ABSTRACT MODEL THEORY WITHOUT NEGATION

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Academic dissertation

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To my mother and the memory of my father.

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Abstract

We study abstract logics that are not necessarily closed under negation. A typical example is the logic of all PC-classes of first-order logic. We show that Lindström's Theorem can be formulated as a separation theorem is abstract model theory without negation. This leads us to study the connections between maximality, model-theoretic characterization and interpolation in the absence of negation. As a case study we investigate closely the family of extensions of first-order logic by unary monotone quantifiers but without negation.

Contents

1. Introduction

There are two different ways of thinking of logics without negation. Firstly, as an extension of a logic with negation, like the logic of PC classes, that extends $\mathcal{L}_{\omega\omega}$, which has negation. And secondly, as a fragment of a logic with negation, as the positive part of $\mathcal{L}_{\omega\omega}$ is a fragment of $\mathcal{L}_{\omega\omega}$. This means that logics without negation can be approximated by logics with negation from inside and from outside. Hence, the extension of the study of classical logics by the study of logics without negation is twofold. In this thesis we explore how several concepts and results in abstract model theory translate when we drop negation out. These model theoretic results for logics without negation are in a way extensions of such results with negation.

There exist many examples that justify a general study of logics without negation. In mathematical logic those are related to restricted quantifier fragments of, for instance, $\mathcal{L}_{\omega\omega}, \mathcal{L}_{\omega\infty}$ or $\mathcal{L}^2_{\omega\omega}$, such as the logic of existential sentences; existential universal FI logic, [PV05]; transfinite game formulas, [Hy90]; existential second order IF logic, [Hi96]; or the logic of positive bounded formulas for Banach structures, [Io1]. There are more varied examples in philosophical logic or logics formalizing phenomena in natural language.

Throughout the effort of outlining a first detailed prospect of the model theory of an abstract logic without negation, we explain the main obstacles, troubles and lacks we had to do with in order to deal with the logic. In the beginning of the study of extensions of first-order logic, the interest in developing such extensions was grounded on the need of studying various mathematical concepts, not definable in first-order logic, that appear especially in some new fields in mathematics -such as being a countable, a well ordered, or a measurable set. But while these extensions have a richer expressive power, it turned out that they lacked interesting properties present in first-order logic. In 1969, Lindström [L69] proved a very important maximality theorem: *'First-order logic is a maximal logic satisfying Compactness and Löwenheim-Skolem theorems'*. That is, any logic expressing more things than first-order logic, will loose at least one of those two model theoretic properties. This phenomenon is interesting by itself; it makes the study of model theoretic languages depart from its original aim, and gives rise to abstract model theory, a new field in model theory that will study these languages concentrating on its model theoretic properties. In this field we are not interested anymore in designing a particular language being able to describe a model as having this or that property. Instead, we are interested in constructing logics with interesting properties, such as satisfying compactness, Löwenheim-Skolem property, Craig's Interpolation Theorem, etc. We are happier if we find a logic to be maximal with respect to these properties. We are even satisfied with just proving the existence of such logics, without ever glancing at how they look like.

We study maximality theorems in logics without negation in the context of their relation to interpolation theorems. A close examination of the ingredients of Lindström's Theorem reveals that when the proof is broken into its parts, a proof of Craig Interpolation Theorem emerges. The framework for this study is hinted at by

Flum, although not fully exploited, in [Fl85]. Caicedo [Ca79], [Ca81], [Ca83] presented several results for extensions of first-order logic with generalized quantifiers that fit within this framework. As he reckons, under very weak assumptions any such extension can be expressed in the form $\mathcal{L}_{\omega\omega}(\overline{Q})$ (where $\mathcal{L}_{\omega\omega}$ denotes first-order logic), for $\overline{Q} = \{Q^i : i \in I\}$ any set of quantifiers. This is also the case for all logics with interpolation. All these logics can be provided with a back-and-forth system [Ca80], so we can find a back-and-forth system for any logic worth to explore[1]. Likewise, all proofs in the above papers concerning interpolation made heavy use of this feature. We therefore use back-and-forth systems as the frame for our investigation in interpolation.

The conclusions of the study of the relation between interpolation and maximality are, in the first place, that for this relation to exist, one should, as suggested by Barwise and van Benthem in [BvB99], understand interpolation theorem for a logic $\mathcal{L}$ with some invariance R as: "If $\mathbf{K}_1, \mathbf{K}_2$ are disjoint classes belonging to (related by R in a way made explicit later) $PC(\mathcal{L})$, then they can be separated by an elementary $\mathcal{L}$-class." -one recovers the usual interpolation when one considers all $PC(\mathcal{L})$-classes[2]. In the second place, that the proof of interpolation by back-and-forth arguments is only possible when the logic is maximal with respect to some model theoretic properties, and when the invariance under R is among these characterizing properties. These conclusions break down in the case the logic is not closed under negation. Basically what happens is that maximality and interpolation theorems can be stated as corollaries for a so called *separation theorem*, and the proofs of these corollaries have different sensibilities to the lack of negation.

1.1. **Structure of the thesis.** In Section 3 we present the study of Lindström's theorem for logics without negation, for fragments as well as for extensions of $\mathcal{L}_{\omega\omega}$.

Section 4 is a presentation of the relation between interpolation and maximality theorems in a general framework with and without negation.

Since Section 4 links the proof of interpolation to maximality, in Section 5 we analyze what individual model theoretic properties give rise to orderings of logics with maximal points. We also investigate how this maximality translates in the case we do not have negation. No proof of maximality for individual properties, with or without negation, uses back-and-forth methods, and although all these logics have

[1]Although it is true that any extension of first-order logic can be written in the form $\mathcal{L}_{\omega\omega}(\overline{Q})$, these extensions are generally divided into two essential kinds: Infinitary logics, $\mathcal{L}_{\kappa\lambda}$ with conjunctions or disjunctions of length $< \kappa$, and strings of existential or universal quantifiers of size less than λ; and extensions by generalized quantifiers, presented for the first time by Mostowski [Mo57], and Lindström [L66]. Infinitary logics have back-and-forth systems [D75] which are described independently of their representation as $\mathcal{L}_{\omega\omega}(\overline{Q})$.

[2]The main reason to do that is that the PC-operation does not preserve the Karp property (partial isomorphism would be a concrete case of the R mentioned above). We want to consider interpolation as a property that emerges from the preservation under PC of certain properties, including invariance under R. When the logic itself is invariant under R, we get a proof of maximality. When the logic has a property that makes R be the relation of isomorphism, we get a proof of full interpolation.

a weaker form of interpolation, proving whether they have interpolation is a very difficult matter: we seem to find ourselves without tools to determine it.

In Section 6, we study extensions of first order logic with upward monotone quantifiers without negation. Flum [Fl85] gave a model theoretic characterization of cardinal quantifiers Q_α in terms of monotonicity and Löwenheim-Skolem theorems. In our framework, a cardinal quantifier splits into four: Q^+_α, Q^-_α, $\tilde{Q}^+_\alpha$, and $\tilde{Q}^-_\alpha$, whose interpretations are $\mathfrak{A} \models Q^+_\alpha x\phi(x)$ iff $|\{a \in A : \mathfrak{A} \models \phi(\bar{a}, a)\}| \geq \aleph_\alpha$, $\mathfrak{A} \models Q^-_\alpha x\phi(x)$ iff $|\{a \in A : \mathfrak{A} \not\models \phi(\bar{a}, a)\}| < \aleph_\alpha$, $\mathfrak{A} \models \tilde{Q}^+_\alpha x\phi(x)$ iff $|\{a \in A : \mathfrak{A} \models \phi(\bar{a}, a)\}| < \aleph_\alpha$, and $\mathfrak{A} \models \tilde{Q}^-_\alpha x\phi(x)$ iff $|\{a \in A : \mathfrak{A} \not\models \phi(\bar{a}, a)\}| \geq \aleph_\alpha$. Only the two first quantifiers are upward monotone. It is proved that $\mathcal{L} = \mathcal{L}_{\omega\omega}(Q_i)_{i=0,1,\ldots,n}$ without negation, where Q_i are upwards monotone quantifiers, is equivalent to $\mathcal{L}_{\omega\omega}(Q^+_{\alpha_j}, Q^-_{\beta_k})_{\substack{j=0,\ldots,m \\ k=m+1,\ldots,n}}$ for some α_j and β_k, and the question arises for what combinations of α_j and β_k does the logic satisfy interpolation and compactness.

It is proved that $\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$ is compact if $\alpha < \beta$, or if $\beta \leq \alpha$ and: 1.) $\aleph_\gamma = \aleph_\gamma^{\aleph_0}, \beta = \gamma + 1$, or 2.) $\aleph_0$ is small for $\aleph_\beta, \beta$ a limit ordinal. However, separation holds if $\beta > \alpha$, and interpolation fails if $\alpha \geq \beta$. The results on interpolation can be regarded as an extension of the results of Caicedo [Ca83], and those on compactness an extension of the result of Shelah [Sh71].

2. Preliminaries

We start by defining the basic concepts and conventions, based on the framework presented at [E85].

Definition 1. *A* vocabulary τ *is a nonempty set that consists of finitary relation symbols $P, R, \ldots,$ and constant symbols $c, d, \ldots$. A* τ-structure $\mathfrak{A}$ *is a pair $\langle A, \nu\rangle$, where A, called the domain of $\mathfrak{A}$, is a nonempty set and ν is a map that assigns to every n-ary relation symbol R in τ, an n-ary relation on A^n, and to every constant symbol in τ an element in A. For any symbol $T \in \tau, T^{\mathfrak{A}}$ denotes the interpretation of T on $\mathfrak{A}$. We denote structures by $\mathfrak{A}[\tau] = \langle A, T_i^{\mathfrak{A}}\rangle_{T_i \in \tau}$. Given a structure $\mathfrak{A}$ of vocabulary τ and a constant symbol $c \notin \tau$, the structure $\langle \mathfrak{A}, a\rangle$ denotes the* expansion *of $\mathfrak{A}$ to a vocabulary $\tau \cup \{c\}$, and a is the interpretation of c in $\mathfrak{A}$.*

Definition 2. *A* renaming *is a map $\rho : \tau \to \sigma$ that is a bijection from a vocabulary τ to a vocabulary σ, that maps relation symbols to relation symbols of the same arity, and constants to constants. Given a renaming ρ and a structure $\mathfrak{A}$ of vocabulary τ, $\mathfrak{B} = \mathfrak{A}^\rho$ is a structure of vocabulary σ with $B = A$ and $\rho(T)^{\mathfrak{B}} = T^{\mathfrak{A}}$, for all symbols $T \in \tau$.*

Let $Str[\tau]$ denote the class of structures of vocabulary τ. Let $\sigma \subseteq \tau$, and let $\mathfrak{A} \in Str[\tau]$. We define $\mathfrak{A} \restriction \sigma$, the *reduct* of $\mathfrak{A}$, to be the structure $\mathfrak{B} = \langle A, T^{\mathfrak{A}}\rangle_{T \in \sigma}$, where $T^{\mathfrak{A}} = T^{\mathfrak{B}}$ for $T \in \sigma$.

Definition 3. *A* logic *is a pair* $\langle \mathcal{L}, \models_{\mathcal{L}} \rangle$ *where* $\mathcal{L}$ *is a map defined on vocabularies such that* $\mathcal{L}[\tau]$ *is a class called the class of* $\mathcal{L}$*-sentences of vocabulary* τ*, and* $\models_{\mathcal{L}}$ *(the* $\mathcal{L}$-satisfaction relation*) is a relation between structures and* $\mathcal{L}$*-formulas such that the following conditions (called closure properties) hold:*

(1) *Inclusion property. If* $\sigma \subseteq \tau$*, then* $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$.
(2) *Isomorphism property. If* $\mathfrak{A} \models_{\mathcal{L}} \varphi$*, and* $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{B} \models_{\mathcal{L}} \varphi$.
(3) *Reduct Property. If* $\varphi \in \mathcal{L}[\tau]$ *and* $\tau \subseteq \tau_{\mathfrak{A}}$*, then* $\mathfrak{A} \models_{\mathcal{L}} \varphi$ *iff* $\mathfrak{A} \upharpoonright \tau \models_{\mathcal{L}} \varphi$.
(4) *Renaming Property. If* $\rho : \sigma \to \tau$ *is a renaming, then for each* $\varphi \in \mathcal{L}[\sigma]$ *there is a* $\psi^\rho \in \mathcal{L}[\tau]$ *such that for all* σ*-structures* $\mathfrak{A}$*,* $\mathfrak{A} \models_{\mathcal{L}} \varphi$ *iff* $\mathfrak{A}^\rho \models_{\mathcal{L}} \varphi^\rho$*;*
(5) *Substitution Property. Suppose* $\sigma \subseteq \tau$*, and* $\varphi \in \mathcal{L}[\tau]$*, and for all* $R_i \in \tau \backslash \sigma$*, we have a sentence* $\varphi_i(d_1^i, \dots, d_{k_i}^i) \in \mathcal{L}[\sigma \cup \{d_1^i, \dots, d_{k_i}^i\}], k_i$ *the arity of* R_i*, and* $d_1^i, \dots, d_{k_i}^i$ *new constants. For any structure* $\mathfrak{A} \in Str[\sigma]$*, let* $\mathfrak{A}^* \in Str[\tau]$ *be such that* $\mathfrak{A}^* \upharpoonright \sigma = \mathfrak{A}$*, and for all* $R_i \in \tau \backslash \sigma, R_i^{\mathfrak{A}^*} = \{\langle a_1, \dots, a_{k_i} \rangle : (\mathfrak{A}, a_1, \dots, a_{k_i}) \models_{\mathcal{L}} \varphi_i(d_1^i, \dots, d_{k_i}^i)\}$*. Then there is* $\varphi^* \in \mathcal{L}[\sigma]$ *such that for all* $\mathfrak{A} \in Str[\sigma], \mathfrak{A} \models_{\mathcal{L}} \psi \leftrightarrow \mathfrak{A}^* \models_{\mathcal{L}} \varphi$*. We say that* ψ *is obtained from* φ *by simultaneously replacing each* $R_i \in \tau \backslash \sigma$ *by* $\varphi_i(d_1^i, d_2^i, \dots, d_{k_i}^i)$[3]*;*
(6) *Atom Property. For all* τ *and atomic* $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ *there is a sentence* $\psi \in \mathcal{L}[\tau]$ *such that* $Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}_{\omega\omega}}^{\tau}(\varphi)$*;*
(7) *Conjunction Property. For all* τ *and all* $\varphi, \psi, \in \mathcal{L}[\tau]$ *there is* $\theta \in \mathcal{L}[\tau]$ *such that* $Mod_{\mathcal{L}}^{\tau}(\varphi) \cap Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}}^{\tau}(\theta)$*;*
(8) *Disjunction Property. For all* τ *and all* $\varphi, \psi, \in \mathcal{L}[\tau]$ *there is* $\theta \in \mathcal{L}[\tau]$ *such that* $Mod_{\mathcal{L}}^{\tau}(\varphi) \cup Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}}^{\tau}(\theta)$*;*
(9) *Particularization Property. If* $c \in \tau$*, then for any* $\varphi \in \mathcal{L}[\tau]$ *there is* $\psi \in \mathcal{L}[\tau \backslash \{c\}]$ *such that for all* $[\tau \backslash \{c\}]$*-structures* $\mathfrak{A}, \mathfrak{A} \models_{\mathcal{L}} \psi$ *iff* $\langle \mathfrak{A}, a \rangle \models_{\mathcal{L}} \varphi$ *for some* $a \in A$*. In a context of a logic with free variables we write* ψ *as* $\exists x \varphi(x)$*;*
(10) *Universalization. If* $c \in \tau$*, then for any* $\varphi \in \mathcal{L}[\tau]$ *there is a sentence* $\psi \in \mathcal{L}[\tau \backslash \{c\}]$ *such that for all* $[\tau \backslash \{c\}]$*-structures* $\mathfrak{A}, \mathfrak{A} \models_{\mathcal{L}} \psi$ *iff* $\langle \mathfrak{A}, a \rangle \models_{\mathcal{L}} \varphi$ *for all* $a \in A$*. In a context of a logic with free variables we write* ψ *as* $\forall x \varphi(x)$.

Further closure properties that a logic may have in the course of this thesis are listed below. Contrary to what happens with the above properties, we don't assume a logic to be closed under these properties, so in the cases where they are present, these closures will be always mentioned explicitly.

[3]It should be mentioned that in a context without negation, we only can make substitutions of P in formulas $\theta(P)$ where all occurrences of P are positive, or in terms of classes, where the class in which we make the substitution is persistent upwards with respect to P, i.e. if $\langle \mathfrak{A}, P, \dots \rangle \in \mathbf{K}$ and $P \subseteq P'$, then $\langle \mathfrak{A}, P', \dots \rangle \in \mathbf{K}$. The role of substitution in abstract logic without negation seems to deserve more study.

Definition 4 (Generalized quantifier)**.** *Let σ be a finite vocabulary. Let $\mathbf{K}$ be a class of σ-structures closed under isomorphism. Suppose for simplicity that $\sigma = \{R, c\}$ with R binary and c a constant. Let Q be a new quantifier symbol. The logic $\mathcal{L}(Q_{\mathbf{K}})$ is obtained as follows:*

$\mathcal{L}(Q_{\mathbf{K}})[\tau]$ is taken as the smallest set (or class) containing $\mathcal{L}[\tau]$ which is closed under properties of definition 3, and which with each ϕ, ξ and for any variables x_0, x_1, y, also contains the formula:

$$\theta = Qx_0x_1y\phi\xi.$$

The meaning of Q is determined by the satisfaction relation:

$$\begin{aligned}
&\mathfrak{A} \models_{\mathcal{L}(Q_{\mathbf{K}})} Qx_0x_1y\phi(x_0,x_1)\xi(y) \\
&\textit{iff there is a } \sigma \textit{ -structure } \mathfrak{C} \in \mathbf{K} \textit{ such that } C = A \\
&R^{\mathfrak{C}} = \{(a,b) \in C \times C : \mathfrak{A} \models_{\mathcal{L}(Q_{\mathbf{K}})} \phi[a,b]\} \\
&\textit{and } \mathfrak{A} \models_{\mathcal{L}(Q_{\mathbf{K}})} \xi[a] \textit{ iff } a = c^{\mathfrak{C}}.
\end{aligned}$$

We usually identify a quantifier $Q_{\mathbf{K}}$ with the class of models $\mathbf{K}$. A quantifier Q may be defined on a set A by a set of subsets of A. So, given a class $\mathbf{K}$, $Q_{\mathbf{K}}(A) = \{X \subseteq A : (A, X) \in \mathbf{K}\}$. The *dual* of Q, in symbols Q^d is defined as $Q^d = \{X \subseteq A : \overline{X} \notin Q(A)$.

Example 5. *The following list associates a quantifier Q with the corresponding class of models $\mathbf{K}$:*

- *$\exists$ for $\mathbf{K} = \{(A, C) : \emptyset \neq C \subseteq A\}$;*
- *$\forall$ for $\mathbf{K} = \{(A, C) : A = C\}$;*
- *The Magidor-Malitz quantifier Q^n_α for $\mathbf{K} = \{(A, D) : D \subseteq A^n,$ and there is $C \subseteq A, |C| \geq \aleph_\alpha$ and $C^n \subseteq D\}$;*
- *The cofinality quantifier $Q^{cf\omega}$ for $\mathbf{K} = \{(A, <^{\mathfrak{A}}) : <^{\mathfrak{A}}$ is a linear ordering on A of cofinality $\omega\}$*

Definition 6 (Further closure properties)**.**

(11) *Negation Property. For all τ and all $\varphi \in \mathcal{L}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau]$ such that $Mod^\tau_{\mathcal{L}}(\varphi) = Str[\tau] \setminus Mod^\tau_{\mathcal{L}}(\psi)$;*

(12) *Relativization Property. If $c \notin \tau \cup \sigma, \xi \in \mathcal{L}[\tau \cup c]$ and $\varphi \in \mathcal{L}[\tau]$, then there is a sentence $\psi \in \mathcal{L}[\tau \cup c]$ such that for any $(\tau \cup \sigma)$-structure $\mathfrak{B}$, if $\xi^{\mathfrak{B}}$ denotes the set $\{b \in B : (\mathfrak{B}, b) \models \xi\}$, then $\mathfrak{B} \models \psi$ iff $(\mathfrak{B} \restriction \tau)|\xi^{\mathfrak{B}} \models \varphi$. Set $X = \xi^{\mathfrak{B}}$, then $\psi := [\phi]^X$ is the relativization of ϕ to X.*

(13) *PC_Δ-operation. If $S = \{\varphi_n : n \in \omega\}$ is a set of $\mathcal{L}[\tau]$-sentences, and if $\sigma \subseteq \tau$, then there is a sentence $\psi \in \mathcal{L}[\sigma]$ such that*

$$Mod^\sigma_{\mathcal{L}}(\psi) = (\bigcap_n Mod^\tau_{\mathcal{L}}(\varphi_n)) \restriction \sigma.$$

If S contains just one sentence, the operation is called PC.

(14) *Q-Projection. Let* $\mathbf{K}$ *be a class of models of vocabulary* $\sigma = \{S_1, \ldots, S_m\}$ *disjoint from* τ*. Then for any* $\varphi_i \in \mathcal{L}[\tau \cup \{c_0, c_1, \ldots, c_{k_i-1}\}]$ *there is* $\psi_i \in \mathcal{L}[\tau]$ *such that* $\mathfrak{A} \in Mod^{\tau}_{\mathcal{L}}(\psi_i)$ *iff*
- *There is a structure* $\mathfrak{C} \in \mathbf{K}$ *with* $C = A$*, and*
- *For all* k_i *and all* k_i*-ary* $S_i \in \sigma$*,* $S_i^{\mathfrak{C}} = \{(a_0, \ldots, a_{k_i-1}) \in C^n : \langle \mathfrak{A}, a_0, a_1, \ldots, a_{k_i-1}\rangle \models \varphi_i\}$*.*

In a context of a logic with free variables we write ψ *as*

$$Q_{\mathbf{K}} x_0^1, \ldots, x_{k_1-1}^1, \ldots, x_0^m, \ldots, x_{k_m-1}^m \varphi_1(x_0^1, \ldots, x_{k_1-1}^1) \ldots \varphi_m(x_0^m, \ldots, x_{k_m-1}^m).$$

If a logic satisfies condition 12, it is called *regular.* Any regular logic contains $\mathcal{L}_{\omega\omega}$. If a logic satisfies condition 13, it is called *relativizing.* We identify sentences with classes of models, and express abstract sentences by means of a generalized quantifier. So, given a logic $\mathcal{L}$ and a class of models $\mathbf{K}$ of an abstract sentence corresponding to a generalized quantifier $Q_{\mathbf{K}}$, the extension $\mathcal{L}(Q_{\mathbf{K}})$ of $\mathcal{L}$ by $Q_{\mathbf{K}}$ is the logic we get when we add $Q_{\mathbf{K}}$ to $\mathcal{L}$ and close under Q-projection.

For φ an $\mathcal{L}$-sentence, and a structure $\mathfrak{A}$, $\mathfrak{A} \models_{\mathcal{L}} \varphi$ is read "$\mathfrak{A}$ is a model of φ". By $Mod^{\tau}_{\mathcal{L}}(\varphi)$ we denote the class of τ-structures $\mathfrak{A}$ such that $\mathfrak{A} \models_{\mathcal{L}} \phi\}$. If $\mathbf{K}$ is a class of models of vocabulary τ, then $\mathbf{K} \restriction \sigma = \{\mathfrak{A} \restriction \sigma : \mathfrak{A} \in \mathbf{K}\}$, and $\overline{\mathbf{K}} = Str[\tau] \setminus \mathbf{K}$.

Definition 7. *Let* τ, τ' *be two countable vocabularies,* $\mathcal{L}$ *a logic, and* $\mathbf{K}$ *a class of* τ*-structures.*

(1) $\mathbf{K}$ *is an* $\mathcal{L}$*-elementary class, in symbols* $\mathbf{K} \in EC(\mathcal{L})$*, or* $\mathbf{K}$ *is EC in* $\mathcal{L}$*, if there is a sentence* $\theta \in \mathcal{L}[\tau]$ *such that* $\mathbf{K} = Mod_{\tau}(\theta)$*.*
(2) $\mathbf{K}$ *is an* $\mathcal{L}$*-elementary class in the wider sense, in symbols* $\mathbf{K} \in EC_{\Delta}(\mathcal{L})$*, or* $\mathbf{K}$ *is* EC_{Δ} *in* $\mathcal{L}$ *if there is a set of sentences* $\Theta \subset \mathcal{L}[\tau]$ *such that* $\mathbf{K} = Mod_{\tau}(\theta)$*.*
(3) $\mathbf{K}$ *is an* $\mathcal{L}$*-pseudo elementary class, in symbols* $\mathbf{K} \in PC(\mathcal{L})$*, or* $\mathbf{K}$ *is PC over* $\mathcal{L}$ *if there is a vocabulary* $\tau' \supseteq \tau$ *and a sentence* $\theta \in \mathcal{L}[\tau']$ *such that* $\mathbf{K} = Mod_{\tau'}(\theta) \restriction \tau$*.*
(4) $\mathbf{K}$ *is an* $\mathcal{L}$*-co-pseudo elementary class, in symbols* $\mathbf{K} \in cPC(\mathcal{L})$*, or* $\mathbf{K}$ *is cPC over* $\mathcal{L}$ *if* $\mathbf{K}$ *is the complement of a* $PC(\mathcal{L})$*-class.*
(5) $\mathbf{K}$ *is an* $\mathcal{L}$*-pseudo elementary class in the wider sense, in symbols* $\mathbf{K} \in PC_{\Delta}(\mathcal{L})$*, or* $\mathbf{K}$ *is* PC_{Δ} *over* $\mathcal{L}$ *if there is a vocabulary* $\tau' \supseteq \tau$ *and a set of sentences* $\Theta \subset \mathcal{L}[\tau']$ *such that* $\mathbf{K} = Mod_{\tau'}(\Theta) \restriction \tau$*.*

Definition 8. *Let* $\mathcal{L}, \mathcal{L}^*$ *be logics. We say that* $\mathcal{L}^*$ (properly) extends $\mathcal{L}$*, in symbols* $\mathcal{L} \leq \mathcal{L}^*$ *(*$\mathcal{L} < \mathcal{L}^*$*), if every EC-class in* $\mathcal{L}[\tau]$*, is an EC-class in* $\mathcal{L}^*[\tau]$ *(and moreover there is an EC-class* $\mathbf{K}$ *in* $\mathcal{L}^*[\tau]$*, such that* $\mathbf{K}$ *is not an EC-class in* $\mathcal{L}[\tau]$*).*

Definition 9. *Let* Q *be a quantifier. The* quantifier rank *of a formula* φ*, in symbols* $qr(\varphi)$ *is defined inductively as follows:*

a) $qr(\varphi) = 0$*, if* φ *is atomic;*
b) $qr(\neg\varphi) = qr(\varphi)$*;*
c) $qr(\varphi \wedge \psi) = max\{qr(\varphi), qr(\psi)\}$*;*
d) $qr(Qx\varphi) = qr(\varphi) + 1$*.*

Definition 10. *A* partial isomorphism *between two models* $\mathfrak{A}, \mathfrak{B}$ *is a function* p *from* $X \subseteq A$ *to* $Y \subseteq B$ *such that the following holds;*

(1) *For all* $n \geq 1$, n*-ary* $R \in \tau$ *and* $a_0, \ldots, a_{n-1} \in X : R^{\mathfrak{A}}(\vec{a})$ *iff* $R^{\mathfrak{B}}(p(\vec{a}))$*;*
(2) *For all* $c \in \tau$ *and* $a \in X : c^{\mathfrak{A}} = a$ *iff* $c^{\mathfrak{B}} = p(a)$.

