_h \left[X \,\middle|\, \hat{\mathcal{F}}_s \right]$$

is a martingale. Therefore we have a large freedom to choose a parameterization. This can be seen in a more general level in the theory of martingales.

Consider the site percolation introduced in the section 2.4. The Cardy–Smirnov formula is the conformally invariant observable. In terms of the curve the event of the Cardy–Smirnov formula can be written as

$$A = \{\gamma : \tau_x = \tau_c\}$$

where τ_x and τ_c are the times that the curve disconnects b from x and c, respectively. So the event occurs when the curve disconnects both points at the same time. By the domain Markov property the conditional expectation can be written as

$$\mathbb{P}_h^{U,a,b}[A|\mathcal{F}_t] = \mathbb{P}_h^{U\setminus\gamma(0,t],\gamma(t),b}[A] = F_h^{U\setminus\gamma(0,t],\gamma(t),b}(x,c)$$

where $F_h^{U,a,b}(x,c)$ is the probability in the Cardy–Smirnov formula. If is some uniform continuity of this quantity with respect to γ, then the limit is a martingale (with respect to $\mathbb{P}$). If $F = F^{\mathbb{H},0,\infty}$ then

$$F(g_t(x), g_t(1))$$

is a martingale. This is enough to determine W_t, since it implies that W_t and $W_t^2 - 6t$ are martingales. By Lévy's characterization of Brownian motion $W_t = \sqrt{6}B_t$ where B_t is a standard, one dimensional Brownian motion. Therefore, if the scaling limit of the percolation interface at criticality exists and can be described by the Loewner equation, then it has to be SLE_6. See more in [22]. The full scaling limit of the critical site percolation is studied in [2].

The critical FK Ising ($q = 2$, $p = p_c(2)$) converges to $\mathrm{SLE}_{16/3}$ based on the convergence of the obsevable of Theorem 2.2 proven in [24] and the a priori bounds proven in [iii] of this thesis $\kappa = 16/3$. Note that the result of [iii] relies on a convergence of another but related observable of [25].

Other models that have proven to converge to SLE_κ are the loop-erased random walk ($\kappa = 2$) and the uniform spanning tree ($\kappa = 8$) [12] and two models the harmonic explorer [17] and the Gaussian free field [18] related to $\kappa = 4$.

4.2 Reversibility and duality

The papers [i] and [ii] of this thesis are best introduced in the context of two symmetries, the reversibility and the duality, of SLE. The methods used in these papers might be even more interesting than the partial results on the reversibility and the duality.

The reversibility was stated as Conjecture 9.10 and the duality as Problem 9.6 in [15]. The reversibility is a property which is immediate for a interface of a statistical physics model, but which doesn't follow easily from the definition of SLE. The reversibility of SLE means that a chordal SLE_κ and its time inverse have the same law. In the notation of the section 3.2, it means that if $\mathbb{P}^{U,a,b}$ is the law of γ and $\hat{\gamma}$ is the time change of γ so that the order of time is inverted, especially $\hat{\gamma}(0) = b$, and $\hat{\gamma}$ is parametrized by the capacity seen from a, then the law of $\hat{\gamma}$ is $\mathbb{P}^{U,b,a}$. Although this definition is exactly formulated, even better formulation is given in terms of $SLE_\kappa(\rho)$ where $\rho = \kappa - 6$. For this see [i] and [ii]. The reversibility was proven recently in [28].

The duality is less exactly formulated symmetry. It states that for $\kappa > 4$, $\mathrm{SLE}_{16/\kappa}$ and the outer boundary (the boundary of the hull of) of SLE_κ look locally the same. Although, it was more exactly formulated [5] in terms of $\mathrm{SLE}_\kappa(\rho_1, \rho_2, \ldots, \rho_n)$-processes, this formulation doesn't shed more light on why the conjecture should hold.

Consider the full trace $\gamma(0, \infty)$ of SLE_κ. For $\kappa < 8$, it almost surely avoids a given point $x \in \mathbb{R}$. To study the component of x in $\mathbb{H} \setminus \gamma(0, \infty)$, use a Möbius transformation ϕ of $\mathbb{H}$ to map x to ∞. Let H be the unbounded component of $\mathbb{H} \setminus \phi(\gamma(0, \infty))$ and let G be the hydrodynamically normalized map from H onto $\mathbb{H}$. This way SLE_κ defines a law for a random hull $K = \mathbb{H} \setminus H$ which describes the whole SLE_κ. This works best for $\kappa \leq 4$ when SLE_κ is a simple curve and hence there are only two components in $\mathbb{H} \setminus \gamma(0, \infty)$. For $\kappa > 4$, the joint law of many components should be considered.

In the paper [ii], the stationarity of SLE is used to formulate the law G as a stationary measure on conformal mappings under the SLE_κ induced flow. This is a novelty in the field: usually martingale methods are used in the SLE papers. The results of [ii] are example calculations done with the method.

In the paper [i], the question of the reversibility and the duality is studied using the fact that local martingales of SLE form a Virasoro module. Consider $\mathrm{SLE}_\kappa(\rho_1, \rho_2, \ldots, \rho_N)$ with marked points $W_t, Y_t^1, \ldots, Y_t^N$ so that the driving process is W_t. The process $\mathrm{SLE}_\kappa(\rho_1, \rho_2, \ldots, \rho_N)$ is a generalization of the chordal SLE_κ, for the definition see [19]. For a smooth function $f(w, y_1, \ldots, y_N)$, the drift of $f(W_t, Y_t^1, \ldots, Y_t^N)$ is given by $(\mathcal{A}f)(W_t, Y_t^1, \ldots, Y_t^N)$, which defines a partial differential operator $\mathcal{A}$.

The Virasoro algebra is a Lie algebra with a basis $\{C\} \cup \{L_n : n \in \mathbb{Z}\}$ so that

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0}\, C \quad \text{and} \quad [L_n, C] = 0.$$

A set of partial differential operators $\mathcal{L}_n$ in the variables $w, y_1, \ldots, y_N$ is defined in the section 3 of [i] and satisfies the commutation relations of the Virasoro algebra as a result of [9].

The special property that connects $\mathcal{L}_n$ to $\mathrm{SLE}_\kappa(\rho_1, \rho_2, \ldots, \rho_N)$ is that they commute with $\mathcal{A}$:

$$[\mathcal{A}, \mathcal{L}_n] = 0.$$

This implies that for any local martingale of the form $f(W_t, Y_t^1, \ldots, Y_t^N)$, f in a suitable function space, any process of the form $(\mathcal{L}_{n_1}\mathcal{L}_{n_2} \ldots \mathcal{L}_{n_k} f)(W_t, Y_t^1, \ldots, Y_t^N)$ is also a local martingale.

This method is used to construct local martingales so that all the expected values of the form $\mathbb{E}(p)$, where p is a polynomial in the coefficients of the expansion of G around ∞, can be determined. The reversibility and the duality are formulated so that two different $\mathrm{SLE}_\kappa(\rho_1, \rho_2, \ldots, \rho_n)$-processes produce the same polynomial expected values.

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