

# ASPECTS OF ATOMIC DECOMPOSITIONS AND BERGMAN PROJECTIONS

TEEMU HÄNNINEN

Academic dissertation

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Department of Mathematics and Statistics  
Faculty of Science  
University of Helsinki

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Teemu Hänninen

## PREFACE

The first part of this thesis is an introduction to the atomic decomposition and the included articles. In section 1 we recall the basics of Bergman spaces and Bergman projections, which are the common denominators of all the included articles. Section 2 has the concept of atomic decomposition developed following its original proof in 1980. In section 3 we take a look at regulated domains and explain shortly some background to article [a].

Later on we introduce the notion of pseudoconvexity in the calculus of several complex variables in section 4 and go briefly through some differences in the theory of Bergman spaces in the complex plane and  $\mathbb{C}^n$  in section 5 which are needed for article [b]. Section 6 develops the subject of locally convex spaces and inductive limits, while section 7 goes through the Köthe sequence spaces, these two subjects being vital for article [c]. Finally in section 8 we sum up articles [b] and [c].

The presentation in this first part of the thesis can be found in the existing literature, the author claims no original ideas thereof.

The second part of the thesis consists of the articles themselves (listed here in order of appearance):

- [a] T. HÄNNINEN: Atomic decomposition in regulated domains. *Note Mat.* **24** (2005), 65–84.
- [b] M. ENGLIŠ, T. HÄNNINEN, J. TASKINEN: Minimal  $L^\infty$ -type spaces on strictly pseudoconvex domains on which the Bergman projection is continuous. *Houston J. Math.* **32** (2006), 253–275.
- [c] T. HÄNNINEN, J. TASKINEN: Atomic decomposition of a weighted inductive limit in  $\mathbb{C}^n$ . *Mediterr. J. Math.* **2** (2005), 277–290.

## 1. THE BERGMAN KERNEL

The theory of Bergman spaces developed in the mid-20th century from several different sources whose primary inspiration was the related theory of Hardy spaces  $H^p$ ,  $0 < p < \infty$ , of functions  $f$  analytic in the unit disk  $\mathbb{D}$  with integrals  $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$  which remain bounded as  $r \rightarrow 1$ .

A significant step forwards was taken when S. Bergman [4] published the first systematic treatment of the Hilbert space of square-integrable analytic functions on a domain with respect to Lebesgue area or volume measure. When attention later shifted to the spaces  $A^p$  on the unit disk, it was natural to call them Bergman spaces  $A^p(\mathbb{D})$ , which, for  $0 < p < \infty$ , consist of all analytic functions such that

$$\|f\|_p = \left( \int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty,$$

where  $dA$  is the area measure normalised so that the area of  $\mathbb{D}$  equals 1. Clearly  $H^p \subset A^p$ . For  $p = \infty$  we denote by  $H^\infty(\mathbb{D}) = A^\infty(\mathbb{D})$  the space of bounded analytic functions on the unit disk with

$$\|f\|_\infty = \sup \{|f(z)| \mid z \in \mathbb{D}\} < \infty.$$

As counterparts to Hardy spaces, Bergman spaces presented analogous problems, however, it soon became evident that they are in many aspects more complicated than Hardy spaces. A good example of the additional complexity of Bergman spaces are the invariant subspaces, which for Hardy spaces were completely characterised already in 1949 by A. Beurling [5], but for Bergman spaces they still remain largely unresolved and are known to be very complicated.

In Bergman's original work the emphasis is on the case of the Hilbert space  $A^2$ . This enables the use of orthogonal systems of functions and ultimately leads to the Bergman kernel function  $K(z, \zeta)$  because if  $\{e_n\}$  denotes an arbitrary orthonormal basis of the space  $A^2(\Omega)$ , then the Bergman kernel, also known as the reproducing kernel, of the domain  $\Omega \subset \mathbb{C}$  has the representation  $K(z, \zeta) = \sum_{n=1}^{\infty} e_n(z) \overline{e_n(\zeta)}$ . On the unit disk this takes the form  $K(z, \zeta) = 1/(1 - z\bar{\zeta})^2$  and it has the reproducing property

$$(1.1) \quad f(z) = \int_{\mathbb{D}} K(z, \zeta) f(\zeta) dA(\zeta), \quad z \in \mathbb{D},$$

for each function  $f \in A^2(\mathbb{D})$ .

Associated with the kernel is the Bergman projection  $P$  which is the integral operator induced by the Bergman kernel. In  $L^2(\mathbb{D})$ ,  $P$  denotes the orthogonal projection of  $L^2$  onto  $A^2$  and although the Bergman projection is originally defined on  $L^2$ , the formula (of which (1.1) is a special case where  $f \in A^2$ )

$$(1.2) \quad Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - \bar{\zeta}z)^2} dA(\zeta),$$

being a well-defined linear operator on  $L^1$ , clearly extends the domain of  $P$  to  $L^1(\mathbb{D})$ . In fact, it can be shown that the Bergman projection is a bounded map

from  $L^p$  onto  $A^p$  for all  $1 < p < \infty$ . It is not, however, bounded for  $p = 1$ , since there exist functions  $f$  in  $L^\infty$  for which  $Pf$  is not essentially bounded for  $\zeta \in \mathbb{D}$ . Thus  $P : L^\infty \rightarrow L^\infty$  is not a bounded operator and hence, the dual space of  $L^1$  being isomorphic to  $L^\infty$ , the operator  $P : L^1 \rightarrow L^1$  cannot be bounded either. There are other continuous projections from  $L^1$  to  $A^1$ , such as (1.4) when  $\alpha > 0$ , but from  $L^\infty$  to  $A^\infty$  no continuous projections exist. To answer this problem a slightly larger space  $L_V^\infty \supset L^\infty$  where the Bergman projection is bounded is introduced by J. Taskinen in [33] and in [b] the results are extended to smoothly bounded strictly pseudoconvex domains in  $\mathbb{C}^n$ . The boundedness of the Bergman projection on  $L^p$  immediately gives the duality between the Bergman spaces, which in the case of weighted Bergman spaces proves to be a very useful property as seen in [a].

For simply connected domains  $\Omega \subset \mathbb{C}$  other than the unit disk the Bergman kernel can be calculated using the relevant Riemann mapping function  $\varphi : \Omega \rightarrow \mathbb{D}$ . The resulting kernel is of the form

$$(1.3) \quad K(z, \zeta) = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{(1 - \varphi(z)\overline{\varphi(\zeta)})^2},$$

where the connection to the geometry of the domain is clearly visible by way of the Riemann mappings present in the formula. The projection associated with the kernel in (1.3) yields the standard orthogonal Bergman projection, but there are also other kernels that render bounded projections from  $L^p(\Omega)$  onto  $A^p(\Omega)$ . The weighted version of the kernel (1.3) can be obtained by applying a simple change of variables in the weighted Bergman projection

$$(1.4) \quad P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - \bar{\zeta}z)^{2+\alpha}} dA_\alpha(\zeta),$$

where the weight function is included in  $dA_\alpha(\zeta) = (\alpha + 1)(1 - |\zeta|^2)^\alpha dA(\zeta)$ , with  $\alpha > -1$ . The formula (1.4) still reproduces analytic functions and thus the reproducing kernel of the weighted Bergman space  $A_\alpha^p(\Omega)$  takes the form

$$(1.5) \quad K(z, \zeta) = \frac{\varphi'(z)\overline{\varphi'(\zeta)}(1 - |\varphi(\zeta)|^2)^\alpha}{(1 - \varphi(z)\overline{\varphi(\zeta)})^{2+\alpha}},$$

which together with a projection operator gives again an orthogonal projection.