$Part(\mathfrak{A}, \mathfrak{B})$ *denotes the set of partial isomorphisms between* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Definition 11. *A* back-and-forth *system for* $(\mathfrak{A}, \mathfrak{B})$ *is a decreasing sequence* $I = (I_\beta)_{\beta \leq \alpha}$ *of subsets of* $Part(\mathfrak{A}, \mathfrak{B})$ *that satisfies the following conditions:*

(i) *Each* I_i *is a set of partial isomorphisms.*
(ii) $\emptyset \in I_\alpha$
(iii) *For* $\beta < \alpha$*, if* $p \in I_{\beta+1}$ *and* $a \in A$*, then there is* $b \in B$ *such that* $p \cup \{\langle a, b\rangle\} \in I_\beta$.
(iv) *For* $\beta < \alpha$*, if* $p \in I_{\beta+1}$ *and* $b \in B$*, then there is* $a \in A$ *such that* $p \cup \{\langle a, b\rangle\} \in I_\beta$.

Suppose F is a function which maps structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same vocabulary to a set $F(\mathfrak{A}, \mathfrak{B}) \subseteq Part(\mathfrak{A}, \mathfrak{B})$. Then we call the functions in $F(\mathfrak{A}, \mathfrak{B})$ simply partial F-isomorphisms. An F-back-and-forth system is a back-and-forth system of partial F-isomorphisms. All notation for partial isomorphisms would be obtained just by ignoring F in the following definitions.

Definition 12. *Let* $\mathcal{L}$ *be a logic. Given two structures* $\mathfrak{A}$, $\mathfrak{B}$*:*

a) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* α*-*F*-isomorphic via* I*, written* $I : \mathfrak{A} \cong^F_\alpha \mathfrak{B}$*, iff* $I = (I_\beta)_{\beta \leq \alpha}$ *is an* F*-back-and-forth system.*
b) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* α*-*F*-isomorphic, written* $\mathfrak{A} \cong^F_\alpha$ *iff there is* I *such that* $I : \mathfrak{A} \cong^F_\alpha \mathfrak{B}$.
c) $I \subseteq F(\mathfrak{A}, \mathfrak{B})$ *has the back (forth) property if for each* $p \in I$ *and* $b \in B (a \in A)$ *there is* $q \in I, p \subseteq q$ *with* $b \in rg(q)(a \in dom(p))$.
d) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* partially F-isomorphic $\mathfrak{A} \cong^F_p \mathfrak{B}$ *if in there is* $I \subseteq F(\mathfrak{A}, \mathfrak{B})$ *with the back-and-forth property.*
e) *We write* $\mathfrak{A} \leq_{\mathcal{L}} \mathfrak{B}$*, if every sentence in* $\mathcal{L}$ *satisfied in* $\mathfrak{A}$ *is also satisfied in* $\mathfrak{B}$*. We write* $\mathfrak{A} \leq^n_{\mathcal{L}} \mathfrak{B}$ *if every sentence of quantifier rank at most* n *in* $\mathcal{L}$ *satisfied in* $\mathfrak{A}$ *is also satisfied in* $\mathfrak{B}$*. If* $\mathfrak{A} \leq_{\mathcal{L}} \mathfrak{B}$ *and* $\mathfrak{B} \leq_{\mathcal{L}} \mathfrak{A}$*, then we write* $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ *and say that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* $\mathcal{L}$*-equivalent. If* $\mathcal{L}$ *is* $\mathcal{L}_{\omega\omega}$ *we say the above models are elementary equivalent.*

Lemma 13. *Let* $F(\mathfrak{A}, \mathfrak{B})$ *be closed under* ω*-chains. Any two countable* F*-partially isomorphic models are* F*-isomorphic.*

Proof. Let $A = \{a_n : n \in \omega\}, B = \{b_n : n \in \omega\}$, and let $I \subseteq F(\mathfrak{A}, \mathfrak{B})$ have the back-and-forth property. Suppose we have defined, for $n \leq k$, $c_{2n} = a_n$, and $d_{2n+1} = b_n$, and a sequence $I_n = \langle q_0, q_1, \ldots, q_{n-1}\rangle \subseteq I$ such that $q_i(c_i) = d_i$. We denote such sequence by $\langle c_n\rangle I_n \langle d_n\rangle$. If $k = 2r$, set $c_k = a_r$. By the forth condition, there is a least index s such that $\langle c_0, \ldots, c_k\rangle I_{n+1}\langle d_0, \ldots, d_{k-1}, b_s\rangle$. Set $d_k = b_s$. Similarly if k is odd. As all functions in I are partial F-isomorphisms, $g = \{\langle c_n, d_n\rangle : n \in \omega\}$ is an F-isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. $\square$

Definition 14 (Model Theoretic Properties)**.**

(1) Compactness: *$\mathcal{L}$ is (countably) compact iff for all (countable) $\Phi \subseteq \mathcal{L}[\tau]$, if each finite subset of Φ has a model, then Φ has a model. Weaker forms of compactness are:*
- Small well ordering number*, where the well ordering number of $\mathcal{L}$ is defined as the supremum of all ordinals α such that for some $\mathcal{L}$-sentence $\phi(<, \dots)$ having only models with well ordered $<$, there is a model of ϕ that is a well-ordering of type α.*
- Boundedness: *$\mathcal{L}$ is bounded if for any $\mathcal{L}$-sentence $\phi(<, \dots)$ having only models with well ordered $<$, there is an ordinal α such that the order type of $<$ is always less than α.*

Without negation, in addition to the previously defined notion of compactness we have:

(a) Dual compactness: *$\mathcal{L}$ is dual compact (DC) if given a countable set of sentences Φ of vocabulary τ such that $\bigcup \Phi = Str[\tau]$, there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\bigcup \Phi_0 = Str[\tau]$.*

(b) $\star$- compactness: *$\mathcal{L}$ is $\star$-compact if given two sets of sentences Φ, and Φ', if for every finite $\Phi_0 \subseteq \Phi$ and every finite $\Phi'_0 \subseteq \Phi'$ there is a model which satisfies each sentence in Φ_0 and no sentence in Φ'_0, then there is a model which satisfies each sentence in Φ, and no sentence in Φ'.*

(2) *$\mathcal{L}$ satisfies the* Downward Löwenhein-Skolem Property down to κ $(LS(\kappa))$ *iff each satisfiable sentence has a model of size $\leq \kappa$. When $\kappa = \omega$ we just write LS. Weaker forms of the Löwenhein-Skolem property are:*
- *Small $l(\mathcal{L})$, where the* Löwenheim number $l(\mathcal{L})$ *of $\mathcal{L}$ is the least cardinal μ such that any satisfiable sentence has a model of power $\leq \mu$. Any logic $\mathcal{L}$ with a set of sentences has $l(\mathcal{L}) = \lambda$ for some cardinal λ. Such logics are called* small *or* set *logics. Otherwise, $l(\mathcal{L}) = \infty$, and $\mathcal{L}$ is called a* big *or* class *logic.*
- *Small $l_\Sigma(\mathcal{L})$, the* Löwenheim number $l_\Sigma(\mathcal{L})$ for countable sets of sentences *of $\mathcal{L}$ is the least cardinal μ such that any countable satisfiable set of sentences has a model of power $\leq \mu$.*

Without negation, in addition to the previously defined notion of LS we have:

(a) *$\mathcal{L}$ satisfies the* Dual Löwenheim-Skolem Property *(DuLS) if any sentence that is true in all countable models, is true in all models.*

(b) *$\mathcal{L}$ has $\star$-LS if every sentence is determined by its countable models, i.e. if given two classes EC-classes $\mathbf{K}_1, \mathbf{K}_2, n(\mathbf{K}_1) = n(\mathbf{K}_2)$ implies $\mathbf{K}_1 = \mathbf{K}_2$, where $n(\mathbf{K})$ denotes the class of countable models in $\mathbf{K}$.*

(3) *$\mathcal{L}$ has the* Karp property *if any two partially isomorphic structures are $\mathcal{L}$-equivalent. Karp property can be generalized to partial F-isomorphisms.*

(4) *$\mathcal{L}$ satisfies the* Separation Theorem *if any two disjoint PC-classes in $\mathcal{L}$ can be separated by an EC-class in $\mathcal{L}$. A weaker form of Separation theorem is:*
- *$\mathcal{L}$ satisfies the* Δ- interpolation theorem *if for each class of models $\mathbf{K} \in \mathcal{L}$, if $\mathbf{K}$ and $\overline{\mathbf{K}}$ are PC in $\mathcal{L}$, then they are EC in $\mathcal{L}$.*

Without negation, in addition to the previously defined notion of the Separation Theorem we have:

(a) Reduction theorem. *If* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *are two* $cPC(\mathcal{L})$*-classes such that* $\mathbf{K}_1 \cup \mathbf{K}_2 = Str[\tau]$*, then there are* $\mathbf{K}'_1$ *and* $\mathbf{K}'_2$ *in* $\mathcal{L}$ *such that* $\mathbf{K}'_1 \cap \mathbf{K}'_2 = \emptyset$, $\mathbf{K}'_1 \cup \mathbf{K}'_2 = Str[\tau]$*, and* $\mathbf{K}'_i \subseteq \mathbf{K}_i, i = 1, 2$.

(b) Craig Interpolation Theorem. *Given* $\varphi, \psi \in \mathcal{L}$*, such that* $\varphi \models \psi$*, there is* $\theta \in \mathcal{L}$ *such that* $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$ *and* $\varphi \models \theta$ *and* $\theta \models \psi$.

The choices of (a) and (b) in both (1) and (2) above are not arbitrary. Dual compactness is the definition of compactness used in topology. Dual Löwenheim-Skolem is an alternative and intuitive way of thinking about this property, and in fact it is the formulation used originally by Löwenheim in his pioneering paper from 1915, [Lö15][4]. The motivation of using $\star$-LS comes from the proof of Wacławek in [W78] about the existence of maximal logics with that property.

We notice that there are two basically different kinds of model theoretic properties. On one hand, we have the properties such that if they are not present in a logic, then no extension of this logic enjoy these properties. That is the case of compactness, Karp, and Löwenheim-Skolem properties. On the other hand, there are properties that can be in principle recovered in an extension of any logic lacking them. This is the case of interpolation. We do not know of any operation on a logic $\mathcal{L}$ with negation and without interpolation that will give an extension with negation and interpolation -without negation there is the trivial one, which consists in adding all $PC(\mathcal{L})$ classes. However, there is a natural closure on Δ-interpolation, introduced by H. Friedman [F71] and J. Barwise [B72], and first referred to in a published paper by J. Barwise [B74].

Definition 15 (The Δ-closure)**.** *Given a logic* $\mathcal{L}$, $\Delta(\mathcal{L})$ *is the logic that has as* EC*-classes just the* $PC(\mathcal{L})$*-classes whose complement is also* $PC(\mathcal{L})$.

The Δ-extension of a logic is deeply studied by J. A. Makowsky et al. in [MShS76]. They prove it is automatically closed under conjunction and Q-projection. It is also closed under negation provided $\mathcal{L}$ is. They also proved that the Δ-extension preserves many model theoretic properties.

3. Lindström's theorem without negation

The first result of this section, Theorem 19, is not on its own a Lindström's theorem but rather an illustration of the relation between this theorem and Craig's Interpolation Theorem. In a context without negation, both theorems share the same form, in the sense that both are covered by the same theorem: the Separation Theorem. This theorem yields Lindström's Theorem when we add negation into the picture, but while the Separation Theorem implies, with or without negation, the

[4]I thank this piece of knowledge to Dr. Juliette Kennedy.

Interpolation Theorem, it does not necessarily imply a characterization theorem if we do not have negation.

3.1. **Lindström's theorem for a fragment of $\mathcal{L}_{\omega\omega}$.**

We present a generalized form of Lindström's Theorem for the fragment of first-order sentences in which a particular predicate occurs only positively. It has as corollary a weaker form of Lyndon's Interpolation theorem.

$\mathcal{L}[\tau], \mathcal{L}^*[\tau], \ldots$ will be used to denote arbitrary logics. However, when the logic or the vocabulary are clear from the context, we use τ-formula, or $\mathcal{L}^*$-formula respectively, instead of the longer notation $\mathcal{L}[\tau]$-formula. If P is a predicate symbol in τ, $\mathcal{L}_{\omega\omega}[\tau, P^+]$ denotes the set of sentences in $\mathcal{L}_{\omega\omega}[\tau]$ in which P occurs only positively. $\mathcal{L}^r_{\omega\omega}[\tau, P^+]$ is the set of formulas in $\mathcal{L}_{\omega\omega}[\tau]$ with free variables among $x_1, \ldots, x_r$ in which P occurs only positively. We call formulas in $\mathcal{L}^r_{\omega\omega}[\tau, P^+]$ *P-positive.* Next definition introduces a relation that is an isomorphism with respect to every symbol in the language except for a particular predicate P.

Definition 16. *Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures. A bijection $\pi : A \to B$ is a P-isomorphism $\mathfrak{A} \to \mathfrak{B}$ if for all r-tuples $\bar{a} \in A$*

$$\mathfrak{A} \models P(\bar{a}) \Rightarrow \mathfrak{B} \models P(\pi(\bar{a}))$$

and

$$\mathfrak{A} \models R_i(\bar{a}) \text{ iff } \mathfrak{B} \models R_i(\pi(\bar{a})) \text{ for } R_i \in \tau \smallsetminus \{P\}.$$

$\mathfrak{A}$ and $\mathfrak{B}$ are P-isomorphic, written $\mathfrak{A} \cong^P \mathfrak{B}$ if there is a P-isomorphism $\mathfrak{A} \to \mathfrak{B}$. An F_P-back-and-forth system is a back-and-forth system in which $F_P(\mathfrak{A}, \mathfrak{B}) = \{p \in Part(\mathfrak{A}, \mathfrak{B}) : p \text{ is a partial } P\text{-isomorphism}\}$.

This definition implies that when we are going F_P-back-and-forth between two models $\mathfrak{A}, \mathfrak{B}$, the image of an element $b \in B$ such that $\mathfrak{B} \not\models P(b)$ is an element a with $\mathfrak{A} \not\models P(a)$ (and agrees in the rest of the predicates with b). But if $\mathfrak{B} \models P(b)$, then one does not need to look whether $\mathfrak{A} \not\models P(a)$ or not (we only have to take care of the rest of the predicates).

Lemma 17. *Every P-positive formula is preserved by P-isomorphisms.*

Proof. Let π be a P-isomorphism $\mathfrak{A} \to \mathfrak{B}$. We prove by induction on the formation of P-positive formulas, that for all assignments s:

$$\mathfrak{A} \models_s \phi \to \mathfrak{B} \models_{\pi \circ s} \phi.$$

The case for a P-positive atomic or negated atomic formulas comes directly from definition, and the case for connectives is easy.

Now suppose $\phi = \exists x \psi(x)$, and $\mathfrak{A} \models_s \phi$. Then there is $a \in A$ such that $\mathfrak{A} \models_{s[a/x]} \psi(x)$. By induction hypothesis, $\mathfrak{B} \models_{\pi \circ (s[a/x])} \psi(x)$. But $\pi \circ (s[a/x]) = (\pi \circ s)[\pi(a)/x]$, so $\mathfrak{B} \models_{(\pi \circ s)[\pi(a)/x]} \psi(x)$, and thus $B \models_{(\pi \circ s)} \exists x \psi(x)$. $\square$

We adapt the following technical notion from [L69]. It is introduced to avoid redundancy in the use of quantifiers or boolean connectives. This prevents formulas from being arbitrarily long, and the number of them becoming infinite.

Definition 18. *An (r,r)-condition is any atomic or negated atomic formula in $\mathcal{L}^r_{\omega\omega}[\tau,P^+]$. A complete (r,i)-condition is any disjunction of conjunctions of (r,i)-conditions. An $(r,i-1)$-condition is a formula of the form $\exists x_j\chi$ or $\forall x_j\chi$, where χ is a complete (r,i)-condition, and $j\in\{1,2,\cdots\}$.*

We will see in Lemmas 22 and 23 below that any $\phi\in\mathcal{L}^r_{\omega\omega}[\tau,P^+]$ of quantifier rank $\leq n$ is equivalent to some $(n+r,r)$-condition. For each different variant of back-and-forth system we have a corresponding variant of the definition of the (r,i)-conditions. Lemma 24 (which is the central part of the proof of Theorem 19) shows the dependency between back-and-forth systems and formulas. In particular, in the proof of $(i)\rightarrow(ii)$ of this lemma it is essential that the set of formulas is finite.

The following theorem is a generalization of Lindström's Theorem with respect to back-and-forth systems. Similar generalizations have been given by Flum [Fl85], Barwise and van Benthem [BvB99], and more recently by Otto [M00]. In those cases, a particular separation theorem for a logic $\mathcal{L}$ was obtained from the interplay between invariant fragments of $\mathcal{L}$ (which amounted to generalizations of Karp property), or of $PC(\mathcal{L})$; and (weaker forms of) compactness. In our particular case, although through the same mechanisms as the mentioned works, we focus on the different implications the negation-less aspect of the logic has to interpolation and maximality.

Theorem 19. *Let τ be a finite and relational vocabulary and P be a predicate symbol in τ. Let $\mathcal{L}^*$ be a logic with the Compactness and Downward Löwenheim-Skolem properties, and closed under isomorphism. Let $\mathcal{L}^*$ also be closed under $\vee,\wedge,\exists,\forall$ (but not necessarily under negation). Assume $\mathcal{L}^*$ contains $\mathcal{L}_{\omega,\omega}[\tau,P^+]$.*

Let ϕ,ψ be τ-sentences in $\mathcal{L}^*$ such that $Mod(\phi)\cap Mod(\psi)=\emptyset$, and $Mod(\phi)$ is closed under P-isomorphisms. Then there is $\theta\in\mathcal{L}_{\omega\omega}[\tau,P^+]$ such that $Mod(\psi)\cap Mod(\theta)=\emptyset$, and $Mod(\phi)\subseteq Mod(\theta)$.

Before proving this theorem, we need some definitions and lemmas.

Contrary to what happens in the first-order case, this theorem cannot be stated in a Lindström like form, for $\mathcal{L}_{\omega\omega}[\tau,P^+]$ is not the strongest logic with the compactness and Löwenheim-Skolem properties that is closed under P-isomorphisms -indeed, $PC(\mathcal{L}_{\omega\omega}[\tau,P^+])$ is a compact extension of $\mathcal{L}_{\omega\omega}[\tau,P^+]$ with the Downward Löwenhein-Skolem property. That means that this generalization for Lindström's theorem as a separation theorem is too wide and does not preserve its character of characterization theorem. In Section 4, Theorem 47 is a generalization that preserves the characterization and the separation character of Lindström's theorem for fragments of first-order logic. There we also discuss why there are such differences between both generalizations, and also what are the implications of this fact to the relation between Lindström's theorem and Craig Interpolation theorem.

Definition 20. *A formula is said to be in* negation normal form *(nnf) if it is built up from atomic formulas and their negations using* $\vee, \wedge, \exists, \forall$.

Every first-order formula is equivalent to an *nnf* formula. We assume throughout this section that first-order formulas are in *nnf*.

Definition 21. *An* assignment s *is a map attributing to each variable* x *a value* a *in the domain of every model. We write* $\mathfrak{A} \models_s \phi$ *to denote* $\langle A, \bar{a}\rangle \models \phi(\bar{a})$ *when* $s(\bar{x}) = \bar{a}$. *Given an assignment* s, *we write* $s[a/x]$ *to denote the assignment that attributes to* x *the value of* a *in* A, *and that coincides with* s *elsewhere.*

Lemma 22. *Let* $\Phi = \{\phi_1, \dots, \phi_n\}$ *be a finite set of* $\mathcal{L}^r_{\omega\omega}[\tau]$*-formulas, and* $\langle\Phi\rangle$ *the least set that contains* Φ *and is closed under* $\vee, \neg$. *Then any formula in* $\langle\Phi\rangle$ *is logically equivalent to some disjunction of conjunctions of the formulas in* $\{\phi_1, \dots, \phi_n, \neg\phi_1, \dots, \neg\phi_n\}$. *In addition, there are only finitely many pairwise logically nonequivalent formulas in* $\langle\Phi\rangle$.

Proof. Suppose each ϕ_i has free variables among $x_1, \dots, x_r$. Given a model $\mathfrak{A}$, and a tuple $\bar{a} = (a_1, \dots, a_r) \in A$, take the conjunction $\xi_{(\mathfrak{A},\bar{a})}(x_1, \dots, x_r)$ of formulas $\xi(x_1, \dots, x_r)$ from $\{\phi_1, \dots, \phi_n, \neg\phi_1, \dots, \neg\phi_n\}$ such that $\mathfrak{A} \models \xi(a_1, \dots, a_r)$. It is clear that the set of satisfiable conjunctions is finite, with cardinality at most 2^n. For any $\psi \in \langle\Phi\rangle$, take the disjunction $\chi(x_1, \dots, x_r) = \bigvee\{\xi_{(\mathfrak{A},\bar{a})}(x_1, \dots, x_r) : \mathfrak{A} \models \psi(a_1, \dots, a_r)\}$. Suppose $\mathfrak{B} \models \chi(b_1, \dots, b_r)$. Then there is $\xi_{(\mathfrak{A},\bar{a})}(x_1, \dots, x_r)$ with $(a_1, \dots, a_r) \in A$ such that $\mathfrak{B} \models \xi_{(\mathfrak{A},\bar{a})}(b_1, \dots, b_r)$ and $\mathfrak{A} \models \psi(a_1, \dots, a_r)$. So, we have $\mathfrak{A}$ such that

$$\mathfrak{A} \models \phi_i(a_1, \dots, a_r) \iff \mathfrak{B} \models \phi_i(b_1, \dots, b_r).$$

Given that $\psi \in \langle\Phi\rangle$, and $\mathfrak{A} \models \psi(a_1, \dots, a_r)$, we conclude that $\mathfrak{B} \models \psi(b_1, \dots, b_r)$. Now suppose $\mathfrak{B} \models \psi(b_1, \dots, b_r)$. Then $\xi_{\mathfrak{B},\bar{b}}$ belongs to the disjunction $\chi(x_1, \dots, x_r)$. Since $\mathfrak{B} \models \xi_{\mathfrak{B},\bar{b}}(b_1, \dots, b_r)$, it follows that $\mathfrak{B} \models \chi(x_1, \dots, x_r)$. $\square$

The representatives of each of the equivalence classes of Lemma 22 can be taken to be the (r, i)-conditions.

Lemma 23. *For any* $n \in \mathbb{N}$ *there are, up to logical equivalence, only finitely many* $\mathcal{L}^r_{\omega\omega}[\tau, P^+]$*-formulas of quantifier rank* $\leq n$.

Proof. By induction on n.

$n = 0$: Let $\psi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ be an atomic or negated atomic formula. Since τ is finite and relational, the set of (r, r)-conditions is finite. The case for quantifier free formulas follows from Lemma 22 if we take Φ to be the set of (r, r)-conditions.

Induction step: Suppose we have already proved there are formulas $\{\phi_1, \dots, \phi_h\} \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ of quantifier rank $\leq n$, and formulas $\{\chi_1, \dots, \chi_k\} \in \mathcal{L}^{r+1}_{\omega\omega}[\tau, P^+]$ of quantifier rank $\leq n$ such that every formula in $\mathcal{L}^r_{\omega\omega}[\tau, P^+]$ (respectively in $\mathcal{L}^{r+1}_{\omega\omega}[\tau, P^+]$) of quantifier rank $\leq n$ is logically equivalent to some ϕ_i (respectively χ_j).

Given a formula in $\psi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ of quantifier rank $\le n+1$, it can be proved by induction on ψ that it is contained in

$$\langle \Phi_1 \rangle = \langle \{\phi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+] : \mathrm{qr}(\phi) \le n\} \cup \{\exists x \chi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+] : \mathrm{qr}(\chi) \le n\} \rangle$$

We prove:

($*$) Any such ψ is logically equivalent to a formula in

$$\langle \Phi_2 \rangle = \langle \{\phi_1, \dots, \phi_h\} \cup \{\exists x_r \chi_1, \dots, \exists x_r \chi_k\} \rangle.$$

But then, by Lemma 22, $\langle \Phi_2 \rangle$ contains only finitely many formulas which are pairwise logically non-equivalent.

