Passive Advection and the Degenerate Elliptic Operators M_n

Ville Hakulinen

Academic dissertation

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Ville Hakulinen
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Contents

Chapter 1

Introduction

1.1 Navier-Stokes Turbulence

This section is adapted from [18]. The Kraichnan model of passive advection is an exactly solvable model that has a very similar phenomenology to the full Navier-Stokes turbulence, but is much simpler in many respects.

The Navier-Stokes equations

$$\begin{cases} \partial_t v = \nu \Delta v - (v \cdot \nabla) v - \nabla p + f \\ \nabla \cdot v = 0 \end{cases} \tag{1.1.1}$$

describe the motion ($v(t,x) \in \mathbb{R}^3$ is the velocity field of the fluid) of incompressible fluid acted on by the external force f. In case of finite volume, boundary conditions should also be specified.

This can be considered as an infinite dimensional dynamical system. If the external force f is zero, then for physical reasons the fluid motion eventually stops (this has not been proven rigorously for any interesting initial data). So if one wants to get a statistical steady state one should look at the forced case. It might be conceivable that this steady state depends strongly on the nature of the forcing, but in fact it is not so and there appears to be a steady state, called isotropic and homogeneous turbulence, which is universal, i.e. has properties relatively independent of the nature of the forcing.

In physical situations one has some (large) scale L where the forcing takes place. For example for underwater golf L would be the diameter of the golf ball or for stirring lentil soup the size of the scoop and the typical variations of the trajectory of the scoop would be of order L. Also one might consider flow on $\mathbb{R}^3$ (or 3-torus) and require that the Fourier transform (or series) of f in spatial coordinates is supported on wavenumbers k with $|k| < L^{-1}$.

For concreteness, let us consider a special situation. Let's take f a Gaussian process with mean zero and covariance

$$\langle f_\alpha(t,x) f_\beta(t',x')\rangle = C_{\alpha\beta}\left(\frac{x-x'}{L}\right) R(t-t'), \tag{1.1.2}$$

where $C_{\alpha\beta}$ is some fixed nice smooth function satisfying $\sum_\alpha \partial_\alpha C_{\alpha\beta}(x) = 0$ and R is say smooth with compact support. Here L corresponds to the spatial scale discussed above.

The Navier-Stokes equations are invariant under certain rescalings. Let

$$\begin{aligned} \tilde{v}(t,x) &= \sigma v(\tau t, sx), \\ \tilde{f}(t,x) &= \sigma f(\tau t, sx) \text{ and} \\ \tilde{p}(t,x) &= \sigma\tau s^{-1} p(\tau t, sx) \end{aligned} \tag{1.1.3}$$

with $\sigma s/\tau = 1$. Then $\tilde{v}$, $\tilde{f}$ and $\tilde{v}$ satisfy (1.1.1) with ν is replaced by $\tilde{\nu} = \tau s^{-2}\nu$. One can then introduce a dimensionless quantity

$$R := \frac{VL}{\nu} \tag{1.1.4}$$

which is called the Reynolds number. Here V stands for the typical value of velocity differences on scales comparable to L. The point of the Reynolds number is that it is invariant under the rescalings (1.1.3).

Experiments show that for small values of R the flows are smooth (laminar) and as R grows the flow goes through a multitude of bifurcations and in the end for large R the flow is very disordered.

Supposing that (1.1.1) has nice enough solutions, one may ask whether the corresponding dynamical system has a unique ergodic invariant measure μ. Supposing this to be the case one may ask about properties of μ. From probabilistic point of view, the standard ones would be the moments of v with respect to μ, but in the hydrodynamics community one usually considers the so called *structure functions*:

$$S_N(x) := \int (\hat{x}\cdot(v(x)-v(0)))^N \, d\mu(v), \tag{1.1.5}$$

where $\hat{x} := \frac{x}{|x|}$.

Observationally, for large R the S_N behave in a range of scales in a power-law fashion:

$$S_N(x) \sim C_N |x|^{\zeta_N}. \tag{1.1.6}$$

A. N. Kolmogorov presented in 1941 a theory (the so called K41-theory) according to which $\zeta_N = \frac{N}{3}$ and the C_N are *universal*. In our concrete model this would mean that C_N depends on $C_{\alpha\beta}R$ only through the mean energy per unit time and volume and nothing else.

Experiments seem to indicate that low order structure functions agree rather well with K41-theory, but for higher ($N \geq 10$) one observes deviations of 10% or more. In particular ξ_N seems not to depend linearly on N, a phenomenon called multiscaling.

For a more detailed discussion of the problematics of turbulece, I can wholeheartedly recommend Uriel Frisch's great book [10].

Unfortunately, there does not seem to be even physicist's arguments to explain the deviations from the K41 theory. It would be nice to have a toy model sharing many characteristics with the full Navier-Stokes turbulence, but which would allow theoretical or even rigorous analysis. This is provided by the Kraichnan model of passive advection.

1.2 Basics of the Kraichnan model

The scalar advection equation

$$\partial_t T = -v \cdot \nabla T + \kappa \Delta T + f. \tag{1.2.1}$$

models a scalar quantity $T : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ (e.g. temperature distribution, concentration of a coloured substance) transported by a velocity field v and pumped in by the forcing f. We want v to mimic turbulent velocities and f to be similar to the f of previous section.

In the Kraichnan model of passive advection [17] one takes v and f random mean zero Gaussians with covariances

$$\langle v^\alpha(t,x) v^\beta(t',x') \rangle = \delta(t-t') \mathcal{D}^{\alpha\beta}(x-x') \tag{1.2.2}$$

and

$$\langle f(t,x) f(t',x') \rangle = \delta(t-t') \mathcal{C}(\frac{x-x'}{L}). \tag{1.2.3}$$

The passivity refers to the fact that there is no feedback from T to v. The $-v \cdot \nabla T$-term is taken in the Stratonovich sense. The reason for this is as follows. If we let δ_ϵ be a sequence of even functions $\mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ so that δ_ϵ weak*-converges to δ, then one can show that if we replace δ by δ_ϵ in the covariance above, then in the limit $\epsilon \to 0$ we get the model above provided that $-v \cdot \nabla T$ is taken in the Stratonovich sense. We shall not discuss this point further and assume from the beginning that the Stratonovich interpretation has been built in.

