Renormalization methods in KAM theory

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ABSTRACT. It is well known that an integrable (in the sense of Arnold-Jost) Hamiltonian system gives rise to quasi-periodic motion with trajectories running on invariant tori. These tori foliate the whole phase space. If we perturb an integrable system, the Kolmogorow-Arnold-Moser (KAM) theorem states that, provided some non-degeneracy condition and that the perturbation is sufficiently small, most of the invariant tori carrying quasi-periodic motion persist, getting only slightly deformed. The measure of the persisting invariant tori is large together with the inverse of the size of the perturbation.

In the first part of the thesis we shall use a Renormalization Group (RG) scheme in order to prove the classical KAM result in the case of a non analytic perturbation (the latter will only be assumed to have continuous derivatives up to a sufficiently large order). We shall proceed by solving a sequence of problems in which the perturbations are analytic approximations of the original one. We will finally show that the approximate solutions will converge to a differentiable solution of our original problem.

In the second part we will use an RG scheme using continuous scales, so that instead of solving an iterative equation as in the classical RG KAM, we will end up solving a partial differential equation. This will allow us to reduce the complications of treating a sequence of iterative equations to the use of the Banach fixed point theorem in a suitable Banach space.

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-Mah...Io dico: perché realizzare un’opera
se è così bello sognarla soltanto?
(Il decameron, Pier Paolo Pasolini)

Chapter 1

Introduction

The Year 1885 is fundamental in the history of the modern theory of dynamical systems: in that year King Oscar II of Sweden and Norway decided to award a prize to the first person who would be able to provide an analytic solution to the n-body problem; the problem read: "Given a system of arbitrarily many mass points that attract each other according to Newton's law, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time and for all whose values the series converges uniformly". The mathematician Henri Poincaré, after three years of hard work, was awarded the prize despite the fact that he couldn't fully accomplish the given task. Even though he was not able to find a complete solution to the n-body problem, the contribution given to the modern understanding of dynamical systems by the research he had done in the attempt to win the prize was inestimable. Later on, gathering his notes, he published the book [**22**] which is considered to be the cornerstone of the modern theory of dynamical systems. The new point of view developed by Poincaré was still in accordance with the assumption that dynamical systems are to be considered deterministic; however his revolutionary idea was that, instead of looking for analytic solutions to the equations governing the motion, one has to start thinking geometrically and quantitatively. In this way, abandoning the goal of finding accurate predictions on the configuration of a system at each time, one can still recover geometrical and quantitative properties which provide a deep insight into the global behavior of the motion. Poincaré's was the first attempt to rigorously define mathematical "chaos" and to deal with it. The reader interested in the historical development of "chaos theory" can read the book [**10**].

KAM Theory can be considered one of the many offsprings of Poincaré's pioneering work. It deals with stability problems that arise in the study of certain perturbed dynamical systems. A brief preliminary discussion is in order: if a dynamical system is very sensitive to the smallest changes in the model used to study it, one has to be careful in understanding whether it is possible to apply the mathematical results to the real world. In fact, whatever model one uses, the latter is necessarily an "approximation" due to the imprecision of measurement instruments, to the idealization of the real model and so on. A very simple example of such "approximations" is the solar system: strictly speaking it is not true that the planets describe elliptical orbits around the sun; that would happen if, studying the motion of a single planet around the sun, one could neglect the perturbative effect produced by the other planets in the solar system; such effect is indeed very small (the masses of the planets are tiny compared to the mass of the sun), but unfortunately not to be neglected: the results of such perturbation can be seen by studying, for instance, the orbits of Venus and Mercury, who describe *slowly processional ellipses*, trajectories that slightly deviate from the Keplerian ellipses at each revolution around the sun. The conclusion we wanted to draw by bringing up the latter example is: the two-body problem (fully described by Keplerian ellipses) is only good as a first approximation of the motion of the planets in the solar system. Keeping that example in mind we can pass to describe the main goal of the KAM theory: if we are given a dynamical system that can be written as a perturbation of a "simpler" one, whose behaviour is well known, we would like to answer the following question: which of the properties of the simple system are preserved under the effect of the perturbation, assuming that the latter is sufficiently small? Returning to the solar system, we can translate the general question above into the following one: if we take into account the gravitational effect of all the planets among each other, will the keplerian ellipses get destroyed? Will periodic motion no longer exist? Will the planets fall into the sun? Will they escape the gravitational attraction of the sun and drift away from the solar system? Leaving these very dramatic questions open [1] we shall now translate this heuristic discussion into the more formal language of mathematics. The natural framework we shall operate in is the theory of Hamiltonian systems (on Hamiltonian systems see for instance [**3**]).

1. The KAM problem

Given a Hamiltonian function $H(p,q) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, it is possible under certain conditions (See [**16**] Appendix A.2) to introduce a special set of canonical coordinates $(I,\theta) \in \mathbb{R}^d \times \mathbb{T}^d$ called *action-angle* variables, so that in the new coordinates the Hamiltonian is a function of the new "momenta" only: $H = H(I)$. In such case the system described by H is called *integrable* and the motion in the new variables is very simple:

[1] To be honest, despite a lot having been written about the solar system's stability, the mutual interactions between the planets are probably too strong for the KAM theorem to be applied directly; nevertheless the example is still very instructive. Also, with the solar system being the main historical reason for studying dynamical systems, we thought it would be good to mention it.

Some interesting results on the stability of the planets of the solar system have been obtained by numerical integrations over large intervals of time: for instance the maximum orbit's eccentricity of the biggest planets (Neptune, Jupiter, Saturn, Uranus) seems to stay virtualy constant; the diffusion of the eccentricity of the Earth and Venus is moderate while that of Mars is large, finally Mercury is the planet with the biggest chaotic zone and its orbit's eccentricity experiences the largest diffusion. (see [**19**])

$$\begin{cases} I(t) = I_0 \\ \theta(t) = \theta_0 + \omega t \quad \text{where} \quad \omega := \frac{\partial H}{\partial I}|_{I=I_0}. \end{cases} \tag{1.1}$$

The trajectories are bound to run on the invariant tori $\mathbb{T}_{I_0} := \{(I_0, \theta) \,|\, \theta \in \mathbb{T}^d\}$. Notice that the frequencies $\omega = \omega_{I_0}$ depend on the particular invariant torus considered. In view of this remark we shall restrict our discussion to the *nondegenerate* case, in which one can number univocally the invariant tori $\mathbb{T}_{I_0}$ with the frequencies ω: the non-degeneracy condition reads

$$\det \left| \frac{\partial \omega}{\partial I} \right| = \det \left| \frac{\partial^2 H}{\partial I^2} \right| \neq 0. \tag{1.2}$$

Using the assumed one to one correspondence between frequencies and invariant tori, we shall call *non resonant* those tori numbered by rationally independent frequencies: $\omega \cdot q \neq 0$ for all $q \in \mathbb{Z}^d \setminus \{0\}$, and in this case the trajectories fill $\mathbb{T}_{I_0}$ densely. Otherwise, if $\exists q \in \mathbb{Z}^d \setminus \{0\}$ s.t. $\omega \cdot q = 0$, $\mathbb{T}_{I_0}$ will be said to be *resonant* and the trajectories will run on a subtorus of dimension $s < d$. We immediately see that the probability of ending up on a resonant torus is zero, hence for almost all the initial conditions the motion is dense on an invariant torus; such trajectories are called *quasi-periodic*.

Unfortunately the problems at our disposal described by integrable Hamiltonians are not numerous. Nevertheless, as pointed out in the heuristic introduction, one can still exploit the knowledge about integrable systems, by considering many important non-integrable systems as "small" perturbations of integrable ones. According to Poincaré (See [**22**]) the "fundamental problem of dynamics" is the study of a Hamiltonian of the form

$$H(I, \theta) = H_0(I) + \lambda V(I, \theta) \tag{1.3}$$

where $\lambda \ll 1$ is a small parameter. Since we already studied and completely solved the integrable case $\lambda = 0$, we are now interested in what happens as $\lambda \neq 0$ and the perturbation is "turned on". Will invariant tori and quasi-periodic motion still exist or will they instead be destroyed by the perturbation? The remarkable discovery of the KAM theory was that a large number of non-resonant invariant tori do not get destroyed, instead they get only deformed a little bit and still carry quasi-periodic motion. More precisely the non resonant tori that survive the perturbation (provided λ is small enough) are those numbered by the so called *diophantine* frequencies, that is, such ω's for which

$$|\omega \cdot q| \geq \gamma |q|^{-\nu} \quad \text{for some} \quad \gamma \in \mathbb{R}, /, \nu > d. \tag{1.4}$$

Hence ω cannot satisfy any resonance relation, not even approximately (the reason of the importance of the condition (1.4) will soon become clear).

Without loss of generality, from now on we shall concentrate on the study of the Hamiltonian function of a perturbed system of rotators:

$$H(I, \theta) = \frac{I^2}{2} + \lambda V(\theta), \tag{1.5}$$

where $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$ are the angles describing the positions of the rotators and $I = (I_1, \ldots, I_d) \in \mathbb{R}^d$ are the conjugated actions. It generates the equations of motion

$$\begin{cases} \dot{\theta}(t) & = I(t) \\ \dot{I}(t) & = -\lambda \partial_\theta V(\theta(t)). \end{cases} \tag{1.6}$$

To look for a "distorted" invariant torus of (1.6) means to find an embedding of the d-dimensional torus in $\mathbb{T}^d \times \mathbb{R}^d$, given by $\mathrm{Id} + X_\lambda : \mathbb{T}^d \to \mathbb{T}^d$, $Y_\lambda : \mathbb{T}^d \to \mathbb{R}^d$, such that the solutions of the differential equation

$$\dot{\varphi} = \omega \tag{1.7}$$

are mapped into the solutions of the equations of motion (1.6), so that the trajectories read

$$\begin{cases} \theta(t) & = \omega t + X_\lambda(\omega t) \\ I(t) & = Y_\lambda(\omega t). \end{cases} \tag{1.8}$$

Plugging (1.8) into (1.6) we get a well known equation for X:

$$\mathcal{D}^2 X(\theta) = -\lambda \partial_\theta V(\theta + X(\theta)), \quad \text{where} \quad \mathcal{D} := \omega \cdot \partial_\theta. \tag{1.9}$$

Trying to invert the operator $\mathcal{D}$ will lead us to deal with the infamous “small denominators”: if we formally write the Fourier expression for $\mathcal{D}^{-1}$, the latter is of the form $\frac{1}{(\omega \cdot q)}$, where $\omega \cdot q$ can become arbitrarily small as q varies in $\mathbb{Z}^d$. As we shall see, the diophantine condition plays a crucial role in controlling the size of such denominators.

2. The "Lindstedt series" and the first KAM proofs

One of the oldest methods of tackling (1.9) is to look for a solution $X(\theta)$ in the form of a λ-*formal power series*. A λ-formal power series expansion of X is a sequence $\{X_k\}_{k\in\mathbb{N}}$, such that $X_k : \mathbb{T}^d \to \mathbb{T}^d$, and it is customary to write $X(\theta) \sim \sum_{k=0}^{\infty} X_k(\theta)\lambda^k$. Expanding both sides of (1.9) in powers of λ one gets an infinite sequence of equations for X_k, $k = 0, 1, 2, \ldots$, which can be solved inductively. The formal power series associated to the problem (1.9) is called the *Lindstedt series*.

However, although this method is old and widely used in perturbation theory, it has a shortcoming: the convergence of the series $\sum_{k=0}^{\infty} X_k \lambda^k$ is not obvious. For instance one can experience that, even in much simpler problems, though the full series stays bounded for all times, if one truncates it up to order N, the truncated series blows up in time, and the blow up gets more and more severe the larger the number of terms N is taken. Nowadays we know that one cannot rely on the predictions given by the truncated series at order

N except for an interval of time much smaller than $\frac{1}{\lambda^N}$. Back in Poincaré's times, when he showed that the solar system is unstable to all orders in perturbation theory, the latter discovery caused consternation, and Poincaré himself became pessimistic about the fact that the perturbative series he was using could converge:

> Il semble donc permis de conclure que les series (2) ne convergent pas.
>
> Toutefois la raisonement qui précède ne suffit pas pour établir ce point avec une rigueur complète.
>
> [...]
>
> Ne peut-il pas arriver que les series (2) convergent quand on donne aux x_i^0 certaines valeurs convenablement choisies?
>
> Supposons, pour simplifier, qu'il y ait deux degrées de liberté les series ne pourraient-elles pas, par example, converger quand x_1^0 et x_2^0 ont été choisis de telle sorte que le rapport $\frac{n_1}{n_2}$ soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport $\frac{n_1}{n_2}$ est assujetti à une autre condition analogue à celle que je viens d'ennoncer un peu au hassard)?
>
> Les raisonnements de ce Chapitre ne me permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invêrsemblable. [2]

In 1954, at the International Mathematical Congress held in Amsterdam, A.N. Kolmogorov presented the paper [**18**] in which he gave a proof of the persistence of quasi-periodic motions for small perturbations of an integrable Hamiltonian. Despite the fact that his proof did not make use of the formal series expansion, the solution was proven to depend analytically on λ, showing

[2] Henri Poincaré, [**22**]

indirectly that the Lindsted series converges. Kolmogorov's result was later improved by V.I.Arnold [**1, 2**] and J.Moser [**20, 21**]: the apparently mysterious letters K, A and M that give the name to the whole theory are the initials of these three mathematicians

3. Inside the Lindstedt series

Even though after Kolmogorov's, Arnold's and Moser's work it was known that the Lindstedt series is convergent, it was only in 1988 that Eliasson, in [**9**] proved it directly. By working on the series terms, Eliasson showed the mechanisms that rely on the compensations that happen inside the series, compensations which counter the effect of the small denominators, and make the series converge. Later on, J. Feldman and F.Trubowitz (see [**11**]) noticed that Eliasson's method could be performed using the same diagrams that physicists had been using since Feynman. Namely one can associate to the Lindstetd series a particular kind of diagrams without loops called *tree graphs*. By means of such graphs one can conveniently express the Fourier coefficients $\widehat{X}_k(q)$ of the terms in the Taylor expansion of the formal solution $\sum_k X_k\lambda^k$. The coefficient $\widehat{X}_k(q)$ will be given by a sum running over all tree graphs with k vertices.

Finally, the analogies between the methods used in Quantum Field Theory and Eliasson's proof of KAM were fully understood by Gallavotti, Chierchia, Gentile et al., who, in many influential papers (see for instance [**7, 6, 14, 13, 12, 15**]), proved the convergence of the Lindstedt series by using a tool of QFT: the *Renormalization Group*. By using RG techniques, one can group the "bad terms" (particular subgraphs called *resonances*, which will be responsible for contributions inside $\widehat{X}_k(q)$ of the order $k!^s$ for $s > 1$.) that plague the Lindstedt series into particular families inside which the diverging contributions compensate each other.

The Renormalization Group has been applied to the KAM problem also by J. Bricmont, K. Gawędzki and A. Kupiainen in [**5**]: here the small denominators are treated separately scale by scale, and the mechanism responsible for the compensations that make the Lindstedt series converge is shown to rely on a symmetry of the problem, expressed by certain identities that are known in QFT: the so called *Ward identities*. The approach adopted in the latter paper is the same we adopt in the present work, for which [**5**] has been the main source of inspiration. By using the Ward identities in a slightly unusual fashion, we shall prove in the first part the KAM theorem in the case of a finitely many times differentiable function; in the second part we shall prove the KAM theorem for an analytic perturbation, using a continuous renormalization scheme.

Part 1

Differentiable perturbation

Chapter 2

The KAM theorem and RG scheme

As said in the Introduction, we are interested in the existence of invariant tori and quasi-periodic solutions of (1.5) for $\lambda > 0$. We shall investigate such problem in the special case of a non analytic perturbation V, the latter being assumed to be $\mathcal{C}^\ell$ for a sufficiently large integer ℓ, whose size will be estimated later on. Even though, as we already said, the main inspiration for this paper has been [**5**], on the case of a non analytic perturbation we are in debt to the papers [**7**] and [**26**] for many fruitful ideas.

From now on, we shall work with Fourier transforms, denoting by lower case letter the Fourier transform of functions of θ, which will be denoted by capital letters:

$$X(\theta) = \sum_{q\in\mathbb{Z}^d} e^{-\mathrm{i}q\cdot\theta} x(q), \quad \text{where} \quad x(q) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{\mathrm{i}q\cdot\theta} X(\theta) d\theta. \tag{2.1}$$

The rest of the first part of this thesis will be devoted to the proof of the following result:

Theorem 1. *Let H be the Hamiltonian* (1.5)*, with a perturbation V such that its Fourier coefficients satisfy $\sum_q |q|^{\ell+1}|v(q)| \leq C$ (i.e. $\partial V \in \mathcal{C}^\ell$), and fix a frequency ω satisfying the diophantine property* (1.4)*. Provided $|\lambda|$ is sufficiently small, if $\ell = \ell(\nu)$ is large enough, then for $s < \frac{2}{3}\ell$ there exists a $\mathcal{C}^s$ embedding of the d-dimensional torus in $\mathbb{T}^d \times \mathbb{R}^d$, given by Id$+X_\lambda : \mathbb{T}^d \to \mathbb{T}^d$, $Y_\lambda : \mathbb{T}^d \to \mathbb{R}^d$, such that the solutions of the differential equation*

$$\dot{\varphi} = \omega \tag{2.2}$$

are mapped into the solutions of the equations of motion generated by H, and the trajectories read

$$\begin{cases} \theta(t) & = \omega t + X_\lambda(\omega t) \\ I(t) & = Y_\lambda(\omega t), \end{cases} \tag{2.3}$$

running quasi-periodically on a d-dimensional invariant torus with frequency ω.

1. Scheme

In view of the discussion at the end of the previous section, let us define

$$W_0(X;\theta) := \lambda \partial_\theta V(\theta + X(\theta)). \tag{2.4}$$

Denote by G_0 the operator $(-\mathcal{D}^2)^{-1}$ acting on $\mathbb{R}^d$-valued functions on $\mathbb{T}^d$ with zero average. In terms of Fourier transforms,

$$(G_0 x)(q) = \begin{cases} \frac{x(q)}{(\omega \cdot q)^2} \quad \text{for} \quad q \neq 0 \\ 0 \quad \text{for} \quad q = 0; \end{cases} \tag{2.5}$$

we know that by inserting (2.3) into the equations of motion we get Eq. (1.9) (see p. 8), so we write the latter as the fixed point equation

$$X = G_0 P W_0(X), \tag{2.6}$$

where P projects out the constants: $PX = X - \int_{\mathbb{T}^d} X(\theta) d\theta$.

As we are not granted analyticity, we are not able to solve (2.6) by using a standard renormalization scheme for analytic perturbations (See for instance [**5**]): we have to proceed by means of analytic aproximations, easier to treat. Let us set for $j = 1, 2, \dots$ the constants $\gamma_j,\ \alpha_j,\ \bar{\alpha}_j$ as follows

$$\begin{aligned} \gamma_j &:= M8^j \\ \alpha_j &:= \frac{1}{\gamma_{j-2}} = \frac{1}{M8^{j-2}} \\ \bar{\alpha}_j &= \frac{1}{\gamma_{j+1}} \end{aligned} \tag{2.7}$$

where M will be a large constant that we shall fix at the end of the proof. We define the analytic approximations

$$V^j(\xi) := \int_{\mathbb{T}^d} V(\theta) D_{\gamma_j}(\xi - \theta) d\theta = \sum_{|q|_\infty \leq \gamma_j} v(q) e^{iq\cdot\xi}. \tag{2.8}$$

where

$$D_N(\theta) = \prod_{i=1}^{d} \frac{\sin{(N + \frac{1}{2})\theta_i}}{\sin{\frac{\theta_i}{2}}} \tag{2.9}$$

is the Dirichlet Kernel (see Fig. 1).

With the latter setup, we get a sequence of "analytically" perturbed Hamiltonians:

$$H(I, \theta) = \frac{I^2}{2} + \lambda V^j(\theta), \tag{2.10}$$

givinge rise to a sequence of "analytic" problems

$$X(\theta) = G_0 P W_0^j(X; \theta). \tag{2.11}$$

where

$$W_0^j(X; \theta) \equiv \lambda \partial_\theta V^j(\theta + X(\theta)) \tag{2.12}$$

For each j using for instance the renormalization scheme in [**5**], one could solve (2.11) for a fixed set of frequencies and for a j-dependent λ, but that would not work, as either λ or the set of allowed frequencies, could shrink to

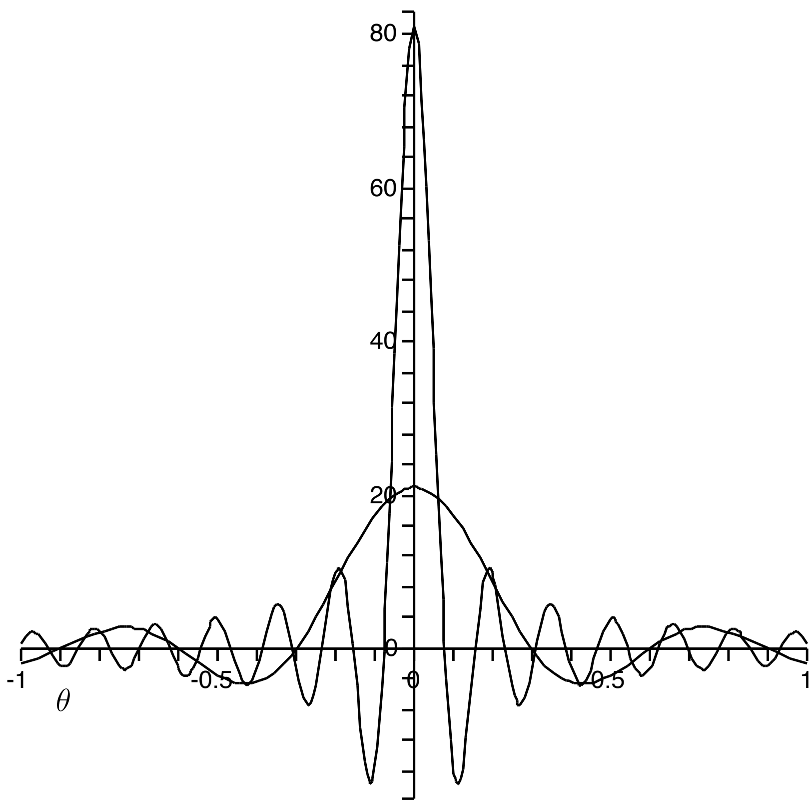

Figure 1. The Dirichlet kernel for $d = 1$ plotted at $N = 10$ and $N = 40$

zero as j grows, making the procedure useless. Instead we shall show that, by a slight modification of the scheme, we obtain a sequence of "approximated" problems, whose solutions will allow us to construct, for ℓ big enough and $|\lambda| \leq \lambda_0$, a sequence (solving (2.11)) converging to a $\mathcal{C}^s$ solution of our original problem, for $s < \frac{\ell}{3}$.

We can assume inductively, as discussed earlier, that for $|\lambda| \leq \lambda_0$ and $k = 0, \ldots j-1$ we have constructed real analytic functions $X_k(\theta)$ such that

$$X_k(\theta) = G_0 P W_0^k(X_k; \theta), \tag{2.13}$$

we shall look for a solution to (2.13) with $k = j$, and in order to do that we shall exploit the fact that X_{j-1} is a good aproximation to it.

From now on we shall write $\bar{X} := X_{j-1} = G_0 W_0^{j-1}(X_{j-1})$ and set

$$\widetilde{W}_0^j(Y) = W_0^j(\bar{X} + Y) - W_0^{j-1}(\bar{X}). \tag{2.14}$$

We notice that if the fixed point equation

$$Y = G_0 \widetilde{W}_0^j(Y) \tag{2.15}$$

has a solution Y_j, then $X_j \equiv \bar{X} + Y_j$, is a solution to (2.11) for $k = j$ that we were looking for.

In this setup we shall start our renormalizative scheme: in the same fashion as in [**5**], we decompose

$$G_0 = G_1 + \Gamma_0 \tag{2.16}$$

where Γ_0 will effectively involve only the Fourier components with $|\omega \cdot q|$ larger than $\mathcal{O}(1)$ and G_1 the ones with $|\omega \cdot q|$ smaller than that.

We want to prove the existence of maps $\widetilde{W}_1^j$ such that

$$\widetilde{W}_1^j(Y) = \widetilde{W}_0^j(Y + \Gamma_0 \widetilde{W}_0^j(Y)). \tag{2.17}$$

Inserting

$$F_1^j(Y) \equiv Y + \Gamma_0 \widetilde{W}_1^j(Y) \tag{2.18}$$

into Eq. (2.15) we notice

$$\begin{aligned} &F_1^j(Y) \quad \text{is a solution to (2.15)} \\ &\iff Y + \Gamma_0 \widetilde{W}_1^j(Y) \\ &= (G_1 + \Gamma_0) P \widetilde{W}_0^j(Y + \Gamma_0 \widetilde{W}_1^j(Y)) \\ &\iff Y = G_1 P \widetilde{W}_0^j(Y + \Gamma_0 \widetilde{W}_1^j(Y)) \\ &\iff Y = G_1 P \widetilde{W}_1^j(Y). \end{aligned} \tag{2.19}$$

Thus (2.15) reduces to (2.19) up to solving the easy large denominators problem (2.17) and to replacing the maps $\widetilde{W}_0^j$ by $\widetilde{W}_1^j$.

After $n-1$ inductive steps, the solution of Eq. (2.15) will be given by

$$F_{n-1}^j(Y) = Y + \Gamma_{n-2} \widetilde{W}_{n-1}^j(Y) \tag{2.20}$$

where Y must satisfy the equation

$$Y = G_{n-1} P \widetilde{W}_{n-1}^j(\bar{X}) \tag{2.21}$$

where G_{n-1} contains only the denominators $|\omega \cdot q| \leq \mathcal{O}(\eta^n)$ where $0 < \eta \ll 1$ is fixed once for all. The next inductive step consists of decomposing $G_{n-1} = G_n + \Gamma_{n-1}$ where Γ_{n-1} involves $|\omega \cdot q|$ of order η^n and G_n the ones smaller than that.

Let's now define $\widetilde{W}_n^j(Y)$ as the solution of the fixed point equation

$$\widetilde{W}_n^j(Y) = \widetilde{W}_{n-1}^j(Y + \Gamma_{n-1} \widetilde{W}_n^j(Y)), \tag{2.22}$$

and set

$$F_n(Y) = F_{n-1}(Y + \Gamma_{n-1} \widetilde{W}_n^j(Y)). \tag{2.23}$$

We infer that $F_n^j(Y)$ is the solution of (2.15) if and only if $Y = G_n P \widetilde{W}_n^j(Y)$, completing the following inductive step.

Finally it is easy to recover the inductive formulae

$$\widetilde{W}_n^j(Y) = \widetilde{W}_0^j(Y + \Gamma_{<n}\widetilde{W}_n^j(Y)) \tag{2.24}$$

$$F_n^j(Y) = Y + \Gamma_{<n}\widetilde{W}_n^j(Y), \tag{2.25}$$

where $\Gamma_{<n}=\sum_{k=0}^{n-1}\Gamma_k$. Using (2.24) and (2.25) we see that, if $F_n^j(0)$ converges for $n \to \infty$ to F^j, we have

$$\begin{aligned} F_n^j(0) &= \Gamma_{<n}\widetilde{W}_n^j(0) \\ &= \Gamma_{<n}\widetilde{W}_0^j(\Gamma_{<n}\widetilde{W}_n^j(0)) \\ &= \Gamma_{<n}\widetilde{W}_0^j(F_n^j(0)), \end{aligned} \tag{2.26}$$

and taking the limit for $n \to \infty$,

$$F^j = G_0\widetilde{W}_0^j(F^j) \tag{2.27}$$

so that F^j is the solution of (2.15) we are looking for.

Chapter 3

Setup and preliminary results

1. Spaces

Let $q \in \mathbb{Z}^d$, $\gamma \in \mathbb{N}^d$, we will use the following notation

$$|q| = \sum_{i=1}^{d} |q_i|, \quad |\gamma| = \sum_{i=1}^{d} |\gamma_i|, \quad \gamma! = \gamma_1! \cdots \gamma_d!, \quad \partial^\gamma X = \frac{\partial^{|\gamma|} X}{\partial\theta_1^{\gamma^1} \cdots \partial\theta_d^{\gamma_d}}; \tag{3.1}$$

Denote by Ξ_α the complex strip

$$\Xi_\alpha := \{\xi \in \mathbb{C}^d \,:\, |\mathrm{Im}\xi| < \alpha\}. \tag{3.2}$$

For $\alpha \geq 0$ we define

$$\mathcal{R}_\alpha(\mathbb{T}^d, \mathbb{R}^N) := \{X \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^N) \text{ with analytic and bounded extension on } \Xi_\alpha\} \tag{3.3}$$

Lemma 2. *We can almost exactly characterize the functions in $\mathcal{R}_\alpha$ in terms of the decay of their Fourier coefficients:*

(i) $X \in \mathcal{R}_\alpha$, *for some* $\alpha > 0 \Longrightarrow |x(q)| \leq Ce^{-\alpha|q|}$

(ii) $|x(q)| \leq Ce^{-\alpha|q|}$, *for some* $\alpha > 0 \Longrightarrow X \in \mathcal{R}_\eta$ *for all* $\eta < \alpha$

Proof. Let $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$, $q = (q_1, \dots, q_d) \in \mathbb{Z}^d$.

(i) If $0 \le \eta \le \alpha$, we have

$$|x(q)| = \left| \int_{\mathbb{T}^d} X(\theta + i\eta \frac{q}{|q|}) e^{iq\cdot(\theta + i\eta\frac{q}{|q|})} d\theta \right|$$
$$\le \int_{\mathbb{T}^d} \left| X(\theta + i\eta \frac{q}{|q|}) \right| d\theta \; e^{-|q|\eta}$$

which yields $|x(q)| \le Ce^{-|q|\eta}$ with $C = \sup_{\xi\in\Xi} |X(\xi)|$.