Proof of ($*$): By induction hypothesis, every $\psi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ of quantifier rank $\le n$ is logically equivalent to some ϕ_i. Now, if $\exists x \chi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ and $\mathrm{qr}(\chi) \le n$, then $\mathfrak{A} \models \exists x \chi(a_1, \dots, a_r, x)$ iff there is $a \in A$ $\mathfrak{A} \models \chi(a_1, \dots, a_r, a_{r+1})$. Now, $\chi \in \mathcal{L}^{r+1}_{\omega\omega}[\tau, P^+]$ and has quantifier rank $\le n$, hence is logically equivalent to some χ_j; thus $\exists x \chi$ is equivalent to $\exists x_r \chi_i$. Finally, it is easy to verify that if every formula in Φ_1 is logically equivalent to a formula in Φ_2, then every formula in $\langle \Phi_1 \rangle$ is logically equivalent to a formula in $\langle \Phi_2 \rangle$. □

Lemma 24. *The following conditions are equivalent:*

(i) *For every r and $\psi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ of quantifier rank $\le n$, $\mathfrak{A} \models \psi \Rightarrow \mathfrak{B} \models \psi$.*

(ii) *There is an F_P-back-and-forth sequence $P_0 \dots P_n$ for $(\mathfrak{A}, \mathfrak{B})$.*

Proof. We prove by induction on $m < n$ that for $(a_1, \dots, a_r) \in A$, and $(b_1, \dots, b_r) \in B$, the following conditions are equivalent:

(i') For all $\psi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$ of quantifier rank $\le m, \mathfrak{A} \models \psi(a_1, \dots, a_r) \Rightarrow \mathfrak{B} \models \psi(b_1, \dots, b_r)$.

(ii') There is a P-back-and-forth sequence $P_0 \supseteq P_1 \supseteq \dots P_m$ for $(\mathfrak{A}, \mathfrak{B})$ such that $\{\langle a_1, b_1 \rangle, \dots, \langle a_r, b_r \rangle\} \in P_0$.

(i') $\Rightarrow$(ii'): Clearly, $m = 0$ gives us a partial P-isomorphism with domain $\langle a_1, \dots, a_r \rangle$ and range $\langle b_1, \dots, b_r \rangle$.

Induction step: Suppose we have proved the claim for $m < n$. Then we have a sequence $P_0 \supseteq P_1 \supseteq, \dots, P_m$ such that $\{\langle a_1, b_1 \rangle, \dots, \langle a_{r+m}, b_{r+m} \rangle\} \in P_0$. Let $p \in P_m$ be such that $p(a_i) = b_i$, for $i = 1, \dots, r{+}m$. Take $\xi(a_1, \dots, a_{r+m}, a)$ to be the $(r{+}m{+}1, r+1)$-condition such that $\mathfrak{A} \models \xi(a_1, \dots, a_{r+m}, a)$. Then $\exists x \xi(a_1, \dots, a_{r+m}, x)$ has quantifier rank m. By induction hypothesis, $\mathfrak{B} \models \exists x \xi(b_1, \dots, b_{r+m}, x)$. Therefore there is $b \in B$ such that $\mathfrak{B} \models \xi(b_1, \dots, b_{r+m}, b)$. Let P_{m+1} be the set of partial P-isomorphisms q between $\mathfrak{A}$ and $\mathfrak{B}$ such that $dom(q) = \{a_1, \dots, a_{r+m}, a\}$ and $\mathfrak{A} \models \xi(a_1, \dots, a_{r+m}, a)$ for ξ a complete $(r{+}m{+}1, r{+}1)$-condition, and $(a_1, \dots, a_{r+m}) = dom(p)$ for some $p \in P_m$. We then have $p \cup \{\langle a, b \rangle\} \in P_{m+1}$. The back condition is proved in a similar way.

(ii') $\Rightarrow$(i'): By induction on the complexity of formulas ψ. Suppose $\psi \in \mathcal{L}^r_{\omega\omega}[\tau, P^+]$, $\mathrm{qr}(\psi) \le n$, $m < n, p \in P_m$, and $a_1, \dots, a_r \in \mathrm{dom}(p)$. The atomic case comes from Definition 16. The connective cases are clear. Suppose $\psi = \exists x \phi$, and

$\mathfrak{A} \models \psi(a_1, \ldots, a_r)$. Then there is an element $a \in A$ such that $\mathfrak{A} \models \phi(a_1, \ldots, a_r, a)$. Using the forth condition of P_m, take the element such that there is $q \in P_{m+1}$, $q \supset p$, $a \in \mathrm{dom}(q)$, and $\mathfrak{A} \models \phi(a_1, \ldots, a_r, a)$. By induction hypothesis $\mathfrak{B} \models \phi(p(a_1), \ldots, p(a_r), q(a))$. Thus $\mathfrak{B} \models \phi(p(a_1), \ldots, p(a_r), b)$, and therefore $\mathfrak{B} \models \psi(p(a_1), \ldots, p(a_r))$. $\square$

Proof of Theorem 19. For each $m \in \omega$, we construct $\theta_m \in \mathcal{L}_{\omega\omega}[\tau, P^+]$ such that $Mod(\phi) \subseteq Mod(\theta_m)$ as

$$\theta_m = \bigvee_{\mathfrak{A} \in Mod(\phi)} \bigwedge \{\chi : \chi \text{ (m,0)-condition, and } \mathfrak{A} \models \chi\}$$

That θ_m is in $\mathcal{L}_{\omega\omega}[\tau, P^+]$ follows from the fact that there are only finitely many $(m, 0)-$conditions. (Lemma 23). Clearly, if $\mathfrak{A} \models \phi$, then $\mathfrak{A} \models \theta_m$ for all m. So $Mod(\phi) \subseteq Mod(\theta_m)$ for all m.

If there is a $k \in \omega$ such that $Mod(\theta_k) \cap Mod(\psi) = \emptyset$, take $\theta = \theta_k$. Suppose to the contrary that there is no such k. In this case, there is $\mathfrak{B}_k \in Mod(\psi)$ such that $\mathfrak{B}_k \models \bigwedge\{\chi : \chi \text{ (k,0)-condition, and } \mathfrak{A}_k \models \chi\}$, for some $\mathfrak{A}_k \in Mod(\phi)$, for each $k \in \omega$. So, we have $\mathfrak{A}_k$ and $\mathfrak{B}_k$ such that for all $(k, 0)$-conditions χ, $\mathfrak{A}_k \models \chi \Rightarrow \mathfrak{B}_k \models \chi$ for each k. By Downward Löwenhein-Skolem property we can take $\mathfrak{A}_k$ and $\mathfrak{B}_k$ countable infinite, and such that $A_k = B_k$. By Lemma 24 there is a sequence of P-isomorphisms $(I_m)_{m \leq k}$ from $\mathfrak{A}_k$ to $\mathfrak{B}_k$.

We now define a new model $\mathfrak{C}_k$ with the same domain as $\mathfrak{A}_k$. First, we choose any mapping from $\bigcup_{m \leq k} I_m$ into A_k. Its value under this mapping will be the code of any partial P-isomorphism in $\bigcup_{m \leq k} I_m$. Let π be a renaming $\pi : \tau \to \tau'$ such that $\pi(S) = S$ for all $S \in \tau \setminus P$, and $\pi(P) = P'$.[5] Let ρ be a renaming from τ' to a disjoint copy τ' of it, $\rho : \tau' \to \tau''$. Let $\tau^* = \tau' \cup \tau'' \cup \sigma$, where σ contains the following relation symbols:

(1) Two unary relations H, U;
(2) two binary relations $<, I$;
(3) one ternary relation G.

Let $\mathfrak{C}_k$ be a $\tau \cup \tau'$-structure with the following interpretations:

(1) $\mathfrak{C}_k \upharpoonright \tau = \mathfrak{A}_k$, $\mathfrak{C}_k \upharpoonright \tau' = \mathfrak{B}_k$,
(2) $U^{\mathfrak{C}_k} = \{0, \ldots, k\}$,
(3) $<^{\mathfrak{C}_k}$ is the natural ordering on $\{0, \ldots, k\}$,
(4) P' is a renaming of P.
(5) $H^{\mathfrak{C}_k} p$ iff $p \in \bigcup_{m \leq k} I_m$,
(6) $I^{\mathfrak{C}_k} mp$ iff $m \leq k$ and $p \in I_m$,
(7) $G^{\mathfrak{C}_k} pab$ iff $p \in \bigcup_{m \leq k} H_m(a)$, and $p(a) = b$.

Then $\mathfrak{C}_k$ is a model of the conjunction θ_k of the following $\mathcal{L}^*[\tau^*]$-sentences:

[5]We make the renaming of P to P' because a logic not closed under negation for P cannot express $\mathfrak{A} \models P(a) \to \mathfrak{B} \models P(b)$, as we want to do in the construction of the P-back-and-forth system for $\mathcal{L}^*$.

(1) ϕ^{π}, ψ^{π}.
(2) " $<$ is a discrete linear ordering with a first element with at least k elements",
(3) "U is the field of $<$",
(4) "each $p \in H$ is a partial P'-isomorphism, that is,

$$\forall p(H(p) \to \forall x \forall y \forall u \forall v(G(p,x,u) \wedge G(p,y,v) \to (x = y \leftrightarrow u = v))),$$

that accounts for the injectivity, together with, for example for T binary

$$\forall p(H(p) \to \forall x \forall y \forall u \forall v(G(p,x,u) \wedge G(p,y,v) \to (T(x,y) \leftrightarrow T(u,v))),$$

for every $T \in \tau'/P'$, and

$$\forall p(H(p) \to \forall x \forall y \forall u \forall v(G(p,x,u) \wedge G(p,y,v) \to (P'(x,y) \to P'(u,v))).$$

(5) "for each $u \in U$ the set I_u is not empty", that is,

$$\forall u(U(u) \to \exists p(H(p) \wedge I(u,p))),$$

(6) "the sequence of I_u's has the forth property", that is,

$$\forall u \forall v(v < u \to \forall p(I(u,p) \to \forall x \exists q \exists y(I(v,q)$$
$$\wedge G(q,x,y) \wedge \forall z \forall w(G(p,z,w) \to G(q,z,w))))),$$

(7) "the sequence of I_u's has the back property", that is,

$$\forall u \forall v(v < u \to \forall p(I(u,p) \to \forall x \exists q \exists y(I(v,q)$$
$$\wedge G(q,y,x) \wedge \forall z \forall w(G(p,z,w) \to G(q,z,w))))),$$

By compactness, the set $\{\theta_k : k \in \omega\}$ has a model, and by Downward Löwenhein-Skolem property a countable model $\mathfrak{D}$. Let $\mathfrak{A} = \mathfrak{D}^{-\pi} \restriction \tau$, and $\mathfrak{B} = (\mathfrak{D}^{-\pi} \restriction \tau')^{-\rho}$. Now let $d_0 >^{\mathfrak{D}^{-\pi}} d_1 >^{\mathfrak{D}^{-\pi}} \ldots$ be the descending chain in $\mathfrak{D}^{-\pi}$. By compactness, we may assume it exists. Let $J = \{p : I^{\mathfrak{D}^{-\pi}} d_k p \text{ for some } k\}$. Since $\mathfrak{D}^{-\pi}$ is a model of $\theta_k^{-\pi}$, by (5) the set I_k is not empty and we can identify p with the partial P-isomorphism $\{(a,b) : G^{\mathfrak{D}^{-\pi}} pab\}$ from $\mathfrak{A}$ to $\mathfrak{B}$. By (6) and (7), J has the F_P-back-and-forth property. Therefore $J : \mathfrak{A} \cong_p^P \mathfrak{B}$. By Lemma 13, $\mathfrak{A} \cong^P \mathfrak{B}$. Since $Mod(\phi)$ is closed under P-isomorphisms, $\mathfrak{A} \models \phi$ implies $\mathfrak{B} \models \phi$, but then $\mathfrak{B} \in Mod(\phi) \cap Mod(\psi)$, a contradiction. □

Let τ be any relational vocabulary and P a predicate symbol in τ. Let $\psi, \phi, \theta \in \mathcal{L}_{\omega\omega}[\tau]$. Theorem 19 has as corollary a weaker form of Lyndon's interpolation theorem [Ly59]:

Corollary 25. *Let ψ, ϕ be such that $\psi \models \phi$. Then there is a sentence θ such that:*

(i) *$\psi \models \theta$ and $\theta \models \phi$.*
(ii) *θ contains only those predicate symbols that occur in both ψ and ϕ.*
(iii) *If ψ is P-positive, then so is θ.*

Proof. Take $\mathcal{L}^*$ in Theorem 19 to be the logic of classes $\mathbf{K}$ such that for some $\chi \in \mathcal{L}_{\omega\omega}[\tau]$, $\mathfrak{A} \in \mathbf{K}$ iff there is $\mathfrak{B}$ such that $\mathfrak{B} \models \chi$ and $\mathfrak{B} \restriction_\tau = \mathfrak{A}$. Let τ_1 be the vocabulary of ψ, and τ_2 that of ϕ, and suppose P occurs only positively in ψ. Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{L}^*$ be such that:

$$\mathfrak{A} \in \mathbf{K}_1 \text{ iff there is } \mathfrak{B} \text{ such that } \mathfrak{B} \models \psi \text{ and } \mathfrak{B} \restriction_{\tau_1} = \mathfrak{A}.$$

$$\mathfrak{A} \in \mathbf{K}_2 \text{ iff there is } \mathfrak{B} \text{ such that } \mathfrak{B} \models \neg\phi \text{ and } \mathfrak{B} \restriction_{\tau_2} = \mathfrak{A}.$$

Then $\mathbf{K_1} \cap \mathbf{K_2} = \emptyset$, and $\mathbf{K_1}$ is closed under P-isomorphisms. By Theorem 19, there is a $\theta \in \mathcal{L}_{\omega\omega}[\tau, P^+]$ such that $\mathbf{K_1} \subseteq Mod(\theta)$ and $\mathbf{K_2} \cap Mod(\theta) = \emptyset$. Then $\psi \models \theta$ and $\theta \models \phi$. $\square$

3.2. **Lindström's theorem for extensions of $\mathcal{L}_{\omega\omega}$.**

Lindström's theorem can be extended so that other than first-order logic can have a model-theoretic characterization. What closure properties must a logic preserve with respect to first-order logic in order to be characterized by compactness and the Downward Löwenhein-Skolem property? Let us look at the proof of Lindström's theorem. It works by contradiction using compactness and Downward Löwenhein-Skolem property in the construction of an isomorphism between two models that belong to disjoint classes. Negation is needed to express the fact that the two models belong to disjoint classes: $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \neg\phi$. Can we work on a proof for Lindström's theorem without negation?

3.2.1. *Countable Extensions without Negation.*

We start with the case of countable logics, those in which $\mathcal{L}[\tau]$ is countable for τ countable. Examples of countable logics are first-order logic, second order logic, the logic of PC classes, and the extension of first-order logic by countably many generalized quantifiers.

We present a countable logic $\mathcal{L}$, not closed under negation, compact and with the Löwenheim-Skolem property, that has infinitely many extensions each of which is compact and has the Löwenheim-Skolem property. In fact, we prove that if a logic $\mathcal{L}$ which is compact and has the Löwenheim-Skolem property is extended countably many times, the extended logic cannot be a maximal logic with respect to those two properties. In particular, if we consider the starting logic to be the logic of PC classes over first-order logic, this counterexample shows that Hintikka's IF logic [Hi96] does not have a Lindström's characterization, in the sense that compactness and Löwenheim-Skolem properties may hold in an extension of it.

Let τ be a vocabulary consisting of two 3-ary relation symbols $\tau = \{P, M\}$, where P stands for addition and M for multiplication. Let T be a positive complete first-order τ-theory extending PA. We can construct such positive theory replacing each $\neg P(x, y, z)$ by $\exists u(P(x, y, u) \wedge u \neq z)$. Now, $u \neq z$ can be written as $\exists v(v > 0 \wedge (P(u, v, z) \vee P(z, v, u))$. We do not have $>$ in our language, so we write $v > 0$ as $\exists w P(1, w, v)$. Now we still have to eliminate the occurrence of constant 1. But

$t = 1$ is equivalent to "$\forall x M(x,t,x)$". The replacement of "$\neg M(x,y,z)$" works in a similar way. It is essential that we get rid of all negations. Let $\mathbf{Q}_T$ be the class of models of T. Consider an abstract sentence ϕ such that $Mod_\tau(\phi) = Mod_\tau(T)$. We write ϕ by means of a generalized quantifier Q_T.

$$\phi := Q_T xyzuvw P(x,y,z) M(u,v,w)$$

The satisfaction relation for Q_T is:

$$\mathfrak{A} \models Q_T xyzuvw \theta(x,y,z)\eta(u,v,w) \leftrightarrow (A, \theta^{\mathfrak{A}}(x,y,z), \eta^{\mathfrak{A}}(u,v,w)) \models T.$$

The picture is as follows: Let $\mathfrak{A}$ be a model in any vocabulary that leads to formulas in three free variables. $\mathfrak{A}$ is a model of $Qxyz\theta(x,y,z)\eta(u,v,w)$, if there is a τ-model of T that has the same domain as $\mathfrak{A}$, and in which the set of triples that satisfies $P(x,y,z)$ (resp. $M(u,v,w)$) is equal to the set of triples that satisfies $\theta(x,y,z)$ (resp. $\eta(u,v,w)$) in $\mathfrak{A}$.

Finally, we define the logic $\mathcal{L}_{Q_T}$ as the logic that results from adding to the logic $PC(\mathcal{L}_{\omega,\omega})$ the class $\mathbf{Q}_T$, and closing under the PC-operation. $\mathcal{L}_{Q_T}$ extends $PC(\mathcal{L}_{\omega,\omega})$.

Proposition 26. *$\mathcal{L}_{Q_T}$ is a proper extension of $PC(\mathcal{L}_{\omega,\omega})$.*

Proof. By Gödel-Rosser incompleteness theorem, which says that no complete extension of PA is recursively enumerable, it is enough to show that PC-classes are recursively enumerable. So let $\tau' \subseteq \tau$ be vocabularies, θ be a τ-sentence, and $K = [Mod(\theta)] \upharpoonright \tau'$. Let T be the set of first-order τ'-sentences that are true in all models in K. Given a first-order τ'-sentence $\psi, \psi \in T \leftrightarrow \ \models \theta \rightarrow \psi$. $\square$

We want to prove that $\mathcal{L}_{Q_T}$ is compact and satisfies the Löwenheim-Skolem property. It is enough to show that $\mathcal{L}_{Q_T}$ is a sublogic of a compact logic that satisfies the Löwenheim-Skolem property (See [BF85], p.45).

Proposition 27. *If $\mathcal{L}$ is a logic such that every class in $EC(\mathcal{L})$ is in $PC_\Delta(\mathcal{L}_{\omega,\omega})$, then $\mathcal{L}$ is compact and satisfies the Löwenheim-Skolem property.*

Proof. To prove compactness, let Φ be a countable set of $\mathcal{L}$-sentences in a vocabulary τ. Suppose Φ is finitely satisfiable. By hypothesis, for every sentence $\phi_i \in \Phi$, there is a first-order theory S_i in an extended vocabulary τ_i' such that $Mod_{\tau_i'}(S_i) \upharpoonright \tau = Mod_\tau(\phi_i)$. For each pair ϕ_i, ϕ_j, we make the extension $\tau_i - \tau$ disjoint from $\tau_j - \tau$. As Φ is finitely satisfiable, any finite intersection of the classes of models of the S_i's is non empty. By compactness of first-order logic, the intersection of them all is non empty, and thus Φ has a model.

For Löwenheim-Skolem, suppose $\phi \in \mathcal{L}$ has an infinite model. Then there is a countable first-order theory S in an extended language whose class of models is a class that contains the expansion to the extended language of each model of ϕ, and therefore it has an infinite model. As first-order logic satisfies Löwenheim-Skolem for countable sets of sentences, S has a countable model, and therefore ϕ has a countable model. $\square$

Proposition 28. *Every τ-elementary class in $\mathcal{L}_{Q_T}$ is in $PC_\Delta(\mathcal{L}_{\omega,\omega})$.*

Proof. We prove by induction on the complexity of formulas, that for every formula in $\mathcal{L}_{Q_T}$, there is a set of formulas $S \subset \mathcal{L}[\tau']$, such that $Mod_\tau(\phi) = \{\mathfrak{A} \restriction \tau : \mathfrak{A} \models S(\tau')\}$, where $\tau \subseteq \tau'$.

For simplicity we assume the language of T consists of the predicate P only. The general case presents no new difficulties. Thus, the new quantifier is $Q_T xyz\phi(x, y, z)$. Suppose also that $\tau' = \tau \cup \{R\}$, with R unary. The sentence $Q_T xyzP(x, y, z)$ is equivalent by assumption to $\exists PT(P)$. Now suppose η is of the form $\exists R \bigwedge S(R)$, we prove $Q_T xyzP(x, y, z) \wedge \eta$ is in PC_Δ. By assumption, $\mathbf{Q}_T$ is axiomatized by $\exists PT(P)$. PC_Δ is closed under conjunction, so we are done. The case of disjunction is proved in a similar way. Now suppose $\eta := Q_T xyz\phi(x, y, z, \vec{u})$. So assume $\phi(x, y, z, \vec{u})$ defines a PC_Δ class. That means that there is a new predicate R, and a set of formulas $S(x, y, z, \vec{u}) = \{\eta_i(x, y, z, \vec{u}) : i \in \omega\}$ such that $Mod_\tau(\phi) = Mod_{\tau\cup\{R\}}(S) \restriction \tau$. On the other hand, by definition, $Q_T xyz\phi(x, y, z, \vec{u})$ is equivalent to $T(\phi(x, y, z, \vec{u})/P)$, where $T(\phi(x, y, z, \vec{u})/P)$ is the theory resulting of replacing everywhere in T P by $\phi(x, y, z, \vec{u})$. By induction hypothesis, if $\mathfrak{A}$ is a model of $\phi(x, y, z, \vec{u})$, $\mathfrak{A}$ is the reduct of a model of $S(x, y, z)$. So if we take all models of $\phi(x, y, z, \vec{u})$ such that $(\mathfrak{A}, \phi^{\mathfrak{A}}) \models T$, these models have an expansion to a model of T', where T' is the result of substituting P in T by $\exists RS(x, y, z)$. We conclude that $Q_T xyz\phi(x, y, z, \vec{x})$ is equivalent to a reduct of a model of T', in the sense that any model of $Q_T xyz\phi(x, y, z, \vec{x})$ has an expansion to a model of T'. The only thing we have to prove is that $\bigwedge_n \eta'_n$, for $\eta'_n \in T'$, defines a PC_Δ, that is, T of the form $\exists R \bigwedge_n \psi_n$. Let $T = \{\eta_n : n \in \omega\}$, and $S = \{\theta_m(x, y, z) : m \in \omega\}$.

Now, to get η'_n, we have to substitute in η_n P by $\exists R \bigwedge_m \theta_m(x, y, z)$. So, as T does not have negation, a typical sentence in T' is of the form

$$\eta'_n = \mathfrak{Q}xyz\exists R \bigwedge_m \theta_m(x, y, z) \tag{1}$$

where $\mathfrak{Q}$ stands for a sequence of quantifiers.

We have to describe how to get a PC_Δ form from (1). Not having negation will allow us to do that. By allowing R to be binary, we can swap its position with the universal quantifier, using the Tarski-Kuratowski algorithm. We prove

$$\mathfrak{A} \models \forall x\exists R \bigwedge_m \theta_m(x, y, z) \text{ iff } \mathfrak{A} \models \exists R_0\forall x \bigwedge_m \theta^*_m(x, y, z) \tag{2}$$

where $\theta^*_m(x, y, z)$ is the result of replacing $R(t)$ by $R_0(x, t)$ in $\theta_m(x, y, z)$

$$\begin{aligned}
&(3) & \mathfrak{A} &\models \forall x \exists R \theta_m(x,y,z) \text{ iff} \\
&(4) & \text{for some } R_0,\ \text{for all } a \in A, \mathfrak{A} &\models \exists R \bigwedge_m \theta_m(a,y,z) \text{ iff} \\
&(5) & \text{for some } R_0,\ \text{for all } a \in A, (\mathfrak{A}, R_0(a,t)) &\models \bigwedge_m \theta^*_m(a,y,z) \text{ iff} \\
&(6) & (\mathfrak{A}, R_0(x,t)) &\models \forall x \bigwedge_m \theta^*_m(x,y,z) \text{ iff} \\
&(7) & \mathfrak{A} &\models \exists R_0 \forall x \bigwedge_m \theta^*_m(x,y,z)
\end{aligned}$$

Which in turn is equivalent to

$$\mathfrak{A} \models \exists R_0 \bigwedge_m \forall x \theta^*_m(x,y,z), \tag{8}$$

which finishes the proof of (2).

The existential second order quantifier commutes with the existential quantifier.

Now we have each η'_n of the form

$$\exists R^n_0, \ldots, R^n_{l_n} \bigwedge_m \xi^n_m.$$

If we take the conjunction of all we get

$$\bigwedge_n \exists R^n_0, \ldots, R^n_{l_n} \bigwedge_m \xi^n_m \tag{9}$$

This is equivalent to (10) below, since each two sequences $R^i_0, \ldots, R^i_{l_i}$ and $R^j_0, \ldots, R^j_{l_j}$ are disjoint.

$$\exists R^0_0, \ldots, R^0_{l_0}, \ldots, R^i_0, \ldots, R^i_{l_i}, \ldots \bigwedge_{n,m} \xi^n_m \tag{10}$$

And we reached the desired PC_Δ form. $\square$

Corollary 29. *$\mathcal{L}_{Q_T}$ satisfies the compactness theorem and the Downward Löwenhein-Skolem properties, and extends PC.* $\square$

Corollary 29 is generalized by the following

Theorem 30. *If $\mathcal{L}$ is any countable logic, then it is not maximal with respect to compactness and the Löwenheim-Skolem property, not even in the family of countable logics.*

Proof. Take T as above to be a complete extension of PA. There are $2^{\aleph_0}$ such complete extensions (see, for instance [Ka], p.39). So, as $\mathcal{L}$ has only countably many classes in the vocabulary $\{P, M\}$, there must be a complete extension T of

PA which is not definable in $\mathcal{L}$. So, we get an extension of $\mathcal{L}$ by adding $\mathbf{Q}_T$ to it. Call this extension $\mathcal{L}^*$. To show that $\mathcal{L}^*$ is compact and have the Löwenheim-skolem property, we just have to find an extension of it with these two properties. Given the closure properties of $\mathcal{L}$, and the form of T (it is a positive theory) the same argument that we used for the proofs of Propositions 27 and 28, shows that $PC_\Delta(\mathcal{L})$ is compact, satisfies the Löwenheim-Skolem property, and that it extends $\mathcal{L}^*$, therefore $\mathcal{L}^*$ is compact and has the Löwenheim-Skolem property. □

3.2.2. *Uncountable Extensions without Negation.*

In this section we introduce the method of partial reduction, which is a method in the study of logics without negation that does not have a clear counter-part in the context where we do have negation. So this is a method characteristic to the special circumstances of our investigation.