The forcing covariance $\mathcal{C}$ will be taken to be a fixed smooth function of compact support and nonnegative Fourier transform. We will discuss the exact form of $\mathcal{D}$ later.

1.3 The Kraichnan model without forcing

The discussion here is adapted from [20]. We investigate

$$\partial_t T = -v \cdot \nabla T + \kappa \Delta T \tag{1.3.1}$$

with the initial condition $T(0,\cdot) = f(\cdot)$ and v Gaussian with mean zero covariance (1.2.2).

Although rigorous, (1.2.2) is obvious physicist notation, so we do the same as [19] and [20] and represent the randomness in (1.3.1) as a countably infinite bunch of independent Brownian motions. Also since the Stratonovich convention is a pain in the neck, we transform our equation into Itô form.

We want to choose our spatial covariance $\mathcal{D}^{\alpha\beta}$ so that it forces our advecting fluid to be incompressible. This is achieved by taking

$$\mathcal{D}^{\alpha\beta}(x) = \int e^{-ik\cdot x} D(|k|) \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right) \frac{dk}{(2\pi)^d} \tag{1.3.2}$$

where D is initially smooth and nonnegative with compact support in $[0,\infty)$, but later we shall discuss more general D which are more realistic for turbulence questions.

Let us denote $D_0 := \mathcal{D}^{\alpha\alpha}(0)$ (it does not depend on α). We shall assume that $D_0 < \infty$. Notice also that $\mathcal{D}^{\alpha\beta} = 0$ for $\alpha \neq \beta$, so $\mathcal{D}(0) = D_0 \mathbb{1}$.

Let $\mathcal{D}$ be the operator associated with the kernel $\mathcal{D}^{\alpha\beta}$, i.e. for $v \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ let $\mathcal{D}v$ be defined by

$$(\mathcal{D}v)^\alpha(x) := \sum_{1\leq\beta\leq d} \int_{\mathbb{R}^d} \mathcal{D}^{\alpha\beta}(x - x') v^\beta(x')\, dx'. \tag{1.3.3}$$

Let $\mathbb{P}$ be the the Gaussian measure with covariance $\mathcal{D}$. We want to find vector fields $\{v_k\}_{k=1}^\infty$ and an inner product $\langle\cdot,\cdot\rangle_D$ so that with probability one the formula

$$X_k := \langle v(\omega), v_k \rangle_D \tag{1.3.4}$$

makes sense, the X_k's are independent Gaussian random variables with unit variance and we have

$$v(\omega, x) = \sum_k X_k v_k \tag{1.3.5}$$

holding almost everywhere.

This is done as follows. Let $\mathcal{P}$ be the following orthogonal projection with kernel

$$\mathcal{P}^{\alpha\beta}(x) = \int e^{-ik\cdot x} \chi_{\operatorname{supp}\mathcal{D}}(|k|) \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right) \frac{dk}{(2\pi)^d}. \tag{1.3.6}$$

Now $\mathcal{P}\mathcal{D} = \mathcal{D}\mathcal{P}$ and $\mathbb{P}(\mathcal{P}[L^2]) = 1$. Moreover $\mathcal{D}$ has an (unbounded) inverse on $\mathcal{P}[L^2]$. Let's define an inner product on $\mathrm{Dom}(\mathcal{D}^{-1})$ by letting

$$\langle v, w \rangle_D = \int_{\mathbb{R}^d} v(x) \cdot (\mathcal{D}^{-1} w)(x)\, dx \tag{1.3.7}$$

Let $\{v_k\}_{i=0}^{k}$ be a maximal orthonormal set on $\mathrm{Dom}(\mathcal{D}^{-1})$ with respect to the inner product above.

The X_k's are obviously Gaussian since $\mathbb{P}$ is Gaussian. Let's compute the covariance of X_k's:

$$\begin{aligned} \mathbb{E}[X_k X_{k'}] &= \mathbb{E}\Big[\int_{\mathbb{R}^d} \sum_\alpha v^\alpha(\omega, x)(\mathcal{D}^{-1} v_k)^\alpha(x)\, dx \cdot \\ &\qquad \cdot \int_{\mathbb{R}^d} \sum_\beta v^\beta(\omega, x)(\mathcal{D}^{-1} v_{k'})^\beta(x)\, dy\Big] \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\alpha\beta} \mathbb{E}[v^\alpha(x) v^\beta(y)](\mathcal{D}^{-1} v_k)^\alpha(x)(\mathcal{D}^{-1} v_{k'})^\beta(y)\, dx\, dy \\ &= \langle \mathcal{D}\mathcal{D}^{-1} v_k, v_{k'} \rangle_D \\ &= \delta^{kk'}. \end{aligned} \tag{1.3.8}$$

So the X_k's are now independent Gaussian random variables with unit variance. So we've gotten what we wanted.

Note that

$$\sum_{k=0}^{\infty} v_k^\alpha(x) v_k^\beta(y) = \mathcal{D}^{\alpha\beta}(x - y). \tag{1.3.9}$$

This is easily seen as follows. Let $f \in \mathcal{S}(\mathbb{R}^d)$ and let $g_\beta \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)$ be defined by $g_\beta^\gamma = \delta^{\gamma\beta} f(y)$. Now an easy computation verifies that

$$\begin{aligned} \int \sum_k v_k^\alpha(x) v_k^\beta(y) f(y)\, dy &= \sum_k v_k^\alpha(x) \langle \mathcal{D} g_\beta, v_k \rangle_D \\ &= (\mathcal{D} g_\beta)^\alpha(x) \\ &= \int \mathcal{D}^{\alpha\beta}(x - y) f(y)\, dy. \end{aligned} \tag{1.3.10}$$

The claim follows.