(ii)

$$\sup_{\xi\in\Xi_\eta} |X(\xi)| = \sup_{\xi\in\Xi_\eta} \left| \sum_{q\in\mathbb{Z}^d} x(q) e^{iq\cdot\xi} \right|$$
$$\le \sup_{\xi\in\Xi_\eta} \sum_{q\in\mathbb{Z}^d} |x(q)| e^{\operatorname{Im}\xi |q|}$$
$$\le \sup_{\xi\in\Xi_\eta} \sum_{q\in\mathbb{Z}^d} C e^{(\operatorname{Im}\xi - \alpha)|q|}$$
$$\le \sum_{q\in\mathbb{Z}^d} C e^{(\eta-\alpha)|q|} < \infty \tag{3.4}$$

□

Recalling the definition (2.8), we write $V^j(\theta) = \sum_q v^j(q) e^{iq\cdot\theta}$ by setting

$$v^j(q) = \begin{cases} v(q) & \text{for} |q| \le \gamma_j \\ 0 & \text{for} |q| > \gamma_j, \end{cases} \tag{3.5}$$

We shall denote

$$\mathcal{H} \equiv \{(w(q))_{q\in\mathbb{Z}} \mid \|w\| := \sum_q |w(q)| < \infty\} \tag{3.6}$$
$$B(r) \equiv \{w \in \mathcal{H} \mid \|w\| \le r\}. \tag{3.7}$$

and let $H^\infty(B(r), \mathcal{H})$ denote the Banach space of analytic functions $w : B(r) \to \mathcal{H}$ equipped with the supremum norm.

From now on we shall write $\bar{x} \equiv x_{j-1}$ for the inductive solution of the $(j-1)$-th analytic problem as discussed in section 1, that is

$$\bar{x} = G_0 w_0^{j-1}(\bar{x}) \quad \bar{x}(0) = 0, \tag{3.8}$$

and assume inductively the following decay:

$$|\bar{x}(q)| \le C\varepsilon A_j \frac{e^{-\frac{|q|}{4\gamma_j}}}{|q|^{\ell/3}} \quad \text{with} \quad A_j := \sum_{k=0}^{j-1} \ell! \left(\frac{4}{M8^{k-5}}\right)^{\frac{\ell}{3}} \quad \text{and} \varepsilon \to 0 \text{ when } |\lambda| \to 0, \tag{3.9}$$

where M is as in (2.7).

From now on $C, C_1, C_2, C_3 \dots$ will denote different constants which can vary from time to time. We can omit their dependence on the parameters when we think it is not important.

2. A priori bounds for the approximated problems

The maps V^j defined in (2.8) clearly belong to $\mathcal{R}_{\gamma_j^{-1}}$, so that there exists $C > 0$ such that for all j

$$\sup_{\xi \in \Xi_{\gamma_j^{-1}}} |V^j(\xi)| \le C \tag{3.10}$$

which implies the following

Lemma 3. *For each $|\sigma| < \frac{1}{4\gamma_j}$, there exists $b > 0$, such that the coefficients $V^j_{n+1}(\theta + \bar{X}(\theta))$ belonging to the space of n-linear maps $\mathcal{L}(\mathbb{C}^d, \dots, \mathbb{C}^d; \mathbb{C}^d)$, of the Taylor expansion*

$$\partial V^j(\theta + \bar{X}(\theta) + Y) = \sum_{n=0}^{\infty} \frac{1}{n!} V^j_{n+1}(\theta + \bar{X}(\theta))(Y, \dots, Y) \tag{3.11}$$

have Fourier coefficients that decay according to the following bound

$$\sum_{q \in \mathbb{Z}^d} e^{\sigma|q|} \|v^j_{n+1}(q; x)\|_{\mathcal{L}(\mathbb{C}^d, \dots, \mathbb{C}^d; \mathbb{C}^d)} < bn!(2\gamma_j)^n. \tag{3.12}$$

Proof. First of all we notice that, if $|\mathrm{Im}\,\xi| \le \frac{1}{4\gamma_j}$ then $|\mathrm{Im}\,(\xi + \bar{X}(\xi))| \le \frac{1}{2\gamma_j}$, in fact

$$\begin{aligned}
&\left|\mathrm{Im}\bar{X}(\xi)\right| = \left|\mathrm{Im}\left(\bar{X}(\xi) - \bar{X}(\mathrm{Re}\,\xi)\right)\right| \\
&\le \left|\bar{X}(\xi) - \bar{X}(\mathrm{Re}\,\xi)\right| \\
&\le \frac{1}{4\gamma_j} \sup_{\xi \in \Xi_{\frac{1}{4\gamma_j}}} \left|\partial_\xi \bar{X}(\xi)\right| \\
&\le \frac{1}{4\gamma_j} \sup_{\xi \in \Xi_{\frac{1}{4\gamma_j}}} \sum_q |q||\bar{x}(q)|e^{iq\cdot\xi} \\
&\le \frac{1}{4\gamma_j} \sum_q |q||\bar{x}(q)|e^{|q|\frac{1}{4\gamma_j}} \\
&\le \frac{1}{4\gamma_j}
\end{aligned} \tag{3.13}$$

using (3.9) for ε (i.e. $|\lambda|$) small enough; hence from the Cauchy estimates for analytic functions we get

$$\|V^j_{n+1}(\theta + X(\theta))\|_{\mathcal{L}(\mathbb{C}^{2d},\ldots,\mathbb{C}^{2d};\mathbb{C}^{2d})} \le Cn!\,(2\gamma_j)^n \quad \exists C \in \mathbb{R} \tag{3.14}$$

and finally using Cauchy Theorem we have for all $\eta \in \mathbb{R}$ such that $|\eta| \le \frac{1}{4\gamma_j}$

$$\begin{aligned}
&\left|v^j_{n+1}(q;x)(Y_1,\ldots,Y_n)\right| = \\
&\quad = \left|\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} V^j_{n+1}(\theta + i\eta + \bar{X}(\theta + i\eta))(Y_1,\ldots,Y_n)e^{iq\cdot(\theta+i\eta)}d\theta\right| \\
&\quad \le \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \left|V^j_{n+1}(\theta + i\eta + \bar{X}(\theta + i\eta))(Y_1,\ldots,Y_n)\right| e^{-q\cdot\eta} \\
&\quad \le Cn!\,(2\gamma_j)^n\,e^{-q\cdot\eta}|Y_1|\cdots|Y_n|
\end{aligned} \tag{3.15}$$

hence

$$\|v^j_{n+1}(q;x)\|_{\mathcal{L}(\mathbb{C}^d,\ldots,\mathbb{C}^d;\mathbb{C}^d)} \le Cn!\,(2\gamma_j)^n\,e^{-q\cdot\eta} \tag{3.16}$$

and taking $\eta = \frac{1}{4\gamma_j}\frac{q}{|q|}$ we get [1]

$$\sum_{q\in\mathbb{Z}^d} e^{\sigma|q|}\|v^j_{n+1}(q;x)\|_{\mathcal{L}(\mathbb{C}^d,\dots,\mathbb{C}^d;\mathbb{C}^d)} \le C \underbrace{\sum_{q\in\mathbb{Z}^d} e^{(\sigma-\frac{1}{4\gamma_j})|q|}}_{:=b<\infty} n!(2\gamma_j)^n \tag{3.17}$$

for all $0<\sigma<\frac{1}{4\gamma_j}$.

□

In view of the latter Lemma, let us introduce a translation τ_β by a vector $\beta\in\mathbb{C}^d$, $(\tau_\beta Y)(\theta) = Y(\theta-\beta)$. On $\mathcal{H}$, τ_β is given by $(\tau_\beta y)(q) = y(q)e^{iq\cdot\beta}$. It induces a map $w\mapsto w_\beta$ from $H^\infty(B(r_0),\mathcal{H})$ to itself if we set

$$w_\beta(y) = \tau_\beta(w(\tau_{-\beta}y)) \tag{3.18}$$

The fixed-point equations, (2.22) and (2.24) may be written in the form

$$\widetilde{w}^j_{n\beta}(y) = \widetilde{w}_{(n-1)\beta}(y+\Gamma_{n-1}\widetilde{w}_{n\beta}(y)) \tag{3.19}$$

$$\widetilde{w}^j_{n\beta}(y) = \widetilde{w}_{0\beta}(y+\Gamma_{<n}\widetilde{w}_{n\beta}(y)) \tag{3.20}$$

Remark 4. Note that, because of the definitions (2.14) and (3.18), one has

$$\widetilde{w}^j_{0\beta}(y) = \tau_\beta w^j_0(\bar{x}+\tau_{-\beta}y) - \tau_\beta w^{j-1}_0(\bar{x}) \tag{3.21}$$

and the right hand side is <u>not</u> $w^j_{0\beta}(\bar{x}+y) - w^{j-1}_{0\beta}(\bar{x})$.

Similarly, the equations (2.23) and (2.25) translate in the Fourier space to the relations

$$f^j_{n\beta}(y) = f_{(n-1)\beta}(y+\Gamma_{n-1}\widetilde{w}^j_{n\beta}(y)) \tag{3.22}$$

$$f^j_{n\beta}(y) = y+\Gamma_{<n}\widetilde{w}^j_{n\beta}(y) \tag{3.23}$$

[1] note that with that choice of η, because of (3.13), $\theta+i\eta+X(\theta+i\eta)$ is in the analyticity strip of the integrand function

Proposition 1. *Let* $|Im\beta| < \frac{1}{8\gamma_j}$*, and* $\|y\| \le \alpha_j^{\frac{2}{3}\ell}$ *(See* (2.7) *at p. 17) we have*

$$\sum_{q\in\mathbb{Z}^d} |\widetilde{w}^j_{0\beta}(y;q)| \le |\lambda| C_{d,\ell}\alpha_j^{\frac{2}{3}\ell} \tag{3.24}$$

and furthermore, writing

$$\widetilde{w}^j_{0\beta}(y) = \widetilde{w}^j_{0\beta}(0) + D\widetilde{w}^j_{0\beta}(0)y + \delta_2\widetilde{w}^j_{0\beta}(y), \tag{3.25}$$

we have

$$|\widetilde{w}^j_{0\beta}(0;q)| \le C_1|\lambda|\frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\frac{\ell}{3}}} \tag{3.26}$$

$$\|D\widetilde{w}^j_{0\beta}(0)y\| \le C_2|\lambda| \tag{3.27}$$

$$\|\delta_2\widetilde{w}^j_{0\beta}(y)\| \le C_3|\lambda|\alpha_j^{\ell} \tag{3.28}$$

Proof. Let us set

$$w_0^{j(n)}(\bar{x};q,q_1,\dots,q_n) \equiv \frac{1}{n!}v^j_{n+1}(\bar{x};q-\sum_j q_j) \tag{3.29}$$

inserting the Fourier expansion of Y, we can compute

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}^d} |\widetilde{w}^j_{0\beta}(y;q)| = \sum_{q\in\mathbb{Z}^d} |\tau_\beta w_0^j(\bar{x}+\tau_{-\beta}y;q) - \tau_\beta w_0^{j-1}(\bar{x};q)| \\
&= |\lambda|\sum_{n=0}^{\infty}\sum_{q,q_1,\dots,q_n}\Big| e^{i\beta\cdot(q-\sum q_j)} w_0^{j(n)}(\bar{x};q,q_1,\dots,q_n)(y(q_1),\dots,y(q_n)) + \\
&\quad - \sum_q e^{i\beta\cdot q} w_0^{j-1}(\bar{x};q)\Big| \\
&= |\lambda|\sum_{n=1}^{\infty}\sum_{q,q_1,\dots,q_n}\Big| e^{i\beta\cdot(q-\sum q_j)} w_0^{j(n)}(\bar{x};q,q_1,\dots,q_n)(y(q_1),\dots,y(q_n)) \\
&\quad + \sum_q \widetilde{w}^j_{0\beta}(0;q)\Big|
\end{aligned} \tag{3.30}$$

from which (3.27) follows immediately from Lemma 3, and (3.28) follows from Lemma 3 and from the fact that $\|y\| \le \alpha_j^{\frac{2}{3}\ell}$

To prove (3.26), for $|\eta| \le \frac{1}{4\gamma_j}$, we use (3.13), the hypotheses on V of Theorem 1 and (2.7) to get

$$\begin{aligned}
&\left|\partial V^j(\theta + i\eta + \bar{X}(\theta + i\eta)) - \partial V^{j-1}(\theta + i\eta + \bar{X}(\theta + i\eta))\right| \\
&= \left| \sum_{\gamma_{j-1} < |q| \le \gamma_j} q v(q) e^{iq\cdot(\theta + i\eta + \bar{X}(\theta + i\eta))} \right| \\
&\le \sum_{\gamma_{j-1} < |q| \le \gamma_j} |q||v(q)| e^{|q|\frac{1}{2\gamma_j}} \\
&\le \frac{1}{\gamma_{j-1}^{\ell}} \sum_{\gamma_{j-1} < |q| \le \gamma_j} |q|^{\ell+1} v(q) e^{|q|\frac{1}{2\gamma_j}} \le C \frac{1}{\gamma_{j-1}^{\ell}}.
\end{aligned} \tag{3.31}$$

Then we choose $|\eta| = \frac{1}{4\gamma_j}\frac{q}{|q|}$ and use (3.31) to proceed as in Lemma 3 in order to get

$$\begin{aligned}
&|\widetilde{w}^j_{\beta 0}(0; q)| = \left| e^{i\beta\cdot q} \left(w_0^j(\bar{x}; q) - w_0^{j-1}(\bar{x}; q) \right) \right| \\
&\le e^{|\mathrm{Im}\,\beta||q|} \frac{\lambda}{(2\pi)^d} \int_{\mathbb{T}^d} \left| (V^j - V^{j-1})(\theta + i\eta + \bar{X}(\theta + i\eta)) \right| e^{iq\cdot(\theta + i\eta)} d\theta \\
&\le |\lambda| C \frac{1}{\gamma_{j-1}^{\ell}} e^{(|\mathrm{Im}\,\beta| - \frac{1}{4\gamma_j})|q|} \le |\lambda| C \frac{(8\gamma_j)^{\ell/3}}{\gamma_{j-1}^{\ell}|q|^{\ell/3}} \le \varepsilon \frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\ell/3}}
\end{aligned} \tag{3.32}$$

for all $|\mathrm{Im}\,\beta| < \frac{1}{8\gamma_j} = \bar{\alpha}_j$.

Finally, in view of (3.25) we combine (3.26), (3.27), (3.28) and for ℓ large enough we obtain (3.24). This concludes the proof of the Lemma.

□

3. Cauchy Estimates

We state now some standard estimates we shall use throughout the paper. Let h, h' be Banach spaces, we define $H^\infty(h; h')$ as the space of analytic functions $w : h \to h'$ equipped with the supremum norm. We shall make use of the following Cauchy estimates throughout the proof:

$$\sup_{\|y\|\le r-\delta}\|Dw(y)\| \le \sup_{\|y\|\le r}\frac{1}{\delta}\|w(y)\| \tag{3.33}$$

$$\sup_{\|y\|\le r'\mu}\|\delta_k w(y)\| \le \frac{\mu^k}{1-\mu}\sup_{\|y\|\le r'}\|w(y)\| \tag{3.34}$$

Furthermore we will also make use of the following estimate: let $w_i \in H^\infty(B(r)\subset h\,;h')$ for $i=1,2$, and $w\in H^\infty(B(r')\subset h'\,;h'')$, then, if $\sup_{\|y\|_h\le r}\|w_i(y)\|_{h'}\le \frac{1}{2}r'$, we have

$$\sup_{\|y\|_h\le r}\|w\circ w_1(y)-w\circ w_2(y)\|_{h''}\le \frac{2}{r'}\sup_{\|y'\|_{h'}\le r'}\|w(y')\|_{h''}\sup_{\|y\|_h\le r}\|w_1(y)-w_2(y)\|_{h'} \tag{3.35}$$

4. The Cutoff and n-dependent spaces

To define the operators Γ_n - that establishes our renormalization- we will divide the real axis in scales. We shall fix $\eta \ll 1$ (once and for all) and introduce the so-called "standard mollifier" by

$$h(\kappa)=\begin{cases} Ce^{\frac{1}{\kappa^2-1}} & \text{if } |\kappa|<1\\ 0 & \text{if } |\kappa|\ge 1\end{cases} \tag{3.36}$$

with the constant C chosen such that $\int_{\mathbb{R}} h dx = 1$. Now let us define $\bar\chi \in \mathcal{C}^\infty(\mathbb{R})$ by

$$\bar\chi(\kappa) := 1-\frac{2}{1-\eta}\int_{\frac{1+\eta}{2}}^{\infty} h\left(\frac{2(|\kappa|-y)}{1-\eta}\right)dy \tag{3.37}$$

so that

$$\bar\chi(\kappa)=\begin{cases} 1 & \text{if } |\kappa|<\eta\\ 0 & \text{if } |\kappa|\ge 1\end{cases} \tag{3.38}$$

and trivially

$$\sup_{\kappa\in\mathbb{R}} |\partial_\kappa \bar{\chi}(\kappa)| \, , \, \sup_{\kappa\in\mathbb{R}} |\partial_\kappa^2 \bar{\chi}(\kappa)| \le C \tag{3.39}$$

$$\bar{\chi}_n(\kappa) = \bar{\chi}(\eta^{-n}\kappa) \tag{3.40}$$

and set

$$\begin{aligned} &\chi_0(\kappa) = 1 - \bar{\chi}_1(\kappa) \\ &\chi_n(\kappa) = \bar{\chi}_n(\kappa) - \bar{\chi}_{n+1}(\kappa) \quad \text{for} \quad n \ge 1. \end{aligned} \tag{3.41}$$

Finally we define the diagonal operator $\Gamma_n : \mathcal{H} \to \mathcal{H}$

$$\Gamma_n(q, q') = \frac{\chi_n(\omega \cdot q)}{(\omega \cdot q)^2} \delta_{q,q'} := \gamma_n(\omega \cdot q)\delta_{q,q'}, \tag{3.42}$$

so that supp $(\Gamma_{n-1}(q)) = \{\eta^{n+1} \le |\omega \cdot q| \le \eta^{n-1}\}$. The formulae coming from our renormalization scheme, suggest us to define n-dependent norms and spaces: for $n \ge 2$ we define the seminorms

$$\|w\|_{-n} = \sum_{|\omega\cdot q| \le \eta^{n-1}} |w(q)|. \tag{3.43}$$

Let $\mathcal{H}_{-n}$ denote the corresponding Banach spaces [2]. Next we consider the projection

$$P_n(y)(q) = \begin{cases} y(q) & \text{if } |\omega \cdot q| \le \eta^{n-1} \\ 0 & \text{otherwise.} \end{cases} \tag{3.44}$$

and define the spaces

$$\mathcal{H}_n \equiv P_n \mathcal{H}, \tag{3.45}$$

[2] In fact, since $\|\|_{-n}$ is a seminorm, $\mathcal{H}_{-n}$ is a Banach space up to identifying the maps $w(q)$ that coincide on the set $\{|\omega \cdot q| \le \eta^{n-1}\}$, but that is all we need.

equipped with the norm inherited from $\mathcal{H}$:

$$\|y\| \equiv \sum_q |y(q)| = \sum_{|\omega\cdot q|\leq\eta^{n-1}} |y(q)|, \tag{3.46}$$

Remark 5. For $y \in \mathcal{H}_n$, $\|y\| = \|y\|_{-n}$, even though in general $\|\cdot\| \neq \|\cdot\|_{-n}$,

Note the natural embeddings for $n \geq 2$:

$$\mathcal{H}_n \to \mathcal{H}_{n-1} \to \mathcal{H} \to \mathcal{H}_{-n+1} \to \mathcal{H}_{-n} \tag{3.47}$$

We shall denote by $B_n^j(r)$ the open ball in $\mathcal{H}_n$ of radius r_j.

If we define the cutoff with "shifted kernel"

$$\Gamma_n[\kappa](q) = \gamma_n(\omega\cdot q + \kappa) \tag{3.48}$$

we can prove the following:

Lemma 6. *For $i = 0, 1, 2$ and $|\kappa| \leq \eta^n$, the cutoff functions obey the following estimates*

$$\|\partial_\kappa^i \Gamma_{n-1}[\kappa]\| \leq C\eta^{-(2+i)n} \tag{3.49}$$

Proof. The proof is trivial, since for $\tilde{\kappa} = \kappa + \omega\cdot q$ we have, by definition, $\Gamma_{n-1}[\kappa](q) = \chi_{n-1}(\tilde{\kappa})/\tilde{\kappa}^2$ and $\chi_{n-1}(\kappa) = 0$ for $|\kappa| \leq \eta^n$. □

5. n-dependent bounds

Our final goal is to show that the maps $\widetilde{w}_n^j$ and f_n^j exist for all j and n, provided λ is small enough in an n-independent way. For later purposes it will be useful to show first some simple n-dependent bounds. Such bounds are carried out quite easily in the next proposition:

Proposition 2. *For any sufficiently small $r > 0$, $|\lambda| \leq \lambda_n$ and $|\mathrm{Im}\,\beta| \leq \alpha_j/2$ the equations* (3.20) *have a unique solution $\widetilde{w}_n^j \in H^\infty(B(\alpha_j^{\frac{2}{3}\ell} r^n), \mathcal{H})$ with*

$$\sup_{y\in B(\alpha_j^{\frac{2}{3}\ell} r^n)} \|w_n^j\| \leq C_{d,\ell}\,\alpha_j^{\frac{2}{3}\ell}|\lambda| \tag{3.50}$$

where $C_{d,\ell}$ is as in Proposition 1. Furthermore the maps $f^j_{n\beta}$ defined by Eqs. (3.23) *belong to $H^\infty(B(\alpha_j^{\frac{2}{3}\ell} r^n), \mathcal{H})$. They satisfy the bounds*

$$\sup_{\|y\| \le \alpha_j^{\frac{2}{3}\ell} r^n} \|f^j_{n\beta}(y)\| \le 2\alpha_j^{\frac{2}{3}\ell} r^n. \tag{3.51}$$

Moreover, $w^j_{n\beta}$ and $f^j_{n\beta}$ are analytic in λ and β and they satisfy the recursive relations (3.19) *and* (3.22)*, respectively.*

Proof. Consider the fixed point equation (3.20) and write it as $w = \mathcal{F}(w)$, for $w = \widetilde{w}^j_{0\beta}$ and

$$\mathcal{F}(w)(y) = \widetilde{w}^j_{0\beta}(y + \Gamma_{<n} w(y)). \tag{3.52}$$

Let

$$\mathcal{B}^j_n = \left\{ w \in H^\infty(B(\alpha_j^{\frac{2}{3}\ell} r^n), \mathcal{H}) \mid \|w\|_{\mathcal{B}^j_n} \equiv \sup_{y \in B(\alpha_j^{\frac{2}{3}\ell} r^n)} \|w(y)\| \le C_{d,\ell} \alpha_j^{\frac{2}{3}\ell} |\lambda| \right\}, \tag{3.53}$$

where $C_{d,\ell}$ is as in Prop. 1. Let us choose λ_n such that $C\eta^{-2n} C_{d,\ell} \lambda_n \le r^n$ for all n, with C as in Lemma 6. It follows from the latter that for $w \in \mathcal{B}^j_n$ and $y \in B(\alpha_j^{\frac{2}{3}\ell} r^n) \subset \mathcal{H}$,

$$\|y + \Gamma_{<n} w(y)\| \le \alpha_j^{\frac{2}{3}\ell} r^n + C\eta^{-2n} C_{d,\ell} \alpha_j^{\frac{2}{3}\ell} |\lambda| \le 2\alpha_j^{\frac{2}{3}\ell} r^n \le \frac{1}{2} \alpha_j^{\frac{2}{3}\ell}, \tag{3.54}$$

so $\mathcal{F}(w)$ is defined in $B(\alpha_j^{\frac{2}{3}\ell} r^n)$ and, by Proposition 1,

$$\|\mathcal{F}(w)\|_{\mathcal{B}^j_n} \le C_{d,\ell} \alpha_j^{\frac{2}{3}\ell} |\lambda|. \tag{3.55}$$

Hence $\mathcal{F} : \mathcal{B}^j_n \to \mathcal{B}^j_n$. For $w_1, w_2 \in \mathcal{B}^j_n$ use (3.35) to conclude that

$$\begin{aligned}
\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{\mathcal{B}^j_n} &= \sup_{\|y\| \le \alpha_j^{\frac{2}{3}\ell} r^n} \|\widetilde{w}^j_{0\beta}(y + \Gamma_{<n} w_1(y)) - \widetilde{w}^j_{0\beta}(y + \Gamma_{<n} w_2(y))\| \\
&\le \frac{2}{\alpha_j^{\frac{2}{3}\ell}} C_{d,\ell} \alpha_j^{\frac{2}{3}\ell} |\lambda| C \eta^{-2n} \|w_1 - w_2\|_{\mathcal{B}^j_n} \\
&\le 2r^n \|w_1 - w_2\|_{\mathcal{B}^j_n} \\
&\le \frac{1}{2} \|w_1 - w_2\|_{\mathcal{B}^j_n}, \qquad (3.56)
\end{aligned}$$

i.e. $\mathcal{F}$ is a contraction. It follows that (3.20) has a unique solution $\widetilde{w}^j_{n\beta}$ in $\mathcal{B}^j_n$ satisfying the bound (3.50), which, besides, is analytic in λ and β.

Consider now for $n \ge 2$ the map $\mathcal{F}'$:

$$\mathcal{F}'(w)(y) = \widetilde{w}_{0\beta}(y + \Gamma_{n-1} \widetilde{w}_{n\beta}(y) + \Gamma_{<n-1} w(y)); \qquad (3.57)$$

again $\mathcal{F}'$ is a contraction in $\mathcal{B}^j_n$ since, for $\|y\| \le \alpha_j^{\frac{2}{3}\ell} r^n$, we have

$$\|y + \Gamma_{n-1} \widetilde{w}_{n\beta}(y) + \Gamma_{<n-1} w(y)\| \le 3\alpha_j^{\frac{2}{3}\ell} r^n \le \frac{1}{2} \alpha_j^{\frac{2}{3}\ell} \qquad (3.58)$$

for r sufficiently small. But from Eqs. (3.20) one deduces that $\widetilde{w}^j_{n\beta}$ and $\widetilde{w}^j_{(n-1)\beta} \circ \left(1 + \Gamma_{n-1} \widetilde{w}^j_{n\beta}\right)$, both in $\mathcal{B}^j_n$, are its fixed points (just plug them into (3.57)), hence by uniqueness they have to coincide, and (3.19) follows.