The first natural uncountable extension to look at is $PC_\Delta(\mathcal{L}_{\omega\omega})$. We prove in this section it is not a maximal logic with respect to compactness and Löwenheim-Skolem properties. Although in a later section we will prove no such extension exists if we restrict to small logics (logics with a set number of model classes), it is still interesting to observe the method we have to use due to the lack of negation. The existence of the largest logic with compactness and Löwenheim-Skolem properties remains an open problem. In a later section we prove no such logic exists if we consider compactness and Löwenheim-Skolem down to $\aleph_1$.

Definition 31. *Let Q be a generalized quantifier. We call $\mathcal{L}_{\omega\omega}(Q)$ the logic obtained by adding the quantifier Q to $\mathcal{L}_{\omega\omega}$ and closing under conjunction, disjunction, existential and universal quantification and Q-projection, but not negation.*

Definition 32. *Suppose $\mathbf{K}$ is a class of models. Suppose $\mathcal{L}$ and $\mathcal{L}'$ are logics. We say that $\mathcal{L}$ is* partially $\mathbf{K}$-reducible *to $\mathcal{L}'$ if for any $\phi \in \mathcal{L}$ there is $\phi^* \in \mathcal{L}'$, such that $\phi \to \phi^*$ in all models, and $\phi \leftrightarrow \phi^*$ in models in $\mathbf{K}$.*

Lemma 33. *Let Q be a quantifier, and $\mathbf{K}$ a class of models. If for any first-order formula $\phi(\vec{x}, \vec{y})$, $Q\vec{x}\phi(\vec{x}, \vec{y})$ is partially $\mathbf{K}$-reducible to first order logic, then so is the whole logic $\mathcal{L}_{\omega\omega}(Q)$.*

Proof. We prove this by induction on the formation of formulas. Suppose $\phi(\vec{x}) \in \mathcal{L}_{\omega\omega}(Q)$ is partially $\mathbf{K}$-reducible. Call $\phi_r(\vec{x})$ its partial $\mathbf{K}$-reduction to first-order logic.

The Boolean cases are clear. Suppose $\eta(\vec{y}) = \exists x\phi(x, \vec{y})$. By induction hypothesis, $\phi(x, \vec{y})$ has a partial $\mathbf{K}$-reduction, $\phi_r(x, \vec{y})$, so we define $\eta_r(\vec{y}) := \exists x\phi_r(x, \vec{y})$. Then $\eta(\vec{y}) \to \eta_r(\vec{y})$ in all models, and the converse is valid in $\mathbf{K}$-models. For Q suppose $\eta(\vec{y}) = Q\vec{x}\phi(\vec{x}, \vec{y})$. We assume $\theta(P)$ to be the partial reduction of $Q\vec{x}P(\vec{x})$, where each occurrence of P in θ is positive. Suppose $\mathfrak{A} \models Q\vec{x}\phi(\vec{x}, \vec{y})$. By induction hypothesis, ϕ has a partial $\mathbf{K}$-reduction ϕ. By definition, there is a model $\mathfrak{C} \in Mod(Q\vec{x}P(\vec{x}))$ with the same domain as $\mathfrak{A}$ such that $\phi^{\mathfrak{A}} = P^{\mathfrak{C}}$. By assumption $\mathfrak{C}$

is a model of $\theta(P)$. For any model $\mathfrak{B}, \phi^{\mathfrak{B}} \subseteq \phi_r^{\mathfrak{B}}$, so by substitution there exists a sentence θ^*, obtained from θ by replacing everywhere P by $\phi_r(\vec{x}, \vec{y})$. Clearly $\mathfrak{A} \models \theta^*$. Now suppose $\mathfrak{A} \models \theta^*$ and $\mathfrak{A} \in \mathbf{K}$. By assumption $\mathfrak{A} \models \theta(P)$ iff $\mathfrak{A} \models Q\vec{x}P(\vec{x})$. By substitution, there is a sentence $Q\vec{x}\phi_r(\vec{x}, \vec{y})$, obtained from $Q\vec{x}P(\vec{x})$. Then $\mathfrak{A} \models \theta^*$ iff $\mathfrak{A} \models Q\vec{x}\phi_r(\vec{x}, \vec{y})$, that in $\mathbf{K}$ is equivalent to say that $\mathfrak{A} \models Q\vec{x}\phi(\vec{x}, \vec{y})$. □

Lemma 34. *Let* $\mathbf{K}$ *be a class of models containing all countable models. Suppose* $\mathcal{L}$ *is partially* $\mathbf{K}$*-reducible to a compact logic with the Downward Löwenhein-Skolem property for countable sets of sentences* $\mathcal{L}'$*. Then* $\mathcal{L}$ *is compact.*

Proof. Let Φ be a finitely satisfiable set of $\mathcal{L}$-sentences. Let Φ^* be the set of $\mathcal{L}'$ -sentences that are the partial reductions of each sentence in Φ. Φ^* is finitely satisfiable, for take any finite set $\Sigma^* \subset \Phi^*$, and look at the corresponding set of sentences $\Sigma \subset \Phi$. By hypothesis, Σ has a model, and it is a model of each of the sentences in Σ^*, so Φ^* is finitely satisfiable. By compactness, it has a model, and by Downward Löwenhein-Skolem theorem for countable sets of sentences, it has a countable model $\mathfrak{M}$. As $\mathfrak{M}$ is a countable model of each sentence in Φ^*, it is a model of each sentence in Φ. □

Lemma 35. *Let* $\mathbf{K}$ *be a class of models containing all countable models. Suppose* $\mathcal{L}$ *is partially* $\mathbf{K}$*-reducible to a logic* $\mathcal{L}'$ *with the Downward Löwenhein-Skolem property. Then* $\mathcal{L}$ *satisfies Downward Löwenhein-Skolem property.*

Proof. Let ϕ be an $\mathcal{L}$-sentence with an infinite model. Let ϕ^* be a partial $\mathbf{K}$-reduction of ϕ to $\mathcal{L}'$. By Downward Löwenhein-Skolem theorem, ϕ^* has a countable model, and by definition of partial reduction, ϕ has also a countable model. □

Definition 36. *Let* Q *be a generalized quantifier. Define* $PC_\Delta(Q)$ *as the logic obtained by adding* Q *to* PC_Δ*, and closing under the* PC_Δ*-operation.*

Lemma 37. *Let* $\mathbf{K}$ *be a model class containing all countable models. If* Q *is partially* $\mathbf{K}$*-reducible to* $L_{\omega\omega}$*, then* $PC_\Delta(Q)$ *is compact and satisfies the Downward Löwenhein-Skolem property.*

Proof. For compactness, let Φ be a finitely satisfiable countable set of $PC_\Delta(Q)$-sentences. That means that there is an extended vocabulary (it can be countable), in which we have a finitely satisfiable countable set of $L_{\omega\omega}(Q)$-sentences. By the compactness result proved above, this set has a model. The reduction of this model to the initial vocabulary is a model for Φ. For Downward Löwenhein-Skolem property, we argue similarly. Let $\phi \in PC_\Delta(Q)[\tau]$, with a model $\mathfrak{M}$. Then there is an extended vocabulary τ' and a sentence $\phi^* \in \mathcal{L}_{\omega\omega}(Q)[\tau']$ that has a model $\mathfrak{M}'$ that is an expansion of $\mathfrak{M}$ to the vocabulary τ'. By the result proved above, this sentence has a countable model $\mathfrak{N}$. The reduction to the initial vocabulary of this model, is a countable model for ϕ. □

Definition 38. *Let Q_1, Q_2 be two quantifiers. We say that Q_1 is* partially **K**-reducible to Q_2 *if for all $\phi(x) \in \mathcal{L}_{\omega\omega}(Q_1)$, $\models Q_1 x\phi(x) \to Q_2 x\phi(x)$ in all models, and $\models Q_2 x\phi(x) \to Q_1 x\phi(x)$ in models in K. If Q_2 is first order definable, we say that Q_1 is* partially **K**-*reducible, or that we can* **K**-*reduce Q_1.*

Proposition 39. *Let* **K** *be a model class that contains all countable models. Let H_1 be such that $H_1 x\phi(x) \leftrightarrow ((Q_1 x\phi(x) \to \forall x\phi(x)) \wedge \exists x\phi(x))$. $\mathcal{L}_{\omega\omega}(H_1)$ is partially* **K**-*reducible to first order logic.*

Proof. H_1 is partially **K**-reducible. Indeed

$$\models ((Q_1 x\phi(x) \to \forall x\phi(x)) \wedge \exists x\phi(x)) \to \exists x\phi(x), \text{ and}$$
$$\models \exists x\phi(x) \to ((Q_1 x\phi(x) \to \forall x\phi(x)) \wedge \exists x\phi(x)) \text{ in countable models.}$$

We can apply this reduction to $H_1 x\phi(x)$ as many times as necessary to get a first-order sentence. Finally, we apply Lemma 33. □

Proposition 40. *$L_{\omega\omega}(H_1)$ is not in PC_Δ.*

Proof. The sentence $\exists x H_1 y(x \neq y)$ has no uncountable models. But PC_Δ has Upward Löwenhein-Skolem property, so $H_1 \notin PC_\Delta$. □

Corollary 41. *PC_Δ is not a maximal logic with respect to Compactness and Downward Löwenhein-Skolem properties.* □

In Section 5, we will prove that actually there is no maximal small logic with respect to compactness and Löwenheim-Skolem properties. The question is open for class logics, although we will also prove there is no such thing as the greatest logic which is compact and satisfies the Löwenheim-Skolem theorem down to $\aleph_1$.

4. Maximality and Interpolation

As said in the introduction, we present the connection between maximality and interpolation in the framework of back-and-forth systems. Two recent works [BvB99], [M00] study the relations between back and forth systems and interpolation. The strategy for proving interpolation theorems begins by finding an appropriate back and forth system for the logic. However, the existence of back and forth systems does not guarantee the success in finding interpolation theorems. As a list of negative results, in the case of extensions with generalized quantifiers, the main result of Caicedo says no extension of first-order logic by means of an arbitrary number of monadic quantifiers satisfies interpolation. Mostowski [Mo68] has a similar result in the case on a finite number of generalized quantifiers of arbitrary type. In this section we address the case of infinitary logics, although the conclusions extend to any logic. $L_{\infty\omega}$ and $L_{\kappa\omega}$ for $\kappa = \beth_\kappa$ both a have a back-and-forth system, a Lindström's type theorem [B71], but not interpolation. On the other hand, $\mathcal{L}_{\omega_1\omega}$ has interpolation but no Lindström's type characterization. We try to give a framework that connects interpolation and maximality and yet it is able to explain all these "anomalies".

We start by deriving Lindström's and interpolation theorems for first-order logic from a common source theorem, that is a generalization of Theorem 19, and that is proved in a similar way:

Theorem 42 (Separation Theorem)**.** *Let $\mathcal{L}^*$ be a compact logic with Downward Löwenhein-Skolem property. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be two disjoint $\mathcal{L}^*$-classes. Then there is a first-order sentence θ in the common language of $\mathbf{K}_1$ and $\mathbf{K}_2$ that separates them.*

Corollary 43 (Lindström's maximality theorem)**.** *$\mathcal{L}_{\omega\omega}$ is a maximal compact logic with negation satisfying the Löwenheim-Skolem property.*

Proof. Let $\mathcal{L}^*$ be closed under negation. □

Corollary 44 (Craig's interpolation theorem)**.** [Cr57] *Any two disjoint $PC(\mathcal{L}_{\omega\omega})$-classes can be separated by an $EC(\mathcal{L}_{\omega\omega})$-class.*

Proof. Take $\mathcal{L}^*$ to be the logic of $PC(\mathcal{L}_{\omega\omega})$ classes. □

There are many compact extensions of $\mathcal{L}_{\omega\omega}$ with LS. Corollary 45 shows that they obey a very strong lack of negation: the only sentences with negation are the first-order ones. Also at this point we notice how important is the fact that first-order logic is closed under negation, for the above corollaries do not behave similarly if we give negation up. Indeed, we saw in Section 3.1 that Theorem 19, analogous to Theorem 42, gives as corollary an interpolation theorem, but no characterization theorem exists for the logic $\mathcal{L}_{\omega\omega}[P^+]$. This logic has separation, but is not maximal with respect to compactness and Löwenheim-Skolem property. We can summarize this phenomena by saying that the PC extension of a compact logic with LS and some Karp property preserve compactness, LS and the Karp property on countable models, hence making the logic to satisfy separation, but new classes can be added freely, since the fact that the logic is not closed under negation does not restrict new classes to existing ones.

Corollary 45. *If $\mathcal{L}^*$ is a compact extension of $\mathcal{L}_{\omega\omega}$ with LS, then the only sentences of $\mathcal{L}^*$ that have negation are the first-order ones.* □

This is related to the fact that the Δ-extension of a $\mathcal{L}_{\omega\omega}$ is closed under Q-projection. That is, any compact and LS class Q whose complement is compact and LS is such that $\mathcal{L}_{\omega\omega}(Q)$ is automatically closed under Q.

Now we try to generalize these ideas for logics other than $\mathcal{L}_{\omega\omega}$. The following table contains a number of logics and their respective satisfaction of interpolation and generalized forms of Lindström's type maximality theorems. The model theoretic properties in the box for LT are the characterizing ones:

Logic	Lindström's theorem	Interpolation theorem
$\mathcal{L}_{\omega\omega}$	Compactness and Löwenheim-Skolem. [L69]	YES [Cr57]
$\mathcal{L}_{\infty\omega}$	Boundedness and Karp property. [B71]	NO [Ma71]
$\mathcal{L}_{\kappa,\omega}$ $(\kappa = \beth_\kappa)$	Well ordering number $\leq \kappa$ and Karp property. [B71]	NO [Ma71]
$\mathcal{L}_{\omega_1\omega}$	NONE	YES [L-E65]
L^1_κ	Löwenheim-Skolem and strong form of undefinability of well order. [ShV04]	YES [ShV04]
$L_{\omega\omega}[P^+]$	NONE	YES

Next definition introduces a relation R involved in the description of the back-and-forth system of a logic.

Definition 46. *Let τ be a vocabulary, R a binary relation between structures, and φ a sentence of a logic $\mathcal{L}$.*

(1) *We say that φ is R-*invariant *if*

$$\mathfrak{A}R\mathfrak{B} \textit{ and } \mathfrak{A} \models \varphi \textit{ imply } \mathfrak{B} \models \varphi$$

Denote the class of R-invariant sentences as $\mathcal{L}^R$. In case $\mathcal{L} = \mathcal{L}^R$, we say $\mathcal{L}$ is a logic of R-invariant sentences.

(2) *We say that φ* entails ψ along R, *written $\varphi \models_R \psi$, iff for all τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A}R\mathfrak{B}$, and $\mathfrak{A} \models \varphi$ then $\mathfrak{B} \models \psi$.*

Next theorem is introduced as a starting point of a generalized study of the Separation Theorem. It appeared in [Fl85].

Theorem 47. *Suppose there is given for any vocabulary τ a set $\Phi^\tau \subseteq \mathcal{L}_{\omega\omega}[\tau]$ and let $\mathcal{R}^\tau = Mod(\Phi^\tau)$. Assume that R is a binary relation between structures such that $\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A}, \mathfrak{B} \in \mathcal{R}$ for some τ. Suppose that*

(1) *R restricted to τ-structures is an equivalence relation.*
(2) *R is invariant under renamings.*
(3) *Given τ, for some $\tau' \supseteq \tau$, there are $\mathcal{L}_{\omega\omega}[\tau']$-sentences $\varphi_0, \varphi_1, \ldots$ such that for any τ-structures $\mathfrak{A}, \mathfrak{B}$, the following hold:*
   a. *$\mathfrak{A}R\mathfrak{B}$ iff $(\mathfrak{A}, \mathfrak{B}, \ldots) \models \{\varphi_i : i \in \omega\}$ for some choice of $\ldots$, and*

b. *for $n \in \omega$ the relation R_n on $\mathcal{R}$ given by*

$$\mathfrak{A} R_n \mathfrak{B} \textit{ iff } (\mathfrak{A}, \mathfrak{B}, \dots) \models \{\varphi_i : i \leq n\} \textit{ for some} \dots$$

has the following two properties:

a.1. *R_n is an equivalence relation on $\mathcal{R}^\tau$;*

a.2. *For $\mathfrak{A} \in \mathcal{R}^\tau$, there is $\psi^n_{\mathfrak{A}} \in \mathcal{L}_{\omega\omega}[\tau]$ such that for $\mathfrak{B} \in \mathcal{R}^\tau$:*

$$\mathfrak{A} R_n \mathfrak{B} \textit{ iff } \mathfrak{B} \models \psi^n_{\mathfrak{A}}.$$

Then

- *If $\mathcal{L}^* \geq \mathcal{L}^R_{\omega\omega}$ is a compact logic of R-invariant sentence not necessarily closed under negation, then any two $\mathcal{L}^*$-classes can be separated by an $\mathcal{L}^R_{\omega\omega}$-class. Hence, among the logics of R-invariant sentences closed under negation, $\mathcal{L}^R_{\omega\omega}$ is a maximal compact logic.*

□

As Flum pointed out, we could generalized this theorem to any logic $\mathcal{L}$ with a given property R, which must satisfy the above plus some further conditions depending on $\mathcal{L}$. So let $\mathcal{L}$ have any given generalized compactness property C, and let $\mathcal{L}^*$ be an extension of $\mathcal{L}^R$ satisfying C, not necessarily closed under negation. Our research program consist in being able to proof the following, that contains a small addenda to Flum's original result:

(F1) If $\mathcal{L}$ is itself a logic of R-invariant sentences, and $\mathcal{L}^*$ is closed under negation, we obtain that among logics of invariant R-sentences, $\mathcal{L}$ is a maximal logic which satisfies C.

(F2) Let $\phi, \psi \in \mathcal{L}^*$ be such that ϕ entails $\neg\psi$ along R (note that $\mathcal{L}^*$ does not necessarily have to be closed under negation. Then there is an $\mathcal{L}^R$-class that separates $Mod(\phi)$ and $Mod(\psi)$.

Example 48 (Lindström's maximality theorem citeLin69)**.** *$\mathcal{L}_{\omega\omega}$ is a maximal compact logic with negation and the Löwenheim-Skolem property.*

Proof. Let $\mathcal{L}$ be $\mathcal{L}_{\omega\omega}$, and R be the relation of partial isomorphism $\cong_p$. Let $\mathcal{L}^*$ be compact and closed under negation. It suffices to show that any logic with Löwenheim-Skolem property has the Karp property. We postpone this proof till Proposition 50.

Corollary 49 (Craig's interpolation theorem [Cr57])**.** *Any two disjoint $PC(\mathcal{L}_{\omega\omega})$-classes can be separated by an $EC(\mathcal{L}_{\omega\omega})$-class.* □

Proof. Let $\mathcal{L}^*$ be the logic of $PC(\mathcal{L}_{\omega\omega})$-classes, and R the relation of partial isomorphisms. PC is not invariant under Karp property, but it is enough for us to prove the following proposition, that it is also the bit of information we need for completing the proof of Example 48.

The following proposition, with a slight variation, appears in [BvB99].

Proposition 50. *Let $\mathcal{L}$ be a regular logic with LS. Let $\phi \in PC(\mathcal{L})$ and $\psi \in cPC(\mathcal{L})$. Then the following are equivalent:*

(1) *ϕ entails ψ along $\cong$.*
(2) *ϕ entails ψ along $\cong_p$.*

Proof. The proof of (2) implies (1) is trivial. Now suppose there are τ-structures such that we have

$$\mathfrak{A} \cong_p \mathfrak{B},\ \mathfrak{A} \models \phi \text{ and } \mathfrak{B} \models \neg\psi.$$

If A and B are countable we are done, since partially isomorphic countable structures are isomorphic. Let I, V, W be new unary predicates; and G be one new ternary relation. Set $\tau' = \tau \cup \{I, V, W.G\}$. Let ξ be the conjunction of the following $\mathcal{L}[\tau']$-sentences:

"V and W are disjoint"
$\phi^{\{x:V(x)\}}$,
$\neg\psi^{\{x:W(x)\}}$,
"each $p \in I$ is a mapping from V to W" that is,

$$\forall p(I(p) \to \forall x\forall y(Gpxy \to (V(x) \wedge W(y)))$$

"each $p \in I$ is a partial injective mapping" that is,

$$\forall p(I(p) \to \forall x\forall y\forall u\forall v(G(p,x,u) \wedge G(p,y,v) \to (x = y \leftrightarrow u = v))),$$

"each $p \in I$ preserves all the symbols in τ" for example, for binary $T \in \tau$,

$$\forall p(I(p) \to \forall x\forall y\forall u\forall v(G(p,x,u) \wedge G(p,y,v) \to (T(x,y) \leftrightarrow T(u,v))),$$

"the set I is not empty"
"the set I has the forth property", that is,

$$\forall p(I(p) \to \forall x(V(x) \to \exists q\exists y(I(q) \wedge G(q,x,y) \wedge \forall z\forall w(G(p,z,w) \to G(q,z,w))))),$$

"the set I has the back property."

Then a model whose relativizations to V and W are isomorphic to $\mathfrak{A}$ and $\mathfrak{B}$, respectively, is a model of ξ, since by hypothesis $\mathfrak{A}$ and $\mathfrak{B}$ are partially isomorphic, $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \neg\psi$. By Downward Löwenhein-Skolem property ξ has a countable model $\mathfrak{C}$. But then we obtain two countable structures $\mathfrak{A}' = \mathfrak{C}^V$ and $\mathfrak{B}' = \mathfrak{C}^W$, that are partially isomorphic, and therefore isomorphic, such that $\mathfrak{A}' \models \phi$ and $\mathfrak{B}' \models \neg\psi$, a contradiction.

If we take $\psi = \phi \in \mathcal{L}$, then we get the proof that regular logics with negation and LS have the Karp property. $\square$

Let us see the case for $\mathcal{L}_{\infty\omega}$. As Flum points out in [Fl85], adding for each ordinal α in Theorem 47, a relation R_α, instead of R_n, with set many equivalence classes, we can prove

Corollary 51 (Barwise's characterization for $\mathcal{L}_{\infty\omega}$)**.** *$\mathcal{L}_{\infty\omega}$ is a maximal bounded logic with negation and the Karp property.*

Proof. Let $\mathcal{L} = \mathcal{L}_{\infty\omega}$. Let R in (F1), page 29, be the relation of partial isomorphism, and $\mathcal{L}^*$ a bounded logic closed under negation. $\square$

The following is the associated interpolation theorem[6]. It is not a full interpolation theorem, for $\mathcal{L}_{\infty\omega}$ does not satisfy LS, hence neither $PC(\mathcal{L}_{\infty\omega})$ does, and thus Proposition 50 does not apply.

Corollary 52 (Barwise-van Benthem interpolation theorem for $\mathcal{L}_{\infty\omega}$)**.** [BvB99] *Given $\psi \in PC(\mathcal{L}_{\infty,\omega})$, and $\phi \in cPC(\mathcal{L}_{\infty,\omega})$, the following are equivalent:*

(1) *ψ entails ϕ along $\cong_p$.*
(2) *There is a sentence θ of $\mathcal{L}_{\infty\omega}$, such that $\psi \models \theta$ and $\theta \models \phi$.*

The proof of this theorem as given in [BvB99] is very similar to the proof of maximality for $\mathcal{L}_{\infty\omega}$. We treat this theorem as an application of point (2) of the generalized program after theorem Theorem 47.

Proof of Corollary 52. Let R in (F2), page 29, be the relation of partial isomorphism, and C boundedness. Let $\mathcal{L}^* = PC(L_{\infty\omega})$. $\square$

This theorem has as corollary a result that already appeared in [MShS76]:

Corollary 53. *If $\mathbf{K}$ is closed under partial isomorphism and $\mathbf{K}$ and $\overline{\mathbf{K}}$ are PC in $\mathcal{L}_{\infty\omega}$, then $\mathbf{K}$ is EC in $\mathcal{L}_{\infty\omega}$.* $\square$

But since Δ-extension does not preserve the Karp property [MShS76], we do not even have Δ-interpolation for $\mathcal{L}_{\infty\omega}$.

On the grounds of the previous considerations, Barwise and van Benthem argue that interpolation[7] theorem should be understood as:

($*$) "A logic $\mathcal{L}$ has interpolation if given two $PC(\mathcal{L})$-classes $\mathbf{K}_1 = Mod(\phi), \mathbf{K}_2 = Mod(\psi)$, if ϕ entails ψ along R, then $\mathbf{K}_1$ and $\mathbf{K}_2$ can be separated by an $EC(\mathcal{L})$-class."

Similar 'interpolation theorems' could be proved for $\mathcal{L}_{\kappa,\omega}, \kappa = \beth_\kappa$ and $\mathcal{L}_{\omega_1\omega}$. From López-Escobar [L-E65], we know the latter has the full interpolation, while the result provided by an adaptation of Corollary 52 for $\mathcal{L}_{\omega_1\omega}$ is only partial. Barwise and van Benthem [BvB99] asked whether it would be possible to make some change in Corollary 52, so that we get the full interpolation theorem for $\mathcal{L}_{\omega_1\omega}$. We see here that this is perhaps not possible, for separation is essentially a Lindström's

[6] The motivation of Barwise and van Benthem in their paper was to state interpolation theorems by finding the absolute version of the predicate R. We do not talk about this here, but rather use their generalization of interpolation to relate interpolation to maximality theorems.