Yet another interesting Bergman kernel can be found by considering the composition operator  $C_\psi : f \mapsto f \circ \psi$ , where  $\psi : \mathbb{D} \rightarrow \Omega$  is a conformal mapping. The operator  $C_\psi$  is a linear homeomorphism from  $L^p(\Omega)$  onto  $L^p(\mathbb{D})$  and from  $A^p(\Omega)$  onto  $A^p(\mathbb{D})$  making the operator  $P_\psi = (C_\psi)^{-1}PC_\psi$ , where  $P$  is the standard unweighted Bergman projection (1.2), again algebraically a projection operator on  $L^p(\Omega)$ . The projection  $P_\psi$  associated with the kernel

$$(1.6) \quad K(z, \zeta) = \frac{|\varphi'(\zeta)|^2(1 - |\varphi(\zeta)|^2)^\alpha}{(1 - \varphi(z)\overline{\varphi(\zeta)})^{2+\alpha}},$$

where  $\varphi = \psi^{-1}$ , is called the conjugated Bergman projection, it plays an essential role in the construction of the atomic decomposition in article [a].

Many operator-theoretic problems in the analysis of Bergman spaces involve estimating integral operators similar to (1.2) or (1.4) whose kernel is a power of the Bergman kernel. This together with the use of the reproducing property of the Bergman kernel brings us to a close relative of the formula (1.1), the atomic decomposition.

For more reading on Bergman spaces we recommend the recent and well-written book by P. Duren and A. Schuster [13] and the slightly more theoretical one by H. Hedenmalm, B. Korenblum and K. Zhu [19].

## 2. BACKGROUND ON ATOMIC DECOMPOSITION

The decomposition of an element of a Banach space on a domain is a widely studied area of modern mathematics of which atomic decomposition is an example. An atomic decomposition consists of a sequence of simple building blocks (called atoms) in the unit ball of the Banach space, such that every element is a linear combination  $\sum_n a_n k(\lambda_n)$  of atoms  $k$  with  $\sum_n |a_n|^p < \infty$  for some  $1 \leq p < \infty$ ,  $a_n \in \mathbb{C}$ . The infimum of the sum of the coefficients  $a_n$  defines the norm or an equivalent one for the Banach space. Thus an atomic decomposition is a sequence which has basis-like properties but which does not need to be a basis.

In general atomic decompositions are overcomplete, the sampling sequences  $(\lambda_n)$  usually contain too many points for the set of atoms  $\{k(\lambda_n)\}$  to be linearly independent in which case it forms a frame instead of a basis. In a frame the representation  $f = \sum_n a_n k(\lambda_n)$  is not unique and there are many possible dual frames such that  $f = \sum_n a_n k(\lambda_n^*)$ . For more details on frames and bases, see [11].

First to come up with the idea of atomic decomposition were Coifman and Rochberg [12] who in 1980 showed that a “decomposition theorem” holds for domains in the Bergman space  $A^p(D, dA)$  of analytic functions on a bounded symmetric domain  $D \subset \mathbb{C}^n$ . Their basic idea is to use the reproducing property (1.1) of the Bergman kernel to represent a function  $f \in A^p$  by an integral. To approximate (1.1) they set up a partition of  $\mathbb{D}$  by covering it with a disjoint union of hyperbolic disks  $D(\lambda_n, r)$  with a constant (hyperbolic) radius  $r$  and points  $\lambda_n$  making up a lattice with respect to the hyperbolic metric.

The integral in (1.1) is then approximated by a Riemannian sum over the partition using the values of  $f$  and the kernel  $K$  at the points  $\lambda_n$  of the lattice. If the partition is sufficiently dense, this will produce a good approximation and an iteration of the process now yields

$$(2.1) \quad f(z) = \sum_{n=1}^{\infty} m(D(\lambda_n, r)) \frac{f(\lambda_n)}{(1 - z\overline{\lambda_n})^2}.$$

Since the functions in question are analytic, then also their building blocks must be analytic. In fact it turns out that the right type of atoms will be comparable to the normalised reproducing kernels  $k_z(w) = (1 - |z|^2)/(1 - \bar{z}w)$  of the Bergman space  $A^2(\mathbb{D})$  (originally in [12] the building blocks for Bergman spaces were called “molecules”). In search for a representation of  $f$  as a linear

combination of atoms this makes sense, since the kernels  $k_z$  are also the unit vectors in  $A^2$  and, in some sense, play the part of an orthonormal basis for  $A^2$  even though they are not mutually orthogonal.

Leaving the atoms and denoting all the rest by the coefficients  $a_n$  the expression (2.1) translates into the atomic decomposition of the function  $f \in A^p(\mathbb{D})$ :

$$(2.2) \quad f(z) = \sum_{n=1}^{\infty} a_n \frac{(1 - |\lambda_n|^2)^{2/q}}{(1 - \bar{\lambda}_n z)^2},$$

valid such that for any  $(a_n) \in \ell^p$ , the function in (2.2) is in  $A^p$  and if  $f \in A^p$ , then there is a sequence  $(a_n) \in \ell^p$  such that (2.2) holds. Here  $\frac{1}{p} + \frac{1}{q} = 1$ . For the coefficients we get

$$(2.3) \quad \sum_{n=1}^{\infty} |a_n|^p < \infty \quad \text{and} \quad \|(a_n)\|_{\ell^p} \asymp \|f\|_{A^p},$$

which can be seen as the discrete analogue to the fact that the integral reproducing formula gives a bounded projection from  $L^p$  onto  $A^p$ . The formal proof for (2.2) is quite technical (see Chapter 4 in [36], or [12]).

In Bergman spaces the atomic decomposition can thus be regarded as a discrete analogue of the reproducing property, where it was derived from initially. The utility of an atomic decomposition is that it is often possible to prove statements about  $A^p$  by verifying them first in the simple special case of atoms and then extending the results to the entire space. Examples of these include the description of the behaviour of Bergman spaces under various integral and differential operators and results about zero sets for holomorphic and harmonic functions. An immediate corollary of the atomic decomposition is that it establishes an isomorphism between  $A^p$  and the sequence space  $\ell^p$ , thus allowing to analyse sequences of numbers instead of functions, which has a potential of easing considerably the time needed for computations on a computer.

The classic atomic decomposition is also the starting point of the first included article [a] where using (2.2) a new atomic decomposition is constructed directly on a regulated domain  $\Omega$ .