By virtue of the estimate (3.54) and definition (3.23),

$$\sup_{\|y\| \le \alpha_j^{\frac{2}{3}\ell} r_j^n} \|f^j_{n\beta}(y)\| = \sup_{\|y\| \le \alpha_j^{\frac{2}{3}\ell} r_j^n} \|y + \Gamma_{<n} \widetilde{w}^j_{n\beta}(y)\| \le 2\alpha_j^{\frac{2}{3}\ell} r^n. \qquad (3.59)$$

The recursion (3.22) follows easily from Eq. (3.19):

$$\begin{aligned}
f^j_{n\beta}(y) &= y + \Gamma_{<n} \widetilde{w}^j_{n\beta}(y) \\
&= y + \Gamma_{n-1} \widetilde{w}^j_{n\beta}(y) + \Gamma_{<n-1} \widetilde{w}^j_{n\beta}(y) \\
&= y + \Gamma_{n-1} \widetilde{w}^j_{n\beta}(y) + \Gamma_{<n-1} \widetilde{w}^j_{(n-1)\beta}(y + \Gamma_{n-1} \widetilde{w}^j_{n\beta}(y)) \\
&= f^j_{(n-1)\beta}(y + \Gamma_{n-1} \widetilde{w}^j_{n\beta}(y)). \qquad (3.60)
\end{aligned}$$

□

Chapter 4

The Ward identities (revised)

We shall prove in this chapter some properties of the maps w_n^j which will be essential in the proof of the main theorem, namely in the part that deals with the compensations of the so-called *resonances*, the latter being the terms that make the convergence of the Lindstedt series problematic. We will prove some idientities, which will be a sort of "modified Ward identities" (for the "standard" Ward identities used to prove a KAM theorem see [**5**]) for the maps $\widetilde{w}_n^j$ that we constructed in Proposition 2. We will omit the indeces j, writing $X = \bar{X}$, $V = V_j$, $\widehat{V} = V_{j-1}$, $W = W^j$ and $U = W^{j-1}$, and the summations over repeated indeces will be understood. The basic identity reads

$$\int_{\mathbb{T}^d} \widetilde{W}_n^\gamma(Y;\theta)d\theta = \int_{\mathbb{T}^d} Y^\alpha(\theta)\partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta$$
$$+ \int_{\mathbb{T}^d} G_n U_0^\alpha(X;\theta)\partial_\gamma \widetilde{W}_n^\alpha(Y;\theta)d\theta. \tag{4.1}$$

Once (4.1) is proven, we can transpose it into the Fourier space language:

$$\widetilde{w}_n^{\gamma}(y;0) = -\sum_{q\neq 0} iq^{\gamma} y^{\alpha}(q) w_0^{\alpha}(x+y+\Gamma_{<n}\widetilde{w}_n(y);-q) \\ -\sum_{q\neq 0} iq^{\gamma}\bar{\chi}_n(\omega\cdot q)\bar{x}^{\alpha}(q)\widetilde{w}_n^{\alpha}(y;-q), \tag{4.2}$$

so it immediately follows that

$$\widetilde{w}_n^{\gamma}(0;0) = -\sum_{q\neq 0} iq^{\gamma}\bar{\chi}_n(\omega\cdot q)\bar{x}^{\alpha}(q)\widetilde{w}_n^{\alpha}(0;-q). \tag{4.3}$$

Differentiating (4.2) with respect to $y^{\alpha}(q)$ and evaluating it at $y=0$, we get

$$\frac{\partial \widetilde{w}_n^{\gamma}(y;0)}{\partial y^{\alpha}(q)}\Big|_{y=0} = -iq^{\gamma} w_0^{\alpha}(x+\Gamma_{<n}\widetilde{w}_n(0);-q) \\ -\sum_{q'\neq 0} iq'^{\gamma}\bar{\chi}_n(\omega\cdot q')\bar{x}^{\beta}(q')\frac{\partial \widetilde{w}_n^{\beta}(y;q')}{\partial y^{\alpha}(q)}\Big|_{y=0} \tag{4.4}$$

Let us finally prove (4.1), starting with $n=0$,

$$\int_{\mathbb{T}^d} \widetilde{W}_0^{\gamma}(Y;\theta)d\theta = \lambda\int_{\mathbb{T}^d}(\partial_{\gamma}V)(\theta+X(\theta)+Y(\theta))d\theta - \int_{\mathbb{T}^d}(\partial_{\gamma}\widehat{V})(\theta+X(\theta))d\theta \\ = \lambda\int_{\mathbb{T}^d}\partial_{\gamma}\left(V(\theta+X(\theta)+Y(\theta))\right)d\theta \\ -\lambda\int_{\mathbb{T}^d}(\partial_{\alpha}V)(\theta+X(\theta)+Y(\theta))\left(\partial_{\gamma}Y^{\alpha}(\theta)+\partial_{\gamma}X^{\alpha}(\theta)\right)d\theta \\ +\lambda\int_{\mathbb{T}^d}\partial_{\gamma}\left(\widehat{V}(\theta+X(\theta))\right)d\theta - \lambda\int_{\mathbb{T}^d}(\partial_{\alpha}\widehat{V})(\theta+X(\theta))\partial_{\gamma}X^{\alpha}(\theta)d\theta. \tag{4.5}$$

The first and the third term in the right hand side vanish, and by integrating the second and the fourth term by parts we get

$$\int_{\mathbb{T}^d} \widetilde{W}_0^{\gamma}(Y;\theta)d\theta = -\lambda\int_{\mathbb{T}^d}\partial_{\gamma}(\partial_{\alpha}V)(\theta+X(\theta)+Y(\theta))\left(Y^{\alpha}(\theta)+X^{\alpha}(\theta)\right)d\theta \\ -\lambda\int_{\mathbb{T}^d}\partial_{\gamma}(\partial_{\alpha}\widehat{V})(\theta+X(\theta))X^{\alpha}(\theta)d\theta. \tag{4.6}$$

Writing $(\partial_\alpha V)(\theta + \bar{X}(\theta) + Y(\theta)) = W_0^\alpha(X+Y;\theta)$, we get :

$$\int_{\mathbb{T}^d} \widetilde{W}_0^\gamma(Y;\theta)d\theta = \int_{\mathbb{T}^d} Y^\alpha(\theta)\partial_\gamma W_0^\alpha(X+Y;\theta)d\theta + \int_{\mathbb{T}^d} X^\alpha(\theta)\partial_\gamma \widetilde{W}_0^\alpha(Y;\theta)d\theta. \tag{4.7}$$

that is (4.1) for $n = 0$, since $X(\theta) = G_0U_0(\bar{X};\theta)$. To prove the claim for $n \geq 1$, we use the relation (2.24):

$$\begin{aligned}
&\int_{\mathbb{T}^d} \widetilde{W}_n^\gamma(Y,\theta)d\theta = \int_{\mathbb{T}^d} \widetilde{W}_0^\gamma(Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\overset{(*)}{=} \int_{\mathbb{T}^d} \Big(Y + \Gamma_{<n}\widetilde{W}_n(Y)\Big)^\alpha \partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\quad + \int_{\mathbb{T}^d} X^\alpha \partial_\gamma \widetilde{W}_n^\alpha(Y;\theta)d\theta \\
&\overset{(**)}{=} \int_{\mathbb{T}^d} Y^\alpha \partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\quad + \int_{\mathbb{T}^d} \Gamma_{<n}W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)\partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\quad - \int_{\mathbb{T}^d} \Gamma_{<n}U_0^\alpha(X;\theta)\partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\quad + \int_{\mathbb{T}^d} X^\alpha \partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\quad - \int_{\mathbb{T}^d} G_0U_0^\alpha(X;\theta)\partial_\gamma U_0^\alpha(X;\theta) \\
&\overset{(***)}{=} \int_{\mathbb{T}^d} Y^\alpha \partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta \\
&\quad + \int_{\mathbb{T}^d} G_nU_0^\alpha(X;\theta)\partial_\gamma W_0^\alpha(X + Y + \Gamma_{<n}\widetilde{W}_n(Y);\theta)d\theta
\end{aligned} \tag{4.8}$$

where (*) comes from (4.7), (**) from $X = G_0 U_0(X)$, (***) from $\bar{X} - \Gamma_{<n} U_0(\bar{X}) = G_n U_0 \bar{X}$ plus

$$\int_{\mathbb{T}^d} \Gamma_{<n}(\theta-\theta') W_0^\alpha(X+Y+\Gamma_{<n}\widetilde{W}_n(Y);\theta)\partial_\gamma W_0^\alpha(X+Y+\Gamma_{<n}\widetilde{W}_n(Y);\theta')d\theta d\theta' - \int_{\mathbb{T}^d} G_0(\theta-\theta')U_0^\alpha(X;\theta)\partial_\gamma U_0^\alpha(X;\theta')d\theta d\theta' = 0 \tag{4.9}$$

which is obtained performing two integrations by parts and using the symmetry of $\Gamma_{<n}$ and G_0; the latter shows that the l.h.s. in (4.9) is equal to its opposite, hence it vanishes.

1. Resonances and compensations

To use the identities we worked out in the last section, we introduce small interpolations of the kernels of the maps $D\widetilde{w}_n$, constructed in Proposition 2 for $|\lambda| \le \lambda_n$. Differentiating (3.20) we get

$$D\widetilde{w}_{n\beta}(y) = [1 - D\widetilde{w}_{0\beta}(y_n)\Gamma_{<n}]^{-1} D\widetilde{w}_{0\beta}(y_n) \quad \text{with} \quad y_n \equiv y + \Gamma_{<n}\widetilde{w}_{n\beta}(y). \tag{4.10}$$

We will show that the diagonal part of the kernel $D\widetilde{w}_{n\beta}(y;q,q)$ depends on q only through $\omega \cdot q$. In order to show this, for $p \in \mathbb{Z}^d$, let $t_p : \mathcal{L}(\mathcal{H};\mathcal{H}) \to \mathcal{L}(\mathcal{H};\mathcal{H})$ be the continuous automorphism that maps $a \in \mathcal{L}(\mathcal{H};\mathcal{H})$ into $t_p a \in \mathcal{L}(\mathcal{H};\mathcal{H})$:

$$(t_p a)(q,q') = a(q+p, q'+p), \tag{4.11}$$

that is, t_p shifts the kernel of the operator a by p. For $n = 0$ we have that $t_p D\widetilde{w}^j_{0\beta} = D\widetilde{w}^j_{0\beta}$ for all $p \in \mathbb{Z}^d$, since the kernel $D\widetilde{w}^j_{0\beta}(y;q,q')$ is function of $q - q'$ only. The latter observation and the definition (3.48) allow us to conclude that, applying t_p to (4.10), we get

$$t_p D\widetilde{w}^j_{n\beta}(y) = \left[1 - D\widetilde{w}^j_{0\beta}(y_n)\Gamma_{<n}(\omega\cdot p)\right]^{-1} D\widetilde{w}^j_{0\beta}(y_n), \tag{4.12}$$

showing that $t_p D\widetilde{w}^j_{n\beta}(y)$ depends on p only through $\omega \cdot p$. Therefore we can define a smooth interpolation of $t_p D\widetilde{w}^j_{n\beta}(y)$ in the following way: denote $\pi^j_{0\beta}(y) = D\widetilde{w}_{0\beta}(y)$ and define for $n \geq 1$ and $|\kappa| \leq \eta^n$,

$$\pi^j_{n\beta}(\kappa; y) = \left[1 - \pi^j_{0\beta}(y_n)\Gamma_{<n}(\kappa)\right]^{-1} \pi^j_{0\beta}(y_n). \tag{4.13}$$

Inequality (3.54) shows that for $y \in B(\alpha_j^{\frac{2}{3}\ell} r^n) \subset \mathcal{H}$, $\|y_n\| \leq \frac{1}{2}\alpha_j^{\frac{2}{3}\ell}$, so Proposition 1 and the Cauchy estimate (3.33) imply for such y

$$\begin{aligned} \|\pi^j_{0\beta}(y_n)\|_{\mathcal{L}(\mathcal{H};\mathcal{H})} &\leq \sup_{\|y\| \leq \frac{1}{2}\alpha_j^{\frac{2}{3}\ell}} \|D\widetilde{w}^j_{0\beta}(y)\|_{\mathcal{L}(\mathcal{H};\mathcal{H})} \\ &\leq \frac{2}{\alpha_j^{\frac{2}{3}\ell}} \sup_{\|y\| \leq \alpha_j^{\frac{2}{3}\ell}} \|\widetilde{w}^j_{0\beta}(y)\| \leq |\lambda| 2C_{d,\ell}. \end{aligned} \tag{4.14}$$

The latter discussion implies that $\pi^j_{n\beta}(\kappa; y)$ is analytic for $|\lambda| \leq \lambda_n$, $|\mathrm{Im}\,\beta| < \bar{\alpha}_j$, $y \in B(\alpha_j^{\frac{2}{3}\ell} r^n) \subset \mathcal{H}$, and $\mathcal{C}^\infty$ for $|\kappa| \leq \eta^n$ with norm, say,

$$\|\pi^j_{n\beta}(\kappa; y)\|_{\mathcal{L}(\mathcal{H};\mathcal{H})} \leq \sqrt{|\lambda|}. \tag{4.15}$$

Furthermore $\pi^j_{n\beta}(\kappa; y)$ is a smooth interpolation of the kernel of $t_p D\widetilde{w}_{n\beta}(y)$, meaning that for $p \in \mathbb{Z}^d$

$$t_p D\widetilde{w}_{n\beta}(y) = \pi^j_{n\beta}(\omega \cdot q; y). \tag{4.16}$$

Differentiating Eq. (4.13) with respect to κ we get the useful identity

$$\partial_\kappa \pi^j_{n\beta}(\kappa; y) = \pi^j_{n\beta}(\kappa; y)\partial_\kappa \Gamma_{<n}(\kappa)\pi_{n\beta}(\kappa; y). \tag{4.17}$$

For $\|y\| \leq \alpha_j^{\frac{2}{3}\ell} r^n$ and $|\kappa| \leq \eta^n$ the following recursive relation holds:

$$\pi^j_{n\beta}(\kappa; y) = \left[1 - \pi^j_{(n-1)\beta}(\kappa; \tilde{y})\Gamma_{n-1}(\kappa)\right]^{-1} \pi^j_{(n-1)\beta}(\kappa; \tilde{y}) \tag{4.18}$$

where $\tilde{y} = y + \Gamma_{n-1}\widetilde{w}_{n\beta}(y)$.

Chapter 5

The Main Proposition

To simplify the notations, we shall denote by B_n^j the open ball in $\mathcal{H}_n^j$ of radius $\alpha_j^{\frac{2}{3}\ell} r^n$ and by $\mathcal{A}_n^j$ the space $H^\infty(B_n^j, \mathcal{H}_{-n})$. Finally Γ will stand for Γ_{n-1}.

Proposition 3. **(a)** *There exist positive constants* r_j, λ_0, *and* $\bar{\alpha}_{j,n}$ *where*

$$\bar{\alpha}_{(j;n)} = \frac{n+2}{2n+2}\bar{\alpha}_j \quad n \geq 1, \tag{5.1}$$

such that, for $|Im\beta| \leq \alpha_{(j;n)}$ *and* $|\lambda| \leq |\lambda_0|$ *there exist solutions* $\widetilde{w}_{n\beta}^j \equiv \widetilde{w}_n^j$ *of Eqs.* (3.19) *such that* $\widetilde{w}_n^j$ *belong to* $\mathcal{A}_n^j$, *and are analytic in* λ.

(b) *Writing*

$$\widetilde{w}_n^j(y) = \widetilde{w}_n^j(0) + D\widetilde{w}_n^j(0)y + \delta_2\widetilde{w}_n^j(y) \tag{5.2}$$

we have

$$|\widetilde{w}_n^j(0;q)| \leq \varepsilon\left(2^{n+1}-1\right)\frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\frac{\ell}{3}}} \quad \textit{for} \quad 0 < |\omega\cdot q| \leq \eta^{n-1} \tag{5.3}$$

$$\|\delta_2\widetilde{w}_n^j\|_{\mathcal{A}_n^j} \leq \varepsilon\alpha_j^\ell r^{\frac{3}{2}n} \tag{5.4}$$

where $\varepsilon \to 0$ *as* $\lambda \to 0$.

(c) *Furthermore*

$$\|D\widetilde{w}_n^j(y)\|_{\mathcal{L}(n;-n)} \leq \varepsilon \eta^{2n} \tag{5.5}$$

1. Proof of (a)

First of all, we show that (5.3) implies for all $n \geq 1$:

$$\|P\widetilde{w}_n^j(0)\|_{-n} \equiv \sum_{|\omega\cdot q|\leq \eta^{n-1}} |\widetilde{w}_n^j(0;q)| \leq \varepsilon \alpha_j^{\frac{2}{3}\ell} r^{2n} \tag{5.6}$$

In fact the diophantine condition (1.4), forces the sum defining the norm to be taken over q such that $|q| \geq \gamma^{\frac{1}{\nu}} \eta^{-\frac{n-1}{\nu}}$, hence we can estimate

$$\begin{aligned}
\sum_{|\omega\cdot q|\leq \eta^{n-1}} |w_n(0;q)| &\leq \sum_{|q|\geq \gamma^{\frac{1}{\nu}}\eta^{-\frac{n-1}{\nu}}} |w_n(0;q)| \\
&\leq \varepsilon \left(2^{n+1}-1\right) \alpha_j^{\frac{2}{3}\ell} \sum_{|q|\geq \gamma^{\frac{1}{\nu}}\eta^{-\frac{n-1}{\nu}}} \frac{1}{|q|^{\frac{\ell}{3}}} \\
&\leq \varepsilon \gamma^{\frac{d-\frac{\ell}{3}}{\nu}} \alpha_j^{\frac{2}{3}\ell} \left(2^{n+1}-1\right) \eta^{\frac{n-1}{\nu}(\frac{\ell}{3}-d)} \\
&\leq \varepsilon \alpha_j^{\frac{2}{3}\ell} r^{2n}
\end{aligned} \tag{5.7}$$

for $\varepsilon = \varepsilon(d,\gamma,\nu)$ and $\ell \geq 12\nu \log_\eta(r/2) + 3d$.

Remark 7. In the diophantine condition (1.4) we would like to take γ as small as possible in order to have more diophantine frequencies ω to which Theorem 1 applies . In order to get (5.7) we got the constraint $\gamma \geq \varepsilon^{f(\ell)}$ where $f \to \infty$ when $\ell \to \infty$. The latter accords with the intuitive fact that as the perturbation grows and the regularity decreases, one expects fewer invariant tori to survive.

Remark 8. Note that (5.6) can be trivially improved with

$$\|\widetilde{w}_m^j(0)\|_{-n} \leq \varepsilon \alpha_j^{\frac{2}{3}\ell} r^{2n} \tag{5.8}$$

for all $m \leq n$. Anyway we shall not need the latter bound and in the following we shall always use (5.6).

Consider now the equation (3.19). The decomposition (5.2) implies

$$\widetilde{w}_n^j(y) = \widetilde{w}_{n-1}^j(0) + D\widetilde{w}_{n-1}^j(0)(y + \Gamma\widetilde{w}_n^j(y)) + \delta_2\widetilde{w}_{n-1}^j(y + \Gamma\widetilde{w}_n^j(y)) \quad (5.9)$$

from which we deduce that

$$\widetilde{w}_n^j(y) = H\widetilde{w}_{n-1}^j(0) + HD\widetilde{w}_{n-1}^j(0)y + u(y) \quad (5.10)$$

where

$$u(y) = H\delta_2\widetilde{w}_{n-1}^j(y + \Gamma\widetilde{w}_n^j(y)) = H\delta_2\widetilde{w}_{n-1}^j(\Gamma H\widetilde{w}_{n-1}^j(0) + \widetilde{H}y + \Gamma u(y)) \quad (5.11)$$

with $H = (1 - D\widetilde{w}_{n-1}^j(0)\Gamma)^{-1}$ and $\widetilde{H} = 1 + \Gamma HD\widetilde{w}_{n-1}^j(0) = (1 - \Gamma D\widetilde{w}_{n-1}^j(0))^{-1}$. The bound (5.5) with n replaced by $n-1$, together with Lemma 6 and the definition of the norms imply

$$\|H\|_{\mathcal{L}(-n+1;-n+1)}\ ,\ \|\widetilde{H}\|_{\mathcal{L}(-n+1;n-1)} \le 1 + C\varepsilon \le 2, \quad (5.12)$$

for $|\lambda|$ small enough.

To solve Eq. (5.11) we use the Banach Fixed Point Theorem. Once u is given, we can recover the existence of $\widetilde{w}_n^j$ solving (5.10). The solution of (5.11) can be given as the fixed point of the map $\mathcal{G}$ defined by

$$\mathcal{G}(u) = H\delta_2\widetilde{w}_{n-1}^j(\tilde{y}) \quad \text{with} \quad \tilde{y} = \Gamma H\widetilde{w}_{n-1}^j(0) + \widetilde{H}y + \Gamma u(y). \quad (5.13)$$

We shall show that $\mathcal{G}$ is a contraction in the ball

$$\mathcal{B}^j = \{u \in H^\infty(B_{n-1}^{j,\delta}, \mathcal{H}_{-n+1}) \mid \|u\|_{\mathcal{B}^j} \equiv \sup_{y \in B_{n-1}^{j,\delta}} \|u(y)\|_{-n+1} \le 2\varepsilon\alpha_j^\ell r^{\frac{3}{2}(n-1)}\}, \quad (5.14)$$

where $B_{n-1}^{j,\delta} \subset \mathcal{H}_{n-1}$ is the open ball of radius $\alpha_j^{\frac{2}{3}\ell} r^{n-\delta}$ for $0 \le \delta < 1$ and $r_j = r_j(\delta)$. Indeed, for $y \in \mathcal{H}_{n-1}$ such that $\|y\|_{n-1} \le \alpha_j^{\frac{2}{3}\ell} r^{n-\delta}$, we get

$\tilde{y} \in \mathcal{H}_{n-1}$ with

$$\begin{aligned}\|\tilde{y}\|_{n-1} &\leq 2C\eta^{-2n}\|w_{(n-1)\beta}(0)\|_{-n+1} + 2\alpha_j^{\frac{2}{3}\ell} r^{n-\delta} + 2C\eta^{-2n}\varepsilon\alpha_j^{\ell} r^{\frac{3}{2}(n-1)} \\ &\leq 2C\eta^{-2n}\varepsilon\alpha_j^{\frac{2}{3}\ell} r^{2(n-1)} + 2\alpha_j^{\frac{2}{3}\ell} r^{n-\delta} + 2C\eta^{-2n}\varepsilon\alpha_j^{\ell} r^{\frac{3}{2}(n-1)} \\ &\leq \frac{1}{2}\alpha_j^{\frac{2}{3}\ell} r^{n-1} \end{aligned} \tag{5.15}$$

for r small enough. Thus $\delta_2\widetilde{w}^j_{n-1}$ is defined at $\tilde{y}$, since the latter is in the domain of definition of $\widetilde{w}^j_{n-1}$. It follows that $\mathcal{G}(u) : B^{j,\delta}_{n-1} \to \mathcal{H}_{-n+1}$. Moreover

$$\|\mathcal{G}(u)(y)\|_{-n+1} \leq 2 \sup_{y\in B^{j,\delta}_{n-1}} \|\delta_2 w^j_{n-1}\|_{-n+1} \leq 2\varepsilon\alpha_j^{\ell} r^{\frac{3}{2}(n-1)}, \tag{5.16}$$

where we used the bounds (5.4) and (5.12). Hence $\mathcal{G} : \mathcal{B}^j \to \mathcal{B}^j$.

To prove that $\mathcal{G}$ is a contraction, we use the estimate (3.35) for

$$\tilde{y}_i(y) = \Gamma H\widetilde{w}^j_{n-1}(0) + \widetilde{H}y + \Gamma u_i(y) \tag{5.17}$$

and $u_i \in \mathcal{B},\ i = 1, 2$. We get immediately that $\tilde{y}_i \in \mathcal{H}_{n-1}$ and by inequality (5.15), $\|\tilde{y}_i\| \leq \frac{1}{2}\alpha_j^{\frac{2}{3}\ell} r^{n-1}$. Hence the bounds (3.35), (5.4), (5.12), together with the relations between the n-dependent spaces and their norms, imply

$$\begin{aligned}\|\mathcal{G}(u_1) - \mathcal{G}(u_2)\|_{\mathcal{B}^j} &= \sup_{y\in B^{j,\delta}_{n-1}} \|H\delta_2\widetilde{w}^j_{n-1}(\tilde{y}_1) - H\delta_2\widetilde{w}^j_{n-1}(\tilde{y}_2)\|_{-n+1} \\ &\leq 4\alpha_j^{-\frac{2}{3}\ell} r^{-n+1} \sup_{y\in B^{j}_{n-1}} \|\delta_2\widetilde{w}^j_{n-1}(y)\|_{-n+1} \sup_{y\in B^{j,\delta}_{n-1}} \|\tilde{y}_1 - \tilde{y}_2\|_{-n+1} \\ &\leq 4\alpha_j^{\frac{1}{3}\ell}\varepsilon r^{\frac{1}{2}(n-1)} \sup_{y\in B^{j,\delta}_{n-1}} \|\tilde{y}_1 - \tilde{y}_2\|_{-n+1} \\ &\leq 4\alpha_j^{\frac{1}{3}\ell}\varepsilon r^{\frac{1}{2}(n-1)} C\eta^{-2n} \sup_{y\in B^{j,\delta}_{n-1}} \|u_1(y) - u_2(y)\|_{-n+1} \\ &\leq \frac{1}{2}\|u_1(y) - u_2(y)\|_{\mathcal{B}^j} \end{aligned} \tag{5.18}$$

for r and ε small enough, proving the contractive property of $\mathcal{G}$ on $\mathcal{B}^j$. Hence the existence of the fixed point $u \in \mathcal{B}^j$ of $\mathcal{G}$ solving the equation (5.11) and providing $\widetilde{w}_n^j : B_{n-1}^{j,\delta} \to \mathcal{H}_{-n+1}$ given by (5.10). Using the natural embeddings we may consider B_n^j a subset of $B_{n-1}^{j,\delta}$, and $\widetilde{w}_n^j$ may be regarded as an element of the space $\mathcal{A}_n^j$. Note also that, since $\tilde{y} = y + \Gamma\widetilde{w}_n^j(y)$ (see (5.11)), the inequality (5.15) can be rewritten as

$$\|y + \Gamma\widetilde{w}_n^j(y)\| \le \frac{1}{2}\alpha_j^{\frac{2}{3}\ell} r^{n-1} \quad \text{for} \quad y \in B_n^j \tag{5.19}$$

which implies that $y + \Gamma\widetilde{w}_n^j(y) \in B_{n-1}^j$ for such y.

2. Proof of (b)

In view of the decomposition (5.10), we write

$$\widetilde{w}_n^j(y) = \widetilde{w}_n^j(0) + D\widetilde{w}_n^j(0)y + \delta_2\widetilde{w}_n^j(y), \tag{5.20}$$

where

$$\begin{aligned} \widetilde{w}_n^j(0;q) &= H\widetilde{w}_{n-1}^j(0;q) + u(0;q) \\ D\widetilde{w}_n^j(0) &= HD\widetilde{w}_{n-1}^j(0) + Du(0) \\ \delta_2\widetilde{w}_n^j(y) &= \delta_2 u(y) \end{aligned} \tag{5.21}$$

Let us first iterate the bound (5.3). Note that, with the projection P defined at page 16

$$P\widetilde{w}_n^j(0;q) = PHP\widetilde{w}_{n-1}^j(0;q) + Pu(0;q) \tag{5.22}$$

since $H = HP$. Since $u \in \mathcal{B}^j$ (See definition (5.14)), we have for $0 < |\omega \cdot q| \le \eta^{n-1}$

$$|u(0;q)| \le \|u(0)\|_{-n+1} \le 2\varepsilon\alpha_j^{\ell} r^{\frac{3}{2}(n-1)}. \tag{5.23}$$

and Eq. (5.22), using the estimate (5.3), yields

$$|\widetilde{w}_n^j(0;q)| \le (2^n - 1)\,\varepsilon\frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\frac{\ell}{3}}} + |u(0;q)|; \tag{5.24}$$

we omitted here the technical details of the estimate of $PHP\widetilde{w}^j_{n-1}(0;q)$, which is obtained by expanding H in a Neumann series; such details are carried out at p. 53 in the estimate of the quantity (5.51). Now the inequality (5.24), in view of (5.23), seems less than what we need to iterate (5.3), but in fact it is much more, as we need a bound only for $|\mathrm{Im}\,\beta| \leq \alpha_{(j;n)}$. For such β, using the estimate (5.23) we get for $0 < |\omega \cdot q| \leq \eta^{n-1}$

$$|u_\beta(0;q)|e^{(\bar{\alpha}_{j,n-1}-\bar{\alpha}_{j,n})|q|} = |u_{\beta'}(0;q)| \leq 2\varepsilon\alpha_j^\ell r^{\frac{3}{2}(n-1)} \tag{5.25}$$

where

$$\beta' = \beta - i\frac{(\bar{\alpha}_{j,n-1} - \bar{\alpha}_{j,n})}{|q|}q \quad \text{so that} \quad |\mathrm{Im}\,\beta'| \leq \bar{\alpha}_{j,n-1}. \tag{5.26}$$

From the definition (5.1) we can write $\alpha_{j,n-1} - \alpha_{j,n} = \frac{\alpha_j}{2n(n+1)}$. It follows from (5.25) that for $0 < |\omega \cdot q| \leq \eta^{n-1}$

$$\begin{aligned} |u_\beta(0;q)| &\leq 2\varepsilon\alpha_j^\ell r^{\frac{3}{2}(n-1)} e^{(\frac{\alpha_j}{2n(n+1)})|q|} \\ &\leq 2\varepsilon\frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\frac{\ell}{3}}} r^{\frac{3}{2}(n-1)} \left[2n(n+1)\right]^{\frac{\ell}{3}} \frac{\ell!}{6} \\ &\leq \varepsilon\frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\frac{\ell}{3}}} \end{aligned} \tag{5.27}$$

for r small enough. Now, combining (5.24) and (5.27) we get the desired bound:

$$|\widetilde{w}^j_n(0;q)| \leq \left(2^{n+1} - 1\right)\varepsilon\frac{\alpha_j^{\frac{2}{3}\ell}}{|q|^{\frac{\ell}{3}}} \quad \text{for} \quad |\omega \cdot q| \leq \eta^{n-1} \tag{5.28}$$

We can now iterate (5.4) for $\delta_2\widetilde{w}^j_n(y) = \delta_2 u(y)$ (See (5.21)). We already proved that for $\|y\|_{n-1} \leq \alpha_j^{\frac{2}{3}\ell} r^{n-\delta}$ we have $\|u(y)\|_{-n+1} \leq 2\varepsilon\alpha_j^\ell r^{\frac{3}{2}(n-1)}$ (see (5.14)). We can apply the estimate (3.34) with $k = 2$ and $\gamma = r^\delta$, so that for$\|y\|_n \le \alpha_j^{\frac{2}{3}\ell} r^n$ we get

$$\|\delta_2 \widetilde{w}^j_{-n}(y)\|_n \le \sup_{\|y\|_{n-1} \le \alpha_j^{\frac{2}{3}\ell} r^n} \|\delta_2 u(y)\|_{-n+1}$$
$$\le \frac{r^{2\delta}}{1-r^\delta} \sup_{\|y\|_{n-1} \le \alpha_j^{\frac{2}{3}\ell} r^{n-\delta}} \|u(y)\|_{-n+1}$$
$$\le \frac{r^{2\delta-\frac{3}{2}}}{1-r^\delta} 2\varepsilon \alpha_j^\ell r^{\frac{3}{2}n}. \tag{5.29}$$

Taking $\delta > \frac{3}{4}$ and r small enough, we infer that $\|\delta_2 \widetilde{w}^j_n(y)\|_{-n} \le \varepsilon \alpha_j^\ell r^{\frac{3}{2}n}$, which concludes the inductive proof of (b).

3. Proof of (c)

This is the part of the proof where the identities introduced in section 1 are needed. We will make use of the maps $\pi_{n\beta} : B(r_j^n) \subset \mathcal{H} \to \mathcal{L}(\mathcal{H};\mathcal{H})$, constructed for $|\lambda| \le \lambda_n$. In view of the embeddings (3.47) such maps can be viewed as

$$\pi^j_{n\beta} : B^j_n \subset \mathcal{H}_n \to \mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n}). \tag{5.30}$$

We shall show that they can be extended to $|\lambda| \le \lambda_0$, and the bound (5.5) will be proven by

Lemma 9. *Denote by D_n the disk $\{\kappa \in \mathbb{C} | |\kappa| \le \eta^n\}$ and splitting $\pi^j_{n\beta}(\kappa;0)$ into its diagonal and off diagonal parts*

$$\pi^j_{n\beta}(\kappa;0) = \sigma^j_{n\beta}(\kappa) + \rho^j_{n\beta}(\kappa), \tag{5.31}$$

where $\sigma^j_{n\beta}(\kappa;q,q') = \pi^j_{n\beta}(\kappa;0;q,q')\delta_{q,q'}$. The maps $\pi^j_{n\beta} : D_n \times B^j_n \to \mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})$ extend analytically to $|\lambda| \le \lambda_0$, their extensions will still be smooth interpolations of the kernel of $t_p D w_{n\beta}(y)$, i.e.

$$t_p D w^j_{n\beta}(y) = \pi^j_{n\beta}(\omega\cdot p; y) \quad \textit{and} \quad t_p \pi^j_{n\beta}(\kappa; y) = \pi^j_{n\beta}(\kappa + \omega\cdot p; y) \tag{5.32}$$

they will depend analytically on β and y and belong to $\mathcal{C}^\infty(D_n)$. For $i = 0, 1, 2$, they obey the bounds

$$\|\partial_\kappa^i \delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \le \varepsilon \alpha_j^{\frac{\ell}{3}} r^{\frac{1}{2+i}n} \tag{5.33}$$

$$\|\partial_\kappa^i \sigma_{n\beta}^j(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \le \varepsilon \eta^{(2-i)n} \tag{5.34}$$

$$|\partial_\kappa^i \rho_{n\beta}^j(\kappa; q, q')| \le \varepsilon \frac{1}{|q-q'|^{\frac{\ell}{3}}}, \tag{5.35}$$

where $\delta_1 \pi_{n\beta}^j(\kappa; y) \equiv \pi_{n\beta}^j(\kappa; y) - \pi_{n\beta}^j(\kappa; 0)$.