Let $\{W^k\}$ be independent Brownian motions with unit variance. Now (1.3.1) can be written as

$$S_t f = f + \sum_k \int_0^t S_s(v_k \cdot \nabla f) \circ dW_s^k + \kappa \int_0^t S_s(\Delta f)\, ds. \tag{1.3.11}$$

We took the liberty of switching from v to $-v$, since this is the convention [20] uses and if we changed that, we would eventually get confused. Of course, this doesn't matter since mean zero Gaussians are invariant under the change of sign (all odd moments are zero).

To convert (1.3.11) to Itô form, we just use the formula

$$\int_0^t X_s \circ dY_s = \int_0^t X_s\, dY_s + \frac{1}{2}\langle X, Y\rangle_t. \tag{1.3.12}$$

and get

$$\begin{aligned} \sum_k \int_0^t & S_s(v_k \cdot \nabla f) \circ dW_s^k \\ &= \sum_k \int_0^t S_s(v_k \cdot \nabla f)\, dW_s^k + \frac{1}{2}\sum_k \langle S_s(v_k \cdot \nabla f), W_k\rangle_t = (*) \end{aligned} \tag{1.3.13}$$

Substituting now (1.3.11) into the last term above and using (1.3.12) again and dropping all terms where we have brackets of processes of bounded variation with W_k, we get

$$\begin{aligned} (*) &= \sum_k \int_0^t S_s(v_k \cdot \nabla f)\, dW_s^k + \\ &\qquad + \frac{1}{2}\sum_{k,k'} \int_0^t S_s((v_k \cdot \nabla)(v_k' \cdot \nabla f))\, d\langle W_k, W_{k'}\rangle_s \\ &= \sum_k \int_0^t S_s(v_k \cdot \nabla f)\, dW_s^k + \frac{1}{2}\sum_k \int_0^t S_s((v_k \cdot \nabla)(v_k \cdot \nabla f))\, ds \\ &= \sum_k \int_0^t S_s(v_k \cdot \nabla f)\, dW_s^k + \frac{1}{2}\sum_k \int_0^t S_s(\nabla \cdot (v_k(v_k \cdot \nabla f))\, ds \\ &= \sum_k \int_0^t S_s(v_k \cdot \nabla f)\, dW_s^k + \frac{1}{2}D_0 \int_0^t S_s(\Delta f)\, ds. \end{aligned} \tag{1.3.14}$$

Thus we arrive at

$$S_t f = f + \sum_k \int_0^t S_s(v_k \cdot \nabla f) dW_s^k + (\kappa + \frac{1}{2}D_0) \int_0^t S_s(\Delta f)\, ds. \tag{1.3.15}$$

Let $\tilde{\kappa} := \kappa + \frac{1}{2}D_0$. The equation (1.3.11) can be solved in a standard way by iterating Duhamel's principle. That is, we try to define S_t using the formula

$$S_t f := \sum_{n=0}^{\infty} \sum_{k_1,\dots,k_n} \int_{0\le s_1 \le \dots \le s_n \le t} e^{\tilde{\kappa} s_1 \Delta}(v_{k_1}\cdot\nabla)e^{\tilde{\kappa}(s_2-s_1)\Delta}\dots \\ \dots(v_{k_n}\cdot\nabla)e^{\tilde{\kappa}(t-s_n)\Delta} f\, dW^{k_1}_{s_1}\dots dW^{k_n}_{s_n}. \tag{1.3.16}$$

To investigate the convergence of (1.3.16) we need the following Lemma, proven by Le Jan and Raimond (see [19], [20]).

Lemma 1.3.1. *Let $S^0_t = e^{\tilde{\kappa} t\Delta}$ and let S^{n+1}_t be defined using following formula*

$$S^{n+1}_t f = e^{\tilde{\kappa} t\Delta} f + \sum_k \int_0^t S^n_s(v_k \cdot \nabla e^{\tilde{\kappa} t\Delta} f)\, dW^k_s. \tag{1.3.17}$$

These satisfy

$$\mathbb{E}[(S^n_t f)^2] \le e^{\tilde{\kappa} t\Delta} f^2 \tag{1.3.18}$$

for every $f \in L^2(\mathbb{R}^d)$ and in particular it converges in $L^2(\mathbb{P})$ to a solution of (1.3.15).

Proof. (Le Jan and Raimond [19]) Let us prove (1.3.18) by induction on n. The case $n = 0$ reads $(e^{\tilde{\kappa} t\Delta} f)^2 \le e^{\tilde{\kappa} t\Delta} f^2$, which follows easily from formulae (1.3.22) and (1.3.23) below. So let's assume it's true for n and we prove it for $n + 1$.

If we square (1.3.17), on the RHS we get a sum of Itô integrals plus

$$(e^{\tilde{\kappa} t\Delta} f)^2 + \Big(\sum_k \int_0^t S^n_s(v_k \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f)\, dW^k_s\Big)^2. \tag{1.3.19}$$

The last term on the RHS is by Itô formula a sum of Itô integrals plus

$$\sum_{k,k'} \int_0^t (S^n_s(v_k \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f))(S^n_s(v_{k'} \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f))\, d\langle W^k, W^{k'}\rangle_s \\ = \sum_k \int_0^t (S^n_s(v_k \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f))^2\, ds. \tag{1.3.20}$$

Taking an expectation, we arrive at

$$\begin{aligned} \mathbb{E}[(S^{n+1}_t f)^2] &= (e^{\tilde{\kappa} t\Delta} f)^2 + \sum_k \int_0^t \mathbb{E}[(S^n_s(v_k \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f))^2]\, ds \\ &\le (e^{\tilde{\kappa} t\Delta} f)^2 + \sum_k \int_0^t e^{\tilde{\kappa} s\Delta}(v_k \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f)^2\, ds \\ &= (e^{\tilde{\kappa} t\Delta} f)^2 + D_0 \int_0^t e^{\tilde{\kappa} s\Delta} |\nabla e^{\tilde{\kappa}(t-s)\Delta} f|^2\, ds = (*). \end{aligned} \tag{1.3.21}$$