[7] This statement is a slight change of what they have. For them, R is the absolute version of the invariance of $\mathcal{L}$, found for the sake of the interpolation theorem -and such that maybe $\mathcal{L}$ is not R-invariant. For us, R is the invariance of $\mathcal{L}$, for the sake of maximality.

theorem, and $\mathcal{L}_{\omega_1\omega}$ does not have any Lindström type characterization[8]. That is, the interpolation theorem can be proved for $\mathcal{L}_{\omega_1\omega}$ per se, but it cannot at present time be understood as a corollary of Theorem 47.

Is there any logic besides first-order with a maximality and a full interpolation theorem? In [ShV04], Shelah and Väänänen construct a new infinitary logic $\mathcal{L}^1_\kappa$ between $\mathcal{L}_{\kappa,\omega}$ and $\mathcal{L}_{\kappa,\kappa}$ characterized by a Löwenheim-Skolem property and a substitute of compactness. Both properties are preserved by the PC operation. They achieve this way a new logic satisfying Lindström's and Craig's theorems.

The Barwise-van Benthem interpolation theorem can be stated in a negationless form.

Definition 54. *Let ϕ and ψ be two sentences in $PC(\mathcal{L}_{\infty\omega})$.*

(1) *We say that ϕ* contradicts ψ along partial isomorphism (p.i.) *if no model of ϕ is partially isomorphic to a model of ψ.*
(2) *We say that ϕ* has a negation along p.i. *if there is another sentence ψ such that no model of ϕ is partially isomorphic to a model of ψ, and also any model is partially isomorphic to either a model of ϕ or a model of ψ.*

Theorem 55. *Let ϕ and ψ be two sentences in $PC(\mathcal{L}_{\infty\omega})$. If ϕ contradicts ψ along p.i. then there is a sentence of $\mathcal{L}_{\infty\omega}$ that separates ϕ and ψ.*

So an abstract version of this theorem says

Theorem 56. *Let $\mathcal{L}^*$ be an abstract logic that is bounded, and $\phi, \psi \in \mathcal{L}^*$. If ϕ contradicts ψ along p.i. then there is a sentence of $\mathcal{L}_{\infty\omega}$ that separates ϕ and ψ.*

As in the case of $\mathcal{L}_{\omega\omega}$, a consequence is that only the sentences in $\mathcal{L}^*$ that are already in $\mathcal{L}_{\infty\omega}$ have a negation along p.i.

Concluding remarks: The (generalizable) schema of the relation between Lindström Characterization Theorem and Craig Interpolation Theorem remains as follows:

(1) $\mathcal{L}_{\omega\omega}$ is the maximal logic with negation, Karp property and Compactness.
(2) With the same argument used to prove last statement, we can prove $PC(\mathcal{L}_{\omega\omega})$ has separation if we restrict to entailment along partial isomorphism.
(3) Because $\mathcal{L}_{\omega\omega}$ has LS, which is preserved by the PC-operation, and LS implies entailment along partial isomorphism is equivalent to entailment along isomorphism, $PC(\mathcal{L}_{\omega\omega})$ has full separation, and hence $\mathcal{L}_{\omega\omega}$ has interpolation.

In the case of logics closed under negation, we only find compact extensions of $\mathcal{L}_{\omega\omega}$ among logics with generalized quantifiers (because infinitary logics are not

[8] In the case of propositional extensions of $\mathcal{L}_{\omega_1,\omega}$, (see [GH76], [Ha80]), Harrington [Ha80] proved there are such extensions that continue to have the same model theoretic properties as $\mathcal{L}_{\omega_1,\omega}$, if we restrict to admissible fragments. Gostanian and Hrbacek proved in [GH76], not restricting to admissible fragments, that among propositional extensions of $\mathcal{L}_{\omega_1,\omega}$, only $\mathcal{L}_{\omega_1,\omega}$ itself warrants interesting model theoretic properties.

compact). But what happens with generalized quantifiers, and their corresponding back-and-forth relation $\cong_Q$, is that their Karp properties are unique for each one, so if $\mathcal{L}(Q)$ is such compact extension, we can prove that $PC(\mathcal{L}(Q))$ has separation if we restrict to entailment along $\cong_Q$. Now the difficult part is to find a property that would make entailment along an analog $\cong_Q$ of partial isomorphism equivalent to entailment along $\cong$, so that the $\mathcal{L}(Q)$ will have full interpolation. Is there a property that would make $\cong_Q$ equivalent to entailment along $\cong$ in this case?

A similar schema for logics without negation can be stated:

(1) $\mathcal{L}_{\omega\omega}$ is the maximal logic with negation, Karp property and Compactness.
(2) With the same argument used to prove last statement, we can prove $PC(\mathcal{L}_{\omega\omega})$ has separation if we restrict to contradiction along partial isomorphism.
(3) Because $\mathcal{L}_{\omega\omega}$ has LS, which is preserved by the PC-operation, and LS implies contradiction along partial isomorphism is equivalent to contradiction along isomorphism, $PC(\mathcal{L}_{\omega\omega})$ has full separation, and hence $\mathcal{L}_{\omega\omega}$ has separation.

5. Other maximality results

We have seen several cases of logics with model theoretic characterizations. We have seen the importance of Karp properties in the relation between interpolation and maximality. We now consider a model theoretic property, and then we try to establish whether it is relevant to interpolation, study the orderings of the family of all logics with respect to this model theoretic property, and look for maximal points. That way we can have an idea of the possible characterizations a logic can be expected to have.

In the literature, there are already some examples. Väänänen and Krynicki [VK82] studied the orderings of all logics; Sgro [Sg77] proved that in the ordering of logics with the Łos' ultraproduct property, there is a maximal logic with this property; Lipparini [Li87] proved that there is a maximal logic that extends a given logic $\mathcal{L}$ and has the same complete extensions as $\mathcal{L}$; Wacławek [W78] proved there is a maximal logic with Löwenheim-Skolem property over any logic with this property. All these maximal logics enjoy the Δ-interpolation theorem, which is a weakening of interpolation theorem. It is an open problem whether there is a maximal logic in the ordering of compact logics.

Of particular interest would be the existence of a maximal logic with Karp property[9]. Since Löwenheim-Skolem property implies Karp property, and the converse is true for logics with interpolation (cf. [Fl85] p. 95), it is also of interest to study the ordering of logics with the Löwenheim-Skolem property. For this reason, as well as for methodological reasons, we reproduce here the proof of Wacławek on the existence of a maximal logic with respect to the Löwenheim-Skolem property. The Δ-interpolation theorem separation theorem for this logic follows straightforwardly.

[9]However, this is an open problem. $\mathcal{L}_{\infty\omega}$ is a maximal logic with respect to Karp property if we chose the relation of extension between two logics to be:"$\mathcal{L} <_{\equiv} \mathcal{L}'$ iff for all τ and all $\mathfrak{A}, \mathfrak{B} \in Str[\tau]$, if $\mathfrak{A} \equiv_{\mathcal{L}'} \mathfrak{B}$ then $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$

We will be able to appreciate the essential differences between these proofs and those of the previous section, as well as to understand what a proof of interpolation would need.

Theorem 57 (Wacławek [W78]). *Let $(LS, \leq)$ be the ordering of logics closed under negation and conjunction that satisfy the Löwenheim-Skolem property. For any logic $\mathcal{L} \in (LS, \leq)$, there is a maximal logic $\mathcal{L}' \in (LS, \leq)$ such that $\mathcal{L} \leq \mathcal{L}'$.*

Proof. Each sentence in a logic $\mathcal{L}$ with the Löwenheim-Skolem property is determined by its countable models, i.e. two different sentences in $\mathcal{L}$ do not have the same countable models, by definition.

We show there are at most $2^{\aleph_0}$ countable non-isomorphic models of finite vocabulary. Let $\tau = \{T_1, \ldots, T_n\}$ be a vocabulary, and let m_i be the arity of T_i. Let A be a countable set, and let $c_i = |\{f : f \text{ is a function from } A^{m_i} \text{ to } \{0, 1\}\}|$. Then the number s of models of vocabulary τ and domain A is $s = \Pi_{i=1,\ldots,n} c_i$ but $c_i = 2^{\aleph_0}$ for all i, so $s = (2^{\aleph_0})^n = 2^{\aleph_0}$.

So there are at most $2^{2^{\aleph_0}}$ possible classes of countable non-isomorphic models, and hence every well-ordered chain in $(LS, \leq)$ has length smaller than $(2^{2^{\aleph_0}})^+$.

Now we show that the union of an increasing sequence of logics $\mathcal{L}'$ with $\mathcal{L} \leq \mathcal{L}'$ is a supremum of this family of logics. Let $\mathcal{L}^* = \bigcup_\alpha \mathcal{L}_\alpha$, where $\mathcal{L}_\alpha \in (LS, \leq)$, and $\mathcal{L}_\delta \leq \mathcal{L}_\gamma$ for $\delta \leq \gamma$, and suppose $\mathcal{L}^* \notin (LS, \leq)$. Then there is some $\theta \in \mathcal{L}^*$ with a model but no countable models. But $\theta \in \mathcal{L}_\alpha$ for some α, a contradiction.

By Zorn's Lemma, there is a maximal logic $\mathcal{L}^{**} \in (LS, \leq)$ extending $\mathcal{L}$. □

The following result is quite obvious but still worth mentioning because we have been unable to decide whether any logic with negation and maximal with respect to the Löwenheim-Skolem property satisfies the interpolation theorem.

Proposition 58. *Any logic with negation and maximal with respect to the Löwenheim-Skolem property satisfies the Δ-interpolation theorem.*

Proof. Let $\mathcal{L}$ be a maximal logic with the Löwenheim-Skolem property, and let $\mathbf{K}$, and $\overline{\mathbf{K}}$ be two PC-classes in $\mathcal{L}$. Suppose $\mathbf{K}$ is not an $EC(\mathcal{L})$-class. Then we can add $\mathbf{K}$ as a new $EC(\mathcal{L})$-class, and get the extension $\mathcal{L}'$ closing under negation and intersection. We prove that $\mathcal{L}'$ satisfies Löwenheim-Skolem theorem, contradicting the hypothesis that $\mathcal{L}$ is a maximal logic in $(LS, \leq)$.

$\mathbf{K}$ has countable models, because it is a class of reducts of models of a sentence of $\mathcal{L}$, and similarly for $\overline{\mathbf{K}}$, $M \cap \mathbf{K}$, and $M \cap \overline{\mathbf{K}}$. This proof involves nothing more than the known fact that the Δ-extension of $\mathcal{L}$ preserves LS.

Corollary 59. *Any logic with negation has a maximal extension with the same Löwenheim number. Moreover, this extension satisfies the Δ-interpolation theorem.*

Proof. Let $\mathcal{L}$ be a logic with Löweheim number κ. Consider the ordering of logics with Löwenheim number κ. By an argument similar to the proof of Theorem 57, there is a maximal logic $\mathcal{L}'$ extending $\mathcal{L}$. By an argument similar to that of the proof of Proposition 58, $\mathcal{L}'$ satisfies the Δ-interpolation theorem. $\square$

Now we study the orderings of logics not necessarily closed under negation, with respect to compactness and and Löwenheim-Skolem properties. We see there is no maximal set logic (cf. Definition 14.*5*) with respect to compactness and Löwenheim-Skolem properties.

Theorem 60. *If we do not assume negation, no set logic is maximal with respect to compactness and Löwenheim-Skolem properties.*

In order to prove Theorem 60, recall the notion of partially $\mathbf{K}$-reducible logic from Definition 32.

Lemma 61. *Let* $\mathbf{K}$ *be a class of models containing all models of cardinality at most* λ*. Suppose a logic* $\mathcal{L}$ *is partially* $\mathbf{K}$*-reducible to a compact logic* $\mathcal{L}'$ *with* $l_\Sigma = \lambda$*. Then* $\mathcal{L}$ *is compact.*

Proof. Let Φ be a finitely satisfiable set of $\mathcal{L}$-sentences. Let Φ^* be the set of $\mathcal{L}'$ -sentences that are the partial reductions of each sentence in Φ. Φ^* is finitely satisfiable, for take any finite set $\Sigma^* \subset \Phi^*$, and look at the corresponding set of sentences $\Sigma \subset \Phi$. By hypothesis, Σ has a model, and it is a model of each of the sentences in Σ^*, so Φ^* is finitely satisfiable. By compactness, it has a model, and as the Löwenheim number for countable sets of sentences of $\mathcal{L}'$ is λ, it has a model $\mathfrak{M}$ of that cardinality. As $\mathfrak{M}$ is a model with cardinality at most λ of each sentence in Φ^*, it is a model of each sentence in Φ. $\square$

Theorem 62. *There is no maximal compact set logic.*

Proof. Let $\mathcal{L}$ be a compact logic not necessarily closed under negation. Let λ be its Löwenheim number. Add to $\mathcal{L}$ the class $\mathbf{K}_\kappa$ of models of the generalized quantifier $Q^{\leq\kappa}x(x = x)$, whose interpretation is “there are at most κ elements", and close under conjunction, disjunction, and existential and universal quantification. We can find $\kappa > \lambda$ such that $\mathbf{K}_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\leq\kappa})$ makes a proper extension of $\mathcal{L}$. We prove $\mathcal{L}(Q^{\leq\kappa})$ is compact. Let $\mathbf{K}$ be the class of models of cardinality λ. Then $\mathcal{L}(Q^{\leq\kappa})$ is partially $\mathbf{K}$-reducible to $\mathcal{L}$ by the sentence $\exists x(x = x)$, and by Lemma 61 it is compact. $\square$

The proof of the following theorem is a slight variation the proof of the same theorem for logics with negation, given by Wacławek in [W78]. He uses $\mathcal{L}_1 = \mathcal{L}_{\omega\omega}(Q_1)$, where Q_1 is the quantifier ”there exists uncountably many”, because in this logic we can characterize countable models. But without negation this is not the case, so we instead use $L_{\omega\omega}(Q^{\leq\aleph_0})$, where $Q^{\leq\aleph_0}$ is the quantifier “there exists countably many”.

Theorem 63. *There is no largest compact logic.*

Proof. The extension $\mathcal{L}_1$ of $L_{\omega\omega}$ by $Q^{\leq\aleph_0}$, without negation, is compact. On the other hand, let Q^{dc}_{ω} be defined as $Q^{dc}_{\omega}xy\phi(x,y,)$ iff ϕ defines a linear ordering without Dedekind cuts of cofinality ω. From [Sh75] it follows that $\mathcal{L}_2 = \mathcal{L}_{\omega\omega}(Q^{dc}_{\omega})$ is compact. Clearly, the ordering $(\omega, <)$ can be characterized in $\mathcal{L}_{\omega\omega}(Q^{dc}_{\omega})$ among countable models. Hence, any common extension of $\mathcal{L}_1$ and $\mathcal{L}_2$ is not compact. Wacławek proof ends here, but we have to make sure we do not need negation for proving the statements related to Q^{dc}_{ω}. First, $\mathcal{L}_{\omega\omega}(Q^{dc}_{\omega})$ without negation is compact because is a sublogic of $\mathcal{L}_{\omega\omega}(Q^{dc}_{\omega})$ with negation. Finally, "the ordering $(\omega, <)$ can be characterized in $\mathcal{L}_{\omega\omega}(Q^{dc}_{\omega})$ among countable models" is expressed by the conjunction σ of the following sentences:

- " $<$ is a linear order with a first element and such that every element except the first has an immediate successor and an immediate predecessor",
- $Q^{\leq\aleph_0}x(x = x)$,
- $Q^{dc}_{\omega}xy(x < y)$.

Clearly, only $(\omega, <)$ is a model of σ, and only the first sentence, which is a first order sentence, uses negation. □

Theorem 64. *There is no maximal set logic with respect to LS.*

Proof. Let $\mathcal{L}$ be as above. Add to $\mathcal{L}$ the class $\mathbf{K}_\kappa$ of models of the generalized quantifier $Q^{\leq\kappa}x(x = x)$, whose interpretation is "there are at most κ elements", and close under conjunction, disjunction, and existential and universal quantification. We can find $\kappa > \omega$ such that $\mathbf{K}_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\leq\kappa})$ makes a proper extension of $\mathcal{L}$.

We show $\mathcal{L}(Q^{\leq\kappa})$ has the Löwenheim-Skolem property. Let $\mathbf{K}$ be the class of models of cardinality $\aleph_0$. Then $\mathcal{L}(Q^{\leq\kappa})$ is partially $\mathbf{K}$-reducible to $\mathcal{L}$ by the sentence $\exists x(x = x)$, and by Lemma 35, it has LS. □

From theorems 62 and 64, we conclude that there is no maximal compact set logic with the Löwenheim-Skolem property. This covers the result for non-maximality of countable logics without negation of last section.

Theorem 65 (Wacławek [W78])**.** *There is no largest logic with respect to LS.*

Sketch of Proof Let Q_B be defined by $\mathfrak{A} \models Q_Bxy\phi(x,y)$ iff $(A, \{(a,b) : \mathfrak{A} \models \phi(a,b))$ is an uncountable well ordering or a well ordering of the type of the set B. Now construct two disjoint classes of well orderings Q_{A_0}, Q_{A_1} such that $\mathcal{L}(Q_{A_i})$ have LS for $i = 0, 1$, but any common extension of both not necessarily closed under negation do not have LS. □

The following theorem is the closest we get to our aim which was to find whether there is a largest logic with respect to Compactness and $LS(\aleph_0)$.

Theorem 66. *There is no largest logic with respect to Compactness and $LS(\aleph_1)$.*

Proof. Let Q_1 and Q_2 be two quantifiers such that $Q_1xF(x)$ says that $|F| < \omega$ or $|F|$ is a limit cardinal, and $Q_2xF(x)$ says that $|F| < \omega$ or $|F|$ is a successor cardinal. We prove that $\mathcal{L}_{\omega\omega}(Q_i), i = 1, 2$ are compact and $LS(\aleph_1)$. Since any common extension of both logics would be able to express F is finite, no such extension is compact. It is easy to see that $Q_i, i = 1, 2$ have $LS(\aleph_1)$. For Compactness, let T be a countable finitely consistent theory in $\mathcal{L}_{\omega\omega}(Q_1)$. Add to the vocabulary τ of T a unary predicate symbol N, a binary relation symbol $<$, and a ternary relation symbol R. For any sentence ϕ, let $[Q_1x\phi(x)]^*$ be the $\tau \cup \{R, N, <\}$-sentence that says that there is x such that $R(x, ., .)$ defines a 1-1 mapping from $\{y : \phi^*(y)\}$ into $\{y \in N : y < z\}$, for some $z \in N$. Let $T^* = \{\phi^* : \phi \in T\} \cup$ " $<$ is a linear order of N. We show that T^* is finitely consistent in first order logic. Let S be a finite sub-theory of T. By assumption, there is a model $\mathfrak{M}$ of S. Let $|\mathfrak{M}| = \kappa$. Expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ of S^* by adding a well-order of type $\kappa + 1$ and letting R map every $\{y : \phi^*(y)\}$ to a respective initial segment of this well-order. By compactness and LS of first-order logic, there is a countable model $\mathfrak{M}''$ of T^*. We get that $\mathfrak{M}''$ is a model of T because every $Q_1xF(x)$ is trivially true, as all subsets of M'' are either finite or of cardinality ω, which is a limit cardinal.

We now show Q_2 is compact. Let T be a countable finitely consistent theory in $\mathcal{L}_{\omega\omega}(Q_2)$. Add to the vocabulary τ of T a unary predicate symbol N, a binary relation symbol $<$, and a ternary relation symbol R. For any sentence ϕ, let $[Q_2x\phi(x)]^*$ be the $\tau \cup \{R, N, <\}$-sentence that says that there is x such that $R(x, ., .)$ defines a 1-1 mapping from $\{y : \phi^*(y)\}$ into $\{y \in N : y < z\}$, for some $z \in N$. Let $T^* = \{\phi^* : \phi \in T\} \cup$ " $<$ is a linear order without a last element of N. We show that T^* is finitely consistent in first order logic. Let S be a finite sub-theory of T. By assumption, there is a model $\mathfrak{M}$ of S. Let $|\mathfrak{M}| = \kappa$. Expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ of S^* by adding a well-order of type $\kappa + 1$ and letting R map every $\{y : \phi^*(y)\}$ to a respective initial segment of this well-order. By compactness we get a model $\mathfrak{M}''$ of T^* of size $\aleph_1$, such that every infinite initial segment of $<$ is uncountable. We get that $\mathfrak{M}''$ is a model of T of size $\aleph_1$ because if $[Q_2x\phi(x)]^*$ holds then ϕ^* holds finitely many x or for exactly $\aleph_1$ many x. Hence $Q_2x\phi(x)$ follows as $\aleph_1$ is a successor cardinal. $\square$

5.1. **Dual Properties.**

If we do not have negation, compactness and Löwenheim-Skolem properties split into three non-equivalent forms (see Definition 14). We study here maximality with respect to these properties.

Given a logic $\mathcal{L}$, we always assume that $Str[\tau]$, the class of all structures of vocabulary τ, and $\mathbf{K}_\emptyset$, the empty class of models, are definable classes in $\mathcal{L}$. That is, we can always assume that there is a valid sentence and an inconsistent sentence. This remark is not necessary when we talk about extensions of first order logic, that is the majority of cases. Recall that $n(\mathbf{K})$ denotes the class of countable models of a class of models $\mathbf{K}$.

Proposition 67. *Let $\mathcal{L}$ be a logic. If $\mathcal{L}$ is $\star$-LS, then it is LS and DuLS. If further we assume $\mathcal{L}$ is closed under negation, then the three properties are equivalent.*

Proof. First we prove $\star$-LS is the strongest condition. $\star$-LS implies LS, since if there is a class $\mathbf{K}$ with $n(\mathbf{K}) = \emptyset$, then $\mathbf{K} = \mathbf{K}_\emptyset$ for otherwise $\mathbf{K}$ and $\mathbf{K}_\emptyset$ would be two different classes with the same countable models. Therefore, any nonempty class has countable models. Now, $\star$-LS implies $DuLS$ since, by the same argument as in the previous case, if there is a class with $n(\mathbf{K}) = n(Str[\tau])$, then $\mathbf{K} = Str[\tau]$.

For the converses, let $\mathcal{L}$ be closed under negation. First we prove that $DuLS$ implies LS. Suppose $\mathcal{L}$ does not satisfy LS. Then there is a nonempty class $\mathbf{K}$ which does not have countable models. Then $n(\overline{\mathbf{K}}) = n(Str[\tau])$, but $\overline{\mathbf{K}} \neq Str[\tau]$. Finally, we prove that $DuLS$ implies $\star$-LS. Let $\mathbf{K}_1, \mathbf{K}_2$ be such that $n(\mathbf{K}_1) = n(\mathbf{K}_2)$ but $\mathbf{K}_1 \neq \mathbf{K}_2$. Suppose there is $\mathfrak{A}$ such that $\mathfrak{A} \in \mathbf{K}_2$, but $\mathfrak{A} \notin \mathbf{K}_1$. Then $\mathbf{K}_1 \cup \overline{\mathbf{K}_2}$ contains all countable models, but it does not contain all models, since it does not contain $\mathfrak{A}$. $\square$

Closure under negation is necessary for the last part of the proof. For instance, we can have $\mathcal{L}$ not closed under negation, e. g. $PC(\mathcal{L}_{\omega\omega})$, such that there is $\phi \in \mathcal{L}$ with $n(Mod(\phi)) = n(Str[\tau])$ and $Mod(\phi) \neq Str[\tau]$, that satisfies Löwenheim-Skolem theorem. It is in principle only necessary that $\neg\phi \notin \mathcal{L}$. Similarly, a logic $\mathcal{L}$ can satisfy $DuLS$ containing a non-empty class without countable models, like for example $cPC(\mathcal{L}_{\omega\omega})$. From these it follows that neither LS nor $DuLS$ imply $\star$-LS. Moreover, not even LS and $DuLS$ together are able to make the logic closed under negation. This is proved by the fact that there is a compact extension of first-order logic with LS and $DuLS$, namely $PC \cap L_{\omega_1\omega}$, a compact fragment of $L_{\omega_1\omega}$ not closed under negation.

Proposition 68. *Let $\mathcal{L}$ be a logic. If $\mathcal{L}$ is $\star$-compact, then it is compact and dual compact. If further we assume $\mathcal{L}$ is closed under negation, then the three properties are equivalent.*

Proof. $\star$-compactness implies compactness: Let Φ' in Definition 14.1.b be inconsistent. $\star$-compactness implies dual-compactness: Let Φ be a set of valid sentences, and suppose no finite subset $\Phi_0 \subseteq \Phi'$ covers everything. Then, by $\star$-compactness, there is a model $\mathfrak{B}$ that is not a model of any sentence in Φ', therefore Φ' does not cover everything.

Now for the converse, suppose $\mathcal{L}$ is closed under negation. We prove first dual compactness implies compactness: Suppose $\mathcal{L}$ is not compact. Let Φ be a countable set of classes with empty intersection. Assume Φ can cover $Str[\tau]$ (we can assume this since we can always add $Str[\tau]$ to Φ). Then, any finite subset Φ_0 of Φ has nonempty intersection. Now consider the set $\overline{\Phi} := \{\neg\phi : \phi \in \Phi\}$. Then $\bigcup\overline{\Phi} = Str[\tau]$, and $\bigcap\overline{\Phi} = \emptyset$ too, but there is no $\overline{\Psi}_0 \subseteq \overline{\Phi}$ that covers $Str[\tau]$, otherwise $\bigcap\Psi_0 = \bigcap\{\neg\psi : \psi \in \overline{\Psi}_0\} = \emptyset$, contradicting the assumption that $\mathcal{L}$ is not compact. Finally, we prove compactness implies $\star$-compactness: Suppose $\mathcal{L}$ is compact, and let Φ, Φ' be two countable sets of sentences such that for every finite $\Phi_0 \subseteq \Phi$ and every finite $\Phi'_0 \subseteq \Phi'$, there is a model $\mathfrak{A}$ such that $\mathfrak{A} \models \bigwedge\Phi_0$ and $\mathfrak{A} \not\models \bigvee\Phi'_0$. Then $\mathfrak{A} \models \bigwedge\tilde{\Phi}'_0$. Clearly $\Phi\cup\tilde{\Phi}'$ is finitely satisfiable, and by compactness there is a model $\mathfrak{B}$ such that $\mathfrak{B} \models \Phi \cup \tilde{\Phi}'$. Hence $\mathfrak{B} \models \bigwedge\Phi$ and $\mathfrak{B} \models \bigwedge\tilde{\Phi}'$, i.e. $\mathfrak{B} \not\models \bigvee\Phi'$. □

Definition 69. *We say that $\mathcal{L}$ is partially $\mathbf{K}$-dual reducible to $\mathcal{L}'$, if for any $\phi \in \mathcal{L}$ there is $\phi^* \in \mathcal{L}'$ such that $\phi^* \to \phi$ in all models, and the converse is true in models in $\mathbf{K}$.*

Definition 70. *The* dual Löwenheim number of $\mathcal{L}$, *in symbols $dl(\mathcal{L})$, is the least κ such that any sentence $\phi \in \mathcal{L}$ valid in all models of size at most κ is valid. The* dual Löwenheim number of $\mathcal{L}$ *for countable sets of sentences, in symbols $dl_\Sigma(\mathcal{L})$, is the least κ such that for any countable Φ, if $Mod(\Phi) = Str[\tau]$, then there is $\Phi_0 \subseteq \Phi$ such that $Mod(\Phi_0) = Str[\tau]$.*

Note that in set logics both numbers clearly exist.