Remark 10. By using the diophantine condition as we did at p. 44 in order to get (5.6), we see that the bound (5.35) implies

$$\|\partial_\kappa^i \rho_{n\beta}^j(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \le \varepsilon r^{\frac{n}{2}}, \tag{5.36}$$

for ℓ large enough.

Taking $p = 0$ in (5.32) and combining Eqs. (5.33), (5.34) and (5.35) we obtain (5.5), so we are only left with

Proof. (Of Lemma 9) Differentiating (3.19) with respect to y we get

$$Dw_n^j(y) = \left(1 - Dw_{n-1}^j(\tilde{y})\Gamma_{n-1}\right)^{-1} Dw_{n-1}^j(\tilde{y}) \tag{5.37}$$

where $\tilde{y} = y + \Gamma_{n-1}\widetilde{w}_{n\beta}^j(y)$. The right hand side is well defined for $y \in B_{n-1}^{j,\delta} \subset \mathcal{H}_{n-1}$, in fact by inequality (5.19), $\tilde{y} \in B_{n-1}^j$ for such y's. Lemma 6 and the inductive hypotheses (5.5) imply that

$$\|D\widetilde{w}_n^j(\tilde{y})\Gamma_{n-1}\|_{\mathcal{L}(H_{-n+1};H_{-n+1})} \le C\varepsilon \tag{5.38}$$

Using the relation (4.18) we define

$$\pi_{n\beta}^j(\kappa; y) = \left[1 - \pi_{(n-1)\beta}^j(\kappa; \tilde{y})\Gamma_{n-1}(\kappa)\right]^{-1} \pi_{(n-1)\beta}^j(\kappa; \tilde{y}). \tag{5.39}$$

The relations (5.32) follow by simply applying t_p to (5.37) and (5.39). By the inductive hypotheses, for $\kappa \in D_{n-1}$ and $y \in B_n^{j,\delta}$, $\pi_{n\beta}^j(\kappa; y) \in \mathcal{L}(\mathcal{H}_n; \mathcal{H}_{-n})$

and it is an analytic function of its arguments. Hence, by induction, it coincides for $|\lambda| \leq \lambda_n$ with the maps $\pi_{n\beta}$ constructed in section 1. Note that

$$\pi^j_{n\beta}(\kappa;0) = \Big[1 - \pi^j_{(n-1)\beta}(\kappa;\tilde{0})\Gamma_{n-1}(\kappa)\Big]^{-1} \pi^j_{(n-1)\beta}(\kappa;\tilde{0}), \tag{5.40}$$

where $\tilde{0} = \Gamma \widetilde{w}^j_{n\beta}(0)$.

To get an a priori bound from (5.39), we formulate an easy Lemma

Lemma 11. *Let* $H^j_n(\kappa, y) \equiv \Big[1 - \pi^j_{(n-1)\beta}(\kappa;y)\Gamma_{n-1}(\kappa)\Big]^{-1}$. *For* $y \in B^j_{n-1}$ *and all* $m \leq n$

$$\|\partial^i_\kappa H^j_m(\kappa, y)\|_{\mathcal{L}(\mathcal{H}_{n-1},\mathcal{H}_{-n+1})} \leq 2\eta^{-i(m-1)} \quad \textit{for} \quad i = 0, 1, 2 \tag{5.41}$$

Proof. For $i = 0$ (5.38) implies trivially that $\|H^j_m(\kappa, y)\|_{\mathcal{L}(\mathcal{H}_{n-1},\mathcal{H}_{-n+1})} \leq 2$. For $i = 1$ we have

$$\begin{aligned} &\|\partial_\kappa H^j_m(\kappa, y)\|_{\mathcal{L}(\mathcal{H}_{n-1},\mathcal{H}_{-n+1})} = \\ &= \|H^j_m(\kappa, y)\partial_\kappa\Big(\pi^j_{(m-1)\beta}(\kappa;y)\Gamma_{m-1}(\kappa)\Big)H^j_m(\kappa, y)\|_{\mathcal{L}(\mathcal{H}_{n-1},\mathcal{H}_{-n+1})} \\ &\leq 2\eta^{-(m-1)}. \end{aligned} \tag{5.42}$$

In the same fashion one gets

$$\|\partial^2_\kappa H^j_m(\kappa, y)\|_{\mathcal{L}(\mathcal{H}_{n-1},\mathcal{H}_{-n+1})} \leq 2\eta^{-2(m-1)} \tag{5.43}$$

□

From the latter Lemma, (5.39) and the inductive hypotheses we get the a priori bound

$$\|\partial^i_\kappa \pi^j_{n\beta}(\kappa;y)\|_{\mathcal{L}(\mathcal{H}_{n-1};\mathcal{H}_{-n+1})} \leq C\varepsilon\eta^{(2-i)(n-1)}. \tag{5.44}$$

To prove (5.33) we note the identity

$$\begin{aligned} H_n(\kappa;\tilde{y})\pi^j_{(n-1)\beta}(\kappa,\tilde{y}) &= \pi^j_{(n-1)\beta}(\kappa,\tilde{y})\Big[1 - \Gamma_{n-1}\pi^j_{(n-1)\beta}(\kappa;\tilde{y})\Big]^{-1} \\ &\equiv \pi^j_{(n-1)\beta}(\kappa,\tilde{y})\widetilde{H}^j_n(\kappa;\tilde{y}), \end{aligned} \tag{5.45}$$

which, for $y \in B_{n-1}^{j,\delta}$ yields

$$\begin{aligned}
&\delta_1 \pi_{n\beta}^j(\kappa; y) = \pi_{n\beta}^j(\kappa; y) - \pi_{n\beta}^j(\kappa; 0) \\
&= H_n^j(\kappa; \tilde{y}) \pi_{(n-1)\beta}^j(\kappa; \tilde{y}) - \pi_{(n-1)\beta}^j(\kappa; \tilde{0}) \widetilde{H}_n^j(\kappa; \tilde{0}) \\
&= H_n^j(\kappa; \tilde{y}) \left[\pi_{(n-1)\beta}^j(\kappa; \tilde{y}) (\widetilde{H}_n^j)^{-1}(\kappa; \tilde{0}) - (H_n^j)^{-1}(\kappa; \tilde{y}) \pi_{(n-1)\beta}^j(\kappa; \tilde{0}) \right] \widetilde{H}_n^j(\kappa; \tilde{0}) \\
&= H_n^j(\kappa; \tilde{y}) \Big[\pi_{(n-1)\beta}^j(\kappa; \tilde{y}) \left(1 - \Gamma_{n-1}(\kappa) \pi_{(n-1)\beta}^j(\kappa; \tilde{0}) \right) \\
&\quad - \left(1 - \pi_{(n-1)\beta}^j(\kappa; \tilde{y}) \Gamma_{n-1}(\kappa) \right) \pi_{(n-1)\beta}^j(\kappa; \tilde{0}) \Big] \widetilde{H}_n^j(\kappa; \tilde{0}) \\
&= H_n^j(\kappa; \tilde{y}) \left[\pi_{(n-1)\beta}^j(\kappa; \tilde{y}) - \pi_{(n-1)\beta}^j(\kappa; \tilde{0}) \right] \widetilde{H}_n^j(\kappa; \tilde{0}) \\
&= H_n^j(\kappa; \tilde{y}) \left[\delta_1 \pi_{(n-1)\beta}^j(\kappa; \tilde{y}) - \delta_1 \pi_{(n-1)\beta}^j(\kappa; \tilde{0}) \right] \widetilde{H}_n^j(\kappa; \tilde{0}). \qquad (5.46)
\end{aligned}$$

From Lemma 11 with $i = 0$ the inductive hypotheses and (5.46) we get the a priori bound for $y \in B_{n-1}^{j,\delta}$

$$\|\delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_n; \mathcal{H}_{-n})} \leq \|\delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_{n-1}; \mathcal{H}_{-n+1})} \leq 8\varepsilon \alpha_j^{\frac{\ell}{3}} r^{\frac{n-1}{2}}. \qquad (5.47)$$

To get (5.33) with $i = 0$, we restrict to $y \in B_n^j$ and using (3.34) we extract

$$\begin{aligned}
&\|\delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_n; \mathcal{H}_{-n})} = \|\delta_1 \delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_n; \mathcal{H}_{-n})} \\
&\leq \sup_{y \in B_n^j} \|\delta_1 \delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_n; \mathcal{H}_{-n})} \\
&\leq \frac{r^\delta}{1 - r^\delta} \sup_{y \in B_{n-1}^{j,\delta}} \|\delta_1 \pi_{n\beta}^j(\kappa; y)\|_{\mathcal{L}(\mathcal{H}_n; \mathcal{H}_{-n})} \\
&\leq \frac{r^\delta}{1 - r^\delta} 8\varepsilon \alpha_j^{\frac{\ell}{3}} r^{\frac{n-1}{2}} \leq \varepsilon \alpha_j^{\frac{\ell}{3}} r^{\frac{n}{2}}. \qquad (5.48)
\end{aligned}$$

To get (5.33) with $i = 1$ we first obtain another a priori bound for $y \in B_{n-1}^{j,\delta}$ by differentiating (5.46) with respect to κ and using (5.44) and the inductive hypotheses:

$$\begin{aligned}
&\|\partial_\kappa \delta_1 \pi^j_{n\beta}(\kappa;y)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} = \\
&= \|\partial_\kappa H^j_n(\kappa;\tilde{y}) \left[\delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{y}) - \delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{0})\right] \widetilde{H}^j_n(\kappa;\tilde{0}) \\
&\quad + H^j_n(\kappa;\tilde{y})\partial_\kappa \left[\delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{y}) - \delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{0})\right] \widetilde{H}^j_n(\kappa;\tilde{0}) \\
&\quad + H^j_n(\kappa;\tilde{y}) \left[\delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{y}) - \delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{0})\right] \partial_\kappa \widetilde{H}^j_n(\kappa;\tilde{0})\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \\
&\le 4\eta^{-n}\varepsilon\alpha_j^{\frac{\ell}{3}} r^{\frac{n-1}{2}} + 4\varepsilon\alpha_j^{\frac{\ell}{3}} r^{\frac{n-1}{3}}\eta^{-n} + 4\eta^{-n}\varepsilon\alpha_j^{\frac{\ell}{3}} r^{\frac{n-1}{2}} \\
&\le 12\varepsilon\alpha_j^{\frac{\ell}{3}} r^{\frac{n-1}{3}},
\end{aligned} \tag{5.49}$$

then we consider again the ball B^j_n to squeeze the correct estimate out:

$$\|\partial_\kappa \delta_1 \pi^j_{n\beta}(\kappa;y)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \le \frac{r^\delta}{1-r^\delta} 12\varepsilon\alpha_j^{\frac{2}{3}\ell} r^{\frac{n-1}{3}} \le \varepsilon\alpha_j^{\frac{2}{3}\ell} r^{\frac{n}{3}}. \tag{5.50}$$

The same procedure (establish an a priori bound, then restrict the domain of y's) yields (5.33) with $i = 2$.

Leaving the more difficult bound (5.34) for last, we can now iterate (5.35). In order to do that inductively, we write

$$\rho^j_{n\beta}(\kappa) = \underbrace{\left[1 - \pi(0;\kappa)^j_{(n-1)\beta}\Gamma_{n-1}(\kappa)\right]^{-1} \pi^j_{(n-1)\beta}(0;\kappa)}_{\Upsilon^j_n(\kappa)} + R^j_n(\kappa) \tag{5.51}$$

where

$$R^j_n(\kappa;q,q') \equiv \left(\frac{1}{1-(\pi(0)+\delta\pi)\Gamma}\delta\pi\Gamma\frac{1}{1-\pi(0)\Gamma}\pi(0) + \frac{1}{1-(\sigma+\delta\pi)\Gamma}\delta\pi\right) \tag{5.52}$$

with $\pi = \pi^j_{(n-1)\beta}(\kappa)$ e $\delta\pi = \delta_1\pi^j_{(n-1)\beta}(\kappa;\tilde{0})$. Using the inductive hypotheses it is not hard to show that

$$\|\partial^i_\kappa R^j_n(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \le \varepsilon\alpha_j^{\frac{\ell}{3}}. \tag{5.53}$$

In order to estimate the first term in (5.51) we notice that it can be written as

$$\Upsilon_n^j(\kappa; q, q') = \sum_{k=0}^{\infty} \sum_{q_1,\ldots,q_k} \pi(0; q, q_1)\Gamma(q_1) \cdots \pi(0; q, q_n)\Gamma_{n-1}(q_n)\pi(0; q_n, q') \tag{5.54}$$

where, again, $\pi = \pi^j_{(n-1)\beta}(\kappa)$. The k-th term in the series reads (leaving the sums over repeated q_j's understood) [1]

$$\sum_{0\le i_1 \le i_2 \cdots \le i_k = k} [\sigma(q)\Gamma(q)]^{i_1} \, \rho(q, q_{i_1})\Gamma(q_{i_1}) \cdots \rho(q_{i_2-1}, q_{i_2})\Gamma(q_{i_2})$$
$$[\sigma(q_{i_2})\Gamma(q_{i_2})]^{i_3 - i_2} \, \rho(q_{i_2}, q_{i_3})\Gamma(q_{i_3}) \cdots \rho(q_{i_4-1}, q_{i_4})\Gamma(q_{i_4})$$
$$\cdots [\sigma(q_{i_4})\Gamma(q_{i_4})]^{i_{k-1} - i_{k-2}} \, \rho(q_{i_{k-2}}, q_{i_{k-1}})\Gamma(q_{i_{k-1}}) \cdots \rho(q_k, q'), \tag{5.55}$$

Using the inductive hypothesis again, and the diophantine condition (1.4), we get

$$\Upsilon_n^j(\kappa; q, q') \le \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{j=1}^{k} \sum_{|\omega\cdot q_i| \le \eta^{n-1}} \frac{\eta^{-2n}}{|q - q_1|^{\frac{\ell}{3}}} \frac{\eta^{-2n}}{|q_1 - q_2|^{\frac{\ell}{3}}} \cdots \frac{\eta^{-2n}}{|q_j - q'|^{\frac{\ell}{3}}} \tag{5.56}$$

$$\overset{(*)}{\le} \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{j=1}^{k} \left[\eta^{-2n} 2^{\ell} \left(2\gamma^{-1}\eta^{n-1} \right)^{\frac{\ell}{\nu}\left(\frac{\ell}{3} - d\right)} \right]^j \frac{1}{|q - q'|^{\frac{\ell}{3}}}$$
$$\le \frac{1}{2}\varepsilon \frac{1}{|q - q'|^{\frac{\ell}{3}}} \tag{5.57}$$

for ℓ large enough and ε small enough. To obtain (*) we repeatedly used the estimate

$$\sum_{|\omega\cdot p| \le \eta^{n-1}} \frac{1}{|q - p|^{\frac{\ell}{3}}} \frac{1}{|p - q'|^{\frac{\ell}{3}}} \le \frac{2^{\ell} \left(2\gamma^{-1}\eta^{n-1}\right)^{\frac{\ell}{\nu}\left(\frac{\ell}{3} - d\right)}}{|q - q'|^{\frac{\ell}{3}}} \tag{5.58}$$

for all $|\omega \cdot q|, \, |\omega \cdot q'| \le \eta^{n-1}$ and $q \ne q'$, which is obtained by using the diophantine condition as in (5.6) and Minkowski inequality for the ℓ^p spaces:

[1] To be *very* exact and consistent with the expression if k is not even, we should take the sum over $0 \le i_1 \le i_2 \cdots \le i_{k+1} = k$, and perform some formal changes in a couple of subindices; we hope the reader will forgive us.

$\|f+g\|_p \le \|f\|_p + \|g\|_p$. Now combining (5.53) and (5.57) we get

$$|\rho_{n\beta}(\kappa;q,q')| \le \varepsilon \alpha_j^{\frac{\ell}{3}} + \frac{1}{2}\varepsilon \frac{1}{|q-q'|^{\frac{\ell}{3}}} \quad \text{for} \quad |\omega\cdot q|,\, |\omega\cdot q'| \le \eta^{n-1}. \tag{5.59}$$

Reasoning exactly in the same way we did at p. 47, we notice that the last bound holds for all $|\mathrm{Im}\,\beta| \le \bar{\alpha}_{(j;n-1)}$, hence we can shift β, and making use of the diophantine property of ω (Cf. p.48) we get for $|\mathrm{Im}\,\beta| \le \bar{\alpha}_{(j;n)}$

$$|\rho_{n\beta}(\kappa;q,q')| \le \varepsilon \frac{1}{|q-q'|^{\frac{\ell}{3}}} \quad \text{for} \quad |\omega\cdot q|,\, |\omega\cdot q'| \le \eta^{n-1}., \tag{5.60}$$

that is, (5.35) for $i=0$. Without any difference one obtains (5.57) for $\partial_\kappa\rho$ and $\partial_\kappa^2\rho$, which combined with (5.53) and the diophantine condition on ω (see (5.59)-(5.60)) yields (5.35) for $i=1,2$.

To prove (5.34) we need to establish a Lemma that will follow from the discussion of chapter 4 as a consequence of the Ward identity (4.4)(the indices j are omitted and the upper indeces stand for the components):

Lemma 12. *The following inequalities hold*

$$|\sigma_{n\beta}(0;0)| \le \varepsilon r^{\frac{n}{2}}, \tag{5.61}$$

$$|\partial_\kappa \sigma_{n\beta}(0;0)| \le \varepsilon \eta^{2n}. \tag{5.62}$$

Proof. Using Eq.(4.4) evaluated at $q=0$, we get

$$\begin{aligned} \sigma_n^{\gamma,\alpha}(\kappa;0)\Big|_{\kappa=0} &= \pi_n^{\gamma,\alpha}(\kappa;y;0,0)\Big|_{\substack{\kappa=0\\ y=0}} = Dw_n^{\gamma,\alpha}(y;0,0)\Big|_{y=0} \\ &= -\sum_{q\in\mathbb{Z}^d} iq^\gamma \bar{\chi}_n(\omega\cdot q)\bar{x}^\beta(q)\rho_n^{\beta,\alpha}(0;-q,0), \end{aligned} \tag{5.63}$$

so (5.61) follows from the decay of the coefficients $\bar{x}(q)$ and from (5.36)

$$|\sigma_{n\beta}(0;0)| \le \|\rho_n(0)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \le \varepsilon r^{\frac{n}{2}}. \tag{5.64}$$

Using (4.17) we get

$$\partial_\kappa \sigma_n(0;0)^{\alpha,\gamma} = \sum_q \pi_n^{\alpha,\delta}(0;0;q,0)\partial_\kappa \gamma_{<n}(\omega\cdot q)\pi_n^{\gamma,\delta}(0;0;-q;0)$$
$$= \sum_q D\widetilde{w}_n^{\alpha,\delta}(0;q,0)\partial_\kappa \gamma_{<n}(\omega\cdot q)D\widetilde{w}_n^{\gamma,\delta}(0;-q;0), \quad (5.65)$$

using (4.4) the latter takes the form

$$\partial_\kappa \sigma_n(0;0) = \mathcal{Z}_n + \mathcal{Q}_n, \quad (5.66)$$

where

$$\mathcal{Z}_n^{\alpha,\gamma} = -\sum_q q^\alpha q^\gamma \left(w_0^\delta(\bar{x} + \Gamma_{<n}\widetilde{w}_n(0);-q)\right)\partial_\kappa \gamma_{<n}(\omega\cdot q)\left(w_0^\delta(\bar{x}+\Gamma_{<n}\widetilde{w}_n(0);-q)\right) \quad (5.67)$$

and

$$\mathcal{Q}_n^{\alpha,\gamma} = \sum_{q,q',q''} iq'^\alpha \bar{\chi}_n(\omega\cdot q')\bar{x}^\beta(q')\pi_n^{\beta,\delta}(0;0;q',q)\partial_\kappa\gamma_{<n}(\omega\cdot q)\cdot \quad (5.68)$$
$$\cdot\, iq''^\gamma \bar{\chi}_n(\omega\cdot q'')\bar{x}^{\beta'}(q'')\pi_n^{\beta',\delta}(0;0;q'',q).$$

The expression summed in the right hand side of (5.67) is odd in q, hence $\mathcal{Z}_n = 0$, so, using Lemma 6 and (5.44), we have

$$|\partial_\kappa\sigma_n(0;0)| = |\mathcal{Q}_n| \le \|\pi_n\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})}\|\|\partial_\kappa\Gamma_{<n}(\kappa)\|\|\|\pi_n\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})}$$
$$\le C\varepsilon^2\eta^{2n-2} \le \varepsilon\eta^{2n} \quad (5.69)$$

for ε small enough. □

Using (5.39) we write

$$\sigma_{n\beta}^j(\kappa) = \underbrace{\left[1-\sigma_{(n-1)\beta}(\kappa)\Gamma_{n-1}(\kappa)\right]^{-1}}_{K_{n\beta}^j(\kappa)}\sigma_{(n-1)\beta}(\kappa) + S_n^j(\kappa) \quad (5.70)$$

where

$$S_n^j(\kappa) \equiv \text{diag}\left(\frac{1}{1-(\sigma+\mathcal{R})\Gamma}\mathcal{R}\Gamma\frac{1}{1-\sigma\Gamma}\sigma + \frac{1}{1-(\sigma+\mathcal{R})\Gamma}\mathcal{R}\right) \quad (5.71)$$

with $\sigma = \sigma^j_{(n-1)\beta}(\kappa)$ e $\mathcal{R} = \rho^j_{(n-1)\beta}(\kappa) + \delta_1 \pi^j_{(n-1)\beta}(\kappa;\tilde{0})$. Using the inductive hypotheses it is not difficult to show that

$$\|\partial^i_\kappa S^j_n(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \leq \varepsilon r^{\frac{n}{2}} \tag{5.72}$$

as $\mathcal{R}$ appears as a factor in both terms of (5.71).

We shall now describe a crucial property of $K^j_n(\kappa)$: fixing n and $|\kappa| \leq \eta^n$, we have that $K^j_m(\kappa;q)$ restricted to the set $\{q \in \mathbb{Z}^d : |\omega\cdot q| \leq \eta^{n-1}\}$, is the identity for all $m \leq n-2$. In fact, for such κ's and q's, we have $|\omega\cdot q+\kappa| \leq \eta^{n-2}$. On the other hand $\Gamma_m(\kappa)$ is supported on the set $|\omega\cdot q+\kappa| \geq \eta^m$, i.e. whenever $\eta^{n-2} \leq \eta^m$, we have $\Gamma_m(\kappa) = 0$. Summarizing for $m \leq n-2$ and $|\kappa| \leq \eta^n$

$$K^j_m(\kappa;q) = \left[1 - \sigma_{(n-1)\beta}(\kappa)\Gamma_{n-1}(\kappa)\right]^{-1}(q) = \mathrm{Id}(q), \quad \text{for} \quad |\omega\cdot q| \leq \eta^{n-1}. \tag{5.73}$$

So, for all $m \leq n-2$ and $|\kappa| \leq \eta^n$ we have

$$\sigma^j_{m\beta}(\kappa;q) = \sigma^j_{(m-1)\beta}(\kappa;q) + R^j_m(\kappa;q) \quad \text{for} \quad |\omega\cdot q| \leq \eta^{n-1}. \tag{5.74}$$

In view of (5.73) we notice that "on the scale n", σ_m stays almost constant until $m = n-2$, in fact if we assume $\|\partial^2\sigma^j_{0\beta}(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \leq \frac{1}{16}\varepsilon$ which we can always do, it follows from (5.74) and (5.72),

$$\|\partial^2_\kappa\sigma^j_{(n-2)\beta}(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \leq \varepsilon\left(\frac{1}{16} + \sum_{k=1}^{n-2} r^{\frac{k}{2}}\right) = \frac{1}{8}\varepsilon. \tag{5.75}$$

Now we can prove (5.34). For $i = 0$ we use (5.70) twice and make use of the fact that for all m $\sigma^j_m(q;\kappa) = \sigma^j_m(0;\tilde{\kappa})$ with $\tilde{\kappa} := \kappa + \omega\cdot q$, so we get

$$\begin{aligned}
\sigma^j_{n\beta}(\kappa) &= K^j_{n\beta}(\kappa)K^j_{(n-1)\beta}(\kappa)\sigma_{(n-2)\beta}(\kappa) + K^j_{n\beta}(\kappa)S^j_{n-1}(\kappa) + S^j_n(\kappa)\\
&= K^j_{n\beta}(\kappa)K^j_{(n-1)\beta}(\kappa)\left(\int_0^{\tilde{\kappa}}\int_0^{\kappa'}\partial^2\sigma_{(n-2)\beta}(\kappa'';0)d\kappa''\,d\kappa' + \tilde{\kappa}\partial\sigma_{(n-2)\beta}(0;0) + \sigma_{(n-2)\beta}(0;0)\right)\\
&\quad + K^j_{n\beta}(\kappa)S^j_{n-1}(\kappa) + S^j_n(\kappa)
\end{aligned} \tag{5.76}$$

from which, using Lemma 11, (5.61), (5.62) and (5.72) we get

$$\|\sigma^j_{n\beta}(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \leq \varepsilon\left(\frac{\tilde{\kappa}^2}{4} + \tilde{\kappa}\eta^{2(n-2)} + r^{\frac{n-2}{2}}\right) + 2\varepsilon r_j^{\frac{n-1}{2}} + \varepsilon r_j^{\frac{n}{2}}$$
$$\leq \varepsilon\eta^{2n}. \tag{5.77}$$

Differentiating (5.76) with respect to κ and using Lemma 11, (5.61), (5.62) and (5.72), we get

$$\partial_\kappa \sigma^j_{n\beta}(\kappa) = \partial_\kappa\left(K^j_{n\beta}(\kappa)K^j_{(n-1)\beta}(\kappa)\right)\sigma_{(n-2)\beta}(\kappa)$$
$$+ K^j_{n\beta}(\kappa)K^j_{(n-1)\beta}(\kappa)\partial_\kappa\sigma_{(n-2)\beta}(\kappa) + \partial_\kappa K^j_{n\beta}(\kappa)S^j_{n-1}(\kappa)$$
$$+ K^j_{n\beta}(\kappa)\partial_\kappa S^j_{n-1}(\kappa) + \partial_\kappa R^j_n(\kappa), \tag{5.78}$$

and proceeding as in (5.77) we get

$$\|\partial_\kappa\sigma^j_{n\beta}(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \leq \varepsilon\eta^n. \tag{5.79}$$

In the same way we get obtain the bound

$$\|\partial^2_\kappa\sigma^j_{n\beta}(\kappa)\|_{\mathcal{L}(\mathcal{H}_n;\mathcal{H}_{-n})} \leq \varepsilon. \tag{5.80}$$

which concludes the proofs of Lemma 9, of **(c)** (p. 44) and, hence, of Proposition 3. □

Chapter 6

Proof of Theorem 1

In this chapter we shall show that $Y_n^j \equiv F_n^j(0)$ converges to an analytic function Y^j with zero average for $n \to \infty$, solving (2.15). Furthermore $X_j \equiv \sum_{i=0}^j Y^i$ converges to a differentiable function X with zero average for $j \to \infty$, solving (2.6), which proves Theorem 1.