### 3. ATOMIC DECOMPOSITION IN REGULATED DOMAINS, SUMMARY OF ARTICLE [a]

The result of Coifman and Rochberg was valid for functions in a bounded symmetric domain  $D \subset \mathbb{C}^n$  with radial weight functions of the form  $K(z, z)^{-\alpha}$ ,  $\alpha > -1$ , where  $K$  is the reproducing kernel of the domain. In one-dimensional complex plane  $\mathbb{C}$  this effectively means (due to the Riemann mapping theorem) that functions in every simply connected domain  $\Omega \subset \mathbb{C}$  have an atomic decomposition. Even though a decomposition resulting from a conformal mapping  $\psi : D \rightarrow \Omega$  is easy to form, it has other problems. Such a decomposition tends to be very implicit and it often lacks any clear connection to the geometry of the domain  $\Omega$  that it has been mapped into, clearly demonstrated by the points of the lattice where the atoms of the decomposition are evaluated. The lattice



usually follows closely the geometry of the original domain  $D$ , but after mapping the domain into another this connection is easily lost and the distribution of points becomes seemingly random. Another problem with forming atomic decompositions by a conformal mapping is that the weight functions which are natural and easy in  $D$  tend to be very unnatural and cumbersome in  $\psi(D)$ .

The above problems pertain to the use of a conformal mapping from the unit disk and can be bypassed by constructing the decomposition directly on the simply connected domain  $\Omega$ , which is what we will do in [a]. Because we need the Bergman projection to be continuous in the domain to obtain a successful decomposition, it turns out that the class of regulated domains with some limitations is suitable for the construction.

A very good treatment of regulated domains can be found in C. Pommerenke's study [28]. We recall the basic definitions. Let  $\varphi : \Omega \rightarrow \mathbb{D}$  and  $\psi := \varphi^{-1} : \mathbb{D} \rightarrow \Omega$  be Riemann mappings and let  $\omega(t) = \psi(e^{it})$ ,  $0 \leq t \leq 2\pi$  be a parametrisation of the boundary curve  $\partial\Omega$ . Now the domain  $\Omega$  is said to be regulated if each point on  $\partial\Omega$  is attained only finitely often by  $\psi$  and if the direction angle

$$(3.1) \quad \beta(t) = \lim_{\tau \rightarrow t^+} \arg(\omega(\tau) - \omega(t))$$

of the forward tangent of the boundary curve  $\omega(t)$  exists for all  $t$  and  $\beta$  defines a regulated function (a function is regulated if it can be uniformly approximated by step functions, that is, for all  $\varepsilon > 0$  there exist  $0 < t_0 < \dots < t_n < 2\pi$  and constants  $\gamma_1, \dots, \gamma_n$  such that

$$|\beta(t) - \gamma_j| < \varepsilon \quad \text{for all } t_{j-1} < t < t_j).$$

Thus  $\Omega$  is regulated if its boundary consists of a finite union of  $C^\infty$ -arcs with a finite number of corners. Forward and backward tangents exist also at these corners.

In [1] and [2] D. Békollé showed that the Békollé–Bonami  $B_p$  condition (see [2], p. 129) for the weight function  $|\psi'|$ ,

$$(3.2) \quad \sup_S \int_S |\psi'| dA_\alpha \left( \int_S |\psi'|^{-q/p} dA_\alpha \right)^{p/q} \leq C m_\alpha(S)^p,$$

where  $S = S(\theta, \rho) = \{re^{it} \in \mathbb{D} \mid 1 - \rho < r < 1, |\theta - t| < 2\pi\rho\}$ , with  $0 \leq \theta \leq 2\pi$  and  $0 < \rho < 1$ , is equivalent to the boundedness of the Bergman projection on the space  $L_\alpha^p(\Omega)$  corresponding to the mapping  $\psi$ . Using Békollé's result J. Taskinen [34] studied the connection between the geometry of a regulated domain  $\Omega$  and the existence of Bergman type projections from  $L_\alpha^p(\Omega)$  to  $A_\alpha^p(\Omega)$ . The relationship was established in Theorem 3.1 of [34] by taking advantage of the possibility in regulated domains to approximate the direction angle of the boundary curve simply by step functions.

In [a] we use this connection to find out what kind of regulated domains are suitable for the construction of an atomic decomposition, i.e. in which domains the Bergman projection associated with (1.6) is continuous. As the weight we use on  $\Omega$ , the power of boundary distance  $(\text{dist}(z, \partial\Omega))^\alpha = d(z)^\alpha$ , corresponds on the unit disk to the weight  $(1 - |z|^2)^\alpha |\psi'(z)|^{2+\alpha}$ , we see that  $|\psi'(z)|^{2+\alpha}$

satisfies a condition similar to (3.2) if the opening angles  $\pi\gamma$  of the corners on the boundary curve  $\partial\Omega$  stay within

$$(3.3) \quad 0 < \pi\gamma < p\pi,$$

where  $p$  is the integration exponent of the Bergman space  $A^p$ . Thus for example in  $A_\alpha^2(\Omega)$  cusps are excluded from the boundary curve.

In the actual construction of the atomic decomposition we may now follow the geometry of  $\Omega$  in the setting up of the lattice of points  $\lambda_{n,k} \in \Omega$  where the atoms are evaluated. This is done by dividing  $\Omega$  into small squares  $Q_{n,k}$  that decrease in size towards the boundary  $\partial\Omega$ . Each of these squares contains one lattice point. Lemma 6 of [a] states that

$$(3.4) \quad \sum_{n,k} \int_{Q_{n,k}} |f(z) - f(\lambda_{n,k})|^p dA_\alpha(z) \leq C \int_\Omega |f(z)|^p dA_\alpha(z),$$

that is, even though in every square  $Q_{n,k}$  the values of a function  $f(z)$  are represented by its value in only one point of the lattice  $f(\lambda_{n,k})$ , the error incurred by all these representations remains small and we will still be able to estimate the behaviour of  $f$ .

Once the lattice has been fixed, the proof that it defines an atomic decomposition follows closely that of K. Zhu [36] and Lusky, Saksman and Taskinen [26]. We define three bounded operators  $R : A_\alpha^p \rightarrow l^p$ ,  $S : A_\alpha^p \rightarrow A_\alpha^p$  and  $T : l^p \rightarrow A_\alpha^p$  with which we are able to show that the space  $A_\alpha^p(\Omega)$  is isomorphic to the sequence space  $l^p$ . The direct consequence of this isomorphism is the atomic decomposition for functions in  $A_\alpha^p(\Omega)$  given in Theorem 3 of [a].

Finally in Section 8 of [a] we compute an example of atomic decomposition in a simple domain on  $\mathbb{C}$ . From the example it becomes clear that the sequence of sampling points  $\lambda_{n,k}$  is quite a bit more dense than its counterpart in the classic atomic decomposition of the unit disk  $\mathbb{D}$ . The reason behind this is the combined effect of the ample security margins in the constants used in the process leading up to the sequence  $(\lambda_{n,k})$ . Aspiring towards a thinner sampling sequence more like those on the unit disk still would not make our main result (constructing a sampling sequence directly on  $\Omega$ ) any better because even in the classic decompositions there are too many points in the sequence to obtain a Schauder basis for the space  $A^p(\mathbb{D})$ , and a frame is formed instead.