Lemma 71. *Let $\mathbf{K}$ be a class of models that contains all models of cardinality at most λ. If $\mathcal{L}$ is partially $\mathbf{K}$-reducible to a dual compact logic $\mathcal{L}'$ with $dl_\Sigma = \lambda$, then $\mathcal{L}$ is dual compact.*

Proof. Let $\Phi \subset \mathcal{L}$ be a countable set of sentences whose classes of models cover everything. Then the set $\Phi^* \subseteq \mathcal{L}'$ of partial reductions of every sentence in Φ covers all models of cardinality at most λ, and by dl_Σ of $\mathcal{L}'$, it covers everything. By dual compactness of $\mathcal{L}'$, there is a finite $\Phi^*_0 \subset \Phi^*$ that covers everything. Let Φ_0 be the finite set of sentences in Φ whose reductions are in Φ^*_0. Clearly Φ_0 covers everything. □

Theorem 72. *There is no maximal set logic with the dual compactness property.*

Proof. Let $\mathcal{L}$ be a dual compact logic with $dl_\Sigma = \lambda$. Add the class $\mathbf{K}_\kappa$ of models of the sentence $Q^{\geq\kappa}x(x = x)$, whose interpretation is "there are at least κ elements", and close under conjunction, disjunction, and existential and universal quantification. We can find $\kappa > \lambda$ such that $\mathbf{K}_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\geq\kappa})$ makes a proper extension of $\mathcal{L}$. We show $\mathcal{L}(Q^{\geq\kappa})$ is dual compact. Let $\mathbf{K}$ be the class of models of cardinality $\leq \lambda$. Then $\mathcal{L}(Q^{\geq\kappa})$ is partially

$\mathbf{K}$-reducible to $\mathcal{L}$ by the sentence $\exists x(x \neq x)$, and by Lemma 71 it is dual compact. $\square$

Lemma 73. *Let $\mathbf{K}$ be a class containing all countable models. If a logic $\mathcal{L}$ is partially $\mathbf{K}$-dual reducible to a dual LS logic $\mathcal{L}'$, then $\mathcal{L}$ is dual LS.*

Proof. Let $\phi \in \mathcal{L}$ be a sentence valid in all countable models. By assumption, there is $\phi^* \in \mathcal{L}'$ such that $\phi \to \phi^*$ in models in K. Clearly, ϕ^* is valid in models in K, and by dual LS it is valid. Now $\phi^* \to \phi$, and therefore ϕ is valid. $\square$

Proposition 74. *There is no maximal set logic with respect to the dual Löwenheim-Skolem theorem.*

Proof. Let $\mathcal{L}$ be a set logic with $DuLS$. Add to $\mathcal{L}$ the class of models of the generalized quantifier $Q^{\geq\kappa}x(x = x)$ whose interpretation is "there are at least κ many elements in the universe", and close under conjunction, disjunction, and existential and universal quantification. We show $\mathcal{L}(Q^{\geq\kappa})$ is partially $\mathbf{K}$-dual reducible to $\mathcal{L}$. Let $\mathbf{K}$ be the class of countable models. Then $\mathcal{L}(Q^{\geq\kappa})$ is partially $\mathbf{K}$-dual reducible to $\mathcal{L}$ by the sentence $\exists x(x \neq x)$, and by Lemma 73 it has dual LS. $\square$

Proposition 75. *There is a maximal set logic with $\star$-LS property.*

Proof. This proof is the same as proof of Theorem 57. $\square$

Next theorems are the natural application of dual properties. We prove a failure of separation and establish a reduction theorem.

Lemma 76. *There are two $cPC(\mathcal{L}_{\omega\omega})$-classes that cannot be separated by any first-order class.*

Proof. Let K_1 be the class of models such that $\langle A, <\rangle \notin \mathbf{K}_1 \leftrightarrow A$ is infinite or $\exists X \subseteq A$ such that X has the first element, the last element, and every second element; and let $\mathbf{K}_2$ be the class of models such that $\langle A, <\rangle \notin \mathbf{K}_2 \leftrightarrow A$ is infinite or $\exists X \subseteq A$ such that X has the first element, every second element, but not the last element. $\mathbf{K}_1$ and $\mathbf{K}_2$ cannot be separated by any first-order sentence because $(2^{n+1}, <) \equiv_n (2^{n+1} + 1, <)$. $\square$

Theorem 77 (Reduction Property)**.** *Let $\mathcal{L}$ be a logic extending $\mathcal{L}_{\omega\omega}$, not necessarily closed under negation, dual compact and duLS. If $\mathbf{K}_1$ and $\mathbf{K}_2$ are two $\mathcal{L}_{\omega\omega}$-classes such that $\mathbf{K}_1 \cup \mathbf{K}_2 = Str[\tau]$, then there is $\mathbf{K}$ in $\mathcal{L}_{\omega\omega}$ such that $\mathbf{K} \subseteq \mathbf{K}_1$ and $\overline{\mathbf{K}} \subseteq K_2$.*

Proof. Let $\mathcal{L}^* = \{\mathbf{K} : \overline{\mathbf{K}} \in \mathcal{L}\}$. We prove that $\mathcal{L}^*$ extends $\mathcal{L}_{\omega\omega}$ and is compact and LS. That $\mathcal{L}^*$ extends $\mathcal{L}_{\omega\omega}$ is clear since $\mathcal{L}$ is assumed to extend it, and $\mathcal{L}_{\omega\omega}$ is closed under negation -so all $\mathcal{L}_{\omega\omega}$ is in $\mathcal{L}^*$. Now we prove $\mathcal{L}^*$ is compact. So let $\bar{\Phi} = \{\overline{\mathbf{K}} : \mathbf{K} \in \Phi\}$, and suppose $\mathcal{L}^*$ is not compact. Then there is a countable Φ with empty intersection, such that for every finite $\Phi_0 \subset \Phi$ we have $\bigcap \Phi_0 \neq \emptyset$. But that means that $\bigcup \bar{\Phi}_0 \neq Str[\tau]$ for all finite $\bar{\Phi}_0 \subset \bar{\Phi}$, while $\bigcup \bar{\Phi} = Str[\tau]$, hence

$\mathcal{L}$ is not dual compact, a contradiction. Now suppose $\mathcal{L}^*$ is not LS. Then there is a nonempty model class $\mathbf{K} \in \mathcal{L}^*$ without countable models. But then $\overline{\mathbf{K}} \in \mathcal{L}$ has all countable models but not all models, and $\mathcal{L}$ is not $duLS$, a contradiction. Now, by Theorem 42, there is $\mathbf{K} \in \mathcal{L}_{\omega\omega}$ that separates $\overline{\mathbf{K}}_1$ and $\overline{\mathbf{K}}_2$. It is easy to see that $\mathbf{K} \in \mathcal{L}_{\omega\omega}$ is as we wanted. □

The above Reduction Property is like the Separation Property (Theorem 42), a negation-free formulation of Lindström's Theorem. It shows that in a dual LS and dual compact extension of first order logic only the first order sentences can have a negation.

6. Extensions by monotone quantifiers

In this section, we study extensions of $\mathcal{L}_{\omega\omega}$ with monadic monotone quantifiers, not closed under negation. We are interested in compactness and interpolation aspects of those logics.

Our first result tries to characterize these quantifiers in terms of cardinal quantifiers, so that it is possible to make the study of compactness and interpolation is terms of the latter.

Definition 78. *Let $\sigma = \{U\}$, U a unary predicate, and let Q be a unary quantifier. Let $Q(A) = \{X \subseteq A : (A, X) \models QyU(y)\}$. We call Q* upwards monotone*, if*

$$X \in Q(A) \text{ and } X \subset Y \subset A \text{ imply } Y \in Q(A).$$

Q is downwards monotone*, if*

$$X \in Q(A) \text{ and } Y \subset X \subset A \text{ imply } Y \in Q(A).$$

Recall from the examples following Definition 4 that a monadic cardinality quantifier Q_α is associated with the class $\mathbf{K} = \{(A, C) : C \subseteq A, |C| \geq \aleph_\alpha\}$. Since we are interested in extensions not closed under negation, in our framework a cardinal quantifier splits into four, Q^+_α, Q^-_α, $\widetilde{Q}^+_\alpha$, and $\widetilde{Q}^-_\alpha$ whose interpretations are:

$$\mathfrak{A} \models Q^+_\alpha x\phi(x) \text{ iff } |\{a \in A : \mathfrak{A} \models \phi(\bar{a}, a)\}| \geq \aleph_\alpha,$$

$$\mathfrak{A} \models Q^-_\alpha x\phi(x) \text{ iff } |\{a \in A : \mathfrak{A} \not\models \phi(\bar{a}, a)\}| < \aleph_\alpha.$$

$$\mathfrak{A} \models \tilde{Q}^+_\alpha x\phi(x) \text{ iff } |\{a \in A : \mathfrak{A} \models \phi(\bar{a}, a)\}| < \aleph_\alpha,$$

$$\mathfrak{A} \models \widetilde{Q}^-_\alpha x\phi(x) \text{ iff } |\{a \in A : \mathfrak{A} \not\models \phi(\bar{a}, a)\}| \geq \aleph_\alpha.$$

Clearly, Q^+_α and Q^-_α are upwards monotone, and $\widetilde{Q}^+_\alpha$ and $\widetilde{Q}^-_\alpha$ are downwards monotone.

We define a weak negation in $\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$ as:

$$\begin{aligned}
{\sim}\phi &= \neg\phi \text{ if } \phi \text{ atomic} \\
{\sim}\phi &= \psi \text{ if } \phi \text{ negated atomic } \neg\psi \\
{\sim}(\phi \wedge \psi) &= {\sim}\phi \vee {\sim}\psi \\
{\sim}(\phi \vee \psi) &= {\sim}\phi \wedge {\sim}\psi \\
{\sim}\exists x\phi &= \forall x \,{\sim}\phi \\
{\sim}\forall x\phi &= \exists x \,{\sim}\phi \\
{\sim}Q^+_\alpha x\phi &= Q^-_\beta x \,{\sim}\phi \\
{\sim}Q^-_\beta x\phi &= Q^+_\alpha x \,{\sim}\phi
\end{aligned}$$

Note that we get closure under negation if $\alpha = \beta$.

Upwards and downwards monotone quantifiers have different back-and-forth systems.

Definition 79 (Back-and-forth systems)**.** *Let τ be a vocabulary not containing S. Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures, $0 \le \gamma \le \omega$, and $I = (I_\delta)_{\delta \le \gamma}$ a sequence of subsets of Part$(\mathfrak{A}, \mathfrak{B})$. We say that I has the $\exists$-back-and-forth property if it satisfies conditions $(i) - (iv)$ of Definition 11.*

We say that I has the Q-forth property iff for all $m < \gamma, p \in I_{m+1}$, and $X \subset A, X \in Q(A)$, there is $Y \subset B, Y \in Q(B)$ such that for all $d \in Y$ there is $q \in I_m$ such that $p \subseteq q, d \in rg(q)$ and $q^{-1}(d) \in X$.

Two structures $\mathfrak{A}, \mathfrak{B}$ are γ, Q-isomorphic via I, written $I : \mathfrak{A} \cong_{\gamma,Q}$ iff $I = (I_\delta)_{\delta \le \gamma}$ is a sequence of length $\gamma + 1$ of nonempty subsets of Part$(\mathfrak{A}, \mathfrak{B})$ having the $\exists$-back-and-forth, and the Q-forth properties. Two structures are γ, Q-isomorphic, written $\mathfrak{A} \cong_{\gamma,Q}$ iff there is I such that $I : \mathfrak{A} \cong_{\gamma,Q} \mathfrak{B}$.

Lemma 80. *Let Q be an upwards monotone quantifier. Let $\mathcal{L}^*$ denote the logic $\mathcal{L}_{\omega\omega}(Q)$. If $\mathfrak{A} \cong_{n,Q} \mathfrak{B}$, then $\mathfrak{A} \le^n_{\mathcal{L}^*} \mathfrak{B}$.*

Proof. By induction on the quantifier rank. The quantifier rank is defined as always. For the case of $Q, qr(Qx\phi) = qr(\phi) + 1$. Let the claim be proved for $qr(\phi) \le m$. Let $I : \mathfrak{A} \cong_{m,Q} \mathfrak{B}$ be given. Let $p \in I_{m+1}$, and $a_0, \ldots, a_{k-1} \in dom(p)$ be given, and assume $\psi = Qx\phi(\bar{a}, x)$, $qr(\phi) \le m$. Suppose $\mathfrak{A} \models Qx\phi(\bar{a}, x)$. Then

$$X = \{c \in A : \mathfrak{A} \models \phi(\bar{a}, c)\} \in Q(A).$$

By the Q-forth property there is $Y \subseteq B, Y \in Q(B)$, such that $\forall d \in Y, \exists q \in I_m$ with $q^{-1}(d) \in X$. We show $Y \subseteq \{b \in B : \mathfrak{B} \models \phi(p(\bar{a}), b)\}$. Let $d \in Y$. Choose q so that $q^{-1}(d) \in X$. Then $\mathfrak{A} \models \phi(\bar{a}, q^{-1}(d))$, and by induction hypothesis $\mathfrak{B} \models \phi(p(\bar{a}), d)$, so we get $d \in \{b \in B : \mathfrak{B} \models \phi(p(\bar{a}), b)\}$ and the claim is proved. Since Q is upwards monotone and $Y \in Q(B)$, by the claim $\{b \in B : \mathfrak{B} \models \phi(p(\bar{a}), b)\} \in Q(B)$ too, and hence $\mathfrak{B} \models Qx\phi(p(\bar{a}), x)$. $\square$

For proving the converse of the Lemma 80, we need the following

Definition 81. *Let $\bar{a} \in A^k$, consider the following formulas:*

$$\begin{aligned}
\phi^0_{\bar{a}} &= \bigwedge\{\phi(\bar{x}) : \phi \text{ atomic or negated atomic }, \mathfrak{A} \models \phi(\bar{x})\}; \\
\phi^{m+1}_{\bar{a}} &= \bigwedge_{c\in A} \exists y \phi^m_{\bar{a},c}(\bar{x}, y) \wedge \forall y \bigvee_{c\in A} \phi^m_{\bar{a},c}(\bar{x}, y) \\
&\wedge \bigwedge_{M\in Q(A)} Qy \bigvee_{c\in M} \phi^m_{\bar{a},c}(\bar{x}, y).
\end{aligned}$$

Lemma 82. *Let τ be a finite vocabulary and Q a monotone quantifier. For each $n, k \in \mathbf{N}$, and every set of variables $\{x_0, x_1, \ldots, x_{k-1}\}$, there are only finitely many non-equivalent formulas of the form $\phi^n_{\bar{x}}$, and they are all in $\mathcal{L}_{\omega\omega}(Q)$.*

Proof. By induction on the quantifier rank of the formulas. It is clear for $n = 0$, for we have finitely many atomic and negated atomic formulas, since the vocabulary is finite. Suppose we have proved the claim for $n = m$. We prove it for $n = m+1$. Now, we prove there are finitely many formulas of the form $\exists y \phi^m_{\bar{a},c}(\bar{x}, y)$. By induction hypothesis, there are finitely many nonequivalent formulas of the form $\phi^m_{\bar{a}a}$, so we can choose finitely many $c \in A$, one for each of them. For the case of Q, we need to consider only finitely many models M. As there are only finitely many c_i, we just need to find at most one M for each c_i. $\square$

Lemma 83. *If $\mathfrak{A} \leq^n_{\mathcal{L}_{\omega,\omega}(Q)} \mathfrak{B}$, then $\mathfrak{A} \cong_{n,Q} \mathfrak{B}$.*

Proof. We check the Q-forth condition. Assume $\mathfrak{A} \leq^n_{\mathcal{L}_{\omega,\omega}(Q)} \mathfrak{B}$, and define

$$I_m = \{p \in Part(\mathfrak{A}, \mathfrak{B}) | dom(p) = \{a_0, \ldots, a_{k-1}\} \text{ and } \mathfrak{B} \models \phi^m_{\bar{a}}(p(\bar{a}))\}$$

Suppose we have $X \subseteq A, X \in Q(A)$. Then we have to prove that we can find $Y \subseteq B, Y \in Q(B)$, and for all $y \in Y$ there is $x \in X$ and $q \supset p$ such that $q(x) = y$. So let $c_0, \ldots, c_m \in X$, be such that

$$X \subseteq \bigcup_{i=0}^{m} \{c \in A | \mathfrak{A} \models \phi^m_{\bar{a},c}(\bar{a}, c_i)\}.$$

Then, $\mathfrak{A} \models Q_{\alpha,\beta} x \bigvee_{i=0}^m \phi^m_{\bar{a},c}(\bar{a}), x)$, and since $qr(\phi) \leq m$, we get, by induction hypothesis $\mathfrak{B} \models Q_{\alpha,\beta} x \bigvee_{i=0}^m \phi^m_{\bar{a},c_i}(p(\bar{a}), x)$. Let

$$Y = \bigcup_{i=0}^{m} \{d \in B | \mathfrak{B} \models \phi^m_{\bar{a},c_i}(p(\bar{a}), d)\}.$$

Then clearly $Y \in Q(B)$, and from the definition of I_m, for all $y \in Y$ there is $x \in X$ and $q \supset p$ such that $q(x) = y$. $\square$

6.1. **Model theoretic characterization.**

In this section we prove that any upwards monotone quantifier Q is equivalent to either Q_α^+ or Q_α^- in models of cardinality $\geq \aleph_\alpha$ for some α. The proof is made along the same lines as Flum's in [Fl85]. The crucial matter that makes his proof works in this context is having at hand a back-and-forth system as defined in the previous section. However, not having negation poses some limitations. Flum is able to say how the quantifier Q_α^+ looks like for models of cardinality equal or bigger than $\aleph_\alpha$ and for models of cardinality smaller than $\aleph_\alpha$. We do not seem to have such possibility.

Let Q be an upwards monotone quantifier. Associate with Q a function g^Q defined on the class of non-zero cardinals which maps each cardinal λ into a pair of cardinals $g(\lambda) = (\mu, \nu)$, where for any set A with $|A| = \lambda$,

$$g^Q(\lambda) = (\lambda, 0) \text{ if } Q(A) = \emptyset,$$

and otherwise,

$$\mu = min\{|X| : X \in Q(A)\}, \text{ and } \nu = sup\{|A \backslash X|^+ : X \in Q(A)\}.$$

Note that for infinite λ, we always have $\mu = \lambda$ or $\nu = \lambda^+$. For the specific case of cardinality quantifiers, we have

$$g^{Q_\alpha^+}(\lambda) = \left\{ \begin{array}{cc} (\lambda, 0) & \lambda < \aleph_\alpha \\ (\aleph_\alpha, \lambda^+) & \lambda \geq \aleph_\alpha \end{array} \right\}$$

and

$$g^{Q_\alpha^-}(\lambda) = \left\{ \begin{array}{cc} (0, \lambda^+) & \lambda < \aleph_\alpha \\ (\lambda, \aleph_\alpha) & \lambda \geq \aleph_\alpha \end{array} \right\}$$

The following theorem is a negation-free version of Theorem 4.1 in [Fl85].

Theorem 84. *Suppose $\mathcal{L}_{\omega\omega}(Q)$ is a relativizing logic properly extending first-order logic with a monotone quantifier. Assume $\mathcal{L}_{\omega\omega}(Q^d)$ is also relativizing. Then there is α such that for models of cardinality equal to or bigger that $\aleph_\alpha$, Q is either Q_α^+ or Q_α^-.*

Proof. Denote g^Q by g. We establish the theorem by showing that $g(\lambda) = g^{Q_\alpha^+}(\lambda)$ or $g = g^{Q_\alpha^-}(\lambda)$ from some cardinality $\lambda_0 = \aleph_\alpha$ onwards. For that, we prove the following claims:

(i) For $\lambda < \mu$, if $g(\lambda) \neq (\lambda, 0)$, then $g(\mu) \neq (\mu, 0)$.

Let $\Lambda = \{\lambda : g(\lambda) = (\mu, \nu) \text{ and } \mu, \nu \geq \omega\}$, and if $\Lambda \neq \emptyset$ let $\lambda_0 = \inf \Lambda$.

(ii) $g(\lambda_0) = (\lambda_0, \lambda_0^+) \vee g(\lambda_0) = (\lambda_0, \lambda_0)$.

(iii) If $g(\lambda_0) = (\lambda_0, \lambda_0^+)$, then for $\lambda \geq \lambda_0, g(\lambda) = (\lambda_0, \lambda^+)$.

Suppose there is no λ_0 such that μ, ν in $g(\lambda_0) = (\mu, \nu)$ are both infinite. In that case, we prove

(iv) Suppose $\lambda \geq \omega$ and $n \in \omega$.
   (a) If $g(\lambda) = (\lambda, n)$, then there is $m_0 \in \omega$ such that for all $m \geq m_0$ $g(m) = (m - n + 1, n)$.
   (b) If $g(\lambda) = (n, \lambda^+)$, then there is $m_0 \in \omega$ such that for all $m \geq m_0$ $g(m) = (n, m - n + 1)$.
   And hence Q or Q^d is equal to $\exists^{\geq n}$.

For example, let Q be an upwards monotone quantifier. By (ii), choose $g^Q(\lambda_0) = (\lambda_0, \lambda_0^+)$. By (iii), for all $\lambda \geq \lambda_0, g(\lambda) = (\lambda_0, \lambda^+)$. Thus for $\lambda_0 = \aleph_\alpha$, we have $g^Q(\lambda)$ is equal to $g^{Q^+_\alpha}(\lambda)$ from some λ onwards. In case $g^Q(\lambda_0) = (\lambda_0, \lambda_0)$, then we have to use the dual Q^d. In this case $g^{Q^d}(\lambda_0) = (\lambda_0, \lambda_0^+)$, and we proceed as we just saw in the previous case, getting $g^Q(\lambda)$ equal to $g^{Q^-_\alpha}(\lambda)$ from some λ onwards.

Proof of (i). Let $g(\lambda) \neq (\lambda, 0)$ and $g(\mu) = (\mu, 0)$ for some $\mu > \lambda$. Let B, B' be two sets of cardinality μ. Let $A \subset B, |A| = \lambda$. Then $\langle B, A, A\rangle \models [QxP(x)]^A$, since $g(\lambda) \neq (\lambda, 0)$. Let $A' \subset B'$ and $|A'| = \mu, |B' \setminus A'| = \mu$. If we can prove $\langle B, A, A\rangle \leq_{\mathcal{L}(Q)} \langle B', A', A'\rangle$, we will get $\langle B', A', A'\rangle \models [QxP'(x)]^{A'}$, contradicting $g(\mu) = (\mu, 0)$. Now, the game cannot start with a choice of a subset $X \in Q(B)$, since $|B| = \mu$ and by assumption $Q(B) = \emptyset$. Then the game restricts to first order moves, and hence $\langle B', A', A'\rangle \models [QxP'(x)]^{A'}$.