First of all in Proposition 2 we constructed for $|\lambda| \leq \lambda_n$ the analytic maps $f_{n\beta}^j$ from $B_n^j \subset \mathcal{H}$ to $\mathcal{H}$, satisfying the relations (3.22) and (3.23) and obeying the bound

$$\sup_{y \in B_n^j} \|f_{n\beta}^j\| \leq 2\alpha_j^{\frac{2}{3}\ell} r^n. \tag{6.1}$$

They may be also viewed as analytic maps from $B_n^j \subset \mathcal{H}_n$ to $\mathcal{H}$. As such they may be analytically extended to $|\lambda| \leq \lambda_0$ for $n \geq n_0$ by iterated use of (3.22) if we recall the bound (5.19). The new maps are clearly bounded uniformly in n (e.g. by $2\alpha_j^{\frac{2}{3}\ell} r_j^{n_0}$). Let us prove now the convergence in $\mathcal{H}$ of $y_{n\beta}^j \equiv f_{n\beta}^j(0)$ obtained this way. The recursion (3.22) implies

$$y_{n\beta}^j = f_{n\beta}^j(0) = f_{(n-1)\beta}^j(\Gamma_{n-1}\widetilde{w}_{n\beta}^j(0)) = y_{(n-1)\beta}^j + \delta_1 f_{(n-1)\beta}^j(\Gamma_{n-1}\widetilde{w}_{n\beta}^j(0)). \tag{6.2}$$

Using Lemma 6, the bound (5.6) and (3.34) we infer

$$\begin{aligned}\|y^j_{n\beta} - y^j_{(n-1)\beta}\| &= \|\delta_1 f^j_{(n-1)\beta}(\Gamma_{n-1}\widetilde{w}^j_{n\beta}(0))\| \\ &\le \sup_{\|y\|\le C\eta^{-2n}\varepsilon\alpha_j^{\frac{2}{3}\ell}r^{2n}} \|\delta_1 f^j_{(n-1)\beta}(y)\| \\ &\le \frac{C\eta^{-2n}\varepsilon r^{n+1}}{1-C\eta^{-2n}\varepsilon r^{n+1}} \sup_{\|y\|\le \alpha_j^{\frac{2}{3}\ell}r^{n-1}} \|f^j_{(n-1)\beta}(y)\| \le C\eta^{-2n}\varepsilon\alpha_j^{\frac{2}{3}\ell}r^n\end{aligned} \tag{6.3}$$

The sequence is hence Cauchy, and therefore it converges in $\mathcal{H}$:

$$y^j_{n\beta} \stackrel{n\to\infty}{\longrightarrow} y^j_\beta \tag{6.4}$$

with

$$\|y^j_\beta\| \le C\varepsilon\alpha_j^{\frac{2}{3}\ell} \tag{6.5}$$

uniformly in the strip $|\mathrm{Im}\,\beta| \le \frac{1}{2}\bar{\alpha}_j$. This last estimate implies that, pointwise,

$$|y^j(q)| \le C\varepsilon\alpha_j^{\frac{2}{3}\ell}e^{-\frac{\bar{\alpha}_j}{2}|q|}. \tag{6.6}$$

For $|\lambda| \le \lambda_n$, Eqs (3.23) and (3.20) imply that

$$y^j_n \equiv f^j_n(0) = \Gamma_{<n}\widetilde{w}^j_0(y^j_n) \quad \text{and} \quad \widetilde{w}^j_0(y^j_n) = \widetilde{w}^j_n(0). \tag{6.7}$$

From the first Eq. in (6.7) we get $y^j_n(q)|_{q=0} = 0$ and from the second one using (4.3) and (5.6) it follows

$$\begin{aligned}&\left|\widetilde{w}^j_0(y^j_n;0)\right| = \left|\widetilde{w}^j_n(0;0)\right| = \left|\sum_{q\ne 0} q\,\bar{\chi}_n(\omega\cdot q)\bar{x}(q)\cdot\widetilde{w}^j_n(0;-q)\right| \\ &\le \|P\widetilde{w}^j_n\|_{-n} \le \varepsilon\alpha_j^{\frac{2}{3}\ell}r^{2n}\end{aligned} \tag{6.8}$$

By analyticity these relations have to hold also for $|\lambda| \le \lambda_0$, so we can take the limit for $n \longrightarrow \infty$ in Eqs. (6.7) and infer that

$$y^j(0) = 0\,, \quad y^j = G_0\widetilde{w}^j_0(q;y^j) \text{ for } q \ne 0. \tag{6.9}$$

Once we have constructed inductively $y^j(0)$ we set $x_j \equiv y^j + x_{j-1}$; using (6.9), the inductive hypotheses on x_{j-1} and (2.14) we get $x_j(q)|_{q=0} = 0$ and for $q \geq 0$,

$$\begin{aligned} x_j = y^j + x_{j-1} &= G_0 \widetilde{w}_0^j(q; y^j) + G_0 w_0^{j-1}(q; x_{j-1}) \\ &= G_0 w_0^j(q; x_{j-1} + y^j) - G_0 w_0^{j-1}(q; x_{j-1}) + G_0 w_0^j(q; x_{j-1}) \\ &= G_0 w_0^j(q; x_j), \end{aligned} \tag{6.10}$$

so x_j solves (2.13) for $k = j$. Furthermore, using (3.9) and (6.6) we get

$$\begin{aligned} |x_j(q)| \leq |y^j(q)| + |x_{j-1}(q)| &\leq C\varepsilon \alpha_j^{\frac{2}{3}\ell} e^{-\frac{\bar{\alpha}_j}{2}|q|} + C\varepsilon A_j \frac{e^{-\frac{|q|}{4\gamma_j}}}{|q|^{\frac{\ell}{3}}} \\ &\leq C\varepsilon \ell! \left(\frac{4\alpha_j^2}{\bar{\alpha}_j}\right)^{\frac{\ell}{3}} \frac{e^{-\frac{\bar{\alpha}_j}{4}|q|}}{|q|^{\frac{\ell}{3}}} + C\varepsilon \sum_{k=0}^{j-1} \ell! \left(\frac{4}{M8^{k-5}}\right)^{\frac{\ell}{3}} \frac{e^{-\frac{|q|}{4\gamma_j}}}{|q|^{\frac{\ell}{3}}} \\ &= C\varepsilon \ell! \left(\frac{4}{M8^{j-5}}\right)^{\frac{\ell}{3}} \frac{e^{-\frac{\bar{\alpha}_j}{4}|q|}}{|q|^{\frac{\ell}{3}}} + C\varepsilon \sum_{k=0}^{j-1} \ell! \left(\frac{4}{M8^{k-5}}\right)^{\frac{\ell}{3}} \frac{e^{-\frac{|q|}{4\gamma_j}}}{|q|^{\frac{\ell}{3}}} \\ &\leq C\varepsilon A_{j+1} \frac{e^{-\frac{|q|}{4\gamma_{j+1}}}}{|q|^{\frac{\ell}{3}}}, \end{aligned} \tag{6.11}$$

that is (3.9) for x_j.

If we can show that x_j converges for $j \to \infty$ to some function x, we can take the limit for $j \to \infty$ on both sides of (6.10) to obtain

$$x(0) = 0\,, \quad x = G_0 w_0(q; x) \text{ for } q \neq 0 \tag{6.12}$$

which is the Fourier transformed version of (2.6). To conclude the proof of Theorem (1) we only have to show that for $j \to \infty$, $x_j(q) \to x(q)$, for all $q \neq 0$, with $\sum_{q \in \mathbb{Z}^d} |q|^s |x(q)| < \infty$ (which implies $X \in \mathcal{C}^s$). In order to do that, we define $u_j := x_j - x_0$ so that

$$\lim_{j\to\infty} x_j - x_0 = \lim_{j\to\infty} u_j = \sum_{j=1}^{\infty} u_j - u_{j-1}, \tag{6.13}$$

and using (6.6) we get, for all s

$$\begin{aligned} |(u_j(q) - u_{j-1}(q))| &= \left|y^j(q)\right| \leq C\varepsilon\alpha_j^{\frac{2}{3}\ell} e^{-\frac{\bar{\alpha}_j}{2}|q|} \\ &\leq 2^{s+d}8^{3(s+d)}(s+d)!C\varepsilon\frac{\alpha_j^{\frac{2}{3}\ell-s-d}}{|q|^{s+d}} \\ &= C_{s,d}\varepsilon\frac{\alpha_j^{\frac{2}{3}\ell-s-d}}{|q|^{s+d}}, \end{aligned} \tag{6.14}$$

from the latter bound we get for $s < \frac{2}{3}\ell$,

$$\begin{aligned} \sum_{q\in\mathbb{Z}^d} |q|^s \lim_{j\to\infty} |u_j(q)| &\leq \sum_{q\in\mathbb{Z}^d}\sum_{j=1}^{\infty} |q|^s \left|u_j(q) - u_{j-1}(q)\right| \\ &\leq \sum_{q\in\mathbb{Z}^d}\sum_{j=1}^{\infty} C_{s,d}\varepsilon\frac{\alpha_j^{\frac{2}{3}\ell-s-d}}{|q|^d} < \infty. \end{aligned} \tag{6.15}$$

Finally

$$\sum_{q\in\mathbb{Z}^d} |q|^s|x(q)| = \sum_{q\in\mathbb{Z}^d} |q|^s \left|\lim_{j\to\infty} x_j(q)\right| = \sum_{q\in\mathbb{Z}^d} |q|^s \left|\lim_{j\to\infty} u_j(q) + x_0(q)\right| < \infty \tag{6.16}$$

which implies that $X \in \mathcal{C}^s$ and proves Theorem 1.

□

Part 2

Continuous Renormalization

Chapter 7

Introduction and continuous RG scheme

Once again, we study the Hamiltonian function

$$H(I,\theta) = \frac{1}{2}I^2 + \lambda V(\theta) \tag{7.1}$$

with $I \in \mathbb{R}^d$, $\theta \in \mathbb{T}^d$, $\lambda \in \mathbb{R}$. We shall relax the hypotheses of V being $\mathcal{C}^{\ell+1}$ as in Part and assume it real analytic in θ. In chapter 1 we studied in detail the flow generated by H in the case $\lambda = 0$

After λ is turned on, we want again to investigate which of the non-resonant invariant tori persist; let me recall that by an invariant torus with frequency ω,we mean an embedding $\mathcal{T}_\omega : \mathbb{T}^N \to \mathbb{T}^N \times \mathbb{R}^N$, $\mathcal{T}_\omega : \varphi \mapsto (\theta(\varphi), I(\varphi))$, where the solutions of

$$\dot{\varphi} = \omega \tag{7.2}$$

are mapped into the solutions of (7.1). More precisely we write the embedding as

$$\mathcal{T}_\omega(\varphi) = (\omega + Y(\varphi), \theta_0 + \varphi + X(\varphi)) \tag{7.3}$$

where $X : \mathbb{T}^N \to \mathbb{T}^N$, $Y : \mathbb{T}^N \to \mathbb{R}^N$ are analytic and $\mathcal{O}(\lambda)$. Let us recall that by plugging the quasiperiodic solutions

$$I(t) = \omega + Y(\omega t) \tag{7.4}$$

$$\theta(t) = \theta_0 + \omega t + X(\omega t), \tag{7.5}$$

into (1.6) we are led, after some straightforward algebra, to the differential equation

$$\mathcal{D}^2 X(\theta) = -\lambda \partial_\theta V(\theta + X(\theta)) \quad \text{where} \quad \mathcal{D} = \omega \cdot \partial_\theta. \tag{7.6}$$

Solving (7.6), as we saw in the first part, turns out to be rather complicated: when we try to invert the operator $\mathcal{D}^2$ in the Fourier space, it has the form $\frac{1}{(\omega \cdot q)^2}$; the denominators $\omega \cdot q$ can become arbitrarily small, causing troubles in the convergence of the formal power series of X. We have a way to cure this: if ω satisfies the diophantine condition (1.4), we can solve (7.6) for $|\lambda|$ sufficiently small. We shall prove the following

Theorem 13 (**K**olmogorov-**A**rnold-**M**oser)**.** *Let V be real analytic in θ and assume that ω satisfies* (1.4)*. Then, if $|\lambda|$ is sufficiently small, Eq.* (7.6) *has a solution X with zero average, analytic in λ and real analytic in θ.*

In order to prove theorem 13 we will split the real axis into t-dependent scales, where $t \in \mathbb{R}$ and it does not have anything to do with the time of the dynamical system; we shall separate small and big denominators and solve at each step only the part containing the large denominators. Iterating this method for bigger scales will lead us to a convergent sequence of problems which will become trivial for $t \to \infty$ and provide us the wanted solution.

1. The continuous scales

To get a scale separating small and large denominators at time t, equal to η^t for some fixed $\eta \ll 1$, we define an operator $\gamma(t)$ using a continuous partition of unity that will divide the real axis in scales.

Let us introduce the so-called "standard mollifier" by

$$\eta(\kappa) = \begin{cases} Ce^{\frac{1}{\kappa^2-1}} & \text{if } |\kappa| < 1 \\ 0 & \text{if } |\kappa| \geq 1 \end{cases} \tag{7.7}$$

the constant C selected such that $\int_{\mathbb{R}} \eta dx = 1$. Now let us define $\chi \in \mathcal{C}^\infty(\mathbb{R})$ by

$$\chi(\kappa) := 1 - \frac{2}{1-\eta} \int_{\frac{1+\eta}{2}}^{\infty} \eta \left(\frac{2(|\kappa| - y)}{1-\eta} \right) dy \tag{7.8}$$

so that

$$\chi(\kappa) = \begin{cases} 0 & \text{if } |\kappa| < \eta \\ 1 & \text{if } |\kappa| \geq 1 \end{cases} \tag{7.9}$$

and trivially

$$|\partial_\kappa \chi(\kappa)| \, , \, |\partial_\kappa^2 \chi(\kappa)| \leq C \quad \exists C < \infty. \tag{7.10}$$

Let us now define

$$\chi_t(\kappa) \equiv \chi(\eta^{-t}\kappa) \tag{7.11}$$

and for $q, q' \in \mathbb{Z}$, $t \in \mathbb{R}$, the kernels of a diagonal linear operator in the Fourier space

$$\gamma_t(q, q') \equiv -\frac{\partial_t \chi_t(\omega \cdot q))}{(\omega \cdot q)^2} \delta(q, q'), \tag{7.12}$$

and for $s \leq t \in \mathbb{R}$

$$\Gamma_{[s,t]}(q, q') = \frac{\chi(\eta^{-t}(\omega \cdot q)) - \chi(\eta^{-s}(\omega \cdot q))}{(\omega \cdot q)^2} \delta(q, q'), \tag{7.13}$$

so that

$$\int_s^t \gamma_\tau(q, q') d\tau = -\Gamma_{[s,t]}(q, q'). \tag{7.14}$$

Furthermore we shall use the notation

$$\Gamma_{<t}(q, q') = \lim_{s \to -\infty} \Gamma_{[s,t]}(q, q') = \frac{\chi(\eta^{-t}(\omega \cdot q))}{(\omega \cdot q)^2} \delta(q, q'), \tag{7.15}$$

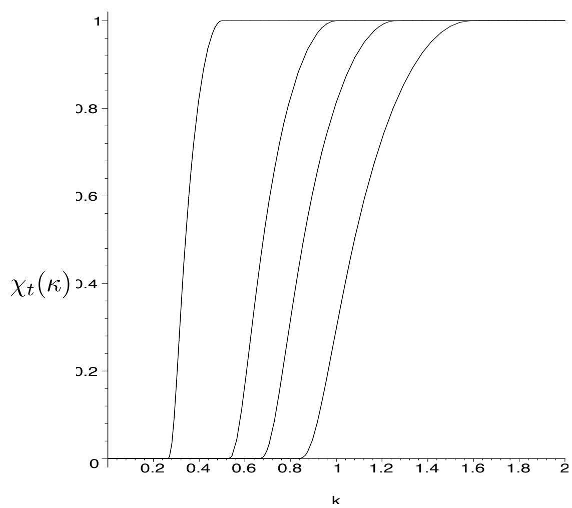

Figure 1. The cutoff function $\chi_t(\kappa)$, with $\eta = \frac{1}{4}$, plotted against κ at different t's

and define the operator $\gamma_t[\kappa]$ with shifted kernel,

$$\gamma_t[\kappa](q, q') \equiv -\frac{\partial_t \chi_t(\omega \cdot q + \kappa))}{(\omega \cdot q + \kappa)^2} \delta(q, q'). \tag{7.16}$$

Lemma 14. *There exists $C > 0$ such that for $i = 0, 1, 2$ and all $\kappa \in \mathbb{R}$*

$$|\kappa^{(2+i)} \partial_\kappa^i \gamma_t[\kappa](0)| \leq C. \tag{7.17}$$

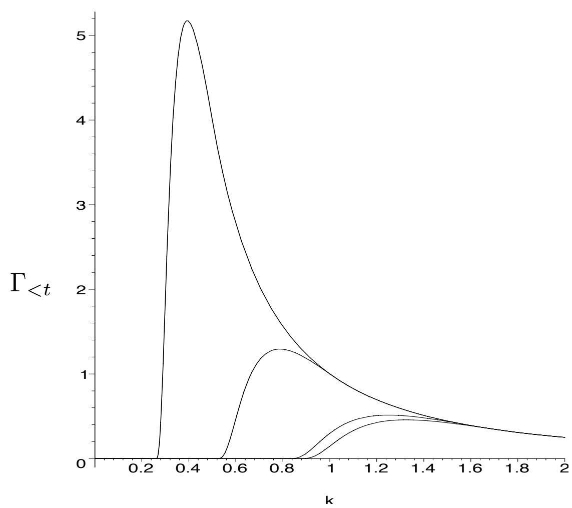

Figure 2. The function $\Gamma_{<t}(q) = \chi_t(\omega \cdot q)/(\omega \cdot q)^2$, with $\eta = \frac{1}{4}$, plotted against $\kappa = \omega \cdot q$ at different times

Proof. Using the definition (7.12) and the bounds (7.10), the estimate is straightforward since $\gamma_t[\kappa](0) = \kappa^{-2}\chi(\eta^{-t}\kappa)$ where $\mathrm{Supp}(\chi) = \mathcal{O}(1)$.

□

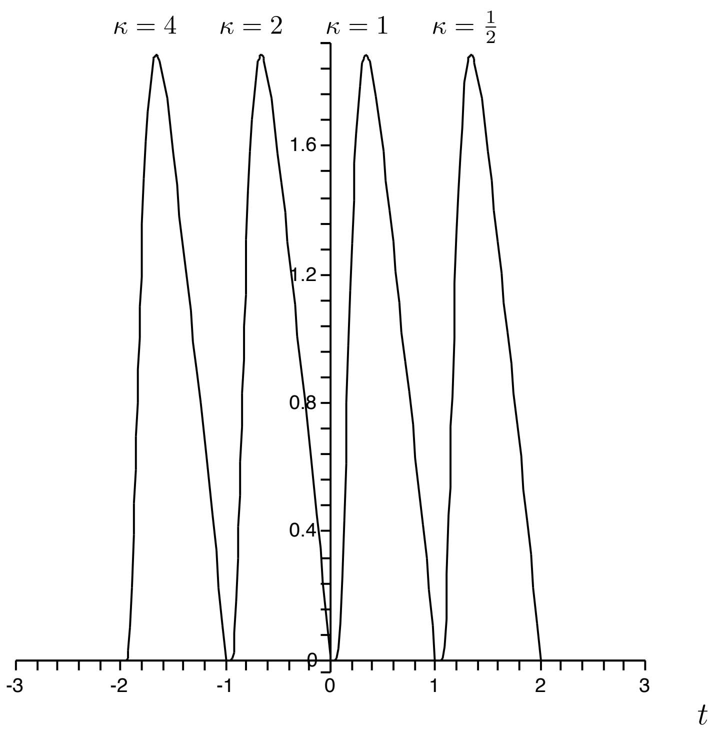

Figure 3. The function $-\gamma.(q)(\kappa)\kappa^2$, with $\eta = \frac{1}{2}$, plotted against t at different $\kappa = \omega \cdot q$

2. Renormalization Group scheme

Returning to the KAM theorem 13, we were left with the problem of finding a solution to Eq. (7.6). We can formally write the latter in the form

$$X = G\overline{W}(X, \theta), \tag{7.18}$$

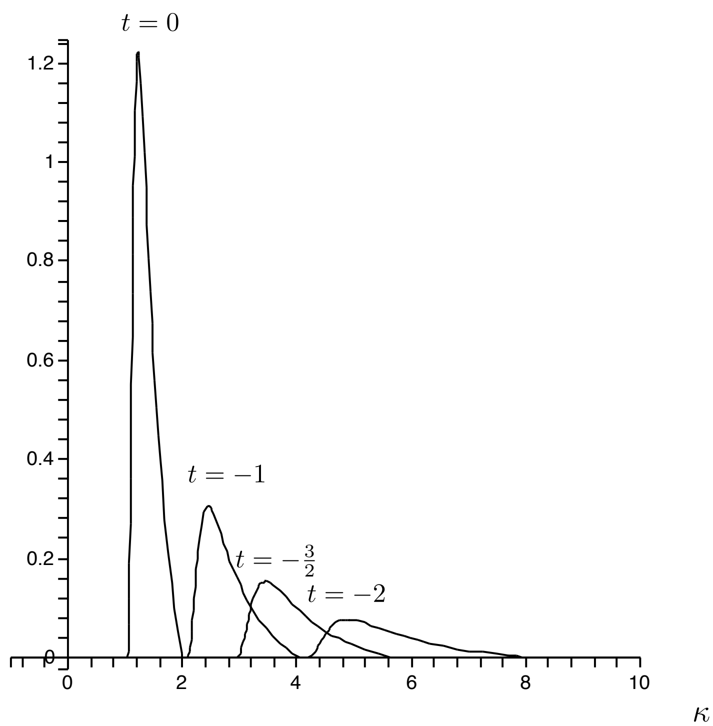

Figure 4. The function $-\gamma_t(q)$, with $\eta = \frac{1}{2}$, plotted against $\kappa = \omega \cdot q$ at different t.

where we defined $G \equiv \mathcal{D}^{-2}$ and $\overline{W} \equiv -\lambda \partial_\theta V(\theta + X(\theta))$. In order to solve Eq. (7.18) we use the cutoff introduced in section 1 and for each $t \in \mathbb{R}$ we split the operator G in two parts

$$G = G_t + \Gamma_{<t}. \tag{7.19}$$

If for all $-\infty \leq s \leq t \in \mathbb{R}$ we can find maps W_t that verify the fixed point equation

$$W_t(Y,\theta) = W_s(Y + \Gamma_{[s,t]} W_t(Y),\theta), \quad \text{with} \quad \lim_{t\to-\infty} W_t(Y,\theta) = \overline{W}(Y,\theta), \tag{7.20}$$

then by writing

$$Z_t(Y,\theta) := \Gamma_{<t} W_t(Y,\theta), \tag{7.21}$$

we see that taking the limit for $s \to -\infty$ in (7.20), $Z_t(Y)$ satisfies

$$Z_t(Y,\theta) = \Gamma_{<t} \overline{W}(Y + Z_t(Y),\theta); \tag{7.22}$$

so, if we split $X(\theta) = Y(\theta) + Z_t(Y,\theta)$, we have

$$\begin{aligned}
& X(\theta) = G\overline{W}(X,\theta) \\
\iff & Y(\theta) + Z_t(Y,\theta) = G_t\overline{W}(Y + Z_t(Y),\theta) + \Gamma_{<t}\overline{W}(Y + Z_t(Y),\theta) \\
\iff & Y(\theta) = G_t\overline{W}(Y + \Gamma_{<t} W_t(Y),\theta) \\
\iff & Y(\theta) = G_t W_t(Y,\theta),
\end{aligned} \tag{7.23}$$

hence $X(\theta) = Y(\theta) + Z_t(Y,\theta) \equiv F_t(Y)$ is a solution of Eq. (7.18) if and only if $Y(\theta) = G_t W_t(Y,\theta)$. Note also the cumulative formulas that follow easily by taking the limit for $s \to -\infty$ in (7.20)

$$W_t(Y) = \overline{W}(Y + \Gamma_{<t} W_t(Y)) \tag{7.24}$$

The main idea is the following: provided that the maps in (7.20) (or equivalently the maps in (7.22)) exist and are analytic for all t in some t-dependent ball, if $W_t(Y,\theta) \stackrel{t\to\infty}{\longrightarrow} 0$ sufficiently fast, then the sequence

$$X_t(\theta) \equiv Z_t(0,\theta). \tag{7.25}$$

has a limit, and $\lim_{t\to\infty} X_t(\theta) = X(\theta)$ will be a solution of (7.18). At an intuitive level, this happens because the operator $G_t W_t$ approaches to a linear operator when t tends to infinity, so $Y = 0$ will satisfy (7.23) for $t \to \infty$. In a formal (and straightforward) way:

$$\begin{aligned} X_t(\theta) &= Z_t(0,\theta) \\ &= \Gamma_{<t}\overline{W}(Z_t(0),\theta) \\ &= \Gamma_{<t}\overline{W}(X_t,\theta), \end{aligned} \tag{7.26}$$

and taking the limit for $t \to \infty$ we get

$$X(\theta) = G_0\overline{W}(X,\theta). \tag{7.27}$$

We reduced our original problem to the existence of analytic maps verifying (7.20), whose decay, for t increasing to infinity, is fast enough to make the sequence (7.25) (whose terms are plagued by small denominators of order η^t) converge. Proving the existence of such analytic maps will be the goal of the rest of the paper.

Chapter 8

Preliminaries

Taking the derivative $\partial_s|_{s=t}$ on both sides of (7.20), we get

$$\partial_t W_t(Y,\theta) = DW_t(Y;\theta)\gamma(t)W_t(Y) \tag{8.1}$$

where DW denotes the Frechet derivative of W with respect to Y and $\gamma_t = -\partial_s \Gamma_{[s,t]}\big|_{s=t}$ like in (7.12).

It will turn out to be useful to introduce the functional $S_t(Y)$:

$$S_t(Y) := -\frac{1}{2}\langle Z_t(Y), \Gamma_{<t}^{-1} Z_t(Y)\rangle_{L^2(\mathbb{T})} + \lambda \int_{\mathbb{T}} V(\theta + Y(\theta) + Z_t(Y,\theta))d\theta \tag{8.2}$$

in order to notice that W_t is its derivative in the following sense:

$$
\begin{aligned}
DS_t(Y)X = &- \int_{\mathbb{T}} \Gamma_{<t} W_t(Y,\theta) DW_t(Y,\theta) X(\theta) d\theta + \\
&+ \lambda \int_{\mathbb{T}} \partial v(\theta + Y(\theta) + Z_t(Y,\theta))(1 + \Gamma_{<t} DW_t(Y,\theta)) X(\theta) d\theta \\
= &- \int_{\mathbb{T}} \Gamma_{<t} W_t(Y,\theta) \cdot DW_t(Y,\theta) X(\theta) d\theta + \\
&+ \int_{\mathbb{T}} W(Y + Z_t(Y), \theta)(1 + \Gamma_{<t} DW_t(Y,\theta)) X(\theta) \\
= &\int_{\mathbb{T}} W_t(Y,\theta) X(\theta) d\theta.
\end{aligned} \tag{8.3}
$$

So $W_t(Y,\theta)$ is the integral kernel of $DS_t(Y)$.

In terms of S Eq. (8.1) reads

$$
\partial_t DS_t(Y;\theta) = D^2 S_t(Y)(\gamma(t) DS_t(Y))(\theta), \tag{8.4}
$$

and writing it in terms of the kernels

$$
\begin{aligned}
\partial_t \frac{\partial S_t(Y)}{\partial Y(\theta)} &= \int_{\mathbb{T}^N \times \mathbb{T}^N} \frac{\partial^2 S_t(Y)}{\partial Y(\theta) \partial Y(\theta')} \gamma_t(\theta', \theta'') \frac{\partial S_t(Y)}{\partial Y(\theta'')} d\theta' d\theta'' \\
&= \frac{1}{2} \frac{\partial}{\partial Y(\theta)} \int_{\mathbb{T}^N \times \mathbb{T}^N} \frac{\partial S_t(Y)}{\partial Y(\theta')} \gamma_t(\theta', \theta'') \frac{\partial S_t(Y)}{\partial Y(\theta'')} d\theta' d\theta'',
\end{aligned} \tag{8.5}
$$

where

$$
\gamma_t(\theta', \theta'') = \sum_q \gamma_t(q) e^{iq\cdot(\theta' - \theta'')}, \tag{8.6}
$$

we can rewrite (8.1) as

$$
\begin{aligned}
\partial_t S_t(Y) &= \frac{1}{2} \int_{\mathbb{T}^N \times \mathbb{T}^N} \frac{\partial S_t(Y)}{\partial Y(\theta')} \gamma_t(\theta', \theta'') \frac{\partial S_t(Y)}{\partial Y(\theta'')} d\theta' d\theta'' \\
\partial_t S_t(Y) &= \frac{1}{2} DS_t(Y) \gamma(t) DS_t(Y).
\end{aligned} \tag{8.7}
$$

Now it is a matter of taste to solve either (8.1) or (8.7); our choice anyway is to tackle Eq.(8.1) keeping in mind (8.3) when needed.

1. Fourier Spaces

We shall work with Fourier transforms, denoting by lower case letter the Fourier transform of functions of θ, which will be denoted by capital letters:

$$X(\theta) = \sum_{q\in\mathbb{Z}^N} e^{-\mathrm{i}q\cdot\theta} x(q), \quad \text{where} \quad x(q) = \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} e^{\mathrm{i}q\cdot\theta} X(\theta)d\theta. \tag{8.8}$$

We write the formal Taylor expansion of $w_t(y;q)$

$$w_t(y;q) \equiv \sum_{n=0}^{\infty}\sum_{\boldsymbol{q}} w_t^{(n)}(q,q_1,\ldots,q_n)(y(q_1),\ldots,y(q_n)). \tag{8.9}$$

in the Fourier variables the equations (7.24) and (8.1) become:

$$w_t(y;q) = \bar{w}(y + \Gamma_{<t} w_t(y);q) \tag{8.10}$$

$$\partial_t w_t(y;q) = \sum_{q'} Dw_t(y;q,q')\gamma_t(q')w_t(y;q') \tag{8.11}$$

Remark 15. We shall adopt the following convention:

$$Dw(y;q,q') \equiv \frac{\partial w(y;q)}{\partial y(q')} = \frac{\partial \widehat{W(Y;\theta)}}{\partial Y(\theta')}\Big|_{q,-q'} \tag{8.12}$$

or equivalently in terms of S

$$Ds(y;q) \equiv \frac{\partial s(y)}{\partial y(q)} = \frac{\widehat{\partial S(Y)}}{\partial Y(\theta)}\Big|_{-q}. \tag{8.13}$$

We can recover some standard but useful bounds from the analyticity of V. The Taylor expansion with respect to $Y \in \mathbb{T}^N$ is

$$\partial V(\theta + Y) = \sum_{n=0}^{\infty}\frac{V_{n+1}(\theta)}{n!}(Y,\ldots,Y), \tag{8.14}$$

We shall use the Cauchy estimates in the following way:

Lemma 16. *There exist $\rho > 0$, $\bar{\alpha} > 0$ and $b < \infty$ such that $v_{n+1}(q)$ satisfy the bound*

$$\sum_{q\in\mathbb{Z}^N} e^{\bar{\alpha}|q|}\|v_{n+1}(q)\|_{\mathcal{L}(\mathbb{C}^d,\ldots,\mathbb{C}^d;\mathbb{C}^d)} \le bn!\rho^{-n}. \tag{8.15}$$

Proof. From the Cauchy estimates for analytic functions we get

$$\|V_{n+1}(\theta)\|_{\mathcal{L}} \le Cn!\rho^{-n} \quad \exists C \in \mathbb{R}, \quad \rho > 0; \tag{8.16}$$

using Cauchy theorem, for all η in the analyticity strip of V

$$\begin{aligned}
&|v_{n+1}(q)(Y_1,\dots,Y_n)| \\
&= \left| \frac{1}{2\pi} \int_{\mathbb{T}^N} V_{n+1}(\theta + i\eta)(Y_1, \dots Y_n)\, e^{iq\cdot(\theta+i\eta)} d\theta \right| \\
&\le \frac{1}{2\pi} \int_{\mathbb{T}^N} |V_{n+1}(\theta + i\eta)(Y_1, \dots Y_n)|\, e^{-q\cdot\eta} d\theta \\
&\le Cn!\rho^{-n} |Y_1| \cdot \ldots \cdot |Y_n| e^{-q\cdot\eta}
\end{aligned} \tag{8.17}$$

hence

$$\|v_{n+1}(q)\|_{\mathcal{L}(\mathbb{C}^d,\dots,\mathbb{C}^d;\mathbb{C}^d)} \le Cn!\rho^{-n} e^{-q\cdot\eta}. \tag{8.18}$$

Taking $\eta = |\eta| \frac{q}{|q|}$ we get for $0 < \bar{\alpha} < |\eta|$

$$\sum_{q\in\mathbb{Z}^N} e^{\bar{\alpha}|q|} \|v_{n+1}(q)\|_{\mathcal{L}(\mathbb{C}^N,\dots,\mathbb{C}^N;\mathbb{C})} \le bn!\rho^{-n} \tag{8.19}$$

□

Taking the Fourier transform of (8.14) we obtain

$$v(y;q) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{q}} \frac{1}{n!} v_n(q - \sum_{j=1}^{n} q_j)(y(q_1), \dots, y(q_n)), \tag{8.20}$$

where $\boldsymbol{q} = (q_1, \dots, q_n)$.