**3.1. Correction.** In Chapter 5 of [a] we mistakenly imply that the result (28) of [a] for analytic functions on  $\mathbb{D}$  in the hyperbolic metric would be used in the argument that then follows. However, the metric in Lemma 6 and Proposition 7 of [a] is Euclidean and hence Corollary 8 of [a] follows from the normal subharmonicity of  $f$ .

On page 73, in the proof of Lemma 9, the disks are erroneously called hyperbolic disks, Euclidean disks are correct.

Additionally, in Proposition 7 the constant 2 in (32) is probably too optimistic and a suitable constant is 7. We present the revised Proposition 7 here with an outline of the proof.

**Proposition 7.** *Let  $Q$  be as in Lemma 6 of [a],  $r = \text{diam}(\varphi(Q))$ . Then there exist disks  $D \subset \mathbb{D}$  such that*

$$(3.5) \quad D(z, \frac{r}{7}) \subset \varphi(Q) \subset D(z, 7r),$$

where  $z = \varphi(x)$ ,  $x \in Q$  and  $x$  is the centre of  $Q$ .

**Proof.** Let

$$(3.6) \quad r := \sup\{|z - w| : z, w \in \varphi(Q)\}$$

be the Euclidean diameter of  $\varphi(Q)$ . From the Koebe distortion theorem we get that

$$\frac{1}{4} \cdot \frac{1 - |\varphi(z)|^2}{d(z)} \leq |\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{d(z)}, \quad z \in \Omega.$$

If now  $\zeta \in Q$  is such that  $d(\zeta) = \min_{z \in Q} d(z)$ , then, by (29) of [a], it follows that  $\max_{z \in Q} d(z) \leq (1 + c)d(\zeta)$ , where  $0 < c \ll 1$ . Hence by (30) and (31) of [a] we have

$$(3.7) \quad 1 \geq \frac{\min_{w \in Q} |\varphi'(w)|}{\max_{w \in Q} |\varphi'(w)|} > \frac{1}{4(1+c)} \cdot \frac{\inf_{z \in Q} (1 - |\varphi(z)|^2)}{\sup_{z \in Q} (1 - |\varphi(z)|^2)} > \frac{1}{4} \cdot \frac{1}{(1+c)^2} > \frac{1}{5},$$

because  $c \ll 1$ . Now for all  $z, x \in Q \subset \Omega$ , it is true that  $|\varphi(z) - \varphi(x)| = |\varphi'(\xi)||z - x|$  for some  $\xi \in Q$ , and thus

$$\frac{1}{5} \max_{w \in Q} |\varphi'(w)||z - x| \leq |\varphi(z) - \varphi(x)| \leq \max_{w \in Q} |\varphi'(w)||z - x|$$

using (3.7). This implies that for points  $\zeta_0, \zeta_1 \in \partial Q$  opposite each other (points for which the straight line that connects them passes through the centre of the square) we get the following inequality:

$$(3.8) \quad \frac{\ell}{5} \max_{w \in Q} |\varphi'(w)| \leq |\varphi(\zeta_0) - \varphi(\zeta_1)| \leq \sqrt{2}\ell \max_{w \in Q} |\varphi'(w)|,$$

since  $\ell \leq |\zeta_0 - \zeta_1| \leq \sqrt{2}\ell$ . Thus  $\varphi(Q)$  contains a disk with radius less than  $r/7$ . More precisely, setting  $f(Q) = \max_{w \in Q} |\varphi'(w)|$  we get that

$$\frac{r}{7} \leq \frac{\ell}{5} f(Q) \leq |\varphi(\zeta_0) - \varphi(\zeta_1)| \leq \sqrt{2}\ell f(Q) \leq 7r.$$

□

Other misprints in atricle [a] include:

- Page 69. The first row of the equation array after equation (19) has an extra  $(\alpha + 1)(\alpha + 2)$ , it should read

$$\langle f' | g \rangle_{\alpha+p} = \int_{\mathbb{D}} f'(z) \overline{g(z)} (1 - |z|^2)^{p/q} (1 - |z|^2)^{p-p/q} dA_{\alpha}(z).$$

- Page 74. On the first row of the equation array instead of  $|f(z)|^p$  there should be  $|f(z)|^1$ .
- Page 75. The line beginning “the adjoining families  $\mathbf{Q}_{\mathbf{n} \pm \mathbf{k}} \dots$ ” should be “the adjoining families  $\mathbf{Q}_{\mathbf{n} \pm 1} \dots$ ”

- Page 76. On the last row of the last equation array, instead of  $dA_{\alpha+p,\varphi}$  there should be  $dA_{\alpha+p,\varphi^p}$ .

#### 4. PSEUDOCONVEX DOMAINS

Let  $\Omega \subset \mathbb{C}^n$  be a domain and  $\mathcal{H}(\Omega)$  be the family of holomorphic functions on  $\Omega$ . Then a *domain of holomorphy* is the proper domain of existence of a holomorphic function which cannot be extended analytically in a neighbourhood of any boundary point. An open set  $U \subset \mathbb{C}^n$  is called a domain of holomorphy if there do not exist nonempty open sets  $U_1, U_2$ , where  $U_2$  is connected,  $U_2 \not\subset U$ ,  $U_1 \subset U_2 \cap U$ , such that for every  $h \in \mathcal{H}(U)$  there is a  $h_2 \in \mathcal{H}(U_2)$  such that  $h = h_2$  on  $U_1$ . The definition is complicated because we have to take into account the possibility that  $\partial U$  may intersect itself. That aside, we see that an open set  $U$  is not a domain of holomorphy if there is an open set  $\hat{U} \supsetneq U$  such that every  $h \in \mathcal{H}(U)$  analytically continues to a holomorphic function  $\hat{h}$  on  $\hat{U}$ .

It is well known that every open subset of  $\mathbb{C}$  is a domain of holomorphy, but for domains in  $\mathbb{C}^n$  the situation is different. For example the Hartogs domain

$$\Omega = D^2(0, 3) \setminus \bar{D}^2(0, 1) \subset \mathbb{C}^2$$

is not a domain of holomorphy, since all holomorphic functions on  $\Omega$  continue to the larger domain  $D^2(0, 3) \supset \Omega$ , where  $D^n(z^0, r) = \{z \in \mathbb{C}^n \mid |z_j - z_j^0| < r, j = 1, \dots, n\}$  denotes the *open polydisk* and  $\bar{D}^n(z^0, r)$  its closure. Thus we may have two connected open sets  $V \subset U \subset \mathbb{C}^n$ ,  $n \geq 2$ , such that every  $f \in \mathcal{H}(V)$  has a unique analytic continuation to  $\mathcal{H}(U)$ . The first to realise this phenomenon was F. Hartogs [18].

The Cartan-Thullen theorem (see [27], p. 52) characterises domains of holomorphy by a convexity property with respect to holomorphic functions by defining that a domain  $\Omega \subset \mathbb{C}$  is *holomorphically convex* or *pseudoconvex* if for every compact subset  $K \subset \Omega$  its holomorphically convex hull

$$(4.1) \quad \hat{K}_\Omega = \{z \in \Omega \mid |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathcal{H}(\Omega)\}$$

is again compact. Consequently every convex domain  $\Omega$  in  $\mathbb{C}^n$  is pseudoconvex.