Next, we'll compute:

$$\begin{aligned} \partial_s e^{\tilde\kappa s\Delta}((e^{\tilde\kappa(t-s)\Delta}f)^2) &= \tilde\kappa\Delta e^{\tilde\kappa s\Delta}((e^{\tilde\kappa(t-s)\Delta}f)^2) \\ &\quad + 2\tilde\kappa e^{\tilde\kappa s\Delta}((e^{\tilde\kappa(t-s)\Delta}f)\Delta e^{\tilde\kappa(t-s)\Delta}f) \\ &= 2\tilde\kappa e^{\tilde\kappa s\Delta}|\nabla e^{\tilde\kappa(t-s)\Delta}f|^2. \end{aligned} \tag{1.3.22}$$

Therefore, since $D_0 \le 2\tilde\kappa$, we have

$$\begin{aligned} (*) &\le (e^{\tilde\kappa t\Delta}f)^2 + \frac{D_0}{2\tilde\kappa}\int_0^t \partial_s e^{\tilde\kappa s\Delta}((e^{\tilde\kappa(t-s)\Delta}f)^2)\,ds \\ &\le (e^{\tilde\kappa t\Delta}f)^2 + e^{\tilde\kappa t\Delta}(f^2) - (e^{\tilde\kappa t\Delta}f)^2 \\ &\le e^{\tilde\kappa t\Delta}(f^2). \end{aligned} \tag{1.3.23}$$

Letting $J_t^n f := S_t^n f - S_t^{n-1} f$ we see that by Itô formula $J_t^n f$ and $J_t^{n'} f$ are independent for $n \ne n'$, i.e. they are orthogonal in $L^2(\mathbb{P})$. Since by (1.3.18) the set $\{S_t^n f : n \in \mathbb{N}\}$ is uniformly bounded in $L^2(\mathbb{P})$ we conclude that the limit

$$S_t f := \lim_{n\to\infty} S_t^n f \tag{1.3.24}$$

exists and this limit solves (1.3.15). □

Next, we'll give a proof by Le Jan and Raimond [19] of the positivity of S_t. Since S_t is linear and $S_t 1 = 1$, we can conclude that S_t is almost surely a contraction on L^∞. For this, we need the following

Lemma 1.3.2. *Let H be a separable Hilbert space and let W be the Gaussian process with covariance $\langle\cdot,\cdot\rangle_H$. Let $V \subseteq H$ be a dense subspace. Then*

$$A = \{\sum_{j=1}^n \alpha_j e^{iW(h^j)} : n \in \mathbb{N}\setminus\{0\}, \alpha \in \mathbb{C}^n \text{ and } h \in V^n\} \tag{1.3.25}$$

is dense in $L^p(\Omega, \Sigma(W), \mathbb{P})$ for all p with $1 \le p < \infty$.

Remark 1.3.3. Actually we are interested only in the case $p = 4$. The Theorem is false for L^∞ as it is not separable.

Proof. (Adapted from [16], p.134) It suffices to prove this for $p > 1$, since then it follows for $p = 1$. This is because if A is dense in some L^p and $x_n \in A$ converges to x in L^p, then it does so also in L^1 by Hölder's inequality and the finiteness of our measure space. Thus L^p is contained in the L^1-closure of A. Since L^p is dense in L^1, we conclude that the L^1-closure of A is the whole of L^1.

Next, let's suppose $p > 1$ and let $q := \frac{p}{p-1}$, i.e. L^q is the dual of L^p. It suffices to show that if $\phi \in L^q$ is such that

$$\int_\Omega (\sum_{j=1}^n \alpha_j e^{iW(h^j)})\phi \, d\mathbb{P} = 0 \tag{1.3.26}$$

for every $n > 0$, $\alpha \in \mathbb{C}^n$ and $h \in V^n$, then $\phi = 0$.

For if A was not dense in L^p, by Hahn-Banach Theorem there would be $\phi \in (L^p)^* = L^q$ so that $f[A] = \{0\}$, but $\phi \neq 0$.

As H is separable we may assume that V is the linear span of a countable set $\{h^j\}_{j=1}^\infty \subseteq H$. Let n be given, let $\beta \in \mathbb{C}^n$ and let $\sigma n = \sigma\{W(h_1), ..., W(h_n)\}$. We have

$$\int_\Omega (\sum_{j=1}^n e^{i\beta_j W(h_j)})\mathbb{E}[\phi|\sigma_n] \, d\mathbb{P} = 0. \tag{1.3.27}$$

Thus by the properties of Fourier Transform we have $\mathbb{E}[\phi|\sigma_n] = 0$. Therefore by letting $n \to \infty$ we conclude that $\mathbb{E}[\phi|\sigma(W)] = \phi = 0$. □

Theorem 1.3.4. *S_t is positive.*

Proof. (Le Jan and Raimond [19]) We prove that for $f \in L^2(\mathbb{R}^d)$ and ϕ in a suitable dense subspace of $L^4(\mathbb{P})$ we have $\mathbb{E}[S_t f|\phi|^2] \geq 0$. Let G be the Hilbert space containing all the v_k's as an orthonormal basis and let H be the Hilbert space corresponding to the countably infinite bunch of independent Brownian motions on $[0, t]$ indexed by the v_k's. It can be constructed for example as follows. Let $\{h_j\}_{j=1}^\infty$ be the Haar basis of $[0, t]$, let f_j be defined by $f_j(s) = \int_0^s h_j(s)\, ds$ and take $(\{f_j\}_{j=1}^\infty \times \{v_k\}_{k=1}^\infty)$ as a basis of H with the inner product

$$\langle (f_j, v_k), (f_{j'}, v_{k'})\rangle_H = \delta^{j,j'}\delta^{k,k'}. \tag{1.3.28}$$

Let V be the space of simple processes defined in $[0, t]$ with values in G. If $h := \sum_{i=1^p} 1_{[t_i,t_{i+1})} g^i$ is in V, then

$$X_h = \sum_k \int_0^t \langle h_s, v_k\rangle_D \, dW_s^k \tag{1.3.29}$$

is a random variable with a preimage (denoted by $W^{-1}h$) in H. Now $W^{-1}X_{[N]}$ is dense in H.