Recalling that from the boundary condition in (7.20) we have, $\lim_{t\to-\infty} W_t \equiv \bar{W}$ where

$$\bar{W}(Y;\theta) = \lambda(\partial V)(\theta + Y(\theta)), \tag{8.21}$$

and taking the Fourier series on both sides and using (8.20), we obtain

$$\begin{aligned}\bar{w}(y;q) &= \lambda \sum_{n=0}^{\infty} \sum_{\boldsymbol{q}} \frac{1}{n!} q v_n (q - \sum_{j=1}^{n} q_j)(y(q_1), \ldots, y(q_n)) \\ &= \sum_{n=0}^{\infty} \sum_{\boldsymbol{q}} \frac{\lambda i^n}{n!} q v (q - \sum_{i=1}^{n} q_i) \prod_{k=1}^{n} (q - \sum_{j=1}^{n} q_j) \cdot y(q_k) \\ &\equiv \sum_{n=0}^{\infty} \sum_{\boldsymbol{q}} \bar{w}^{(n)}(q; q_1, \ldots, q_n)(y(q_1), \ldots, y(q_n)). \end{aligned} \tag{8.22}$$

The formula (8.22) implies that one can consider the map $\bar{w}$ as an analytic function of y, where y belongs to a suitable Banach Space. For the sake of convenience we denote

$$h \equiv \ell^1 = \{y = \{y(q)\}_{q\in\mathbb{Z}}\,,\ y(q) \in \mathbb{C}^N\ :\ \|y\| \equiv \sum_q |y(q)| < \infty\} \tag{8.23}$$

Let $B(\bar{r})$ be the open ball of radius $\bar{r}$ in h centered at zero and let $H^\infty(B(\bar{r}), h)$ the Banach space of analytic functions $w : B(\bar{r}) \to h$ equipped with the supremum norm.

In order to encode the decay property of the kernels $\bar{w}^{(n)}$ inherited from the analyticity of V, properties which we shall exploit later on, let τ_β denote the translation by $\beta \in \mathbb{R}^N$, $(\tau_\beta Y)(\theta) = Y(\theta - \beta)$. On h, τ_β is realized by $(\tau_\beta y)(q) = e^{i\beta\cdot q} y(q)$. It induces a map $s \mapsto s_\beta$ from $\mathcal{H}$ to itself if we set

$$u_\beta(y) = \tau_\beta u(\tau_{-\beta} y). \tag{8.24}$$

On the kernels $u^{(n)}$ this is given by

$$u_\beta^{(n)}(q_1, \ldots, q_n) = e^{i\beta\cdot(-\sum q_j)} u^{(n)}(q_1, \ldots, q_n), \tag{8.25}$$

and makes sense also for $\beta \in \mathbb{C}^N$. We have

$$\sup_{\|y\|\le\bar{r}} \|\bar{w}_\beta\| \le \sum_{n=0}^{\infty} \sup_{q_1,\ldots,q_n} \sum_q e^{-\mathrm{Im}\,\beta\cdot(q-\sum q_j)} |\bar{w}^{(n)}(q; q_1, \ldots, q_n)| \bar{r}^n \tag{8.26}$$

Combining this with the bound (8.19) we get

Proposition 4. *There exist $\bar{r}$, $\bar{\alpha} > 0$ and $D < \infty$, such that $\bar{w}_\beta \in H^\infty(B(\bar{r}), h)$ and it extends to an analytic function of β in the region $|Im\,\beta| \le \bar{\alpha}$ with values in $H^\infty(B(\bar{r}), h)$ satisfying the bound*

$$\sup_{\|y\|\le\bar{r}} \|\bar{w}_\beta\| \le D|\lambda|. \tag{8.27}$$

Let us finally state some standard estimates that we shall use throughout the proof. Let h, h' be Banach spaces, and we define $H^\infty(h; h')$ as the space of analytic functions $w : h \to h'$ equipped with the supremum norm. We shall make use of the following Cauchy estimates throughout the proof:

$$\sup_{\|y\|\le r-\delta} \|Dw(y)\| \le \sup_{\|y\|\le r} \frac{1}{\delta}\|w(y)\|, \tag{8.28}$$

$$\sup_{\|y\|\le r'\gamma} \|\delta_k w(y)\| \le \frac{\gamma^k}{1-\gamma} \sup_{\|y\|\le r'} \|w(y)\|. \tag{8.29}$$

Furthermore we will also make use of the following estimate: let $w_i \in H^\infty(B(r) \subset h\,; h')$ for $i = 1, 2$, and $w \in H^\infty(B(r') \subset h'\,; h'')$; then, if $\sup_{\|y\|_h\le r} \|w_i(y)\|_{h'} \le \frac{1}{2}r'$, we have

$$\sup_{\|y\|_h\le r} \|w \circ w_1(y) - w \circ w_2(y)\|_{h''} \le \frac{2}{r'} \sup_{\|y'\|_{h'}\le r'} \|w(y')\|_{h''} \sup_{\|y\|_h\le r} \|w_1(y) - w_2(y)\|_{h'} \tag{8.30}$$

2. A temporary solution

We shall construct now a solution of (8.1); the inconvenient is that for the latter to be defined until t we shall need to take $|\lambda| \le \lambda_t$ with $\lambda_t \overset{t\to\infty}{\longrightarrow} 0$. This preliminary result will allow us, by choosing a sufficiently large index T_0, to start with a “shifted” initial condition defined for $|\lambda| \le |\lambda_{T_0}|$. From such a new initial condition we shall be able to extend (uniformly in λ) the solution to all $t \ge T_0$. In the following Lemma we will only show how to construct solutions for all $t \in \mathbb{R}$ with $|\lambda| \le \lambda_t$; the index T_0 will be chosen later on.

Proposition 5. *For all t and for any sufficiently small $r > 0$, $|\lambda| \le \lambda_t$ and $|\mathit{Im}\,\beta| \le \bar{\alpha}$ the equations* (7.24) *have a unique solution $w_t \in H^\infty(B(r^t), h)$ with*

$$\sup_{y \in B(r^t)} \|w_t\| \le D|\lambda| \tag{8.31}$$

where D is as in Proposition 4. Moreover, $w_{t\beta}$ is analytic in λ and β and it satisfies the recursive relation (7.20)

Proof. Consider the fixed point equation (7.24) and write it as $w = \mathcal{F}(w)$, for $w = w_{t\beta}$ and

$$\mathcal{F}(w)(y) = \bar{w}_\beta(y + \Gamma_{<t} w(y)). \tag{8.32}$$

Let

$$\mathcal{B}_t = \left\{ w \in H^\infty(B(r^t), h) \mid \|w\|_{\mathcal{B}_t} \equiv \sup_{y \in B(r^t)} \|w\| \le D|\lambda| \right\}, \tag{8.33}$$

where D is as in Prop. 4. Choose λ_t so that $C\eta^{-2t} D\lambda_t \le r^t$, with C as in Lemma 1. It follows from the latter that for $w \in \mathcal{B}_t$ and $y \in B(r^t) \subset h$,

$$\|y + \Gamma_{<t} w(y)\| \le r^t + C\eta^{-2t} C|\lambda| \le 2r^t \le \frac{1}{2}\bar{r}, \tag{8.34}$$

so $\mathcal{F}(w)$ is defined in $B(r^t)$ and, by Proposition 4,

$$\|\mathcal{F}(w)\|_{\mathcal{B}_t} \le D|\lambda|. \tag{8.35}$$

Hence $\mathcal{F} : \mathcal{B}_t \to \mathcal{B}_t$. For $w_1, w_2 \in \mathcal{B}_t$ we use (8.30) to conclude that

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{\mathcal{B}_t} &= \sup_{\|y\| \le r^t} \|\bar{w}_\beta(y + \Gamma_{<t} w_1(y)) - \bar{w}_\beta(y + \Gamma_{<t} w_2(y))\| \\ &\le \frac{2}{\bar{r}} C\eta^{-2t} D|\lambda| \|w_1 - w_2\|_{\mathcal{B}_t} \\ &\le \frac{2r^t}{\bar{r}} \|w_1 - w_2\|_{\mathcal{B}_t} \\ &\le \frac{1}{2} \|w_1 - w_2\|_{\mathcal{B}_t}, \end{aligned} \tag{8.36}$$

i.e., $\mathcal{F}$ is a contraction. It follows that (7.24) has a unique solution $w_{t\beta}$ in $\mathcal{B}_t$ satisfying the bound (8.31) which, besides, is analytic in λ and β. Consider now the map $\mathcal{F}'$:

$$\mathcal{F}'(w)(y) = \bar{w}_\beta(y + \Gamma_{[s,t]} w_{t\beta}(y) + \Gamma_{<s} w(y)); \tag{8.37}$$

again $\mathcal{F}'$ is a contraction in $\mathcal{B}_t$ since, for $\|y\| \le r^t$, we have

$$\|y + \Gamma_{[s,t]} w_{t\beta}(y) + \Gamma_{<s} w(y)\| \le 3r^t \le \frac{1}{2}\bar{r} \tag{8.38}$$

for r sufficiently small. But from Eqs. (7.24) one deduces that $w_{t\beta}$ and $w_{s\beta} \circ \big(1 + \Gamma_{[s,t]} w_{t\beta}\big)$, both in $\mathcal{B}_t$, are its fixed points (just insert them into (8.37)), hence by uniqueness they have to coincide, and (7.20) follows. □

3. t-dependent Banach Spaces

Let us first introduce the projection

$$P_t(y)(q) = \begin{cases} y(q) & \text{if } |\omega \cdot q| \le \eta^t \\ 0 & \text{otherwise.} \end{cases} \tag{8.39}$$

We define now the t-dependent spaces (see the footnote at p. 31)

$$h_{-t} := \{u(q) : \|u\|_{-t} \equiv \sum_{q\in\mathbb{Z}} |P_t u(q)| = \sum_{|\omega\cdot q|\le\eta^t} |u(q)|\}, \tag{8.40}$$

$$h_t := P_t h, \tag{8.41}$$

so that we have, for $s \le t$, the obvious inclusions:

$$h_t \subset h_s \subset h \subset h_{-s} \subset h_{-t} \tag{8.42}$$

Remark 17. The spaces h_t as subset of h, will naturally "inherit" the same t-dependent seminorms defined in (8.40): for $y \in h_t$

$$\|y\| = \sum_{|\omega\cdot q|\le\eta^t} |y(q)| = \|y\|_{-t}. \tag{8.43}$$

We shall also write $\|L\|_{t;-t} := \|L\|_{\mathcal{L}(h_t;h_{-t})}$ for the norm of a linear operator L from h_t to h_{-t}.

We shall adopt the notation $B_t \equiv \{y \in h_t : \|y\| < r^t\}$, and define a continuous automorphism $t_p : \mathscr{L}(h;h) \to \mathscr{L}(h;h)$, for $p \in \mathbb{Z}^N$, shifting both arguments of the kernel of an operator A in the following way:

$$t_p A(q,q') = A(q+p, q'+p). \tag{8.44}$$

We shall define $\mathcal{H}_t$ as the space of functions $u : B_t \to h_{-t}$,

$$u(y;q) = \sum_{n=0}^{\infty} \sum_{q_1,\dots,q_n} u^{(n)}(q;q_1,\dots,q_n)(y(q_1),\dots,y(q_n)), \tag{8.45}$$

such that the kernel $Du_t(y;q,q')$, on the diagonal $q = q'$, depends on q only through $\omega \cdot q$, that is, for $\kappa \in \mathbb{R}$, there exists a function $\Pi(\kappa;y) \in \mathcal{L}(h_t;h_{-t})$, twice differentiable in $k \in \mathbb{R}$ and $y \in B_t$, such that

$$t_p Du(y;q,q') = \Pi(\omega \cdot p; y, q, q') \quad \text{and} \quad \Pi(\kappa;y) = \mathcal{O}(\kappa^2) \tag{8.46}$$

Let us now write $\Pi_t(\kappa;0)$ as the sum of its diagonal and off-diagonal part $\Pi_t(0;\kappa) = \sigma_t(\kappa) + \rho_t(\kappa)$, where

$$\begin{aligned} \sigma_t(\kappa;q,q') &\equiv \Pi_t(0;\kappa;q,q')\delta(q,q') \\ \rho_t(\kappa;q,q') &\equiv \Pi_t(0;\kappa;q,q') - \sigma_t(\kappa,q,q'). \end{aligned} \tag{8.47}$$

and equip $\mathcal{H}_t$ with the norm

$$\begin{aligned} \|u\|_{\mathcal{H}_t} = r^{-2t}\|u(0)\|_{-t} + \sup_{i=0,1,2} \Big(& 2^i \eta^{(i-2)t} \|\partial_\kappa^i \sigma_t(\kappa)\|_{t;-t} \\ & + r^{-\frac{t}{2+i}} \|\partial_\kappa^i \rho_t(\kappa)\|_{t;-t} + \sup_{p\geq 1} r^{(p-\frac{3-i}{4})t} \frac{\|\partial_\kappa^i D^p \Pi(0;\kappa)\|_{t;-t}}{(p-1)!} \Big). \end{aligned} \tag{8.48}$$

4. The Banach Space $\mathcal{H}$

Let us fix $T_0 > 0$ once and for all, and define $\mathcal{H}$ as the space consisting of all functions $u : (T_0, \infty) \to \cup_{t \geq T_0} \mathcal{H}_t$, such that $u : t \mapsto u_t \in \mathcal{H}_t$, obeying the following condition

$$u_t(y;0)^\gamma = -i \sum_q q^\gamma \; y(q)^\alpha u_t(y;-q)^\alpha. \tag{8.49}$$

$\mathcal{H}$ will be endowed with the norm

$$\|u\|_{\mathcal{H}} = \sup_{t \geq T_0} \sup_{|\mathrm{Im}\beta| \leq \alpha_t} \|u_{t\beta}\|_{\mathcal{H}_t} \tag{8.50}$$

where

$$\alpha_t \equiv \frac{t+1}{2t}\alpha_0 \qquad \alpha_0 \equiv \frac{2T_0}{T_0+1}\bar{\alpha}. \tag{8.51}$$

The condition (8.49) is called *Ward identity*, which, expressing a translation symmetry of the original problem, allows the compensations needed in the so called *resonances* (we shall mention them later on in a slightly unusual fashion. For the classical definition of "resonance" (see [**6, 9, 14, 13, 12**]), such compensations will overcome the small denominators problem.

Chapter 9

Properties of w (Ward Identities)

1. Ward Identities

Let us show now that the maps w_t constructed in Proposition 5 for $|\lambda| \leq \lambda_t$ obey the Ward identity (8.49).

We notice that the scalar function

$$S(Y) = \lambda \int_{\mathbb{T}^N} v(\theta + Y(\theta)) d\theta \tag{9.1}$$

is invariant under translations of the type

$$T_\beta : Y(\theta) \mapsto Y_\beta(\theta) \quad Y_\beta(\theta) = Y(\theta + \beta) + \beta. \tag{9.2}$$

This means

$$\frac{\partial}{\partial \beta^\gamma}\Big|_{\beta=0} \lambda \int_{T^N} v(\theta + Y_\beta(\theta)) d\theta = 0; \tag{9.3}$$

from which we get the equation [1]

$$\int_{T^N} \bar{W}(Y;\theta)^\gamma d\theta = -\int_{\mathbb{T}^N} \bar{w}(Y;\theta)^\alpha \partial_\gamma Y(\theta)^\alpha d\theta. \tag{9.4}$$

[1] The summations over the repeated index α are understood

Integration by parts of the right hand side yelds the basic identity:

$$\int_{\mathbb{T}^N} \bar{W}(Y;\theta)^\gamma = \int_{\mathbb{T}^N} Y^\alpha \partial_\gamma \bar{W}(Y;\theta)^\alpha d\theta. \qquad (9.5)$$

Let us now show that equation (9.5) holds also for W_t constructed for $|\lambda| \leq \lambda_t$, in fact using (7.24) we obtain

$$\begin{aligned} &\int_{T^N} W_t(Y;\theta)^\gamma d\theta = \int_{T^N} \bar{W}(Y + \Gamma\!< tW_t(Y);\theta)^\gamma d\theta \\ &= (Y^\alpha + (\Gamma\!< tW_t)^\alpha(Y))(\theta)\partial_{\theta^\gamma}\bar{W}(Y + \Gamma\!< tW_t(Y);\theta)^\gamma d\theta \\ &= \int_{T^N} Y^\alpha \partial_{\theta^\gamma} W_t(Y;\theta)^\alpha d\theta + \int_{T^N} (\Gamma\!< tW_t(Y))^\alpha \partial_{\theta^\gamma} W_t(Y;\theta)^\alpha d\theta. \qquad (9.6) \end{aligned}$$

The last integral, after two integrations by parts, turns out to be equal to its opposite, hence it vanishes, yielding (9.5) for W_t. In the Fourier representation it is

$$w_t(y;0)^\gamma = -i \sum_q q^\gamma \; y(q)^\alpha w_t(y;-q)^\alpha. \qquad (9.7)$$

To derive a first consequence of the Ward identity (9.7), which will be used later, we evaluate it at $y = 0$ to get the following

Lemma 18. *The maps w_t constructed in Proposition 5 for $|\lambda| \leq \lambda_t$ satisfy*

$$w_t(0;0) = 0 \qquad (9.8)$$

It is also very important to notice the following:

Lemma 19. *The derivative $t_p D w_{t\beta}(y)$ depends on p only through $\omega \cdot p$.*

Proof. First of all, we have

$$t_p D\bar{w}(y;q,q') = D\bar{w}(y;q,q'), \qquad (9.9)$$

which follows easily writing the explicit expansion for $D\bar{w}(y)$ from (8.22):

$$D\bar{w}(y;q,q')^{\gamma\beta} = \sum_{m=0}^{\infty} \frac{\lambda i^{m+2}}{m!} v(q-q' - \sum_{j=0}^{m} q_j)$$
$$\times (q-q' - \sum_{j=3}^{n+2} q_j)^{\gamma} (q-q' - \sum_{j=3}^{n+2} q_j)^{\beta} \prod_{k=1}^{n} (q-q' - \sum_{j=0}^{n} q_j) \cdot y(q_k), \qquad (9.10)$$

this is a function of $q - q'$ only, hence (9.9) follows. To finish the proof it is enough to differentiate (8.10) to get

$$t_p D w_{t\beta}(y) = t_p \left([1 - D\bar{w}_\beta(y_t)\Gamma_{<t}]^{-1} D\bar{w}_\beta(y_t) \right)$$
$$= [1 - D\bar{w}_\beta(y_t)\Gamma_{<t}[\omega \cdot p]]^{-1} D\bar{w}_\beta(y_t) \qquad (9.11)$$

where $y_t \equiv y + \Gamma_{<t} w_{t\beta}(y)$, and (9.11) depends on p only through $\omega \cdot p$ as claimed. □

Lemma 20. *According to the discussion above, let us write, for $|\lambda| \le \lambda_t$, the kernels*

$$\pi_t(\omega \cdot p; y; q, q') \equiv t_p D w_{t\beta}(y; q, q'), \qquad (9.12)$$

and denote their smooth interpolations $\pi_t(\kappa; y; q, q')$ for $\kappa \in \mathbb{R}$. For $i = 0, 1, 2,\ y \in B(r^{T_0})$ and$|\kappa| \le \eta^{T_0}$ we obtain

$$\|\partial_\kappa^i \pi_{T_0}(\kappa; y)\|_{\mathcal{L}(h;h)} \le |\lambda|^{1/2}; \qquad (9.13)$$

furthermore,

$$\pi_{T_0}(\kappa; y; 0, 0)|_{y=0} = \mathcal{O}(\kappa^2) \qquad (9.14)$$

The proof of Lemma 20 is straightforward and follows exactly [**5**] section 5. Taking $|\lambda| \le \lambda_{T_0}$ small enough, we have

Corollary 21. *In view of Proposition 5, for $|\lambda| \le \lambda_{T_0}$ and $|Im\,\beta| \le \bar{\alpha}$, we have $w_{T_0\beta} \in \mathcal{H}_{T_0}$, furthermore $\|w_{T_0\beta}\|_{\mathcal{H}_{T_0}} \le \varepsilon$, where $\varepsilon \to 0$ when $\lambda \to 0$*

Chapter 10

The integral operator Φ

In order to solve equation (8.11) we define the operator $\Phi : u \mapsto \Phi(u)$ on $\mathcal{H}$ such that for $t \geq T_0$, $\Phi(u) : t \mapsto \Phi(u)_t \in \mathcal{H}_t$ in the following way:

$$\Phi(u)_{t\beta}(y) = w_{T_0}(y) + \int_{T_0}^{t} Du_\tau(y)\gamma_\tau u_\tau(y)d\tau. \tag{10.1}$$

If Φ has a fixed point w, the latter will solve (7.20) for all $t \geq T_0$; in order to show that such fixed point exist, we shall prove that Φ is a contraction in $\mathcal{H}$. We shall divide the proof in several lemmata. The next remark is also important:

Remark 22. Since the zero function belongs to our ball, we can always assume that u is such that

$$Du_t(y; q, q') = Du_t(y; -q', -q), \tag{10.2}$$

as Φ preserves such property. The latter claim is easy to check: first of all

$$D\bar{w}(y; q, q') = D\bar{w}(y; -q', -q), \tag{10.3}$$

since $\bar{w}(q) = \frac{\partial s(y)}{\partial y(-q)}$; by differentiating Eq. (8.10) it is easily seen that also $Dw_{T_0}(y; q, q') = Dw_{T_0}(y; -q', -q)$; finally using Eq. (10.1) the claim is proven.

1. Φ preserves the properties of the the functions in $\mathcal{H}$

We shall show that Φ preserves the properties (8.49) and (8.46).

Lemma 23. *Let $u \in \mathcal{H}$; then, for all $t \geq T_0$, $\Phi(u)_t$ obeys the Ward identity*

$$\Phi(u)_t(y;0)^\gamma = -i \sum_q q^\gamma \; y(q)^\alpha \Phi(u)_t(y;-q)^\alpha. \tag{10.4}$$

Furthermore, for $\kappa \in \mathbb{R}$, there exist functions $\Pi'_t(\kappa;y) \in \mathcal{L}(h_t;h_{-t})$, twice differentiable in $k \in \mathbb{R}$ and $y \in B_t$, such that

$$t_p D\Phi(u)_t(y;q,q') = \Pi'_t(\omega \cdot p; y, q, q') \quad \textit{and} \quad \Pi'_t(\kappa;y;0,0)|_{y=0} = \mathcal{O}(\kappa^2), \tag{10.5}$$

that is

$$\Pi'_t(\kappa;y;0,0)|_{\substack{\kappa=0\\y=0}} = 0, \tag{10.6}$$

$$\partial_\kappa \Pi'_t(\kappa;y;0,0)|_{\substack{\kappa=0\\y=0}} = 0. \tag{10.7}$$

Proof. Let $u \in \mathcal{H}$. Differentiating (8.49) with respect to $y(q')^\delta$ we get

$$Du_t(y;0,q')^{\gamma\delta} = -iq'^\gamma u_t(y;-q')^\delta - i \sum_q q^\gamma y(q)^\alpha Du_t(y;-q;q')^{\alpha\delta}, \tag{10.8}$$

and we can use (10.1) to get

$$\begin{aligned}
\Phi_{t\beta}(u)(y;0)^\gamma &= w_{T_0\beta}(y;0)^\gamma + \int_{T_0}^t \sum_{q'} Du_{\tau\beta}(y;0,q')^{\gamma\delta}\gamma_\tau(q')u_{\tau\beta}(y;q')^\delta d\tau \\
&= -i\sum_q q^\gamma y(q)^\alpha w_{T_0\beta}(y;-q)^\alpha - \int_{T_0}^t \underbrace{\sum_{q'} iq'^\gamma u_{\tau\beta}(y;-q')^\delta \gamma_\tau(q) u_{\tau\beta}(y;q')^\delta}_{=0} \\
&\quad - i\sum_{q',q} q^\gamma y(q)^\alpha Du_t(y;-q;q')^{\alpha\delta}\gamma_\tau(q')u_{\tau\beta}(y;q')^\delta d\tau \\
&= -i\sum_q q^\gamma y(q)^\alpha \left(w_{T_0\beta}(y;q)^\alpha + \int_{T_0}^t \sum_{q'} Du_t(y;-q;q')^{\alpha\delta}\gamma_\tau(q')u_{\tau\beta}(y;q')^\delta d\tau \right) \\
&= -i\sum_q q^\gamma y(q)^\alpha \Phi(u)_{t\beta}(y;-q)^\alpha,
\end{aligned} \tag{10.9}$$

which is (8.49) for $\Phi(u)_t$ and it proves the first part of the claim.

Let us now prove (10.5):

$$
\begin{aligned}
t_p D\Phi(u)_t(y;q,q') &= t_p Dw_{T_0}(y;q,q') + \int_{T_0}^{t} \sum_{q''} t_p D^2 u_\tau(y;q,q',q'')\gamma_\tau(q'')u_\tau(y;q'') \\
&\quad + \sum_{q} Du_\tau(y;q+p,q'')\gamma_\tau(q'')Du_\tau(y;q'',q'+p)d\tau \\
&= \pi_{T_0}(\omega\cdot p;y;q,q') + \int_{T_0}^{t} \sum_{q''} D\Pi_\tau(\omega\cdot p;y;q,q',q'')\gamma_\tau(q'')u_\tau(y;q'') \\
&\quad + \sum_{q''} \Pi_\tau(\omega\cdot p;y;q,q'')\gamma_\tau[\omega\cdot p](q'')\Pi_\tau(\omega\cdot p;y;q'',q')d\tau \\
&\equiv \Pi'_t(\omega\cdot p;y;q,q'). \qquad (10.10)
\end{aligned}
$$

We interpolate (10.10), defining

$$
\begin{aligned}
\Pi'_t(\kappa;y;q,q') \equiv \pi_{T_0}(\kappa;y;q,q') &+ \int_{T_0}^{t} \sum_{q''} D\Pi_\tau(\kappa;y;q,q',q'')\gamma_\tau(q'')u_\tau(y;q'') \\
&+ \sum_{q''} \Pi_\tau(\kappa;y;q,q'')\gamma_\tau[\kappa](q'')\Pi_\tau(\kappa;y;q'',q')d\tau \qquad (10.11)
\end{aligned}
$$

which is smooth in κ and yields (10.5).

Differentiating (10.9) w.r.t. y and evaluating it at $y=0$ we get

$$D\Phi(u)_t(0;0,q')^{\gamma\delta} = -iq'^{\gamma}\Phi(u)_t(0;-q')^{\delta}, \qquad (10.12)$$

and setting $q'=0$ we obtain

$$D\Phi(u)_t(y;0,0)|_{y=0} = \Pi'_t(\kappa;y;0,0)|_{\substack{\kappa=0\\ y=0}} = 0, \qquad (10.13)$$

that is, (10.6).

Next, using (10.11), we have

$$
\begin{aligned}
\partial_\kappa\Pi'_t(\kappa;y;0,0) &= \partial_\kappa\pi_{T_0}(\kappa;y;0,0) + \int_{T_0}^{t} \sum_{q} \partial_\kappa D\Pi_t(\kappa;y;0,0,q)\gamma_\tau(q)u_\tau(y;q) \\
&\quad + \partial_\kappa \sum_{q} \Pi_\tau(\kappa;y;0,q)\gamma_\tau[\kappa](q)\Pi_\tau(\kappa;y;q,0).
\end{aligned}
$$

When evaluating this at $\kappa = 0, y = 0$ the first term vanishes because of lemma 20. The second term is zero as well, since $D\Pi_t(\kappa; y; 0, 0, q)$ is differentiable at zero and even in κ for all $q \in \mathbb{Z}^N$. The third term vanishes as the expression inside the parentheses is once again differentiable at zero and even in κ; to check the latter claim it is enough to use two facts: $\Pi(\kappa, y, q, q') = \Pi(-\kappa, y, -q', -q)$ (see the discussion in Remark 22 and the definition (8.46)) and $\gamma[\kappa](q) = \gamma[-\kappa](-q)$, so that we get

$$\begin{aligned}
&\sum_q \Pi_\tau(\kappa; y; 0, q)\gamma_\tau[\kappa](q)\Pi_\tau(\kappa; y; q, 0) \\
&= \sum_q \Pi_\tau(-\kappa; y; -q, 0)\gamma_\tau[\kappa](q)\Pi_\tau(-\kappa; y; 0, -q) \\
&= \sum_q \Pi_\tau(-\kappa; y; 0, q)\gamma_\tau[\kappa](q)\Pi_\tau(-\kappa; y; q, 0) \qquad (10.14)
\end{aligned}$$

showing that (10.14) is even in κ. Thus we have obtained

$$\partial_\kappa \Pi_t'(\kappa; y; 0, 0)|_{\substack{\kappa=0 \\ y=0}} = 0, \qquad (10.15)$$

which proves (10.7).