The definition (4.1) of pseudoconvexity uses only the internal structure of  $\Omega$ , but it is quite difficult to verify in general. It turns out that for domains  $\Omega \subset \mathbb{C}^n$  whose topological boundary is smooth (of class  $C^\infty$ ) there exists an equivalent condition in terms of the differential geometry of the manifold  $\partial\Omega$  which is easier to verify.

Let  $\Omega \subset \mathbb{C}^n$  be a bounded domain with a smooth boundary. A real-valued function  $\rho \in C^\infty(\mathbb{C}^n)$  is a *defining function* for  $\Omega$  if

$$\Omega = \{z \in \mathbb{C}^n \mid \rho(z) < 0\}, \quad \bar{\Omega}^{\mathbb{C}} = \{z \in \mathbb{C}^n \mid \rho(z) \geq 0\}$$

and  $\|\text{grad } \rho(z)\| \neq 0$  for all  $z \in \partial\Omega$ . The negative signed distance  $\text{dist}(\cdot, \partial\Omega)$  always has these properties, but it is often convenient to use other choices. Then the boundary

$$\partial\Omega = \{z \in \mathbb{C}^n \mid \rho(z) = 0\}$$

is a smooth real submanifold of  $\mathbb{C}^n \sim \mathbb{R}^{2n}$  whose real tangent space at  $z \in \partial\Omega$  can be identified with the real subspace

$$T_z^{\mathbb{R}}(\partial\Omega) = \{ u \in \mathbb{C}^n \mid \langle \text{grad } \rho(z) \mid u \rangle_{\mathbb{R}^{2n}} = 0 \}$$

of  $\mathbb{C}^n$ . The complex subspace

$$\mathcal{T}_z(\partial\Omega) = \left\{ u \in \mathbb{C}^n \mid \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z) u_j = 0 \right\} = T_z^{\mathbb{R}}(\partial\Omega) \cap iT_z^{\mathbb{R}}(\partial\Omega)$$

is called the *complex tangent space* at  $z \in \partial\Omega$ . It is the largest complex subspace contained in  $T_z^{\mathbb{R}}(\partial\Omega)$  in the sense that if  $S$  is a real linear subspace of  $T_z^{\mathbb{R}}(\partial\Omega)$  that is closed under multiplication by  $i$ , then  $S \subset \mathcal{T}_z(\partial\Omega)$ .

Now consider, for every  $z \in \partial\Omega$ , the sesquilinear form

$$(4.2) \quad \langle u \mid v \rangle_z = \partial\bar{\partial}\rho(z)(u, v) := \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z) u_i \bar{v}_j$$

defined for  $u, v \in \mathbb{C}^n$ . Since  $\rho$  is real-valued, we have  $\overline{\langle u \mid v \rangle_z} = \langle v \mid u \rangle_z$ , that is, (4.2) is a Hermitian form at  $z$  called the *Levi form* associated with  $\Omega$ . The domain  $\Omega$  is called pseudoconvex if at every boundary point  $z \in \partial\Omega$  the Levi form (4.2) is positive semi-definite when restricted to the complex tangent space  $\mathcal{T}_z(\partial\Omega)$ , i. e.,

$$(4.3) \quad \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z) w_i \bar{w}_j \geq 0, \quad \forall w \in \mathcal{T}_z(\partial\Omega).$$

If the expression on the left side of (4.3) is strictly positive definite (positive whenever  $w \neq 0$ ,  $w \in \mathcal{T}_z(\partial\Omega)$ ) for all  $z \in \partial\Omega$ , then  $\Omega$  is said to be *strictly pseudoconvex*. Note that in one complex dimension pseudoconvexity is not an interesting condition because  $\mathcal{T}_z(\partial\Omega) = \{0\}$  at every boundary point and thus any domain in  $\mathbb{C}$  is vacuously pseudoconvex. The condition (4.3) was discovered by E.E. Levi [24] in 1910 in the case of two variables.

For example the unit ball  $\mathbb{B}^n = \{ z \in \mathbb{C}^n \mid z \cdot \bar{z} < 1 \}$  is strictly pseudoconvex since  $\rho(z) = z \cdot \bar{z} - 1$  is a smooth defining function with the Levi form  $\partial\bar{\partial}\rho(z)(u, v) = u \cdot \bar{v}$  for all  $u$  and  $v$ , where  $z \cdot \bar{\zeta} = z_1 \bar{\zeta}_1 + z_2 \bar{\zeta}_2 + \dots + z_n \bar{\zeta}_n$  is the usual scalar product for  $z, \zeta \in \mathbb{C}^n$ . On the other hand, the polydisk  $D^n(0, 1) = \{ z \in \mathbb{C}^n \mid \|z\|_{\infty} < 1 \}$  is pseudoconvex but not strictly pseudoconvex if  $n > 1$ .

It is known that pseudoconvexity is independent of the choice of the defining function  $\rho$ , and if  $\Omega$  is strictly pseudoconvex, then it is possible to even choose a  $\rho$  such that the Levi form is positive definite not only on the tangent space but on the whole of  $\mathbb{C}^n$ , for all  $z \in \partial\Omega$ , i. e., one can choose  $\rho$  to be strictly plurisubharmonic on  $\partial\Omega$ .

For more details on pseudoconvex domains see for instance S.G. Krantz [23], R.M. Range [29] or H. Upmeyer [35].

5. BERGMAN TYPE PROJECTIONS IN  $\mathbb{C}^n$ 

As in the one dimensional case, define the Bergman space  $A^2(\Omega)$ ,  $\Omega \subset \mathbb{C}^n$  to be the space of all holomorphic functions  $f$  on  $\Omega$  such that

$$\|f\|_{A^2} = \left( \int_{\Omega} |f(z)|^2 dV(z) \right)^{1/2} < \infty,$$

where  $dV(z) = (1/2i)^n (d\bar{z}_1 \wedge dz_1) \wedge \dots \wedge (d\bar{z}_n \wedge dz_n)$  is the standard Euclidean volume form on  $\mathbb{C}^n$ . Then the Bergman kernel  $K(z, \zeta)$  is the uniquely determined function of  $A^2(\Omega)$  in the variable  $z$ , which is conjugate symmetric  $K(z, \zeta) = \overline{K(\zeta, z)}$  and has the reproducing property

$$f(z) = \int_{\Omega} K(z, \zeta) f(\zeta) dV(\zeta)$$

for all  $f \in A^2(\Omega)$ . For example, for  $z, \zeta$  in the unit ball  $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$  we have

$$K_{\mathbb{B}^n}(z, \zeta) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \bar{\zeta})^{n+1}}.$$

However, unlike in the complex plane, in  $\mathbb{C}^n$  the Bergman kernel can almost never be calculated explicitly. Unless the domain  $\Omega$  has a great deal of symmetry, so that a useful orthonormal basis for  $A^2(\Omega)$  can be obtained, there are few techniques for determining  $K_{\Omega}(z, \zeta)$ .