Proof of (ii). First we prove that the first component of $g(\lambda_0)$ must be λ_0. So suppose $g(\lambda_0) = (\mu, \lambda_0^+), \mu < \lambda_0$. By definition of $\lambda_0, \mu \geq \omega$. Let B, B' be two sets of cardinality λ_0. Let $A \subset B, |A| = \lambda_0, |B \setminus A| = \lambda_0$, and $P \subset A, |P| = \mu$ -see Figure 1. Then $\langle B, A, P\rangle \models [QxP(x)]^A$, since $g(\lambda_0) = (\mu, \lambda_0^+)$. Let $A' \subset B', |A'| = \mu$, and $P' \subset A', |P'| = \mu, |A' \setminus P'| = \mu$. If we can prove that

$$(\star) \qquad \langle B, A, P\rangle \leq_{\mathcal{L}(Q)} \langle B', A', P'\rangle,$$

we will get $\langle B', A', P'\rangle \models [QxP'(x)]^{A'}$, hence contradicting the minimality of λ_0. **Proof of** $(\star)$. Let I be the set of finite partial isomorphisms between $\langle B, A, P\rangle$ and $\langle B', A', P'\rangle$, $p = \{(a_0, b_0), (a_1, b_1), \ldots, (a_{m-1}, b_{m-1})\} \in I, a_i \in B, b_i \in B'$, and let $X \in Q(B)$. Choose $Y \subseteq B'$, as described below, such that $|X| = |Y|, |\overline{X}| = |\overline{Y}|$. For a monotone quantifier Q, the fact that a subset $X \subseteq A$ is in $Q(A)$ depends only on $|A|, |X|$, and $|A \setminus X|$, hence $Y \in Q(B')$ follows immediately from $X \in Q(B)$. Now, there are two general cases for the election of Y:

(1) First suppose -see Figure 2- that $X \subset P$, then take $Y \subset P'$, and be sure that for every a_i if $a_i \in X$, then $b_i = p(a_i) \in Y$. Since $|X| = |Y|$, any choice of $y \in Y$ can be replied with $x \in X$.
(2) Now suppose -see Figure 3- $X \subset A$, but there is an $x \in P$ such that $x \notin X$. Then take Y of the same form and such that for every $a_i \in X$, the

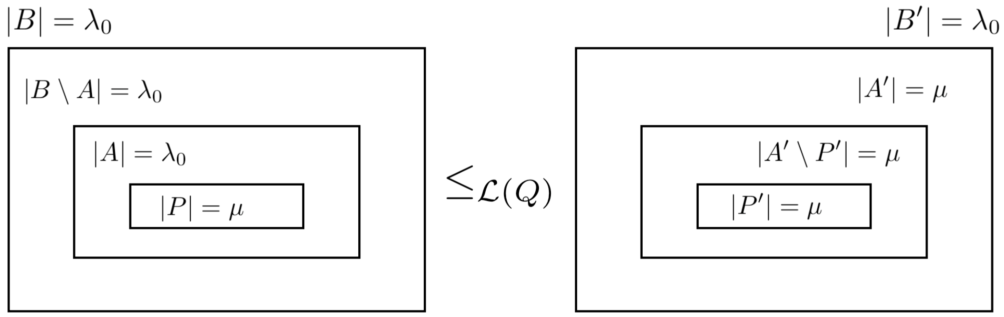

FIGURE 1

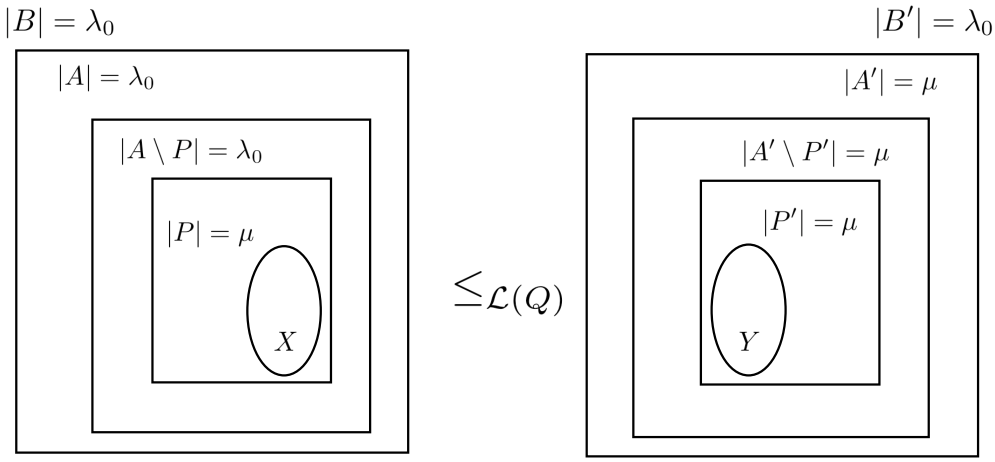

FIGURE 2

corresponding $b_i \in Y$. Since there have been a finite number of choices, and $|A \setminus P|$ is infinite, any choice in Y can be replied in X.

(3) The other cases for choosing Y are similar.

Now, we prove the second component is either λ_0 or λ_0^+. Let $\mu^+ < \lambda_0$ and assume $g(\lambda_0) = (\lambda_0, \mu^+)$. Let B, B' be two sets of cardinality λ_0. Let $A \subset B, |A| = \lambda_0$, and $|B \setminus A| = \lambda_0$; and $P \subset A, |P| = \lambda_0$, such that $|A \setminus P| = \mu$. Then $\langle B, A, P\rangle \models [QxP(x)]^A$. Let $A' \subset B', |A'| = \mu$, and $P' \subset A', |P'| = \mu$, such that $|A' \setminus P'| = \mu$ -see Figure 4. If we can prove

$$(\star\star) \qquad \langle B, A, P\rangle \leq_{\mathcal{L}(Q)} \langle B', A', P'\rangle,$$

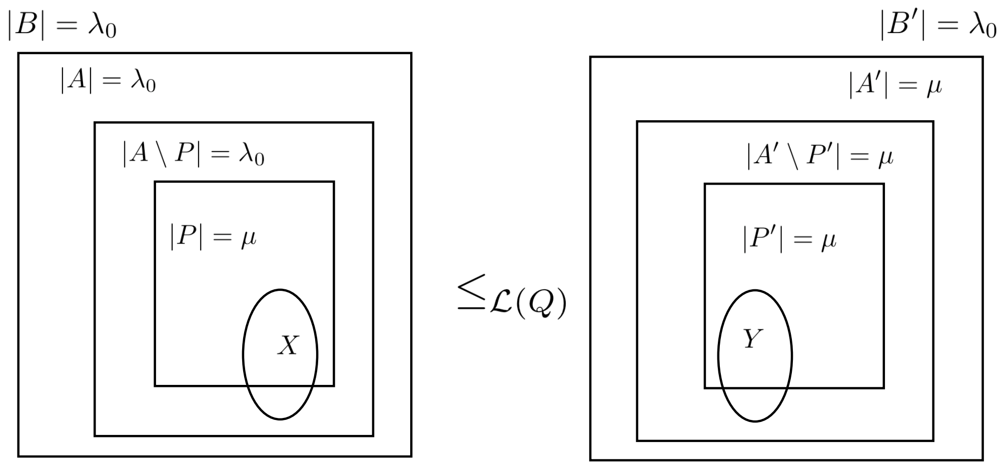

FIGURE 3

we will get $\langle B', A', P'\rangle \models [QxP'(x)]^{A'}$, contradicting the minimality of λ_0. The procedure is the same in case $g(\lambda_0) = (\lambda_0, \mu)$, and μ is a limit cardinal. The proof of $(\star\star)$ works in the same way as the proof of $(\star)$.

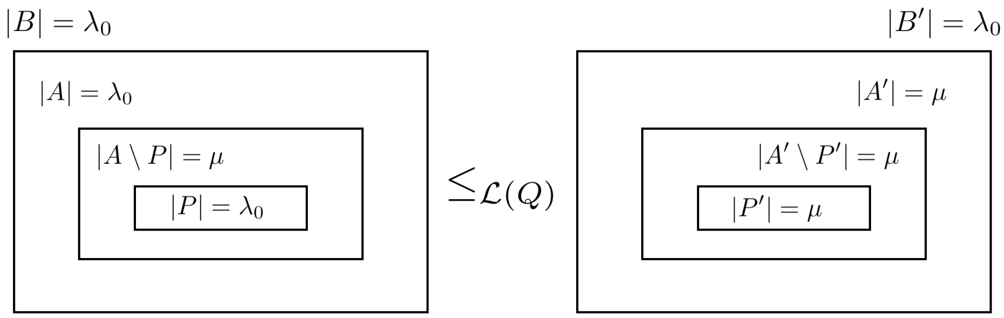

FIGURE 4

Proof of (iii). Let $g(\lambda_0) = (\lambda_0, \lambda_0^+)$. First we prove that in a set A of size $\lambda > \lambda_0$ $Q(A)$ contains subsets of size λ_0. So consider two sets B, B' of cardinality λ. Let $A \subset B$ of cardinality λ_0, and $P \subset A$ of cardinality λ_0, such that $|A \setminus P| = \lambda_0$. Since $g(\lambda_0) = (\lambda_0, \lambda_0^+)$, it means that $\langle B, A, P\rangle \models [QxP(X)]^A$. Let $A' \subset B'$ of cardinality $\lambda, |B' \setminus A'| = \lambda$, and $P' \subset A'$ of cardinality λ_0 -see figure 5. If we can prove that

$$(\star\star\star) \qquad \langle B, A, P\rangle \leq_{\mathcal{L}(Q)} \langle B', A', P'\rangle,$$

then we will get $\langle B', A', P'\rangle \models [QxP'(X)]^{A'}$, demonstrating that $Q(A')$ contains a set of size λ_0.

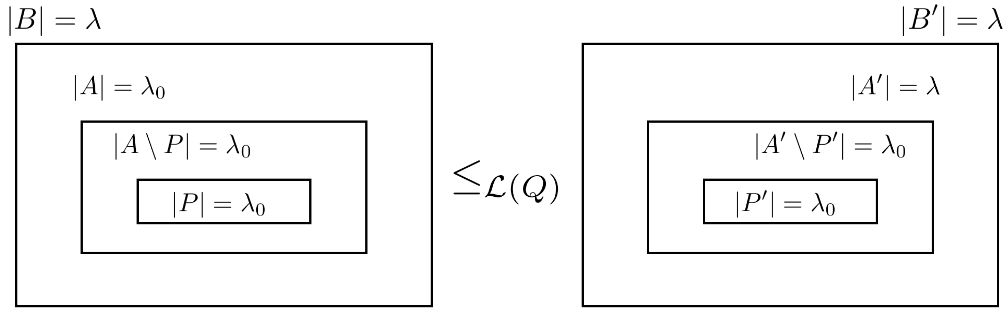

FIGURE 5

Again, $(\star\star\star)$ is proved like $(\star)$. In the same way as in (ii), we can prove that $Q(A)$ does not contain subsets of size less than λ_0.

Proof of (iv). (a). We show if $g(\lambda) = (\lambda, n)$, there is $m_0 \in \omega$ such that for all finite $m \geq m_0, g(m) = (m - n + 1, n)$. Let k be the quantifier rank of the sentence $[QxP(x)]^A$. Suppose that $m \geq k, g(\lambda) = (\lambda, n), \lambda \geq \omega, n \in \omega$, but $g(m) = (m - l + 1, l), m > n, l > n, n \in \omega$. Let B, B' be two sets of cardinality λ. Let $A \subset B, |A| = m$, and $P \subset A, |A \setminus P| = l - 1$ -see Figure 6. Then $\langle B, A, P\rangle \models [QxP(x)]^A$. Let $A' \subset B', |A'| = \lambda$, and $P' \subset A', |A' \setminus P'| = l - 1$. If we can prove $(4\star) : \langle B, A, P\rangle \leq_{\mathcal{L}(Q)^k} \langle B', A', P'\rangle$, we will get $\langle B', A', P'\rangle \models [QxP'(x)]^{A'}$, hence contradicting the maximality of n. The proof of $(4\star)$ is based in the same argument as the proof of $(\star)$.

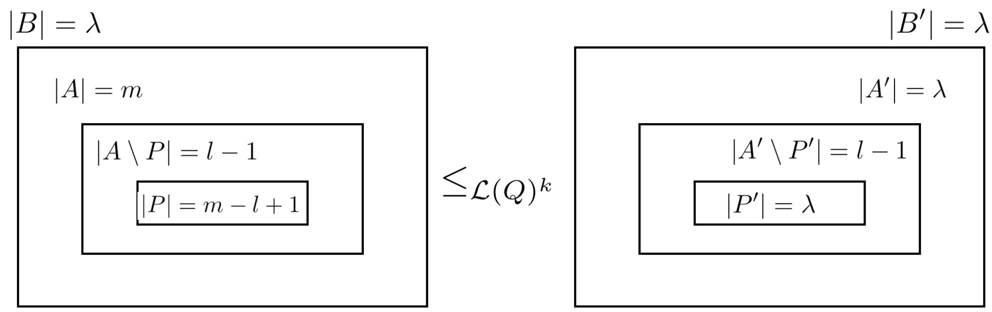

FIGURE 6

(b). We show if $g(\lambda) = (n, \lambda^+)$, then there is $m_0 \in \omega$ such that for all $m \geq m_0$, $g(m) = (n, m-n+1)$. Let again k be the quantifier rank of the sentence $[QxP(x)]^A$. So suppose $m \geq k, g(\lambda) = (n, \lambda^+), \lambda \geq \omega, n \in \omega$, and $g(m) = (l, m-l+1), m > n, l < n, n \in \omega$. Let B, B' be two sets of cardinality λ. Let $A \subset B, |A| = m$, and $P \subset A, |P| = l$ -see Figure 7. Then $\langle B, A, P\rangle \models [QxP(x)]^A$, since $g(m) =$

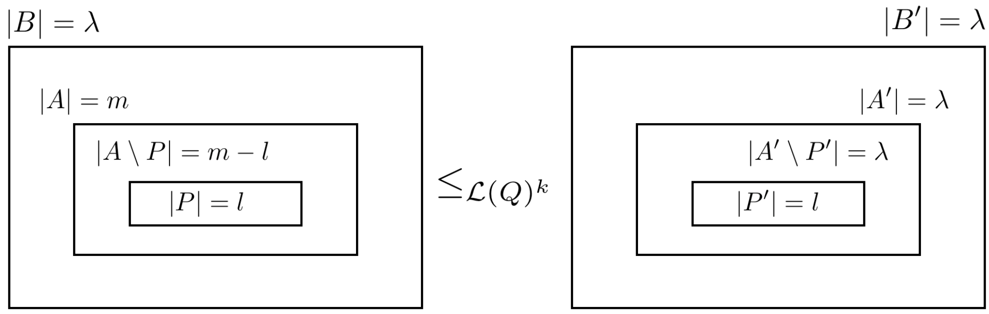

FIGURE 7

$(l, m-l+1)$. Let $A' \subset B', |A'| = \lambda$, and $P' \subset A', |P'| = l$. If we can prove that

$$(5\star) \qquad \langle B, A, P\rangle \leq_{\mathcal{L}(Q)^k} \langle B', A', P'\rangle,$$

we will get $\langle B', A', P'\rangle \models [QxP'(x)]^{A'}$, hence contradicting the minimality of n, in view of $g(\lambda) = (n, \lambda^+)$.

Again, the proof of $(5\star)$ is based in the same argument as the proof of $(\star)$. $\square$

Proposition 85. *Let Q be an upwards monotone quantifier such that $\mathcal{L}_{\omega\omega}(Q)$ and $\mathcal{L}_{\omega\omega}(Q^d)$ relativize and have Löwenheim number κ. Then α in Theorem 84 is such that $\kappa = \aleph_\alpha$.*

Proof. If Q is as in the hypothesis, then Theorem 84 gives us an α such that if we restrict $\mathcal{L}_{\omega\omega}(Q)$ to models of size $\geq \aleph_\alpha$,

$$\mathcal{L}_{\omega\omega}(Q) \equiv \mathcal{L}_{\omega\omega}(Q^+_\alpha) \text{ or } \mathcal{L}_{\omega\omega}(Q) \equiv \mathcal{L}_{\omega\omega}(Q^-_\alpha). \tag{11}$$

Now, because of the Löwenheim number of $\mathcal{L}_{\omega\omega}(Q)$, we know in every nonempty model class in $\mathcal{L}_{\omega\omega}(Q)$ there is a model of size $\leq \kappa$, and because of (11), $\kappa \geq \aleph_\alpha$, since $L_{\omega\omega}(Q^+_\alpha)$ has $LS(\aleph_\alpha)$. To prove that $\kappa = \aleph_\alpha$, suppose $\kappa > \aleph_\alpha$. Now that contradicts the Löwenheim-Skolem property of $L_{\omega\omega}(Q^+_\alpha)$. $\square$

The above analysis of $L(Q)$ for Q upwards monotone, can be extended to extensions of first order logic by a finite number of upwards monotone quantifiers. This analysis will be pursued in another paper.

6.2. **Compactness.**

In this section we explore the conditions on α, β for $\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$ to satisfy compactness. The case for an arbitrary number of quantifiers turns out to be very complicated. We follow the works of Shelah [Sh71] and Fuhrken [F69].

The main result of the section is Theorem 102, that says $\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$ is compact if $\alpha < \beta$, or if $\beta \leq \alpha$ and: 1.) $\aleph_\gamma = \aleph_\gamma^{\aleph_0}, \beta = \gamma + 1$, or 2.) $\aleph_0$ is small for $\aleph_\beta, \beta$ a limit ordinal. We first study the case when β is a successor ordinal, and then the case when it is a limit ordinal.

Definition 86. *Let $\lambda \geq \mu, \nu$, be infinite cardinals. A (λ, μ, ν)-model is a model of cardinality λ in which two distinguished predicates P, Q are interpreted as a set of at least μ elements and a set of at most ν elements respectively.*

We first look at the case $\mu \leq \nu$.

Proposition 87. *Let $\lambda = \mu \leq \nu$. Let T be a first-order theory in a vocabulary $\tau, |\tau| \leq \mu$, such that every finite $T_0 \subset T$ has a (λ, μ, ν)-model. Then T has a (λ, μ, ν)-model.*

Proof. Add to τ μ-many constants and let $S = \{P(c_i) : i < \mu\}$. By assumption $T \cup S$ is finitely satisfiable, and by compactness $T \cup S$ is satisfiable, and hence has a model in which $|P| \geq \mu$. By Löwenheim-Skolem theorem, T has a model of cardinality $\lambda = \mu \leq \nu$ in which $|P|$ is still of cardinality $\geq \mu$, and in which obviously $|Q| \leq \nu$. $\square$

Now we turn to the case in which $\mu > \nu$. In here we can adapt Shelah's method in [Sh71]. Let T be a theory in a vocabulary τ, and let $A = \{a_i : i < \lambda\}$ be a set of λ new individual constants. Consider the following condition $(*)$ on T:

$(*)$ "Let $\nu = \nu_1 + 1$. There is an equivalence relation E on $\bigcup_{n\in\omega} A^n$, with ν_1 equivalence classes, such that equivalent sequences are of the same length and the following sentences are consistent with T:

(i) $a_i \neq a_j$, where $i \neq j$ for $\nu_1 \leq i < j < \mu$;
(ii) $P(a_i)$, where $\nu_1 \leq i < \mu$
(iii) $Q(a_i)$, where $i < \nu_1$;
(iv) $\tau(\bar{b}) = \tau(\bar{c}) \vee (\neg Q(\tau(\bar{b})) \wedge \neg Q(\tau(\bar{c})))$, where, for some n, τ is an n-place term of τ, $\bar{b}, \bar{c} \in A^n$, and $\bar{b} E \bar{c}$."

Lemma 88. *Let $\lambda > \mu > \nu$, be as above. Let T be a first-order theory in a vocabulary τ, such that $|T| \leq \nu_1$, T has names for Skolem functions and T satisfies $(*)$. Then T has a (λ, μ, ν)-model.*

Proof. We can consider the vocabulary σ of T, to be of cardinality $\leq \nu_1$. By $(*)$, there is a model $\mathfrak{M}$ in the language $\sigma' = \sigma \cup A$ that satisfies $(i) - (iv)$ above. By

LS, we can assume $\mathfrak{M}$ be of cardinality λ. To simplify notation, we can identify each a_i with the element of $\mathfrak{M}$ that is denoted by it, and thus we have $A \subset M$.

Let $\mathfrak{N}$ be the elementary submodel of $\mathfrak{M}$, such that N is the union of A and the set of elements of the form $\tau_{\mathfrak{M}\restriction\sigma}(b_1, \ldots, b_n)$,[10] where $b_1, \ldots b_n \in A$. Since there are λ finite strings $b_1, \ldots b_n$ in A, and there are at most ν_1 ($\mathfrak{M} \restriction \sigma$)-terms, there are $\lambda + \lambda * \nu_1 = \lambda$ elements in N. Since in A we already have at least μ elements in P, we are certain $|P^{\mathfrak{N}}| \geq \mu$. We only have to prove that $|Q^{\mathfrak{N}}| \leq \nu_1$. Now, $Q^{\mathfrak{N}}$ consists of all $a_i, i < \nu_1$, as well as the elements of the form $\tau_{\mathfrak{M}\restriction\sigma}(b_1, \ldots, b_n)$ which are in $Q^{\mathfrak{M}}$. But the sentences in (iv) imply that for every term τ, and each n, each equivalence class contributes at most once to elements in $Q^{\mathfrak{N}}$. Since there are ν_1 $\mathfrak{M} \restriction \sigma$-terms, each equivalence class contributes with ν_1 elements to Q, and since there are ν_1 equivalence classes, we get $|Q^{\mathfrak{N}}| = \nu_1$. Hence, $\mathfrak{N} \restriction \sigma$ is a (λ, μ, ν)-model of T. $\square$

Lemma 89. *Let $\mu > \nu$. Let T be a first-order theory. If T has a (λ, μ, ν)-model, $\nu_1 = \nu_1^{\aleph_0}$, and $|T| \leq \aleph_0$, then T satisfies $(*)$.*

Proof. Let $\mathfrak{M}$ be a (λ, μ, ν)-model of T. Expand $\tau(\mathfrak{M})$ with the set of new constants A, and interpret the a_i so that they are the names of all the elements of $\mathfrak{M}$ and so that $a_i, i < \nu_1$ are the names of all the elements of $Q_{\mathfrak{M}}$, and $a_i, \nu_1 < i < \mu$ are all the elements of $P_{\mathfrak{M}}$. So we have $M = A, Q_{\mathfrak{M}} = \{a_i : i < \nu_1\}$ and $P_{\mathfrak{M}} = \{a_i : \nu_1 < i < \mu\}$. Let $\mathfrak{M}' = \mathfrak{M} \restriction (L(T) \bigcup A)$. It is obvious that $\mathfrak{M}'$ is a model of T and that $(i) - (iii)$ hold in it. Now define an equivalence relation E by: For any pair of n-tuples $\bar{b}, \bar{c}$ of A, $\bar{b}E\bar{c}$ if, for every n-place $\mathfrak{M}'$-term τ, either $\tau(\bar{b}) = \tau(\bar{c})$ or both are not in $Q_{\mathfrak{M}'}$. It is obvious that (iv) holds, and it remains to show that there are ν_1 equivalence classes. It is only necessary to check this for terms in $L(T)$, since for the constants in A, the requirement holds. So let Trm_n be the set of all n-place terms of $\tau(T)$. For every $\bar{b} \in A^n$, let $f_{\bar{b}}$ be the function from Trm_n to $\{a_i : i < \nu_1\} \cup \{e\}$ (where e is a new individual) defined as follows:

$f_{\bar{b}}(\tau) = a_i$ if $\tau(\bar{b}) = a_i$ and $i < \nu_1$
$f_{\bar{b}}(\tau) = e$ otherwise.

Clearly $\bar{b}E\bar{c}$ iff $f_{\bar{b}} = f_{\bar{c}}$. Hence, since the number of such functions is $\nu_1^{\aleph_0}$, there are at most $\nu_1^{\aleph_0} = \nu_1$ equivalence classes. On the other hand, each $a_i, i < \nu_1$ forms an equivalence class for the term v, where v is an individual variable. Hence, there are at least ν_1 equivalence classes and the lemma follows. $\square$

[10] Given a model $\mathfrak{A}$ of vocabulary σ and set of elements $S \subseteq A$, an $\mathfrak{A}$-term of σ over S, denoted as $\tau_{\mathfrak{A}}(s_1, \ldots, s_n), \bar{s} \subseteq S$ is defined as follows:

(1) Given an assignment π, a variable x whose image under $\pi, \pi(x) = s$ for some $s \in S$, is an $\mathfrak{A}$-term of σ over S.
(2) Every constant $c \in \sigma$ whose interpretation $\sigma^{\mathfrak{A}} = s$ for some $s \in S$, is an $\mathfrak{A}$-term of σ over S.
(3) If $n > 0$, f is and n-ary function symbol in σ, nad $\tau_1, \ldots, \tau_n$ are $\mathfrak{A}$-terms of σ over S, then $f(\tau_1, \ldots, \tau_n)$ is an $\mathfrak{A}$-term of σ over S.

Lemma 90. *If $\nu_1 = \nu_1^{\aleph_0}$, T first-order and at most countable, and every finite subtheory of T satisfies $(*)$, so does T.*

Proof. Clearly T is consistent with $(i), (ii)$. Let $T = \{\phi_0, \phi_1, \dots\}$, and let E_i be equivalence relations such that $(i) - (iv)$ are consistent with $\phi_0 \wedge \cdots \wedge \phi_i$. Let E be defined as:

$$\bar{b}E\bar{c} \text{ if, for every } i < \omega, \bar{b}E_i\bar{c}.$$

Then clearly T is consistent with (iii). Now, each E_i has ν_1 equivalence classes, and if we superpose the grids of each E_i, since there are $\aleph_0$ of them, we could get at most $\nu_1^{\aleph_0} = \nu_1$ equivalent classes, that is, in case no E_i has an equivalence class equal to some equivalence class in some E_j. And we could get at least ν_1 equivalence classes, that is, in case all E_i provide the same equivalence classes. $\square$

Lemma 91. *A first-order theory T satisfies $(*)$ iff every countable subtheory of T satisfies $(*)$.*

Proof. This proof is exactly the same as Shelah's in [Sh71], only modifying that the permutations on the relation symbols keep fixed also P.

The above lemmas prove the following

Proposition 92. *Let $\mu > \nu$. If every finite subtheory of a first-order theory T, with $|T| \leq \nu_1 = \nu_1^{\aleph_0}$ has a (λ, μ, ν)-model, then T has a (λ, μ, ν)-model.*

We now adapt Fuhrken's method in [F69] for translating an $\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$-theory into a first order one, and then be ready to apply Proposition 92 to it.

Let τ be a vocabulary, and U, V, F_1, F_2 be respectively two unary and two ternary predicates not in τ. Let $\mathcal{L}^*_{\omega\omega} = \mathcal{L}_{\omega\omega}[\tau \cup \{U, V, F_1, F_2\}]$.

For any formula $\varphi \in \mathcal{L}_{\omega\omega}[\tau](Q^+_\alpha, Q^-_\beta)$ define $\varphi^* \in L^*_{\omega\omega}$:

(1) $\varphi^* = \varphi$ if φ is atomic.
(2) $(\neg\varphi)^* = \neg\varphi^*$ if $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$
(3) $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$
(4) $(\exists x\varphi(x))^* = \exists x\varphi^*(x)$
(5) $(Q^+_\alpha v_n\varphi(v_n))^* = \exists v_k \forall v_{k+1} \exists v_n(U(v_{k+1}) \to (\varphi^*(v_n) \wedge F_1(v_k, v_{k+1}, v_n)))$
(6) $(Q^-_\beta v_n\varphi(v_n))^* = \exists v_k \forall v_{k+1} \exists v_n(V(v_{k+1}) \vee (\varphi^*(v_n) \wedge F_2(v_k, v_{k+1}, v_n)))$.

We associate with any set $\Sigma \subseteq L_{\omega\omega}(Q^+_\alpha, Q^-_\beta)[\tau]$ a set $\Sigma^* \subseteq \mathcal{L}^*_{\omega\omega}$ which consists of the following sentences:

(a) All the sentences σ^*, for $\sigma \in \Sigma$
(b) The sentence which says "F_1, F_2 are indexed collections of one-to-one functions".