□

2. Φ preserves the balls in $\mathcal{H}$

In view of the results obtained in Lemma 23, we prove the following result

Proposition 6. *Let $\mathcal{B}$ be the ball in $\mathcal{H}$ of radius ε (where ε is as in Corollary 21), then Φ preserves $\mathcal{B}$, that is: $\Phi : \mathcal{B} \to \mathcal{B}$*

Proof. Let $u \in \mathcal{B}$; we already know from Lemma 23 that, for $t \geq T_0$, $\Phi(u)_t$ satisfies (10.4) and (10.6). We are left to show that $\|\Phi\|_{\mathcal{H}} \leq \varepsilon$. We shall estimate the different terms in (8.50) separately, sorting them in an increasing order of difficulty.

2.1. Estimate of high orders. We shall start with the high order terms of the norm (8.50), using the fact that $u \in \mathcal{B}$ implies for $|\kappa| \le \eta^t$ and $|\mathrm{Im}\,\beta| \le \alpha_t$

$$r^{-2t}\|u_{t\beta}(0)\|_{-t} \le \varepsilon \tag{10.16}$$

$$\eta^{-2t}\|\Pi_{t\beta}(\kappa;0)\|_{-t} \le \varepsilon \tag{10.17}$$

$$\sup_{p\ge 1} r^{(p-\frac{1}{1+i})t}\frac{\|\partial_\kappa^i D^p\Pi_t(0;\kappa)\|_{t;-t}}{(p-1)!} \le \varepsilon. \tag{10.18}$$

Keeping in mind Corollary 21, the relation $Du_t(y) = \Pi_t(0;y)$ and writing $D^{-1}\Pi_\tau(0;y) \equiv u_\tau(y)$, for $p \ge 1$, $i = 0, 1, 2$ and $r < \eta^4$ we estimate [1]

$$\begin{aligned}
&\|\partial_\kappa^i D^p\Pi_t'(\kappa;0)\|_{t;-t} \le \|\partial_\kappa^i D^p\pi_{T_0}(\kappa;0)\|_{-t}\\
&\quad + \int_{T_0}^t \sum_{j=0}^p \binom{p}{j}\|\partial_\kappa^i D^{j+1}\Pi_\tau(\kappa;0)\|_{-\tau}\|\gamma_\tau\|\|D^{p-j-1}\Pi_\tau(0;0)\|_{-\tau}d\tau\\
&\quad + \sum_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|=i}}\frac{i!}{\alpha!}\int_{T_0}^t\sum_{j=0}^p\binom{p}{j}\|\partial_\kappa^{\alpha_1}D^j\Pi_{-\tau}(\kappa;0)\|_\tau\|\partial_\kappa^{\alpha_2}\gamma_{-\tau}[\kappa]\|\|\partial_\kappa^{\alpha_3}D^{p-j}\Pi_\tau(\kappa;0)\|_{-\tau}d\tau\\
&\le \|\partial_\kappa^i D^p\pi_{T_0}(\kappa;0)\|_{-T_0} + (p-1)!\,\varepsilon^2\sum_{j=0}^{p-2}\frac{p}{(p-j)(p-j-1)}\int_{T_0}^t r^{-(p-\frac{3-i}{4}-1)\tau}\eta^{-2\tau}d\tau\\
&\quad + (p-1)!\,\varepsilon^2\int_{T_0}^t p\,r^{-(p-\frac{3-i}{4})\tau}d\tau + (p-1)!\varepsilon^2\int_{T_0}^t p\,r^{-(p-\frac{3-i}{4}-1)\tau}\eta^{-2\tau}d\tau\\
&\quad + \sum_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|=i}}\frac{i!}{\alpha!}(p-1)!\,\varepsilon^2\sum_{j=1}^{p-1}\frac{p}{j(p-j)}\int_{T_0}^t r^{-(p-\frac{6-\alpha_1-\alpha_3}{4})\tau}\eta^{(-2-\alpha_2)\tau}d\tau\\
&\quad + \sum_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|=i}}\frac{i!}{\alpha!}2(p-1)!\varepsilon^2\int_{T_0}^t r^{-(p-\frac{3-\alpha_3}{4})\tau}\eta^{(-\alpha_1-\alpha_2)\tau}d\tau
\end{aligned}$$

[1] The subindex β does not play any role here, hence it is understood, and for $p = 1$ the sum $\sum_{j=0}^{p-2}$ is to be considered zero.

$$
\begin{aligned}
&\overset{(*)}{\le} \frac{\varepsilon}{4} + (p-1)!\varepsilon^2 2p \int_{T_0}^{t} r^{-(p-\frac{3-i}{4}-1)\tau}\eta^{-2\tau}d\tau + (p-1)!\varepsilon^2 \int_{T_0}^{t} p r^{-(p-\frac{7-i}{4})\tau}d\tau \\
&\quad + \sum_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|=i}} \frac{i!}{\alpha!}(p-1)!\varepsilon^2 p \int_{T_0}^{t} r^{-(p-\frac{3-i}{4}-\frac{3+\alpha_2}{4})\tau}\eta^{(-2-\alpha_2)\tau}d\tau \\
&\quad + \sum_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|=i}} \frac{i!}{\alpha!}(p-1)!\varepsilon^2 \int_{T_0}^{t} r^{-(p-\frac{3-i}{4})\tau}d\tau \\
&\overset{(**)}{\le} \frac{\varepsilon}{4} + (p-1)!\varepsilon^2 21p \int_{T_0}^{t} r^{-(p-\frac{3-i}{4})\tau}d\tau \\
&\le \frac{\varepsilon}{4} + (p-1)!\varepsilon^2 21p \frac{r^{-(p-\frac{3-i}{4})t} - r^{-(p-\frac{3-i}{4})T_0}}{\ln(\frac{1}{r})(p-\frac{3-i}{4})} \\
&\le \frac{1}{3}(p-1)!\varepsilon r^{-(p-\frac{3-i}{4})t}, \qquad (10.19)
\end{aligned}
$$

where we obtained (*) by simply noticing $\sum_{j=0}^{p-2}\frac{1}{(p-j)(p-j-1)} \le \sum_{p=1}^{\infty}\frac{1}{(p+1)p} \le 1$ and $\sum_{j=1}^{p-1}\frac{1}{j(p-j)} \le 1$, and to get (**) we used $\sup_{i=0,1,2}\sum_{\substack{\alpha\in\mathbb{N}^3\\|\alpha|=i}}\frac{i!}{\alpha!}(p-1)! = 9$.

2.2. Estimate of $\Phi(u)_t(0)$ (using the diophantine condition). The quantity whose norm we want to estimate is

$$\Phi(u)_{t\beta}(0) = w_{T_0\beta}(0) + \int_{T_0}^{t} Du_{\tau\beta}(0)\gamma_\tau u_{\tau\beta}(0)d\tau, \qquad (10.20)$$

Let $\beta \in \mathbb{C}^N$ such that $|\mathrm{Im}\beta| \le \alpha_t$, and shift it to $\beta' = \beta - i\frac{(\bar\alpha-\alpha_t)}{|q_1|}q$, so that $|\mathrm{Im}\beta'| \le \bar\alpha$. Using Corollary 21, we get, for $q \ne 0$,

$$|w_{T_0\beta}(0;q)|e^{(\bar\alpha-\alpha_t)|q|} \le \|w_{T_0\beta'}(0)\|_{-T_0} \le \frac{r^{T_0}}{2}\varepsilon \qquad (10.21)$$

which implies, using Definition (8.51), [2]

$$\sum_{\substack{|\omega\cdot q|<\eta^t\\ q\ne 0}} |w_{T_0\beta}(0;q)| \le \frac{r^{T_0}}{2}\varepsilon \sum_{|\omega\cdot q|<\eta^t} e^{-\frac{t-T_0}{2t^2}\bar\alpha|q|}. \qquad (10.22)$$

The diophantine condition forces $|q|$ to be large when $|\omega\cdot q|$ is small, i.e., (1.4) yields $|\omega\cdot q| \le \eta^t \Rightarrow |q| \ge C\eta^{-\frac{t}{\nu}}$, hence we can extract a super-exponential

[2] Note that $|\alpha_\tau - \alpha_t| \le \frac{t-\tau}{2t^2}$ and $\bar\alpha = \alpha_{T_0}$.

factor from the sum:

$$\begin{aligned}\sum_{\substack{|\omega\cdot q|<\eta^t\\ q\neq 0}}|w_{T_0\beta}(0;q)| &\leq \frac{r^{T_0}}{2}\varepsilon e^{-\frac{t-T_0}{4t^2}\bar{\alpha}C\eta^{-\frac{t}{\nu}}}\sum_{|q|\geq C\eta^{-\frac{t}{\nu}}} e^{-\frac{t-T_0}{4t^2}\bar{\alpha}|q|}\\ &\leq \frac{1}{6}\varepsilon r^{2t}. \end{aligned} \tag{10.23}$$

As for the second term in (10.20), we set, for all $\tau \leq t$, $\beta_\tau = \beta - i\frac{(\alpha_\tau-\alpha_t)}{|q'|}q'$, in order to get $|\mathrm{Im}\beta_\tau| \leq \alpha_\tau$ and $-(\mathrm{Im}\beta_\tau - \mathrm{Im}\beta)\cdot q' = (\alpha_\tau-\alpha_t)|q'|$, so, carrying out all the details, we obtain for $q \neq 0$

$$\begin{aligned}&\left|\sum_{q'} Du_{\tau\beta}(0;q,q')\gamma_\tau(q')u_{\tau\beta}(0;q')\right| e^{(\alpha_\tau-\alpha_t)|q|}\\
&= \left|\sum_{q'} Du_{\tau\beta}(y;q,q')\gamma_\tau(q')u_{\tau\beta}(0;q')\right| e^{-(\mathrm{Im}\beta_\tau-\mathrm{Im}\beta)\cdot q}\\
&= \left|\sum_{q'} Du_{\tau\beta}(0;q,q')\gamma_\tau(q')u_{\tau\beta}(0;q')e^{(\mathrm{Im}\beta_\tau-\mathrm{Im}\beta)\cdot q'}e^{-(\mathrm{Im}\beta_\tau-\mathrm{Im}\beta)\cdot q'}\right| e^{-(\mathrm{Im}\beta_\tau-\mathrm{Im}\beta)\cdot q}\\
&\leq \sum_{q'} |Du_{\tau\beta}(0;q,q')|e^{-(\mathrm{Im}\beta_\tau-\mathrm{Im}\beta)\cdot(q-q')}|\gamma_\tau(q')||u_{\tau\beta}(0;q')|e^{-(\mathrm{Im}\beta_\tau-\mathrm{Im}\beta)\cdot q}\\
&= \sum_{q'} |Du_{\tau\beta_\tau}(0;q,q')||\gamma_\tau(q')||u_{\tau\beta_\tau}(0;q')|\\
&\leq \|Du_{\tau\beta_\tau}(0)\|_{\tau;-\tau}\|\gamma_\tau\|\|u_{\tau\beta_\tau}(0)\|_{-\tau}\\
&\leq \frac{1}{6}\varepsilon r^{2\tau}.\end{aligned} \tag{10.24}$$

From the latter estimate, using again the diophantine condition as in (10.23) and $|\mathrm{Im}\beta| \leq \alpha_t$, we can squeeze out the bound we need: the condition (1.4) yields $|\omega\cdot q| < 2\eta^t \Rightarrow |q| > C\eta^{-\frac{t}{\nu}}$ and keeping this in mind we use (10.24) to get

$$\begin{aligned}
&\sum_{\substack{|\omega\cdot q|<2\eta^t\\ q\neq 0}} \int_{T_0}^t \left|\sum_{q'} Du_{\tau\beta}(q.q')\gamma_\tau(q')u_{\tau\beta}(q')d\tau\right| \\
&\leq \frac{1}{6}\varepsilon \int_{T_0}^t e^{-\frac{t-\tau}{2(t+1)(\tau+1)}C\eta^{\frac{-t}{\nu}}} r^{2\tau} d\tau \\
&\leq \frac{1}{6}\varepsilon \int_{T_0}^t e^{-\frac{t-\tau}{2(t+1)^2}C\eta^{\frac{-t}{\nu}}} r^{2\tau} d\tau \\
&\leq \frac{1}{6}\varepsilon e^{-\frac{tC\eta^{\frac{-t}{\nu}}}{2(t+1)^2}} \int_0^t e^{\tau\left(\frac{C\eta^{\frac{-t}{\nu}}}{2(t+1)^2}-2\ln(1/r)\right)} d\tau \\
&= \frac{1}{6}\varepsilon e^{-\frac{tC\eta^{\frac{-t}{\nu}}}{2(t+1)^2}} \frac{1-r^{2t}e^{\frac{tC\eta^{\frac{-t}{\nu}}}{2(t+1)^2}}}{2\ln(1/r)-\frac{C\eta^{\frac{-t}{\nu}}}{2(t+1)^2}} \\
&\leq \frac{1}{6}\varepsilon e^{-\frac{tC\eta^{\frac{-t}{\nu}}}{2(t+1)^2}} \\
&\leq \frac{1}{6}\varepsilon r^{2t},
\end{aligned} \tag{10.25}$$

where T_0 is chosen large enough, so that

$$2\ln(1/r) - \frac{C\eta^{\frac{-T_0}{\nu}}}{2(T_0+1)^2} \geq 1. \tag{10.26}$$

We have hence proved

$$\sum_{|\omega\cdot q'|<\eta^t} \int_{T_0}^t \left|\sum_q u_{\tau\beta}(0;q)\gamma_\tau(q)Du_{\tau\beta}(0;-q,q')d\tau\right| \leq \frac{1}{6}\varepsilon r^{2t}. \tag{10.27}$$

Finally we notice that due to the Ward identity (10.9), it follows that $\Phi_{t\beta}(u)(0;0) = 0$ for all $t \geq T_0$. Hence combining (10.23) and (10.27) we get

$$\sup_{t\geq T_0} \sup_{|\mathrm{Im}\,\beta|\leq\alpha_t} \|\Phi(u)_t(0)\|_{-t} \leq \frac{1}{3}\varepsilon \tag{10.28}$$

2.3. Estimate of the linear term Π' (using diophantine and Ward). The bound on Π' is the most complicated to achieve and its proof is where the consequences of the Ward identities are needed. The actual difficult part is in estimating the norm of σ' i.e., the diagonal part of Π'. In the KAM literature it corresponds to what in the tree graphs language is called a *resonance* , i.e., for the acquainted reader, σ' being a multiple of $\delta_{q,q'}$ corresponds to a subtree carrying the same incoming and outgoing "momentum" (see for instance [**6, 14, 13, 12**]).

We shall write, as on p. 83, $\Pi'_t(\kappa;0) = \sigma'_t(\kappa) + \rho'_t(\kappa)$. When needed we shall use the notation

$$\sigma_t(\tilde{\kappa}) \equiv \sigma_t(\tilde{\kappa};0) = \sigma_{t\beta}(\kappa, q), \tag{10.29}$$

where $\tilde{\kappa} = \kappa + \omega \cdot q$, so that we can leave the q dependence in σ understood.

Let us first see how one succeds in extracting the right bound for σ'. Using (10.11) we write

$$\sigma'_t(\kappa;q) = \pi_{T_0}(\kappa;y;q,q)|_{y=0} + \int_{T_0}^t \sigma_\tau(\kappa;q)\gamma_\tau[\kappa](q)\sigma_\tau(\kappa;q)d\tau + \mathcal{R}_t(\kappa;q) \tag{10.30}$$

where we denoted the rest $\mathcal{R}_t(\kappa;q)$ as

$$\mathcal{R}_t(\kappa;q) \equiv \int_{T_0}^t \sum_{q'} D\Pi_\tau(\kappa;q;q;q')\gamma_\tau(q')u_\tau(0;q') + \sum_{q'} \rho_\tau(\kappa;q,q')\gamma_\tau[\kappa](q')\rho_\tau(\kappa;q',q)d\tau \tag{10.31}$$

As we shall see, estimating $\mathcal{R}_t(\kappa;q)$ will not be difficult. What will require a special treatment instead, are the first two terms in (7.23): intuitively, in fact, the solution of an equation of the type

$$\dot{\sigma}(t) = \sigma(t)^2 \tag{10.32}$$

would blow up in finite time. Furthermore, to extract the right bound for

$$\pi_{T_0}(\kappa;0;q,q)+\int_{T_0}^{t}4\sigma_\tau(\kappa;q)\gamma_\tau[\kappa](q)\sigma_\tau(\kappa;q)d\tau \tag{10.33}$$

we cannot even use the trick of shrinking the analyticity strip by shifting β, since as $q=q'$, a factor $e^{-C\cdot|q-q'|}=1$ will not provide any benefit. Albeit the situation might look hopeless, we can point out a crucial observation: with fixed κ and q in a suitable t-dependent way, the integrand function (10.33) is non-zero only in a small interval around t. Let us explain why: the function $\gamma_\tau[\kappa](q)$ can be written as $\gamma_\tau[\tilde\kappa](0)$, where $\tilde\kappa=\kappa+\omega\cdot q$, and $\gamma_\tau[\tilde\kappa]\equiv\gamma_\tau[\tilde\kappa](0)$ is supported in the interval $\eta^\tau\le|\tilde\kappa|\le\eta^{\tau-1}$. If we fix $|\kappa|\le\eta^t$ and $|\omega\cdot q|\le\eta^t$, then $|\tilde\kappa|\le 2\eta^t$, so, for all $\tau\le t-1$, we have $2\eta^t\le\eta^\tau\Rightarrow|\kappa|\le\eta^\tau\Rightarrow\gamma_\tau[\tilde\kappa]=0$. We hence proved that the integral in (10.33) can be taken over the interval $(t-1,t)$. However, the latter remark is not enough, as we still get large numbers for small $\tilde\kappa$'s, due to the large size of $\gamma_t[\tilde\kappa]$; here the consequences of the Ward identities come into play: since $\sigma_\tau(\kappa)=\mathcal{O}(\kappa^2)$, the compensation we need follows from Lemma 14, where we showed that $\tilde\kappa^{2+i}\partial^i_{\tilde\kappa}\gamma_\tau[\kappa]$ stays uniformly bounded! (see also Fig. 3 at p. 70)

Let us now fill in the details: in order to estimate (10.33), we recall that since $u\in\mathcal{H}$ we have $\sigma_t(\tilde\kappa)=\mathcal{O}(\tilde\kappa^2)$, with the trivial bounds

$$|\sigma_t(\tilde\kappa)|=\left|\int_0^{\tilde\kappa}\left(\int_0^{\kappa'}\partial^2_\kappa\sigma_t(\bar\kappa)d\bar\kappa\right)d\kappa'\right|\le|\tilde\kappa|^2\sup_{0\le|\bar\kappa|\le|\tilde\kappa|}|\partial^2_\kappa\sigma_t(\bar\kappa)|\le\varepsilon|\tilde\kappa|^2$$

$$|\partial_\kappa\sigma_t(\tilde\kappa)|=\left|\int_0^{\tilde\kappa}\partial^2_\kappa\sigma_t(\bar\kappa)d\bar\kappa\right|\le|\tilde\kappa|\sup_{0\le|\bar\kappa|\le|\tilde\kappa|}|\partial^2_\kappa\sigma_t(\bar\kappa)|\le\varepsilon|\tilde\kappa|, \tag{10.34}$$

In view of (10.34) (that holds for σ' as well thanks to Lemma 23), we shall need to get an estimate for $\partial^2_\kappa\sigma'$ only, as the bounds for $\partial_\kappa\sigma'$ and σ' will easily follow. Following the discussion at page 97 and using (9.13) we have, for $\tilde\kappa\le\eta^t$,

$$
\begin{aligned}
&\left|\partial_\kappa^2 \sigma_t'(\tilde\kappa) - \partial_\kappa^2 \mathcal{R}_t(\tilde\kappa)\right| \\
&\le \partial_\kappa^2 \pi_{T_0}(\tilde\kappa; y; 0,0)|_{y=0} + \int_{T_0}^t |\partial_\kappa \sigma_\tau(\tilde\kappa)|^2 \gamma_\tau[\tilde\kappa] + 2|\sigma_\tau(\tilde\kappa)||\partial_\kappa^2 \sigma_\tau(\tilde\kappa)|\gamma_\tau[\tilde\kappa] \\
&\quad + 2|\sigma_\tau(\tilde\kappa)||\partial_\kappa \sigma_\tau(\tilde\kappa)||\partial_\kappa \gamma_\tau[\tilde\kappa]| + |\sigma_\tau(\tilde\kappa)|^2 |\partial_\kappa^2 \gamma_\tau[\tilde\kappa]| d\tau \\
&\le \frac{\varepsilon}{96} + \varepsilon^2 \left(\int_{t-1}^t \tilde\kappa^2 \gamma_\tau[\tilde\kappa] d\tau + \int_{t-1}^t \tilde\kappa^3 \partial_\kappa \gamma_\tau[\tilde\kappa] d\tau + \int_{t-1}^t \tilde\kappa^4 \partial_\kappa^2 \gamma_\tau[\tilde\kappa] d\tau \right) \\
&\le \frac{\varepsilon}{96} + \frac{3}{2}\varepsilon^2 C \\
&\le \frac{\varepsilon}{48},
\end{aligned} \tag{10.35}
$$

where we used the important bound of Lemma 14:

$$
\left|\kappa^{2+i} \partial_\kappa^i \gamma_t[\kappa]\right| \le C, \tag{10.36}
$$

for all κ and t. The estimate of the rest in (7.23) is conceptually easier but more tedious:

Lemma 24. *The operator $\mathcal{R}_t(\kappa)$ defined in* (10.31) *obeys the bound*

$$
\sup_{|\kappa| \le \eta^t} \|\partial_\kappa^2 \mathcal{R}_t(\kappa)\|_{t;-t} \le \frac{\varepsilon}{48}. \tag{10.37}
$$

Proof. Writing as usual $\tilde\kappa = \kappa + \omega \cdot q_1$, we shall estimate the norm of the second derivative w.r.t. κ of the linear operator with kernel

$$
\begin{aligned}
\mathcal{R}_t(\kappa; q) \equiv \mathcal{R}_t(\tilde\kappa) = &\int_{T_0}^t \sum_{q'} D\sigma_\tau(\tilde\kappa; q')\gamma_\tau(q') u_\tau(0; q') \\
&+ \sum_{q' \ge 0} \rho_\tau(\tilde\kappa; 0, q')\gamma_\tau[\tilde\kappa](q')\rho_\tau(\tilde\kappa; q', 0) d\tau.
\end{aligned} \tag{10.38}
$$

Taking the second derivative w.r.t. κ of the first term we get

$$\begin{aligned}
&\sup_{|\tilde{\kappa}|\leq 2\eta^t}\left|\int_{T_0}^{t}\sum_{q'}\partial_\kappa^2 D\sigma_\tau(\tilde{\kappa};0,q')\gamma_\tau(q')u_\tau(0;q')\right| \\
&\leq \varepsilon^2 \sup_{|\tilde{\kappa}|\leq 2\eta^t}\int_{\mathbb{T}_0}^{t} r^{5/4\tau}\eta^{-2\tau}d\tau \\
&\leq \varepsilon^2\int_{\mathbb{T}_0}^{t}\eta^{3\tau}d\tau \\
&\leq \frac{1}{96}\varepsilon
\end{aligned} \tag{10.39}$$

for $r \ll \eta^4$ and $\varepsilon \ll 1$.

We can take the second derivative w.r.t. κ of the second term in (10.38):

$$\begin{aligned}
&\sup_{|\tilde{\kappa}|\leq 2\eta^t}\left|\partial_\kappa^2\int_{T_0}^{t}\sum_{q'}\rho_\tau(\tilde{\kappa};0,q')\gamma_\tau[\tilde{\kappa}](q')\rho_\tau(\tilde{\kappa};q',0)d\tau\right| \\
&\leq \sup_{|\tilde{\kappa}|\leq 2\eta^t}\int_{\mathbb{T}_0}^{t}\sum_{q}\Big|2\partial_\kappa\rho_\tau(\tilde{\kappa};0,q')\partial_\kappa\gamma_\tau[\tilde{\kappa}](q')\rho_\tau(\tilde{\kappa};q',0) \\
&2\partial_\kappa\rho_\tau(\tilde{\kappa};0,q')\gamma_\tau[\tilde{\kappa}](q')\partial_\kappa\rho_\tau(\tilde{\kappa};q',0)+2\rho_\tau(\tilde{\kappa};0,q')\partial_\kappa\gamma_\tau[\tilde{\kappa}](q')\partial_\kappa\rho_\tau(\tilde{\kappa};q',0) \\
&\rho_\tau(\tilde{\kappa};0,q')\gamma_\tau[\tilde{\kappa}](q')\partial_\kappa^2\rho_\tau(\tilde{\kappa};q',0)+\rho_\tau(\tilde{\kappa};0,q')\partial_\kappa^2\gamma_\tau[\tilde{\kappa}](q')\rho_\tau(\tilde{\kappa};q',0) \\
&+\partial_\kappa^2\rho_\tau(\tilde{\kappa};0,q')\gamma_\tau[\tilde{\kappa}](q')\rho_\tau(\tilde{\kappa};q',0)\Big|d\tau \\
&\leq \int_{T_0}^{t}C\varepsilon^2\left(r^{\frac{5}{6}\tau}\eta^{-3\tau}+r^{\frac{2}{3}\tau}\eta^{-2\tau}+r^{\frac{5}{6}\tau}\eta^{-3\tau}+r^{\frac{3}{4}\tau}\eta^{-2\tau}+r^{\tau}\eta^{-4\tau}+r^{\frac{3}{4}\tau}\eta^{-2\tau}\right)d\tau \\
&\leq 4C\varepsilon^2\int_{\mathbb{T}_0}^{\infty}\eta^{\tau}d\tau \\
&\leq \frac{1}{96}\varepsilon,
\end{aligned} \tag{10.40}$$

for $r \ll \eta^5$ and $\varepsilon \ll 1$. Putting toghether (10.39) and (10.40) gives us (10.37) and finishes the proof of the lemma. □

For $\partial_\kappa \sigma'$ and σ' we simply use Lemma 23, more exactly $\sigma' = \mathcal{O}(\kappa^2)$, and we get for $\tilde\kappa = \kappa + \omega \cdot q$

$$\begin{aligned}&\sup_{|\kappa|\le\eta^t} \|\partial_\kappa \sigma'_t(\kappa; q)\|_{-t} \\ &= \sup_{|\tilde\kappa|\le 2\eta^t} |\partial_\kappa \sigma'_t(\tilde\kappa)| \le \sup_{\tilde\kappa\le 2\eta^t} |\tilde\kappa||\partial^2_\kappa \sigma'(\tilde\kappa)| \le \frac{\varepsilon}{12}\eta^t \end{aligned} \tag{10.41}$$

and

$$\begin{aligned}&\sup_{|\kappa|\le\eta^t} \|\partial^2_\kappa \sigma'_t(\kappa; q)\|_{-t} \\ &= \sup_{|\tilde\kappa|\le 2\eta^t} |\sigma'_t(\tilde\kappa)| \le \sup_{\tilde\kappa\le 2\eta^t} |\tilde\kappa|^2 \left|\partial^2_\kappa \sigma'(\tilde\kappa)\right| \le \frac{\varepsilon}{6}\eta^{2(t-1)}. \end{aligned} \tag{10.42}$$

The estimates (10.35), (10.37), (10.41) and (10.42) establish for $|\kappa| \le \eta^t$

$$\sup_{i=0,1,2} 2^i \eta^{(i-2)t} \|\partial^i_\kappa \sigma'(\kappa)\|_{t;-t} \le \frac{\varepsilon}{6}. \tag{10.43}$$

We can now prove a bound for ρ'. The kernel of the operator $\rho'_t(\kappa)$ for $q \neq q'$ reads

$$\begin{aligned}\rho'_{t\beta}(\kappa; q, q') = \pi_{T_0}(\kappa; y, q, q')|_{y=0} + &\int_{T_0}^t \sum_{q''} D\Pi_{\tau\beta}(\kappa; q, q', q'')\gamma_\tau(q'')u_{\tau\beta}(0; q'') \\ &+ \sum_{q''} \Pi_{\tau\beta}(\kappa; 0; q, q'')\gamma_{\tau\beta}[\kappa](q'')\Pi_{\tau\beta}(\kappa; 0; q'', q')d\tau. \end{aligned} \tag{10.44}$$

We shall estimate ∂^i_κ of (10.44) term by term, for $i = 0, 1, 2$. In order to obtain a bound for the first term we simply use (9.13) and shrink again the strip of analyticity by shifting β: we fix $\beta \in \mathbb{C}^N$ such that $|\mathrm{Im}\beta| \le \alpha_t$ and then we set $\beta' = \beta - i\frac{(\alpha_0-\alpha_t)}{|q-q'|}(q - q')$, so that $|\mathrm{Im}\beta'| \le \alpha_0$, and for $i = 0, 1, 2$

we get

$$\begin{aligned}
&|\partial_\kappa^i \pi_{T_0\beta}(\kappa;0;q,q')|e^{(\alpha_0-\alpha_t)|q-q'|} = |\partial_\kappa^i \pi_{T_0\beta}(\kappa;0;q,q')|e^{-(\mathrm{Im}\beta'-\mathrm{Im}\beta)\cdot(q-q')}\\
&= |\partial_\kappa^i \pi_{T_0\beta'}(\kappa;0;q,q')|\\
&\le \|\partial_\kappa^i \pi_{T_0\beta'}(\kappa;0)\|_{\mathcal{L}(h;h)}\\
&\le \frac{\varepsilon}{18}, \qquad (10.45)
\end{aligned}$$

where we used (21). Now we can use the diophantine condition as in (10.23) to get $|q-q'| \ge b\eta^{-\frac{t}{\nu}}$, thus it follows that

$$\begin{aligned}
&\|P\partial_\kappa^i \pi_{T_0\beta}(\kappa;0)P\|_{t;-t} = \sup_{|\omega\cdot q|\le\eta^t} \sum_{\substack{|\omega\cdot q'|\le\eta^t\\ q\ne q'}} |\partial_\kappa^i \pi_{T_0\beta'}(\kappa;0;q,q')|\\
&\le \frac{\varepsilon}{18} \sup_{\substack{|\omega\cdot q|,|\omega\cdot q'|\le\eta^t\\ q\ne q'}} e^{-(\alpha_0-\alpha_t)|q-q'|}\\
&\le \frac{\varepsilon}{18} e^{-(\alpha_0-\alpha_t)b\eta^{-\frac{t}{\nu}}}\\
&\le \frac{\varepsilon}{18} r^{t/2}. \qquad (10.46)
\end{aligned}$$

For the second term operator in (10.44) we fix $\beta \in \mathbb{C}^N$ such that $|\mathrm{Im}\beta| \le \alpha_t$ and then we set $\beta_\tau = \beta - i\frac{(\alpha_\tau-\alpha_t)}{|q-q'|}(q-q')$, so that $|\mathrm{Im}\beta_\tau| \le \alpha_\tau$, and for $i = 0, 1, 2$ and $q \ne q'$ we get

$$\begin{aligned}
&\left|\sum_{q''} \partial_\kappa^i D\Pi_{\tau\beta}(\kappa;q,q',q'')\gamma_\tau(q'')u_{\tau\beta}(0;q'')\right| e^{(\alpha_\tau-\alpha_t)|q-q'|}\\
&\le \sum_{q''} \left|\partial_\kappa^i D\Pi_{\tau\beta_\tau}(\kappa;q,q',q'')\gamma_\tau(q'')u_{\tau\beta_\tau}(0;q'')\right|\\
&\le \|u(0)\|_{-\tau}\|\gamma_\tau\| \sup_{q''} \left|\partial_\kappa^i D\Pi_{\tau\beta_\tau}(\kappa;q,q',q'')\right|\\
&\le \varepsilon\eta^{-2\tau} r^{2\tau} \sup_{q''} \left|\partial_\kappa^i D\Pi_{\tau\beta_\tau}(\kappa;q,q',q'')\right|\\
&\le \varepsilon r^{3/2\tau} \sup_{q''} \left|\partial_\kappa^i D\Pi_{\tau\beta_\tau}(\kappa;q,q',q'')\right|. \qquad (10.47)
\end{aligned}$$

This implies

$$\left|\sum_{q''} \partial_\kappa^i D\Pi_{\tau\beta}(\kappa;q,q',q'')\gamma_\tau(q'')u_{\tau\beta}(0;q'')\right| \\ \leq \varepsilon r^{3/2\tau} \sup_{q''} \left|\partial_\kappa^i D\Pi_{\tau\beta_\tau}(\kappa;q,q',q'')\right| e^{-(\alpha_\tau-\alpha_t)|q-q'|} \tag{10.48}$$

so, in the same fashion as in (10.27), we conclude, for $|\mathrm{Im}\beta| \leq \alpha_t$,

$$\sup_{|\omega\cdot q|\leq\eta^t} \sum_{|\omega\cdot q'|\leq\eta^t} \left|\int_{T_0}^t \sum_q \partial_\kappa^i D\Pi_{\tau\beta}(\kappa;q,q',q'')\gamma_\tau(q'')u_{\tau\beta}(0;q'')\right| \leq \frac{\varepsilon}{18} r^{\frac{t}{i+1}}. \tag{10.49}$$

In fact the constraints $|\omega\cdot q|$, $|\omega\cdot q'| \leq \eta^t$ imply $|\omega\cdot(q-q')| \leq 2\eta^t$, so the diophantine condition (1.4) forces the bound $|q-q'| \geq C\eta^{-t/\nu}$, and we use (10.48) to conclude

$$\begin{aligned} &\sum_{|\omega\cdot q|\leq\eta^t} \sup_{|\omega\cdot q'|\leq\eta^t} \left|\int_{T_0}^t \sum_{q''} \partial_\kappa^i D\Pi_{\tau\beta}(\kappa;q,q',q'')\gamma_\tau(q'')u_{\tau\beta}(0;q'')\right| \\ &\leq \int_{T_0}^t \varepsilon \sum_{|\omega\cdot q|\leq\eta^t} \sup_{|\omega\cdot q'|\leq\eta^t} \sup_{|\omega\cdot q''|\leq\eta^t} \left|\partial_\kappa^i D\Pi_{\tau\beta}(\kappa;q,q',q'')\right| r^{3/2\tau} e^{-\frac{t-\tau}{2t\tau}C\eta^{-t/\nu}} d\tau \\ &\leq C\varepsilon^2 \int_{T_0}^t e^{-\frac{t-\tau}{2t^2}C\eta^{-t/\nu}} r^{(3/2-i/1+i)\tau} d\tau \\ &\leq \frac{1}{18}\varepsilon r^{(3/2+\frac{i}{i+1})t} \\ &\leq \frac{1}{18}\varepsilon r^{\frac{t}{i+1}}, \end{aligned} \tag{10.50}$$

where the estimate of the integral is obtained in the same way as in (10.25).