In 1974 C. Fefferman [14] (see also L. Boutet de Monvel and J. Sjöstrand [10]) introduced a new technique for obtaining an asymptotic expansion for the Bergman kernel on a large class of domains, which is used in [b] in the more general case of weighted Bergman spaces. This enabled rather explicit estimations of the Bergman kernel and opened up an entire branch of analysis on domains in  $\mathbb{C}^n$ .

The lack of an exact formula for the Bergman kernel also contributes to difficulties in establishing the boundedness of the Bergman projection, which was resolved also in 1974 by F. Forelli and W. Rudin in [16]. Their idea was to imbed the unit ball  $\mathbb{B}^n$  of  $\mathbb{C}^n$  into the unit ball  $\mathbb{B}^{n+s}$  of  $\mathbb{C}^{n+s}$  via  $i(z) = (z, 0)$  and use the reproducing property of the Bergman kernel of  $\mathbb{B}^{n+s}$  to obtain a new reproducing kernel on  $\mathbb{B}^n$ . Namely, if  $z, w \in \mathbb{B}^n$ , then for each complex  $s = \sigma + it$ ,  $\sigma > -1$  there is an associated kernel

$$(5.1) \quad K_s(z, w) = \frac{(1 - |w|^2)^s}{(1 - z \cdot \bar{w})^{n+1+s}}.$$

When  $s = 0$ , then (5.1) is the classical Bergman kernel for  $\mathbb{B}^n$  up to normalisation. Using ordinary Lebesgue measure, the kernel  $K_s$  induces an integral operator  $T_s$ ,

$$T_s = \binom{n+s}{s} \int_{\mathbb{B}^n} K_s(z, w) f(w) dV(w)$$

on functions defined on  $\mathbb{B}^n$ . The main theorem of [16] (see also [30], Chapter 7) states that, for  $1 \leq p < \infty$ ,  $T_s$  is bounded on  $L^p(\mathbb{B}^n)$  if and only if  $(1 + \sigma)p > 1$ , in which case  $T_s$  projects  $L^p(\mathbb{B}^n)$  onto  $A^p(\mathbb{B}^n)$ .

## 6. LOCALLY CONVEX INDUCTIVE LIMITS

A locally convex space  $(E, p)$  is a Hausdorff topological vector space whose topology is defined by a family of seminorms  $\{p\}$  such that the neighbourhood filter at zero (and thus at any point) has a basis consisting of open convex sets  $B_p(x, r) = \{y \in E \mid p(x - y) < r\}$ . If in addition the family of seminorms is countable the resulting space, when complete, is called a Fréchet space. Locally convex spaces are generalisations of seminormed spaces, also every Banach space is a locally convex space and as such their theory generalises parts of the theory of Banach spaces. An important difference between Banach and Fréchet spaces is that unlike for Banach spaces the strong dual of a Fréchet space is not metrisable in general, but instead a (DF)-space, a class introduced by A. Grothendieck [17].

A directed index set  $A$  is a partially-ordered set with an upper bound for any pair of elements, that is, given  $i, j \in A$  there exists a  $k \in A$  such that  $i \leq k$  and  $j \leq k$ . The set of positive integers with its natural ordering is the simplest example of a directed set.

A locally convex inductive system  $\{E_\alpha, f_{\alpha\beta}\}_{\alpha, \beta \in A}$  is a family of locally convex spaces  $\{E_\alpha \mid \alpha \in A\}$  indexed by a directed set  $A$  with a collection of linking maps

$$f_{\alpha\beta} : E_\alpha \longrightarrow E_\beta \quad \text{when } \alpha < \beta$$

which satisfy the compatibility condition

$$(6.1) \quad f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta} \quad \text{when } \alpha < \beta < \gamma.$$

The inductive limit  $\text{ind}_{\alpha \rightarrow} \{E_\alpha, f_{\alpha\beta}\}$  of the system is now defined as follows: We form the disjoint union of the locally convex spaces  $E_\alpha$

$$U = \bigsqcup_{\alpha \in A} E_\alpha$$

and define an equivalence relation in  $U$  by putting  $x \sim y$  for  $x \in E_\alpha$  and  $y \in E_\beta$  if there exists a  $\gamma \geq \alpha, \beta$  such that  $f_{\alpha\gamma}x = f_{\beta\gamma}y$ . Then we get

$$(6.2) \quad E = \text{ind}_{\alpha \rightarrow} \{E_\alpha, f_{\alpha\beta}\} = U / \sim.$$

The space  $E$  is now endowed with the finest locally convex topology that makes all the mappings  $f_\alpha : E_\alpha \longrightarrow E$  continuous.

If a family of locally convex spaces  $\{E_n \mid n \in \mathbb{N}\}$  has  $\mathbb{N}$  with its natural ordering as its index set the resulting space  $E = \text{ind}_{n \rightarrow} \{E_n, f_{nm}\}$  is called a *countable* inductive limit. In many cases the family  $\{E_n\}_{n \in \mathbb{N}}$  can be organised in an increasing order  $\dots \subset E_n \subset E_{n+1} \subset E_{n+2} \subset \dots$  creating natural inclusions  $f_{nm} : E_n \hookrightarrow E_m$ ,  $n < m$ . For the inclusions we get  $f_{nm}x = x$  for all  $x \in E_n$  and hence  $x \sim y$  in the sense of (6.2) if and only if  $x = y$ . Then the countable inductive limit of the spaces  $E_n$  reduces simply to

$$E = \text{ind}_{n \rightarrow} \{E_n, f_{nm}\} = \bigcup_{n=1}^{\infty} E_n$$

equipped with the finest locally convex topology that makes all the mappings  $f_\alpha : E_\alpha \longrightarrow E$  continuous.

Likewise, if the index set of the locally convex family is larger than  $\mathbb{N}$ , then we will have an *uncountable* locally convex inductive system  $\{E_\alpha\}_{\alpha \in A}$ . The difference in the size of the index set is remarkable as nearly all existing positive results about inductive limits concern countable cases and a general theory of uncountable inductive limits appears to be practically impossible to develop.

A locally convex *projective* system  $\{E_\alpha, f_{\alpha\beta}\}_{\alpha, \beta \in A}$  is, similarly to inductive systems, again a family of locally convex spaces indexed by a directed set  $A$  with a collection of maps  $f_{\alpha\beta} : E_\beta \rightarrow E_\alpha$ ,  $\alpha < \beta$  which satisfy the compatibility condition (6.1). The projective limit of the system  $\{E_\alpha, f\}$  is now defined as a subspace of the product space  $\prod_\alpha E_\alpha$  such that

$$\text{proj}_{\leftarrow \alpha} E_\alpha = E = \left\{ (x_\alpha)_\alpha \in \prod_{\alpha \in A} E_\alpha \mid f_{\alpha\beta} x_\beta = x_\alpha \text{ for all } \beta \geq \alpha \right\}$$

endowed with the induced topology from the product space. Algebraically it is the set of all the vectors from the product whose position in the product space commutes with the mapping  $f$ . Again, if the index set  $A = \mathbb{N}$ , then usually the family  $\{E_n\}_{n \in \mathbb{N}}$  can be organised in a decreasing order and the countable projective limit of the spaces  $E_n$  becomes  $E = \text{proj}_{\leftarrow n} E_n = \bigcap_{n=1}^{\infty} E_n$ .