Therefore it suffices to show, by Lemma 1.3.2, that for all $f \geq 0$ in L^2, $\alpha \in \mathbb{C}^p$ and $h \in V^p$ we have

$$\mathbb{E}[S_t f \sum_{j,j'} \alpha_j \overline{\alpha_{j'}} e^{i \sum_k \int_0^t \langle h_s^j - h_s^{j'}, v_k\rangle_D \, dW_s^k}] \geq 0. \tag{1.3.30}$$

This is readily computed using Itô's formula and a dirty trick. Denote $X_t := e^{i\sum_k \int_0^t \langle h_s^j - h_s^{j'}, v_k\rangle_D \, dW_s^k}$, By Itô's formula one sees that

$$X_t = 1 + i\sum_k \int_0^t X_s \langle h_s^j - h_s^{j'}, v_k\rangle_D \, dW_s^k - \frac{1}{2}\int_0^t X_s ||h_s^j - h_s^{j'}||_D^2 \, ds. \tag{1.3.31}$$

This is now a finite-dimensional SDE with the unique solution

$$X_t = e^{-\frac{1}{2}\int_0^t ||h_s^j - h_s^{j'}||_D^2 \, ds} \cdot \sum_{n\geq 0} i^n \sum_{k_1,\dots,k_n} \int_{0\leq s_1 \leq \dots \leq s_n \leq t} \prod_{m=1}^n \langle h_{s_m}^j - h_{s_m}^{j'}, v_k\rangle_D \, dW_{s_m}^{k_m}. \tag{1.3.32}$$

as is easily computed by taking a differential of the RHS.
So, we see that the LHS of (1.3.30) equals

$$\sum_{j,j'} \alpha_j \overline{\alpha_{j'}} e^{-\frac{1}{2}\int_0^t ||h_s^j - h_s^{j'}||_D^2 \, ds} \cdot \mathbb{E}\Big[S_t f \sum_{n\geq 0} i^n \sum_{k_1,\dots,k_n} \int_{0\leq s_1 \leq \dots \leq s_n \leq t} \prod_{m=1}^n \langle h_{s_m}^j - h_{s_m}^{j'}\rangle_D \, dW_{s_m}^{k_m}\Big]. \tag{1.3.33}$$

Let us denote the expectation above with $(*)$. Define R_t^n as $R_t^0 := 1$ and

$$R_t^{n+1} := \sum_k \int_0^t R_s^n i \langle h_s^j - h_s^{j'}, v_k\rangle_D \, dW_s^k. \tag{1.3.34}$$

So $R_t := \sum_{n\to\infty} R_t^n$ is just the infinite sum inside $(*)$. Write also $S_t f = \sum_{n=0}^\infty J_t^n f$ with $J_t^0 f := e^{\tilde{\kappa} t \Delta} f$ and

$$J_t^{n+1} f := \sum_k \int_0^t J_s^{n+1}(v_k \cdot e^{\tilde{\kappa}(t-s)\Delta} f) \, dW_s^k \tag{1.3.35}$$

Now a direct computation using (1.3.34) and (1.3.35) shows (observing that $\mathbb{E}[J_t^m f R_t^{m'}] = 0$ for $m \neq m'$) that $\mathbb{E}[S_t f R_t]$ is the sum as $n \to \infty$ of $\mathbb{E}[J_t^0 f R_t^0] = e^{\tilde{\kappa} t \Delta} f$ and

$$\begin{aligned} \mathbb{E}[J^{n+1} t R^{n+1} f] &= \mathbb{E}\Big[\sum_{k,k'} \int_0^t J_s^n (v_k \cdot \nabla e^{\tilde{\kappa}(t-s)} f) \, dW_s^k \cdot \\ &\qquad \int_0^t R_s^n i \langle h_s^j - h_s^{j'}, v_k\rangle_D dW_s^{k'}\Big] \\ &= \mathbb{E}\Big[\sum_k \int_0^t J_s^n (v_k \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f) R_s^n \langle v_k, i(h_s^j - h_s^{j'})\rangle_D \, ds\Big] \\ &= \int_0^t J_s^n (i(h_s^j - h_s^{j'}) \cdot \nabla e^{\tilde{\kappa}(t-s)\Delta} f) R_s^n \, ds. \end{aligned} \tag{1.3.36}$$

Write $(*) := Q_t f := \sum_{n\geq 0} Q_t^n f$ with $Q_t^0 f := e^{\tilde\kappa t\Delta} f$ and

$$Q_t^{n+1} f := \tilde\kappa^{-1} \int_0^t Q_s^n(i(h_s^j - h_s^{j'}) \cdot \nabla e^{\tilde\kappa(t-s)\Delta} f)\, ds. \tag{1.3.37}$$

Let X_t be a Brownian motion of variance $2\tilde\kappa$ on $\mathbb{R}^d$, independent of W.