Finally we estimate the third term in (10.44): for $\tau \leq t$ we shift β to β_τ in the usual way, and in view of the bounds (10.36) we obtain, for $|\kappa| \leq \eta^t$, $q \neq q'$ and $i = 0$,

$$\left|\sum_{q''} \Pi_{\tau\beta}(\kappa;0;q,q'')\gamma_\tau[\kappa](q'')\Pi_{\tau\beta}(\kappa;0;q'',q')\right| e^{(\alpha_\tau-\alpha_t)|q-q'|} \\ \leq \eta^{-2\tau} \sum_{|\omega\cdot q''|\leq\eta^\tau} \Pi_{\tau\beta_\tau}(\kappa;0;q,q'')\Pi_{\tau\beta_\tau}(\kappa;0;q'',q'), \tag{10.51}$$

which implies

$$\begin{aligned}
&\sup_{|\omega\cdot q|\le\eta^t}\sum_{|\omega\cdot q'|\le\eta^t}\left|\int_{T_0}^t\sum_{q''}\Pi_{\tau\beta}(\kappa;0;q,q'')\gamma_\tau(q'')\Pi_{\tau\beta}(\kappa;0;q'',q')d\tau\right|\\
&\le\int_{T_0}^t\eta^{-2\tau}\sup_{|\omega\cdot q|\le\eta^t}\sum_{\substack{|\omega\cdot q'|\le\eta^t\\|\omega\cdot q''|\le\eta^\tau}}\left|\Pi_{\tau\beta_\tau}(\kappa;0;q,q'')\Pi_{\tau\beta_\tau}(\kappa;0;q'',q')\right|e^{-(\alpha_\tau-\alpha_t)|q-q'|}d\tau\\
&\le\varepsilon\int_{T_0}^t\eta^{2\tau}\sup_{|q-q'|\ge C\eta^{-\tau/\nu}}e^{-(\alpha_\tau-\alpha_t)|q-q'|}\\
&\le\varepsilon\int_{T_0}^t\eta^{2\tau}e^{-\frac{t-\tau}{2t^2}C\eta^{-t/\nu}}\\
&\le\frac{\varepsilon}{18}r^t.
\end{aligned}\tag{10.52}$$

Furthermore we get

$$\begin{aligned}
&\left|\partial_\kappa\left(\sum_{q''}\Pi_{\tau\beta}(\kappa;q,q'')\gamma_\tau[\kappa](q'')\Pi_{\tau\beta}(\kappa;q'',q')\right)\right|e^{(\alpha_\tau-\alpha_t)|q-q'|}\\
&\le\sum_{|\omega\cdot q''|\le\eta^\tau}\Bigg(\eta^{-2\tau}\partial_\kappa\Pi_{\tau\beta}(\kappa;q,q'')\Pi_{\tau\beta}(\kappa;q'',q')+\eta^{-3\tau}\Pi_{\tau\beta}(\kappa;q,q'')\Pi_{\tau\beta}(\kappa;q'',q')\\
&\quad+\eta^{-2\tau}\Pi_{\tau\beta}(\kappa;q,q'')\partial_\kappa\Pi_{\tau\beta}(\kappa;q'',q')\Bigg)
\end{aligned}\tag{10.53}$$

which, as in (10.52), implies

$$\begin{aligned}
&\sup_{|\omega\cdot q|\le\eta^t}\sum_{|\omega\cdot q'|\le\eta^t}\left|\int_{T_0}^t\partial_\kappa\left(\sum_{q''}\Pi_{\tau\beta}(\kappa;q,q'')\gamma_\tau(q'')\Pi_{\tau\beta}(\kappa;q'',q')\right)\right|\\
&\le\frac{\varepsilon}{18}r^{\frac{t}{2}}.
\end{aligned}\tag{10.54}$$

In the same way one derives the estimate for the second derivative w.r.t. κ:

$$\begin{aligned}
&\sup_{|\omega\cdot q|\le\eta^t}\sum_{|\omega\cdot q'|\le\eta^t}\left|\int_{T_0}^t\partial_\kappa^2\left(\sum_{q''}\Pi_{\tau\beta}(\kappa;q,q'')\gamma_\tau(q'')\Pi_{\tau\beta}(\kappa;q'',q')\right)\right|\\
&\le\frac{\varepsilon}{18}r^{\frac{t}{3}}.
\end{aligned}\tag{10.55}$$

Combining (10.46), (10.49), (10.52), (10.54) and (10.55) we get the bound

$$\sup_{i=0,1,2} r^{-\frac{1}{2+i}} \|\partial_\kappa^i \rho' \kappa\|_{t;-t}. \tag{10.56}$$

2.4. Conclusion. Summarizing the results obtained so far, we have that if $u \in \mathcal{B}$ then $\Phi(u) \in \mathcal{H}$; furthermore, combining (10.28),(10.43),(10.56) and(10.19), we get

$$\|\Phi(u)\|_{\mathcal{H}} \leq \varepsilon. \tag{10.57}$$

Hence under the action of Φ the ball of radius ε in $\mathcal{H}$ is preserved, which is what we had to prove.

□

3. Φ is a contraction in $\mathcal{B}$

We are left with showing that Φ is a contraction on $\mathcal{B}$. We shall prove the following

Proposition 7. *There exists $0 < \mu < 1$ such that, for all $u, v \in \mathcal{B}$, we have*

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{H}} \leq \mu \|u - v\|_{\mathcal{H}}. \tag{10.58}$$

Proof. Let us first show that for all $u, v \in \mathcal{B}$ we have

$$\sup_{\substack{|\mathrm{Im}\beta| \leq \alpha_t \\ t \geq T_0}} r^{-2t} \|\Phi(u)_{\beta t}(0) - \Phi(u)_{\beta t}(0)\|_{-t} \leq \varepsilon \|u - v\|_{\mathcal{H}}. \tag{10.59}$$

We get (see estimate (10.24) for the definition of β_τ)

$$
\begin{aligned}
&\|\Phi(u)_{\beta t}(0)-\Phi(u)_{\beta t}(0)\|_{-t}\\
&=\|\int_{T_0}^{t} Du_{\tau\beta}(0)\gamma_\tau u_{\tau\beta}(0)-Dv_{\tau\beta}(0)\gamma_\tau v_{\tau\beta}(0)d\tau\|_{-t}\\
&\le \sum_{|\omega\cdot q|\le\eta^t}\int_{T_0}^{t}\sum_{q'}|Du_{\tau\beta}(0;q,q')|\,|\gamma_\tau(q')|\,|(u_{\tau\beta_\tau}-v_{\tau\beta_\tau})(0;q')|\,e^{-(\alpha_\tau-\alpha_t)|q|}d\tau\\
&\quad+\sum_{|\omega\cdot q|\le\eta^t}\int_{T_0}^{t}\sum_{q'}|v_{\tau\beta}(0;q')|\,|\gamma_\tau(q')|\,|(Du_{\tau\beta_\tau}-Dv_{\tau\beta_\tau})(0;q',q)|\,e^{-(\alpha_\tau-\alpha_t)|q|}d\tau\\
&\le\int_{T_0}^{t}\big(\|(u_{\tau\beta_\tau}-v_{\tau\beta_\tau})(0)\|_{-\tau}+\eta^{-2\tau}r^{2\tau}\|(Du_{\tau\beta_\tau}-Dv_{\tau\beta_\tau})(0)\|_\tau\big)\sup_{|\omega\cdot q|\le\eta^t}e^{-(\alpha_\tau-\alpha_t)|q|}d\tau\\
&\le\varepsilon\sup_{\substack{|\mathrm{Im}\beta|\le\alpha_t\\ t\ge T_0}}\big(r^{-2t}\|u_{t\beta}(0)-v_{t\beta_t}(0)\|_{-t}+\eta^{-2t}\|(Du_{t\beta}-Dv_{t\beta_t})(0)\|_{-t}\big)\cdot\\
&\cdot\int_{T_0}^{t}r^{2\tau}e^{-(\alpha_\tau-\alpha_t)C\eta^{-t/\nu}}d\tau\\
&\overset{(*)}{\le}\varepsilon r^{2t}\|u-v\|_{\mathcal H},
\end{aligned}
\tag{10.60}
$$

where the steps leading to the estimate (*) have been carried out in (10.25). Trivially (10.60) implies

$$\sup_{\substack{|\mathrm{Im}\beta|\le\alpha_t\\ t\ge T_0}} r^{-2t}\|\Phi(u)_{\beta t}(0)-\Phi(u)_{\beta t}(0)\|_{-t}\le\varepsilon\|u-v\|_{\mathcal H}. \tag{10.61}$$

Next we estimate the other term of the norm by setting

$$D^i\Pi^u(\omega\cdot p):=t_pD^{i+1}\Phi(u)(y)|_{y=0}\quad\text{and}\quad D^i\Pi^v(\omega\cdot p):=t_pD^{i+1}\Phi(v)(y)|_{y=0}$$

and we interpolate in the usual fashion to define $\Pi^u(\kappa)$ and $\Pi^v(\kappa)$ on the whole real line. We have

$$
\begin{aligned}
&(\Pi^u_{t\beta})'(\kappa) - (\Pi^v_{t\beta})'(\kappa) \\
&= \int_{T_0}^t D\Pi^u_{\tau\beta}(\kappa)\gamma_\tau u_{\tau\beta}(0) - D\Pi^v_{\tau\beta}(\kappa)\gamma_\tau v_{\tau\beta}(0) d\tau \\
&\quad + \int_{T_0}^t \Pi^u_{\tau\beta}(\kappa)\gamma_\tau[\kappa]\Pi^u_{\tau\beta}(\kappa) - \Pi^v_{\tau\beta}(\kappa)\gamma_\tau[\kappa]\Pi^v_{\tau\beta}(\kappa) d\tau \\
&= \int_{T_0}^t \sum_{q''} D\Pi^u_{\tau\beta}(\kappa; q, q', q'')\gamma_\tau(q'')\left(u_{\tau\beta}(0; q'') - v_{\tau\beta}(0; q'')\right) d\tau \\
&+ \int_{T_0}^t \sum_{q''} \left(D\Pi^u_{\tau\beta}(\kappa; q, q', q'') - D\Pi^v_{\tau\beta}(\kappa; q, q', q'')\right)\gamma_\tau(q'')v_{\tau\beta}(0; q'')d\tau \\
&+ \int_{T_0}^t \sum_{q''} \left((\Pi^u_{\tau\beta} + \Pi^v_{\tau\beta})(\kappa; q, q'')\right)\gamma_\tau[\kappa](q'')\left((\Pi^u_{\tau\beta} - \Pi^v_{\tau\beta})(\kappa; q'', q')\right) d\tau.
\end{aligned}
\tag{10.62}
$$

From the latter we can easily write the expression for the norm of its diagonal part $\|(\sigma^u_{t\beta})'(\kappa) - (\sigma^v_{t\beta})'(\kappa)\|_{-t}$, and of its off-diagonal part $\|(\rho^u_{t\beta})'(\kappa) - (\rho^v_{t\beta})'(\kappa)\|_{-t}$. Let us consider the diagonal (and the only "significant") part. If we can prove that for $t \geq T_0$

$$
\sup_{|\kappa| \leq \eta^t} \|\partial^2_\kappa(\sigma^u_{t\beta})'(\kappa; q) - \partial^2_\kappa(\sigma^v_{t\beta})'(\kappa; q)\|_{-t} \leq \frac{\varepsilon}{4}\|u - v\|_{\mathcal{H}} \tag{10.63}
$$

then using $(\sigma^u_{t\beta})'(\kappa; 0) = \mathcal{O}(\kappa^2) = (\sigma^v_{t\beta})'(\kappa; 0)$, we get, for $|\kappa| \leq \eta^t$, $t \geq T_0$ and $i = 0, 1, 2$ (see the analogous discussion at Pag. 98)

$$
\begin{aligned}
&\|\partial^i_\kappa(\sigma^u_{t\beta})'(\kappa; q) - \partial^i_\kappa(\sigma^v_{t\beta})'(\kappa; q)\|_{t;-t} = \sup_{|\omega\cdot q| \leq \eta^t}\left|(\partial^i_\kappa(\sigma^u_{t\beta})' - \partial^i_\kappa(\sigma^v_{t\beta})')(\kappa + \omega\cdot q; 0)\right| \\
&\leq \sup_{|\omega\cdot q|, |\kappa| \leq \eta^t} |\kappa + \omega\cdot q|^{2-i}\left|\partial^2_\kappa(\sigma^u_{t\beta})'(\kappa; q) - \partial^2_\kappa(\sigma^v_{t\beta})'(\kappa; q)\right| \leq \varepsilon\eta^{(2-i)t}\|u - v\|_{\mathcal{H}}.
\end{aligned}
\tag{10.64}
$$

In order to show that (10.63) holds, we shall only sketch a part of the proof and leave the rest to the interested (or skeptical) reader, since it involves methods we already used extensively. Namely we shall only consider the diagonal part of the third term in (10.62), by writing it as:

$$\int_{T_0}^{t}\sum_{q''\neq q}\left(\Pi^u_{\tau\beta}(\kappa;q,q'')+\Pi^v_{\tau\beta}(\kappa;q,q'')\right|\gamma_\tau[\kappa](q'')\left(\Pi^u_{\tau\beta}(\kappa;q'',q)-\Pi^v_{\tau\beta}(\kappa;q'',q)\right)d\tau$$
$$+\int_{T_0}^{t}\left(\sigma^u_{\tau\beta}(\kappa;q)+\sigma^v_{\tau\beta}(\kappa;q)\right)\gamma_\tau[\kappa](q)\left(\sigma^u_{\tau\beta}(\kappa;q)-\sigma^v_{\tau\beta}(\kappa;q)\right)d\tau. \tag{10.65}$$

As an example on how to proceed (and to please the skeptical reader mentioned above), we shall estimate the norm of the second derivative w.r.t. κ of the last term in (10.65), which is (as we have already pointed out earlier) the only "interesting" part:

$$\sup_{|\omega\cdot q|\le\eta^t}\left|\partial^2_\kappa\int_{T_0}^{t}\left(\sigma^u_{\tau\beta}(\kappa;q)+\sigma^v_{\tau\beta}(\kappa;q)\right)\gamma_\tau[\kappa](q)\left(\sigma^u_{\tau\beta}(\kappa;q)-\sigma^v_{\tau\beta}(\kappa;q)\right)d\tau\right|$$
$$=\sup_{|\omega\cdot q|\le\eta^t}\left|\partial^2_\kappa\int_{T_0}^{t}\left(\sigma^u_{\tau\beta}(\tilde\kappa)+\sigma^v_{\tau\beta}(\tilde\kappa)\right)\gamma_\tau[\tilde\kappa]\left(\sigma^u_{\tau\beta}(\tilde\kappa)-\sigma^v_{\tau\beta}(\tilde\kappa)\right)d\tau\right| \tag{10.66}$$

where $\tilde\kappa=\kappa+\omega\cdot q$, $\sigma^u_{\tau\beta}(\tilde\kappa)=\sigma^u_{\tau\beta}(\tilde\kappa;0)$ and $\gamma_\tau[\tilde\kappa]=\gamma_\tau[\tilde\kappa](0)$. Now we use the same observations as made on page 98, and estimate

$$\sup_{|\omega\cdot q|,|\kappa|\le\eta^t}\left|\int_{T_0}^{t}\sum_{i+\ell+j=2}\left(\partial^i_\kappa\sigma^u_{\tau\beta}(\tilde\kappa)+\partial^i_\kappa\sigma^v_{\tau\beta}(\tilde\kappa)\right)\partial^\ell_\kappa\gamma_\tau[\tilde\kappa]\left(\partial^j_\kappa\sigma^u_{\tau\beta}(\tilde\kappa)-\partial^j_\kappa\sigma^v_{\tau\beta}(\tilde\kappa)\right)d\tau\right|$$
$$\le\sum_{i+\ell+j=2}\Bigg(\sup_{\substack{|\mathrm{Im}\beta|\le\alpha_t\\ t\ge T_0}}\sup_{|\omega\cdot q|,|\kappa|\le\eta^t}\eta^{(j-2)t}\left|\partial^j_\kappa\sigma^u_{\tau\beta}(\tilde\kappa)-\partial^j_\kappa\sigma^v_{\tau\beta}(\tilde\kappa)\right|\Bigg)\cdot$$
$$\cdot\sup_{|\omega\cdot q|,|\kappa|\le\eta^t}\int_{t-1}^{t}\eta^{(2-j)\tau}\left(\partial^i_\kappa\sigma^u_{\tau\beta}(\tilde\kappa)+\partial^i_\kappa\sigma^v_{\tau\beta}(\tilde\kappa)\right)\partial^\ell_\kappa\gamma_\tau[\tilde\kappa]d\tau$$
$$\le\|u-v\|_{\mathcal H}\sum_{i+\ell+j=2}\sup_{|\omega\cdot q|,|\kappa|\le\eta^t}\int_{t-1}^{t}\eta^{(2-j)\tau}\tilde\kappa^{-i-\ell}\left|\partial^2_\kappa\sigma^u_{\tau\beta}(\tilde\kappa)+\partial^2_\kappa\sigma^v_{\tau\beta}(\tilde\kappa)\right|\tilde\kappa^{\ell+2}\partial^\ell\gamma_\tau[\tilde\kappa]d\tau$$
$$\overset{(*)}{\le}\|u-v\|_{\mathcal H}\sum_{i+\ell+j=2}C\varepsilon\int_{t-1}^{t}d\tau$$
$$\le C\varepsilon\|u-v\|_{\mathcal H} \tag{10.67}$$

where to get (*) we used Lemma 14.

By using the same methods we used several times throughout the paper (shifting of β, diophantine condition etc.), in the same fashion as in section 2 we can show that for $u\,,\ v\in\mathcal{B}$ we have

$$r^{-\frac{1}{2+i}}\|\rho_t^u(\kappa)-\rho_t^v(\kappa)\|_{t;-t}\le\varepsilon\|u-v\|_{\mathcal{H}} \tag{10.68}$$

$$\sup_{p\ge1} r^{(p-\frac{1}{1+i})t}\frac{\|\partial_\kappa^i D^p\Pi^u(0;\kappa)-\partial_\kappa^i D^p\Pi^v(0;\kappa)\|_{t;-t}}{(p-1)!}\le\varepsilon\|u-v\|_{\mathcal{H}}. \tag{10.69}$$

Putting together (10.61), (10.67), (10.68) and (10.69), and taking ε small enough we get

$$\|\Phi(u)-\Phi(v)\|_{\mathcal{H}}\le\mu\|u-v\|_{\mathcal{H}} \tag{10.70}$$

for $u\,,\ v\in\mathcal{B}$ and $0<|\mu(\varepsilon)|<1$. This concludes the proof of the theorem. □

We can now put together all the results and obtain the corollary:

Corollary 25. *The operator Φ defined in* (10.1) *has a fixed point in $\mathcal{B}$.*

Proof. Trivial: use Propositions 6 and 7, and the Banach Fixed Point Theorem. □

Chapter 11

Proof of the KAM theorem

Let us return to (7.25), where we defined the sequence

$$X_t(\theta) \equiv Z_t(0,\theta), \tag{11.1}$$

we will show that it converges to a real analytic function with zero average for $t \to \infty$, such that the limit will solve Eq. (7.18):

$$X = GW(X,\theta). \tag{11.2}$$

In the Fourier space we get

$$x_t(q) = \Gamma_{<t}(q) w_t(0;q); \tag{11.3}$$

to show that this converges in h for $t \to \infty$ we take its time derivative and show that it decays to zero:

$$\begin{aligned} \partial_t x_{t\beta} &= \gamma_t w_{t\beta}(0) + \Gamma_{<t} \partial_t w_{t\beta}(0) \\ &= \gamma_t w_{t\beta}(0) + \Gamma_{<t} D w_{t\beta}(0) \gamma_t w_{t\beta}(0), \end{aligned} \tag{11.4}$$

and writing this in terms of q's we have

$$\partial_t x_{t\beta}(q) = \gamma_t(q) w_{t\beta}(0;q) + \Gamma_{<t}(q) \sum_{q'} D w_{t\beta}(0;q,q') \gamma_t(q') w_{t\beta}(0;q'), \tag{11.5}$$

so that

$$\|\partial_t x_{t\beta}\|_h \le \eta^{-2t} \sum_{|\omega\cdot q|\le\eta^t} |w_{t\beta}(0;q)| + \eta^{-4t} \sup_{|\omega\cdot q'|\le\eta^t} \sum_{|\omega\cdot q|\le\eta^t} |Dw_{t\beta}(0;q,q')| \times \\ \times \sum_{|\omega\cdot q'|\le 2\eta^t} |w_{t\beta}(q')| \le \varepsilon\eta^{-2t}r^{2t} + \varepsilon^2\eta^{-2t}r^{2t} \stackrel{t\to\infty}{\longrightarrow} 0 \tag{11.6}$$

which proves that $\exists x \in h$ such that

$$x_{t\beta} \stackrel{t\to\infty}{\longrightarrow} x_\beta \tag{11.7}$$

and such that $\|x_\beta\|_h \le C\varepsilon$ uniformly in the strip

$$|\mathrm{Im}\beta| < \frac{1}{2}\alpha_0. \tag{11.8}$$

The latter estimate implies that, pointwise one has

$$|x(q)| < C\varepsilon e^{-1/2\alpha_0|q|}, \tag{11.9}$$

hence X, the Fourier transform of x is real analytic. Furthermore (11.3) implies

$$x_t(q)|_{q=0} = 0 \tag{11.10}$$

and, as we have

$$x_t = \Gamma_{<t} w_t(0) = \Gamma_{<t}\bar{w}(x_t), \tag{11.11}$$

the Ward identity (9.8) gives

$$\bar{w}(x_t;q)|_{q=0} = 0; \tag{11.12}$$

taking the limit for $t \longrightarrow \infty$ in (11.3) and (11.11) we get

$$x(0) = 0, \quad x(q) = G\bar{w}(x;q) \text{ for } q \ne 0, \quad \bar{w}(x;0) = 0; \tag{11.13}$$

the second of these equations is the Fourier transform of (7.18). This solution $X(\theta;\lambda)$ is analytic for $|\lambda| \le \lambda_0$ and vanishes for $\lambda = 0$. Its uniqueness, up to translations of the kind T_β (see (9.2)), follows from the fact that (7.18) completely determines the Taylor coefficients in powers of λ of its solutions. This concludes the proof of Theorem 13.

□

Bibliography

1. V.I. Arnol'd, *Proof of a theorem by A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian*, Russ. Math. Surv. **18** (1963), no. 5, 9–36.
2. ______, *Small denominators and problems of stability of motion in classical and celestial mechanics*, Russ. Math. Surv. **18** (1963), no. 6, 9–36, [Corrigenda (in Russian): *Uspekhi Math. Nauk.* **23** (1968) 216.].
3. ______, *Mathematical Methods of Classical Mechanics*, I ed., Springer, 1978.
4. ______, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1987.
5. J. Bricmont, K. Gawędzki, and A. Kupiainen, *KAM Theorem and Quantum Field Theory*, Mathematical Physics Preprint Archive (1999).
6. L. Chierchia, *Lezioni di Analisi matematica 2*, I ed., Aracne, September 1996.
7. L. Chierchia and C.Falcolini, *A Direct Proof of a Theorem by Kolmogorov in Hamiltonian Systems*, Mathematical Physics Preprint Archive (1993).
8. W. Ditrich and M.Reuter, *Classical and Quantum Dynamics*.
9. L. H. Eliasson, *Absolutely convergent series expansions for quasi periodic motions*, Reports Department of Math., Univ. of Stockholm, Sweden (1988), 1–31.
10. P. Holmes F. Diacu, *Celestial Encounters*, II ed., Princeton University Press, 1999.
11. J. Feldman and E. Trubowitz, *Renormalization in classical mechanics and many body quantum field theory*, 1992.

12. G.Gallavotti, *Twistless KAM tori.*, Comm. Math. Phys. **164** (1994), no. 1, 145–156.
13. ______, *Twistless KAM tori, quasi flat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review.*, Rev. Math. Phys. **6** (1994), no. 3, 343–411.
14. ______, *Lindstedt series and Kolmogorov theorem.*, Hamiltonian systems with three or more degrees of freedom (C. Simó, ed.), NATO Science Series C: Math.Phys.Sci., vol. 533, Springer, first ed., June 1999, pp. 62–71.
15. G.Gentile and V.Mastropietro, *Methods for the analysis of the lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications*, Rev. Math. Phys. **8** (1996), 393–444.
16. H. Hofer and Eduard Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Verlag, 1994.
17. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
18. A. N. Kolmogorov, *On the conservation of conditionally periodic motions for a small change in Hamilton's function*, Dokl. Akad. Nauk. SSSR **98** (1954), 525–530, in Russian; English translation in *LNP* **93** (1979) 51-56.
19. J. Laskar, *Large schale chaos and marginal stability in the solar system*, Proceedings XIIth Int. Congress of Math. Phys. (Paris 1994), Int. Press Cambridge, 1995.
20. J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen, Math-phys. **Kl. II** (1962), no. 1, 1–20.
21. ______, *Convergent series expansions for quasi-periodic motions*, Math. Ann. **169** (1967), 136–176.
22. H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, vol. 1-3, Paris: Gauthier-Villars, 1892/1893/1899.
23. J. Pöschel, *Über invariante tori in differenzierbaren Hamiltonschen Systemen*, Bonn. Math. Schr. **120** (1980), 1–103.
24. ______, *Integrability of Hamiltonian systems on Cantor sets*, Pure Appl. Math. **35** (1982), 653–695.
25. ______, *A Lecture on the Classical KAM Theorem*, Proc. Symp. Pure Math. **69** (2001), 707–732.

26. Dietmar A. Salamon, *The Kolmogorov-Arnold-Moser theorem*, Mathematical Physics Electronic Journal (2004).
27. W Thirring, *Classical Dynamical Systems*, Springer-Verlag, 1978.