On the duality of inductive and projective limits we get (see [15]) that for any inductive system  $(E_\alpha)_{\alpha \in A}$  of locally convex spaces,  $(E'_\alpha)_{\alpha \in A}$  is a projective system and thus

$$(6.3) \quad (\text{ind}_{\alpha \rightarrow} E_\alpha)' = \text{proj}_{\leftarrow \alpha} E'_\alpha$$

algebraically. If the system  $(E_\alpha)_{\alpha \in A}$  is regular, that is, if for each bounded set  $B \subset \text{ind}_{\alpha \rightarrow} E_\alpha$  there exists an  $\alpha = \alpha(B) \in A$  such that  $B \subset E_\alpha$  and  $B$  is bounded in  $E_\alpha$ , then (6.3) holds also topologically.

Each Fréchet space is the projective limit (usually an intersection) of a countable collection of Banach spaces, of which a good example is the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$  of smooth rapidly decreasing functions. On the other hand, a countable inductive limit of Fréchet spaces (resp. Banach spaces) is called an (LF)-space (resp. (LB)-space), some of the most important examples of locally convex inductive limits belong to one of these classes. A Fréchet space  $E$  is called Fréchet–Schwartz (FS), if the linking maps are compact in the sense that for each  $n \in \mathbb{N}$  there is an  $m > n$  such that  $f_{nm} : E_m \rightarrow E_n$  is compact, or equivalently, if for each  $n \in \mathbb{N}$  there is an  $m > n$  such that for each  $\varepsilon > 0$  there is a finite set  $F$  with  $U_m \subset F + \varepsilon U_n$ , where  $U_n = \{f \in F \mid p_n(f) \leq 1\}$  is the unit ball with respect to the corresponding seminorm  $p_n$ .

## 7. KÖTHE SEQUENCE SPACES

Let the *Köthe matrix*  $A = (a_n)_{n \in \mathbb{N}}$  be an increasing sequence of strictly positive functions on an arbitrary index set  $I$ . The *echelon space*  $\lambda_p = \lambda_p(A)$  of order  $p$  corresponding to each Köthe matrix  $A$  and  $1 \leq p < \infty$  is defined to be

$$\lambda_p(A) = \{x = (x(i))_{i \in I} \in \mathbb{C}^I \mid \forall n \in \mathbb{N}, (a_n(i)x(i))_{i \in I} \text{ is } p\text{-absolutely summable}\},$$



that is, for all  $n \in \mathbb{N}$ ,  $q_n^p(x) = \left(\sum_{i \in I} (a_n(i)|x(i)|)^p\right)^{1/p} < \infty$ . We also have

$$\lambda_\infty(A) = \{x \in \mathbb{C}^I \mid \forall n \in \mathbb{N}, q_n^\infty(x) = \sup_{i \in I} a_n(i)|x(i)| < \infty\},$$

$$\lambda_0(A) = \{x \in \mathbb{C}^I \mid \forall n \in \mathbb{N}, (a_n(i)x(i))_{i \in I} \text{ converges to } 0\}.$$

The echelon spaces  $\lambda_p(A)$  are Fréchet spaces with the sequence of norms  $q_n = q_n^p$ ,  $n = 1, 2, \dots$ . If  $A$  consists of a single strictly positive function  $a = (a(i))_{i \in I}$ , we may write  $\ell_p(a)$  instead of  $\lambda_p(A)$ ,  $1 \leq p \leq \infty$ , and  $c_0(a)$  instead of  $\lambda_0(A)$ . Of course, if  $I = \mathbb{N}$  and  $a \equiv 1$ , we obtain the familiar sequence spaces  $\ell_p$  and  $c_0$ . The elements of the echelon spaces can be considered as generalised sequences and for instance  $\ell_p(a)$  is a diagonal transform via  $a$  of the space  $\ell_p(I)$  of all  $p$ -absolutely summing sequences on  $I$ . Thus formally  $\ell_p(a) = \lambda_p(C_n)$  for the constant Köthe matrix  $C_n$  on  $I$  consisting of the single function  $a$  and hence  $\lambda_p(A) = \text{proj}_{\leftarrow n} \ell_p(a)$  algebraically and topologically.

For a Köthe matrix  $A = (a_n)_n$ , let  $V = (v_n)_n$  denote the associated decreasing sequence of strictly positive functions  $v_n = 1/a_n$ . In the notation introduced above we put

$$k_p(V) = \text{ind}_{n \rightarrow} \ell_p(v_n) \quad \text{for } 1 \leq p \leq \infty$$

and  $k_0(V) = \text{ind}_{n \rightarrow} c_0(v_n)$ . That is,  $k_p(V)$  is an increasing union of the Banach spaces  $\ell_p(v_n)$  endowed with the strongest locally convex topology under which the injection from each of these Banach spaces is continuous. The spaces  $k_p(V)$  are called *co-echelon spaces* of order  $p$  and by the duality of projective and inductive limits (6.3) we get algebraically that

$$\lambda_p(A)' = k_q(V), \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad q = 1 \quad \text{for } p = 0.$$

Köthe echelon and co-echelon spaces are among the most important examples of Fréchet and (DF)-spaces, respectively.

For a given decreasing sequence  $V = (v_n)_n$  of strictly positive functions on  $I$  or for the corresponding Köthe matrix  $A = (a_n)_n$ ,  $a_n = 1/v_n$ , we denote by  $\bar{V} = \bar{V}(V)$  the uncountable system

$$\lambda_\infty(A)_+ = \left\{ \bar{v} = (\bar{v}(i))_{i \in I} \in \mathbb{R}_+^I \mid \forall n \in \mathbb{N} : \sup_{i \in I} \frac{\bar{v}(i)}{v_n(i)} = \sup_{i \in I} a_n(i)\bar{v}(i) < \infty \right\}$$

of non-negative generalised sequences. Even though all the functions  $v_n$  are assumed to be strictly positive, the system  $\bar{V}$  need not contain any strictly positive elements (see [7], Example 1.6). However, if  $I$  is countable, then  $\bar{V}$  always contains strictly positive functions  $\bar{v} \in \bar{V}$  and we can restrict our attention to such functions.

Then, for  $1 \leq p \leq \infty$  we associate with  $\bar{V}$  the spaces  $K_p$ , where

$$K_p(\bar{V}) = \{x = (x(i))_{i \in I} \in \mathbb{C}^I \mid \forall \bar{v} \in \bar{V} : r_{\bar{v}}^{(p)}(x) = \left(\sum_{i \in I} (\bar{v}(i)|x(i)|)^p\right)^{1/p} < \infty\},$$

as well as

$$K_\infty(\bar{V}) = \{x \in \mathbb{C}^I \mid \text{for each } \bar{v} \in \bar{V} : r_{\bar{v}}^\infty(x) = \sup_{i \in I} \bar{v}(i)|x(i)| < \infty\},$$

$$K_0(\bar{V}) = \{x \in \mathbb{C}^I \mid \text{for each } \bar{v} \in \bar{V} : (\bar{v}(i)x(i))_{i \in I} \text{ converges to } 0 \text{ on } I\}.$$

These spaces are equipped with the complete locally convex topology given by the seminorms  $r_{\bar{v}}^{(p)}$ ,  $\bar{v} \in \bar{V}$ . The notation suggests that  $K_p(\bar{V})$  is in some sense related to  $k_p(V)$  and in fact it is easily seen that  $k_p(V)$  is continuously embedded in  $K_p(\bar{V})$ ,  $p = 0$  or  $1 \leq p \leq \infty$  and that for  $1 \leq p \leq \infty$ ,  $k_p(V) = K_p(\bar{V})$  algebraically, that is, the spaces are equal as linear spaces and have the same bounded sets.