We now claim that

$$Q_t^n f = \frac{1}{(2\tilde\kappa)^n} \mathbb{E}^x \left[f(x_t) \int_{0\leq s_1 \leq \ldots \leq s_n \leq t} \prod_{m=1}^n i(h_{s_m}^j - h_{s_m}^{j'})(x_{s_m}) \cdot dX_{s_m} \right] \tag{1.3.38}$$

This is easy to prove using induction. First of all clearly

$$Q_t^0 f(x) = \mathbb{E}^x[f(X_t)]. \tag{1.3.39}$$

Next, suppose (1.3.38) holds and we compute Q_t^{n+1}. Below, let $h := h^j - h^{j'}$.

$$\begin{aligned} Q_t^{n+1} f(x) &= \frac{1}{(2\tilde\kappa)^n} \int_0^t \mathbb{E}^x \Bigg[\nabla e^{\tilde\kappa(t-s_{n+1})\Delta} f \cdot i h_{s_{n+1}}(X_{s_{n+1}}) \\ &\qquad \int_{0\leq s_1\leq\ldots\leq s_{n+1}} \prod_{m=1}^n i h_{s_m}(X_{s_m}) \cdot dX_{s_m} \Bigg] ds_{m+1} \\ &= \frac{1}{(2\tilde\kappa^n)} \int_0^t \mathbb{E}^x \Bigg[\mathbb{E}^{X_{s_{n+1}}}[\nabla f(X_t)] \cdot i h_{s_{n+1}}(X_{s_{n+1}}) \\ &\qquad \int_{0\leq s_1\leq\ldots\leq s_{n+1}} \prod_{m=1}^n i h_{s_m}(X_{s_m}) \cdot dX_{s_m} \Bigg] ds_{m+1} \\ &= \frac{1}{(2\tilde\kappa^n)} \mathbb{E}^x \Bigg[\int_0^t \nabla f(X_{s_{n+1}}) \cdot i h_{s_{n+1}}(X_{s_{n+1}}) \\ &\qquad \int_{0\leq s_1\leq\ldots\leq s_{n+1}} \prod_{m=1}^n i h_{s_m}(X_{s_m}) \cdot dX_{s_m}\, ds_{m+1} \Bigg] \\ &= \frac{1}{(2\tilde\kappa^n)} \mathbb{E}^x \Bigg[\int_0^t \nabla f(X_{s_{n+1}}) \cdot dX_{s_{n+1}} \\ &\qquad \int_{0\leq s_1\leq\ldots\leq s_{n+1}\leq t} \prod_{m=1}^n i h_{s_m}(X_{s_m}) \cdot dX_{s_m} \Bigg] \\ &= \frac{1}{(2\tilde\kappa^{n+1})} \mathbb{E}^x \left[f(X_t) \int_{0\leq s_1\leq\ldots\leq s_{n+1}\leq t} \prod_{m=1}^n i h_{s_m}(X_{s_m}) \cdot dX_{s_m} \right], \end{aligned} \tag{1.3.40}$$

which was to be proven.

Let

$$Z_t := \sum_{n\geq 0} \frac{1}{2\tilde{\kappa}} \int_{0\leq s_1\leq \ldots \leq s_n \leq t} \prod_{m=1}^{n} i(h^j_{s_m}(X_{s_m}) - h^{j'}_{s_m}(X_{s_m})) \cdot dX_{s_m} \tag{1.3.41}$$

Now it is easy to verify that Z_t satisfies the following SDE

$$Z_t = 1 + \frac{1}{2\tilde{\kappa}} \int_0^t Z_t i(h^j_s - h^{j'}_s) \cdot dX_s. \tag{1.3.42}$$

It is immediately verified using Itô formula that this has the unique solution

$$Z_t = e^{i\frac{1}{2\tilde{\kappa}}\int_0^t (h^j_s(X_s)-h^{j'}_s(X_s))\cdot dX_s + \frac{1}{4\tilde{\kappa}}\int_0^t |h^j_s(X_s)-h^{j'}_s(X_s)|^2\, ds}. \tag{1.3.43}$$

Thus

$$Q_t f(x) = \mathbb{E}^x\left[f(X_t) e^{i\frac{1}{2\tilde{\kappa}}\int_0^t (h^j_s(X_s)-h^{j'}_s(X_s))\cdot dX_s + \frac{1}{4\tilde{\kappa}}\int_0^t |h^j_s(X_s)-h^{j'}_s(X_s)|^2\, ds} \right]. \tag{1.3.44}$$

Denote now

$$\gamma^{jj'} := e^{\frac{1}{2}\int_0^t ||h^j_s - h^{j'}_s||_D - \frac{1}{4\tilde{\kappa}}|h^j_s(X_s)-h^{j'}_s(X_s)|^2\, ds}. \tag{1.3.45}$$

Now (1.3.33) can be written as

$$\mathbb{E}^x\left[f(X_t) \sum_{j,j'} \alpha_j e^{i\frac{1}{2\tilde{\kappa}}\int_0^t h^j_s(X_s)\cdot dX_s} \gamma^{jj'} \overline{\alpha_{j'}} e^{-i\frac{1}{2\tilde{\kappa}}\int_0^t h^{j'}_s(X_s)\cdot dX_s} \right]. \tag{1.3.46}$$

Being obviously real and symmetric, it suffices to show that the random variable γ is a positive matrix almost surely.

Now we claim that

$$(\pi_x h)^\beta(y) := D_0^{-1} \sum_\alpha \mathcal{D}^{\alpha\beta}(x-y) h^\alpha(x) \tag{1.3.47}$$

is an orthogonal projection with kernel G_x, the vector fields in G vanishing at x. This is seen as follows. First of all $(\pi_x h)^\beta(x) := h^\beta(x)$, so $h - \pi_x h \in G_x$. Secondly we have $\langle \pi_x h, g\rangle_D = D_0^{-1}\sum_k v_k(x)\cdot h(x) D_0^{-1}\langle v_k, g\rangle_D = D_0^{-1} g(x)\cdot h(x)$, so for $g \in G_x$ this equals 0 and also for all h we have $|h(x)|^2 = \langle \pi_x h, h\rangle_D$. Therefore we conclude that

$$\gamma^{jj'} = e^{-\frac{1}{2}\int_0^t \langle (1-\frac{D_0}{2\tilde{\kappa}}\pi_{X_s})(h^j_s(X_s)-h^{j'}_s(X_s)), h^j_s(X_s)-h^{j'}_s(X_s)\rangle_D\, ds}. \tag{1.3.48}$$