Lemma 93. *A structure $\mathfrak{A}$ of cardinality λ is a model of Σ iff there are subsets X, Y, such that $|X| \geq \aleph_\alpha, |Y| < \aleph_\beta$, and some ternary relations Z_1, Z_2 on A such that $\langle \mathfrak{A}, X, Y, Z_1, Z_2 \rangle$ is a model of Σ^*.*

Proof. Suppose first that $\mathfrak{A}$ is a model of Σ. Let X, Y be subsets of A of cardinalities $\aleph_\alpha, \aleph_\gamma$ respectively. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of all subsets S_1 (resp. S_2) of A such that for some subformula $Q^+_\alpha x\psi(\bar{a}, x)$ (resp. $Q^-_\beta x\theta(\bar{a}, x)$) of a sentence of Σ true in $\mathfrak{A}$,

$$S_1 = \{a \in A : \mathfrak{A} \models \psi(\bar{a}, a)\}$$

respectively

$$S_2 = \{a \in A : \mathfrak{A} \models \theta(\bar{a}, a)\}.$$

For every $S_1 \in \mathcal{A}$, let h_1 be a function that maps one-to-one the elements of X into the elements of S_1, so $h_1 : X \to S_1$. Let $H_1 = \{h_1 : S_1 \in \mathcal{A}\}$

For every $S_2 \in \mathcal{B}$, let h_2 be a function that maps one-to-one the elements of $\overline{Y}$ into the elements of S_2, so $h_2 : \overline{Y} \to S_2$. Let $H_2 = \{h_2 : S_2 \in \mathcal{B}\}$.

The sets $\mathcal{A}, \mathcal{B}$ are composed of subsets of A definable by formulas of the above form with parameters. So $|\mathcal{A}|, |\mathcal{B}| \leq |A|$, and therefore $|H_1|, |H_2| \leq |A|$, and we can index both sets by one-to-one functions $f_i : H_i \to A$.

Let Z_1, Z_2 be ternary relations defined on A by

$$< x, y, x >\in Z_1 \text{ iff for some } h_1 \in H_1, f_1(h_1) = x \text{ and } h_1(y) = z$$

and

$$< x, y, x >\in Z_2 \text{ iff for some } h_2 \in H_2, f_2(h_2) = x \text{ and } h_2(y) = z.$$

Clearly $\mathfrak{A}^+ = \langle \mathfrak{A}, X, Y, Z_1, Z_2 \rangle$ is a model of the sentences in (b). Now we have to prove that it is a model of the sentences $\sigma^* \in \Sigma^*$, for $\sigma \in \Sigma$. We prove by induction on the formation of formulas that

$$(\star) \qquad \mathfrak{A} \models \varphi \to \mathfrak{A}^+ \models \varphi^*.$$

The only interesting cases are those with the generalized quantifier.

So suppose $\bar{a} \in A$, and $\mathfrak{A} \models Q^+_\alpha v_n \varphi(\bar{a}, v_n)$. Let $S_1 \in H_1$ be

$$S_1 = \{a \in A : \mathfrak{A} \models \varphi(\bar{a}, a)\},$$

Then $|S_1| \geq \alpha$, and there is a function $h_1 : X \to S_1$. Let $f_1(h_1) = b$. Then, for all $c \in X$, there is some $a \in S_1$ such that $< b, c, a >\in Z_1$. So $\langle \mathfrak{A}, X, Z_1 \rangle \models \exists v_k \forall v_{k+1} \exists v_n F_1(v_k, v_{k+1}, v_n)$. Now, by induction hypothesis, we have for all $a \in S_1$ that $\mathfrak{A}^+ \models \varphi^*(\bar{a}, a)$. It follows that

$$\mathfrak{A}^+ \models \exists v_k \forall v_{k+1} \exists v_n (U(v_{k+1}) \to (\varphi^*(\bar{a}, v_n) \wedge F_1(v_k, v_{k+1}, v_n)))$$

that is

$$\mathfrak{A}^+ \models (Q^+_\alpha v_n \varphi(\bar{a}, v_n))^*.$$

Now let $\mathfrak{A} \models Q^-_\beta v_n \varphi(\bar{a}, v_n)$. Let

$$S_2 = \{a \in A : \mathfrak{A} \models \varphi(\bar{a}, a)\},$$

Then $|\bar{S_2}| < \aleph_\beta$, and there is a function $h_2 : \bar{Y} \to S_2$. Let $f_2(h_2) = b$. Then, for all $a \in \overline{Y}$, there is some $c \in S_2$ such that $< b, a, c >\in Z_2$. So $\langle \mathfrak{A}, Y, Z_2 \rangle \models$

$\exists v_k \forall v_{k+1} \exists v_n F_2(v_k, v_{k+1}, v_n)$. Now, by induction hypothesis, we have for all $c \in S_2$ that $\mathfrak{A}^+ \models \varphi^*(c)$. It follows that

$$\mathfrak{A}^+ \models \exists v_k \forall v_{k+1} \exists v_n (V(v_{k+1}) \vee (\varphi^*(\bar{a}, v_n) \wedge F_2(v_k, v_{k+1}, v_n)))$$

that is

$$\mathfrak{A}^+ \models (Q^-_\beta v_n \varphi(\bar{a}, v_n))^*.$$

Conversely, assume now that $\mathfrak{A}$ is a τ-structure of cardinality $|A| = \lambda$, $X, Y \subset A$ with $|X| \geq \aleph_\alpha, |Y| < \aleph_\beta$, and Z_1, Z_2 ternary relations in A such that $\mathfrak{A}^+ = \langle \mathfrak{A}, X, Y, Z_1, Z_2 \rangle$ is a model of Σ^*. We show that $\mathfrak{A}$ is a model of Σ. We do it by induction on the formation of formulas, proving that for any formula $\varphi \in \Sigma^*$

$$(+)\ \mathfrak{A}^+ \models \varphi^* \to \mathfrak{A} \models \varphi.$$

Again the only interesting case is the generalized quantifier. Let us use henceforth the notation:

$$\begin{aligned} S &= \{a \in A : \mathfrak{A} \models \varphi(a)\}, \\ S^* &= \{a \in A : \mathfrak{A}^+ \models \varphi(a)^*\}. \end{aligned}$$

First suppose $\mathfrak{A}^+ \models (Q^+_\alpha v_n \varphi(\bar{a}, v_n))^*$. By (5), it means $\mathfrak{A}^+ \models \exists v_k \forall v_{k+1} \exists v_n$ $(U(v_{k+1}) \to (\underline{\varphi^*(v_n)} \wedge F_1(v_k, v_{k+1}, v_n)))$, that is, there is a one-to-one function mapping X into $\overline{S^*}$. By induction hypothesis, $S^* = S$, so $|S| \geq |X| \geq \alpha$, whence $\mathfrak{A} \models Q^+_\alpha v_n \varphi(\bar{a}, v_n)$. Similarly for the case $\mathfrak{A}^+ \models (Q^-_\beta v_n \varphi(\bar{a}, v_n))^*$. And this completes the proof of the theorem. $\square$

Now we consider the case in which β is a limit. Again we follow Shelah's work on [Sh71].

Definition 94. *A* (λ, μ, ν)-like ordered model *is a model* $\langle A, R, P, Q \ldots \rangle$ *with* $|A| = \lambda$*,* R *binary and* P *and* Q *unary, such that* $|P^{\mathfrak{A}}|$ *is at least* μ*, and such that* $R^{\mathfrak{A}}$ *is a* ν*-like ordering of* $Q^{\mathfrak{A}}$*.*

Definition 95. *Given two ordinals* α, β*, we say that* α *is* small for β *if* $\kappa^\alpha < \beta$ *for all* $\kappa < \beta$*, and the cofinality of* β *is greater than* α*.*

We have to prove the following

Proposition 96. *Let* T *be a first-order theory in a vocabulary* σ*, and* $\lambda \geq \mu$*, with* $\aleph_0$ *small for* ν*. If every finite subtheory of* T *has a* (λ, μ, ν)*-like ordered model, then* T *has a* (λ, μ, ν)*-like ordered model.*

Let $\lambda \geq \mu > \nu$ be infinite cardinals, ν limit, T a first-order theory in a vocabulary σ of cardinality at most ν, A a set $\{a_i : i < \lambda\}$ of constants, and consider the following condition $(\star\star)$ on T:

$(\star\star)$: There is a family $\{E_l : l < \nu\}$ of equivalence relations on $\bigcup_{n<\omega} A^n$ such that each E_l has less than ν equivalence classes, and a function $h : \bigcup_{n<\omega} \rho^n \to$ field$(<)$ such that such that $|$ field$(<)| = \rho \leq \lambda$; and the set of all the following sentences is consistent with T:

(i) $a_i \neq a_j$ for $i \neq j$,
(ii) $a_i < a_j$ for $i < j < \nu$,
(iii) $P(a_i)$, where $\nu \leq i < \mu$,
(iv) $Q(a_i)$, where $i < \nu$,
(v) $Q(\tau(a_{i_1}, \dots, a_{i_n})) \to \tau(a_{i_1}, \dots, a_{i_n}) < a_{h(i_1,\dots,i_n)}$, where τ is a term of σ,
(vi) $\tau(\bar{b}) = \tau(\bar{c}) \vee (a_l < \tau(\bar{b}) \wedge a_l < \tau(\bar{c}))$, where, for some n, τ is an n-place term of σ, $\bar{b}, \bar{c} \in A^n$, and $\tau(\bar{b}) E_l \tau(\bar{c})$,
(vii) $<$ is a linear order of Q.

Lemma 97. *Let $\lambda > \mu > \nu$ be infinite cardinals, ν a limit, and T a first order theory with $|T| \leq \nu$. Suppose T has names for Skolem functions and satisfies $(\star\star)$. Then T has a (λ, μ, ν)-like ordered model.*

Proof. We can consider the vocabulary σ of T, to be of cardinality $\leq \nu$. By $(**)$, there is a model $\mathfrak{M}$ in the language $\sigma' = \sigma \cup \{A, <'\}$ that satisfies $(i) - (vi)$ above. By LS, we can assume $\mathfrak{M}$ to be of cardinality λ. To simplify notation, we can identify each a_i with the element of $\mathfrak{M}$ that is denoted by it, and thus we have $A \subset M$.

Let $\mathfrak{N}$ be the elementary submodel of $\mathfrak{M}$, such that N is the union of A and the set of elements of the form $\tau_{\mathfrak{M}\restriction\sigma}(b_1, \dots, b_n)$, where $b_1, \cdots_n \in A$. Since there are λ finite strings $b_1, \cdots_n$ in A, and there are at most ν $(\mathfrak{M} \restriction \sigma)$-terms, there are $\lambda + \lambda * \nu = \lambda$ elements in N. Since in A we already have at least μ elements in P, we are certain $|P^{\mathfrak{N}}| \geq \mu$. Now we have to show that $Q^{\mathfrak{N}}$ is ν-like ordered by $<$. Condition (vi) says that for each a_l in Q, there is an equivalence relation E_l (with less than ν-many equivalence classes) such that the interpretation of each new term (there are ν-many) over the tuples of the same equivalence class is the same, or otherwise every different image is bigger than a_l, which makes a_l have less than ν-many predecessors. Then, $(Q^{\mathfrak{N}}, <^{\mathfrak{N}})$ is a ν-like order, and $\mathfrak{N} \restriction \sigma$ is a (λ, μ, ν)-like ordered model. $\square$

Lemma 98. *Let $\lambda \geq \mu > \nu$ be infinite cardinals and ν a limit with $\aleph_0$ small for ν. Let T be a first order theory in a vocabulary σ, with $|T| \leq \aleph_0$. If T has a (λ, μ, ν)-like ordered model, then T satisfies $(\star\star)$.*

Proof. Let $\mathfrak{M}$ be a (λ, μ, ν)-like ordered model of T. Expand the vocabulary of $\mathfrak{M}$ with the set of new constants A, and interpret each a_i so that they are the names of all the elements of M, and so that $a_i, i < \nu$ are the names of all the elements of $Q^{\mathfrak{M}}$, and $a_i, \nu < i < \mu$ are the names of the elements of $P^{\mathfrak{M}}$. Let $\mathfrak{M}'$ be the expansion of $\mathfrak{M}$ such that $M' = M = A, Q^{\mathfrak{M}'} = \{a_i : i < \nu\}$ and $P^{\mathfrak{M}'} = \{a_i : \nu < i < \mu\}$. Set $\mathfrak{M}'' = \mathfrak{M}' \restriction (\sigma \cup A)$. It is obvious that $\mathfrak{M}''$ is a model of T and that (i)-(iv) hold in it. Then we can define a function h so that in condition (v), $\tau(a_{i_1}, \dots, a_{i_n}) < a_{h(i_1,\dots,i_n)}$, if $\tau(a_{i_1}, \dots, a_{i_n})$ is in the domain of $<$.

Now define a set $\{E_l : l < \nu\}$ of equivalence relations such that for any pair of n-tuples $\bar{a}, \bar{b}$ of A, and for some l, $\bar{a} E \bar{b}$ if, for every n-place $\mathfrak{M}''$-term τ, either $\tau(\bar{a}) = \tau(\bar{b})$ or $a_l < \tau(\bar{a}) \wedge a_l < \tau(\bar{b})$. It is clear that (vi) holds, and it remains to show that each E_l has less than ν-many equivalence classes. Let Trm_n be the set of all n-terms of σ. For each $\bar{a} \in A^n$, let $f_{\bar{a}}$ be the function from Trm_n to $\{a_i : i < \nu\} \cup \{e\}$ (where e is a new individual constant) defined as:

$$f_{\bar{a}}(\tau) = \left\{ \begin{array}{cc} a_i & \text{if } \tau(a_i) \leq a_l, l < \nu \\ e & \text{otherwise} \end{array} \right\}$$

There are at most $\aleph_0$ terms, so there are at most $\kappa^{\aleph_0} < \nu$ equivalence classes, for some $\kappa < \nu$. $\square$

Lemma 99. *If $\nu = \nu^{\aleph_0}$, T is an at most countable first-order theory, and every finite subtheory of T satisfies $(\star\star)$, so does T.*

Proof. This proof is like the proof of lemma 93. $\square$

Lemma 100. *T satisfies $(\star\star)$ iff every countable subtheory of T satisfies $(\star\star)$.*

Proof. Again this proof is as in Shelah's [Sh71], only modifying that the permutations on the relation symbols keep fixed P and $<$. $\square$

All the above lemmas prove Proposition 96.

We now adapt Fuhrken translation to this case, because we cannot chose for V a cardinality $\kappa < \aleph_\beta$ such that being less than $\aleph_\beta$ implies being less or equal to κ. Instead, we choose an $\aleph_\beta$- like ordering P in A (an ordering in which every initial segment is of size less than $\aleph_\beta$), and for every definable subset X with complement of cardinality $< \aleph_\beta$ in A, we choose a one-to-one function that maps the complement of an initial segment of P into X.

Let τ be a vocabulary, and U, V, P, F_1, F_2 be respectively two unary, one binary and two ternary predicates not in τ. Let $\mathcal{L}^*_{\omega\omega} = \mathcal{L}_{\omega\omega}[\tau \cup \{U, V, P, F_1, F_2\}]$. For any formula $\phi \in \mathcal{L}_{\omega\omega}[\tau](Q^+_\alpha, Q^-_\beta)$, let ϕ^* be:

(1) $\varphi^* = \varphi$ if φ is atomic.
(2) $(\neg\varphi)^* = \neg\varphi^*$ if $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$
(3) $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$
(4) $(\exists x \varphi(x))^* = \exists x \varphi^*(x)$
(5) $(Q^+_\alpha v_n \varphi(v_n))^* = \exists v_k \forall v_{k+1} \exists v_n (U(v_{k+1}) \to ((\varphi(v_n))^* \wedge F_1(v_k, v_{k+1}, v_n)))$
(6) $(Q^-_\beta v_n \varphi(v_n))^* = \exists v_k \forall v_{k+1} \exists v_n (P(v_{k+1}, v_{k+2}) \vee ((\varphi(v_n))^* \wedge F_2(v_k, v_{k+1}, v_n)))$.

We associate with any set $\Sigma \subseteq L_{\omega\omega}(Q^+_\alpha, Q^-_\beta)[\tau]$ a set $\Sigma^* \subseteq \mathcal{L}^*_{\omega\omega}$ which consists of the following sentences:

(a) All the sentences σ^*, for $\sigma \in \Sigma$
(b) The sentence which says "F_1, F_2 are indexed collections of one-to-one functions".
(c) The sentences which says that P totally orders V.

Lemma 101. *A relational structure $\mathfrak{A}$ of cardinality λ is a model of Σ iff there are two subsets X, Y of A, $|X| \geq \aleph_\alpha$ a binary relation W on Y which is a $\aleph_\beta$-like ordering, and two ternary relations Z_1, Z_2 on A such that $\mathfrak{A}^+ = \langle \mathfrak{A}, X, Y, W, Z_1, Z_2 \rangle$ is a model of Σ^*.*

Proof. We only prove the part concerning the quantifier Q^-_β. Suppose first that $\mathfrak{A}$ is a model of Σ. Let Y be a subset of A, and W an $\aleph_\beta$-like ordering of Y. Let $\mathcal{A}$ be the set of all subsets S of A such that for some n, $\bar{a} \in A^n$, and subformula $Q^-_\beta v_n \phi(\bar{a}, v_n)$ of a sentence of Σ true in $\mathfrak{A}$,

$$S = \{a \in A : \mathfrak{A} \models \phi(\bar{a}, a)\}$$

For every $S \in \mathcal{A}$, let h be a one-to-one function that maps the complement of an initial segment of W into elements of S. Since there are as many functions h as definable subsets in A, we can index these functions by elements of A.

Let Z_1 be a ternary relation defined on A by:

$$< x, y, z >\in Z_2 \text{ iff for some } h \in H, f(h) = x \text{ and } h(y) = z.$$

Clearly $\mathfrak{A}^+$ is a model of $(a) - (c)$ above. We have to prove that it is a model of the sentences σ^* for $\sigma \in \Sigma$. We prove by induction on the formation of formulas that

$$(\star) \qquad \mathfrak{A} \models \phi \rightarrow \mathfrak{A}^+ \models \phi^*$$

Suppose $\mathfrak{A} \models Q^-_\beta v_n \phi(\bar{a}, v_n)$. Let $S = \{a \in A : \mathfrak{A} \models \phi(\bar{a}, a)\}$. Then, $|\overline{S}| < \aleph_\beta$, and there is a one-to-one function h mapping the complement of an initial segment of W into S. Therefore $(\mathfrak{A}^+, S) \models \exists v_k \exists v_{k+2} \forall v_{k+1} \exists v_n (P(v_{k+1}, v_{k+2}) \vee (S(v_n) \wedge F_2(v_k, v_{k+1}, v_n)))$. Now, by induction hypothesis, we have that for all $a \in S, \mathfrak{A}^+ \models \phi^*(\bar{a}, a)$. It follows that $\mathfrak{A}^+ \models \exists v_k \exists v_{k+2} \forall v_{k+1} \exists v_n (P(v_{k+1}, v_{k+2}) \vee (\phi^*(\bar{a}, v_n) \wedge F_2(v_k, v_{k+1}, v_n)))$. That is

$$\mathfrak{A}^+ \models (Q^-_\beta v_n \phi(\bar{a}, a))^*.$$

□

Theorem 102. *$\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$ is compact, if $\alpha < \beta$, or if $\beta \leq \alpha$ and $\aleph_0$ is small for $\aleph_\beta$.*

Proof. Cases $\alpha < \beta$ and $\beta < \alpha$ follow from Propositions 87 and 92, and Lemmas 93 and 101. The case $\beta = \alpha$ has negation and thus is Shelah's result in [Sh71]. □

Compactness of logics obtained by adding various quantifiers Q^+_α and Q^-_β to first order logic reduces to the above analysis if all the relevant cardinals $\aleph_\beta$ are below all the relevant cardinals $\aleph_\alpha$, or vice versa, but the general case seem quite difficult to tackle.

6.3. **Separation.**

Separation theorem in logics $\mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$ depends on the relation between α and β. Only when $\beta > \alpha$ does separation hold for $\mathcal{L} = \mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$. Since by the results of last section this logic is also compact, we get that without negation, there are compact extensions of first-order logic that satisfy separation.

Theorem 103. *If $\mathcal{L} = \mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$, and $\alpha \geq \beta$, then $\mathcal{L}$ does not satisfy the interpolation theorem.*

Proof. Let R be an equivalence relation, and let

$$\mathbf{K}_1 = \{(A, R) : R \text{ has at least } \aleph_\alpha \text{ classes.}\}$$

and

$$\mathbf{K}_2 = \{(A, R) : R \text{ has less than } \aleph_\beta \text{ classes.}\}$$

$\mathbf{K}_1$ and $\mathbf{K}_2$ are PC-classes in $\mathcal{L}$. Indeed let $\mathbf{K}_i = Mod(\phi_i) \restriction R$, and the vocabulary of ϕ_i be $\{R, P\}$, where P is a unary predicate. Let $\phi_1 =$ the conjunction of

(1) R is an equivalence relation;
(2) "There are at least $\aleph_\alpha$ elements in P", that is, $Q^+_\alpha x P(x)$;
(3) "P is a set of non-equivalent elements." That is

$$\forall x \forall y((P(x) \wedge P(y)) \rightarrow \neg R(x, y))$$

.

Now let ϕ_2 be the conjunction of

(1) R is an equivalence relation;
(2) "There are less than $\aleph_\beta$ elements in $\neg P$", that is, $Q^-_\beta P(x)$;
(3) "$\neg P$ meets every equivalence class." That is, $\forall x \exists y(\neg P(y) \wedge R(x, y))$

Then, clearly $\mathbf{K}_1$ and $\mathbf{K}_2$ are as we wanted, and are disjoint. The proof is completed by the following theorem of Caicedo

Theorem 104 (Caicedo [Ca79])**.** *Let $\mathcal{L}_{\infty\omega}(Mon)$ denotes $\mathcal{L}_{\infty\omega}$ with any finite number of monadic quantifiers. Let (κ, μ) be a model for an equivalence relation R, with cardinality κ and μ many equivalence classes.*
Then $(\kappa, \mu_1) \equiv_{\mathcal{L}_{\infty\omega}(Mon)} (\kappa, \mu_2)$ whenever $\kappa \geq \mu_1 \geq \mu_2 \geq \omega$. $\square$

A weaker version of the above theorem was proved by Väänänen in [V74].

Theorem 105. *If $\mathcal{L} = \mathcal{L}_{\omega\omega}(Q^+_\alpha, Q^-_\beta)$, and $\alpha < \beta$, then $\mathcal{L}$ satisfies the separation theorem.*

Proof. Let $\tilde{\phi}$ be the sentence that results from $\phi \in \mathcal{L}$ when we replace Q^+_α everywhere by Q^+_0.

Let $\varphi \in \mathcal{L}[\tau_1]$ and $\psi \in \mathcal{L}[\tau_2]$, with $\tau_i \supset \tau$ for $i = 1, 2$ such that $Mod(\varphi) \restriction \tau \cap Mod(\psi) \restriction \tau = \emptyset$.

We show $Mod(\tilde{\varphi}) \restriction \tau \cap Mod(\tilde{\psi}) \restriction \tau = \emptyset$. Let τ'' be a disjoint copy of $\tau' = (\tau_1 \cap \tau_2) \setminus \tau$, and $\rho : \tau' \rightarrow \tau''$ be a renaming. Suppose the claim does not hold. Then there is a model $\mathfrak{M}$ of $\tilde{\varphi} \wedge \tilde{\psi}^\rho$, where the possible common symbols in $(\tau_1 \cap \tau_2) \setminus \tau$ has been renamed in $\tilde{\psi}$ in order to avoid conflict. Add to τ a set $C = \{c_i : i \in \aleph_\alpha\}$ of $\aleph_\alpha$ many constants, and let $T = Th(\mathfrak{M})$. Now consider the set of sentences $S = \{\xi_j(c_i) : i \in \aleph_\alpha, j \text{ such that } \xi \text{ is a subformula of } \varphi \text{ or } \psi^\rho \text{ and } \xi_j^{\mathfrak{M}} \text{ is infinite}\}$. Then by compactness $S \cup T$ has a model $\mathfrak{N}$ of size $\aleph_\alpha$, which is a model of $\varphi \wedge \psi^\rho$.

Hence, the restriction of $\mathfrak{N}$ to τ is in $Mod(\varphi) \restriction \tau$ as well as in $Mod(\psi^\rho) \restriction \tau$, which is the same as $Mod(\psi) \restriction \tau$, since the renaming only affects symbols outside τ. That makes $Mod(\varphi) \restriction \tau \cap Mod(\psi) \restriction \tau$ nonempty, a contradiction.

Now we show that there is a first order sentence that separates $Mod(\tilde{\varphi}) \restriction \tau$ and $Mod(\tilde{\psi}) \restriction \tau$ -and hence $Mod(\varphi) \restriction \tau$ and $Mod(\psi) \restriction \tau$. Suppose that no such sentence exists. Adding extra predicates to code partial isomorphisms, and using compactness and Löwenheim-Skolem properties of $\mathcal{L}_{\omega\omega}(Q_0^+)$ (as is done in, for example, [Fl85]), we get two countable models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \tilde{\varphi}$ and $\mathfrak{B} \models \tilde{\psi}$, and there is a sequence of length ω of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$. Since partially isomorphic countable models are isomorphic, and $Mod(\tilde{\varphi})$ is closed under isomorphisms, $\mathfrak{A} \models \tilde{\varphi}$ implies $\mathfrak{B} \models \tilde{\varphi}$, but then $\mathfrak{B} \in Mod(\tilde{\varphi}) \cap Mod(\tilde{\psi})$ -a contradiction. $\square$

In the case we have several cardinal quantifiers, $\mathcal{L} = \mathcal{L}_{\omega\omega}(Q^+_{\alpha_j}, Q^-_{\beta_k})$, for $j = 0, \ldots, m$ and $k = m+1, \ldots, n$, has separation only if every β_k is bigger than every α_j, in which case we should argue as in Theorem 105. Otherwise, Theorem 103 applies.

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