Whether this equality holds also topologically is an interesting question, since in general the inductive limit topology of  $k_p$  is strictly finer than the weighted topology of  $K_p$ . It turns out that for  $1 \leq p < \infty$  we do have  $k_p(V) = K_p(\bar{V})$  algebraically and topologically and in particular the inductive limit topology of  $k_p(V)$  is given by the system  $(r_{\bar{v}}^{(p)})_{\bar{v} \in \bar{V}}$  of seminorms and  $k_p$  is always complete. However, for  $p = 0$  and  $p = \infty$  the topological equality is not true in general: when  $p = 0$ , the inductive limit topology of  $k_0$  is the one induced by the system  $(r_{\bar{v}}^{(0)})_{\bar{v} \in \bar{V}}$  of seminorms, but  $k_0$  can be a proper subspace of  $K_0$ , whereas for  $p = \infty$  the topologies of  $k_\infty$  and  $K_\infty$  do not always agree. To establish when an inductive limit can be identified algebraically and topologically with its associated weighted space is called the *projective description problem*, see for instance [9].

In the literature a thorough introduction to Köthe sequence spaces by K. Bierstedt, R. Meise and W. Summers can be found in [7] with the main results being summarised in [8] from the point of view of Fréchet spaces. In [6] the emphasis is on inductive limits but the theory of sequence spaces is also developed as an example of the applicability of projective and inductive limits.

The echelon and co-echelon spaces have been named after G. Köthe who studied them (with O. Toeplitz) already before the development of the tools available through the present day theory of topological vector spaces. Köthe's early work with sequence spaces helped the development of the general theory of locally convex spaces by often simplifying proofs of old theorems and making important generalisations of others. Even today echelon and co-echelon spaces are still a useful source of examples and counterexamples which mark out the boundaries of possible theorems, see [22].

## 8. SUMMARY OF ARTICLES [b] AND [c]

It is well known that the Bergman projection is not bounded on the space  $L^\infty(\mathbb{D})$  of bounded measurable functions. In [33] J. Taskinen introduced the weighted locally convex spaces  $L_V^\infty(\mathbb{D})$  of measurable and  $H_V^\infty(\mathbb{D})$  of analytic functions on the open unit disk. They are both (LB)-spaces containing the spaces  $L^\infty(\mathbb{D})$  and  $H^\infty(\mathbb{D})$ , respectively, and the Bergman projection is continuous from  $L_V^\infty(\mathbb{D})$  onto  $H_V^\infty(\mathbb{D})$ . Considering the continuity of the Bergman projection the space  $H_V^\infty$  is in some sense the smallest possible substitute to  $H^\infty$ .

This result was extended to the unit ball of  $\mathbb{C}^n$  by M. Jasiczak in [20], then further generalised to a smoothly bounded strictly pseudoconvex domain  $\Omega \subset \mathbb{C}^n$  by M. Engliš, the author and J. Taskinen in [b] and also, independently and more or less at the same time, by Jasiczak in [21].

Let  $\Omega \subset \mathbb{C}^n$  be a smoothly bounded strictly pseudoconvex domain with a defining function  $r$  such that  $r > 0$  on  $\Omega$ . Denote by  $V$  the family of strictly positive functions  $v : (-\infty, 1) \rightarrow \mathbb{R}_+$  such that  $\sup_{t < 1} v(t)l(t)^n < \infty$  for all  $n = 0, 1, 2, \dots$ , where  $l(t) = \max\{1, -\log(1-t)\}$ . Then a function  $w$  on  $\Omega$  with a defining function  $r$  belongs to  $V_{\Omega, r}$  if  $w(z) = v(1-r(z))$  for some  $v \in V$ . Let also  $H_{V, r}^\infty(\Omega)$  be the space of all holomorphic functions  $f$  on  $\Omega$  such that  $\|f\|_w = \sup_{z \in \Omega} w(z)|f(z)|$  is finite for all  $w \in V_{\Omega, r}$ , equipped with the topology induced by the seminorms  $\|\cdot\|_w$ . Define  $L_{V, r}^\infty(\Omega)$  in the same way replacing "holomorphic" by "measurable" and sup by ess sup.

In the article [b] we show that, firstly, the weighted Bergman projection  $P_\alpha : L_\alpha^2(\Omega) \rightarrow A_\alpha^2(\Omega)$ ,

$$P_\alpha f(z) = \int_{\Omega} K_\alpha(z, \zeta) f(\zeta) r(\zeta)^\alpha dA(\zeta),$$

is a continuous operator on  $L_{V, r}^\infty$ , here  $K_\alpha(z, \zeta)$  is the associated reproducing kernel and  $dA$  is the Lebesgue measure on  $\mathbb{C}^n$ . Secondly, we show that  $L_{V, r}^\infty$  is the smallest locally convex space  $\mathcal{X}$  for which a)  $L^\infty(\Omega) \subset \mathcal{X}$ , b) the unweighted Bergman projection  $P$  is bounded on  $\mathcal{X}$  and c) the topology in  $\mathcal{X}$  is given by a family of radially weighted sup-norms. This, as well as [21], generalises the result of [33].

In both cases the proof of the continuity is based on generalised Forelli-Rudin estimates while the proof of minimality uses peaking functions and a construction of functions inspired by S. Bell (Lemma 2 in [3]).

In [32] Taskinen showed that the space  $H_{V, r}^\infty(\mathbb{D})$  admits an atomic decomposition and this result is generalised in [c] by proving an atomic decomposition result for the space  $H_{V, r}^\infty(\Omega)$ , where  $\Omega \subset \mathbb{C}^n$  is again a smoothly bounded strictly pseudoconvex domain. Every function  $f \in H_{V, r}^\infty(\Omega)$  can be presented as an infinite linear combination of atoms on  $\Omega$  such that the coefficient sequence belongs to a suitable Köthe co-echelon space.

The construction of the atomic decomposition in [c] follows that of [32] with some technical modifications due to higher dimensions. The general outline of the proof is the same as the one used in [a], dating back to Coifman and Rochberg [12] and Zhu [36]. This includes the setting up of a lattice for the atoms to be evaluated in and defining three continuous operators which together make up a continuous projection in the Köthe sequence space  $K_\infty$  thus implying that the space  $H_{V, r}^\infty$  and a complemented subspace of  $K_\infty$  are isomorphic to each other.

The notation used in [c] for the Köthe sequence spaces differs slightly from the traditional one used by Bierstedt and above in Section 7, where  $a_n(i)$  denotes the same sequence as  $a_k(n)$  in [c], equation (4.2).

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