This is obviously a positive matrix as the Gaussian is a positive definite function on $\mathbb{R}^p$. □

Next, we jump to analyzing the n-point functions of the passive scalar. Let

$$\mathcal{M}_n := -\frac{1}{2}\sum_{i,j=1}^{n}\sum_{\alpha,\beta}\mathcal{D}^{\alpha\beta}(x_i - x_j)\frac{\partial}{\partial x_i^\alpha}\frac{\partial}{\partial x_j^\beta} - \kappa\sum_{i=1}^{n}\Delta_i. \tag{1.3.49}$$

Theorem 1.3.5. *For all $f_i \in L^\infty(\mathbb{R}^d) \cap \mathrm{Dom}(\Delta)$, $1 \le i \le n$ we have*

$$\mathbb{E}[\bigotimes_{i=1}^{n}(S_t f_i)] = e^{-t\mathcal{M}_n}\bigotimes_{i=1}^{n} f_i. \tag{1.3.50}$$

Proof. (Le Jan and Raimond [20]) By Itô formula, we have

$$\begin{aligned}\bigotimes_{i=1}^{n}(S_t f_i) = \bigotimes_{i=1}^{n} f_i &+ \sum_{i=1}^{n}\sum_{k}\int_0^t S_s f_1 \otimes ... \otimes S_s(v_k \cdot \nabla f_i) \otimes ... \\ &\otimes S_s f_n \, dW_s^k \\ &+ \tilde{\kappa}\sum_{i=1}^{n}\int_0^t S_s f_1 \otimes ... \otimes S_s \Delta f_i \otimes ... \otimes S_s f_n) \, ds \\ &+ \sum_{i\le i<j\le n}\sum_{k}\int_0^t S_s f_1 \otimes ... \otimes S_s(v_k \cdot \nabla) f_i \otimes ... \\ &\otimes S_s(v_k \cdot \nabla) f_j \otimes ... \otimes S_s f_n \, ds.\end{aligned} \tag{1.3.51}$$

Let's show that in the above equation the stochastic integrals make sense. First of all a single integral makes sense, since S_t is almost surely a contraction on L^∞ and thus it suffices to show

$$\mathbb{E}\left[\int_0^t (S_s(v_k \cdot \nabla f))^2 \, ds\right] \in L^1(\mathbb{R}^d). \tag{1.3.52}$$

By Fubini's Theorem one can take the expectation inside and an application of (1.3.18) we get that the LHS above is less than

$$\int_0^t e^{\tilde{\kappa} s \Delta}(v_k \cdot \nabla f)^2 \, ds. \tag{1.3.53}$$

As $f_i \in \mathrm{Dom}(\Delta)$, $v_k \cdot \nabla f \in L^2(\mathbb{R}^d)$ and as $e^{\tilde{\kappa} t \Delta}$ is contraction on L^1 for all $t \ge 0$ we can conclude (1.3.52). So it suffices to show that the infinite k-sum converges in $L^1(\mathbb{P})$ in order to conclude that

$$\mathbb{E}[\sum_k \int_0^t ... \, dW_s^k] = \sum_k \mathbb{E}[\int_0^t ... \, dW_s^k] = 0. \tag{1.3.54}$$

It is easy to see that the k-sum converges in $L^2(\mathbb{P})$ and therefore in $L^1(\mathbb{P})$ by the finiteness of $\mathbb{P}$. Indeed, as the stochastic integrals for different k are orthogonal, by ignoring the $S_t f_i$'s we get

$$
\begin{aligned}
\mathbb{E}\Big[\big(\sum_k \int_0^t S_s(v_k\cdot\nabla f_i)\,dW_s^k\big)^2\Big] &= \sum_k \mathbb{E}\Big[\int_0^t (S_s(v_k\cdot\nabla f_i))^2\,ds\Big]\\
&= \sum_k \int_0^t \mathbb{E}[(S_s(v_k\cdot\nabla f_i))^2]\,ds\\
&\le \sum_k \int_0^t e^{\tilde\kappa s\Delta}(v_k\cdot\nabla f_i)^2\,ds\\
&= D_0\int_0^t e^{\tilde\kappa s\Delta}|\nabla f_i|^2\,ds \in L^1(\mathbb{R}^d).
\end{aligned}
\tag{1.3.55}
$$

Using now (1.3.9) we see that (1.3.49) holds. □

1.4 The Kraichnan model with forcing

Now if we put the forcing on, we get the so called Hopf equations. Suppose for simplicity that the initial condition is zero. Now an application of Duhamel's priciple to (1.2.1) for single force yields

$$
T_t = \int_{t_0}^t \int S_{t-s} f_{s-t_0}\,ds. \tag{1.4.1}
$$

Now if we look at the correlators, we see that

$$
\begin{aligned}
\mathbb{E}[\bigotimes_{i=1}^n T_t] &= \mathbb{E}\Big[\int_{t_0}^t \bigotimes_{i=1}^n S_{t-s_i} f_{s_i-t_0} \prod_{i=1}^n ds_i\Big]\\
&= \int_{t_0}^t \mathbb{E}[\bigotimes_{i=1}^n S_{t-s_i} f_{s_i-t_0}] \prod_{i=1}^n ds_i =: (*).
\end{aligned}
\tag{1.4.2}
$$

Let us process the integrand. Assume for simplicity that $s_1 \le ... \le s_n$. Then by the semigroup property of S we have

$$
\begin{aligned}
&\mathbb{E}\big[(\bigotimes_{i=1}^n S_{t-s_i}) f_{s_i-t_0}\big]\\
&= \mathbb{E}\big[e^{-(t-s_n)\mathcal{M}_n}(e^{-(s_n-s_{n-1})\mathcal{M}_{n-1}}(...(e^{-(s_2-s_1)\mathcal{M}_1} f_{s_1-t_0})\otimes ...)\otimes f_{s_n-t_0})\big]\\
&=: (**)_n.
\end{aligned}
\tag{1.4.3}
$$

As f is mean zero Gaussian and delta-correlated in time, we get 0 for odd correlators and for even correlators we get inductively

$$(1.4.4)\qquad (**)_2(x_1,x_2) := \int_{t_0}^{t} ds \int d\mathbf{y}\, e^{-(t-s)\mathcal{M}_2}(\mathbf{x},\mathbf{y})\mathcal{C}(y_1-y_2)$$

and

$$(1.4.5)\qquad \begin{aligned}(**)_{2n} = \int_{t_0}^{t} ds \int d\mathbf{y}\, e^{-(t-s)\mathcal{M}_{2n}}(\mathbf{x},\mathbf{y})\cdot \\ \cdot (**)_{2n-2}(y_1,...,y_{2n-2})\mathcal{C}(y_{2n-1}-y_{2n}).\end{aligned}$$

Thus by sending $t_0 \to -\infty$ we end up with the formulae

$$(1.4.6)\qquad \mathcal{F}_2 = \int d\mathbf{y}\,(\mathcal{M}_2)^{-1}(\mathbf{x},\mathbf{y})\mathcal{C}(y_1-y_2)$$

$$(1.4.7)\qquad \mathcal{F}_{2n} = \sum_{1\le i<j\le 2n} \int (\mathcal{M}_{2n})^{-1}(\mathbf{x},\mathbf{y})\mathcal{F}_{2n-2}(y_1,\underset{\hat{i}\hat{j}}{\ldots},y_{2n})\mathcal{C}(y_i-y_j)\,d\mathbf{y}.$$

For $\kappa > 0$, using standard machinery for elliptic operators one can show that

$$(1.4.8)\qquad F_{2n}(\mathbf{x}) \le C_{n,\kappa}\sum_{\pi}\prod_{\{i,j\}\in\pi}(1+|x_i-x_j|)^{2-d}.$$

We just give the crux of the argument. For $\kappa > 0$ we have $\epsilon_\kappa\Delta \le \mathcal{M}_n$. So morally $(\mathcal{M}_n)^{-1}(x,y) \le C'_{n,\kappa}(-\Delta)^{-1}(x-y)$ with $C'_{n,\kappa} \to \infty$ as $\kappa \to 0$.

Denote $C(y_1,y_2) = \mathcal{C}(y_1-y_2)$. Specializing in a fixed pairing and denoting the kd-dimensional Laplacian by Δ_k we can therefore conclude that

$$(1.4.9)\qquad \begin{aligned}&\mathcal{M}_{2n}^{-1}(\mathcal{M}_{2n-1}^{-1}(...(M_2^{-1}C\otimes C)...)\otimes C)\\ &\qquad \le C''_{2n,\kappa}(-\Delta_{2n})^{-1}((-\Delta_{2n-2})^{-1}(...(\Delta_2^{-1}C\otimes C)...)\otimes C).\end{aligned}$$

Let's now define $\mathcal{G}_{2n}$ inductively as follows:

$$(1.4.10)\qquad \mathcal{G}_2(y_1,y_2) := (-\Delta_2)^{-1}(y_1,y_2)$$

and

$$(1.4.11)\qquad \mathcal{G}_{2n}(y_1,...,y_{2n}) := (-\Delta_{2n})^{-1}\left(\sum_{i=1}^{n} G_{2n-2}(y_1,...,2i-2,2i+1,...,y_{2n})\right).$$

Thus G_{2n} gives an estimate for the unordered pairing

$$\Big\{\{1,2\},...,\{2n-1,2n\}\Big\}, \tag{1.4.12}$$

so summing over all unordered pairings gives an estimate for $\mathcal{F}_{2n}$.

An easy formal computation on the Fourier side gives (k_i's are $2d$-dimensional Fourier coordinates).

$$\begin{aligned}&\big(\sum_{i=1}^{n}|k_i|^2\big)^{-1}\Big(\sum_{i=1}^{n}|k_1|^{-2}...|k_{i-1}|^{-2}|k_{i+1}|^{-2}...|k_n|^{-2}\Big)\\ &=|k_1|^{-2}...|k_n|^{-2}.\end{aligned} \tag{1.4.13}$$

Therefore we "can" conclude that

$$\mathcal{F}_{2n}(\mathbf{x})\leq C_n\sum_{\pi}\prod_{\{i,j\}\in\pi}(1+|x_i-x_j|)^{2-d}. \tag{1.4.14}$$

1.5 Analysis results

The trouble with the $\kappa\to 0$ limit is as follows: As $\kappa\to 0$ the operators $\mathcal{M}_n$ become degenerate elliptic and thus the argument given above does not work.

For turbulence questions the exact form of $\mathcal{D}^{\alpha\beta}$ is important. We would like $\mathcal{D}^{\alpha\beta}$ to mimic turbulent velocities as much as possible. To achieve this, we let

$$\mathcal{D}(x):=\int_{\mathbb{R}^d}\frac{e^{-ik\cdot x}}{(|k|^2+m^2)^{\frac{d+\xi}{2}}}(\mathbb{1}-\hat{k}\otimes\hat{k})\,dk. \tag{1.5.1}$$

Here $m>0$ is a parameter called ultraviolet cutoff and $\xi\in(0,2)$ is a parameter corresponding to the Kolmogorov scaling exponent of the two-point function with $\xi=\frac{4}{3}$ corresponding to the value given by Kolmogorov theory.

First we discuss the $m\to 0$ limit. If $a\in\mathbb{R}^d$, we denote the vector $(x_i+a)_{i=1}^n$ by $\mathbf{x}+a$. We call a function $f:\mathbb{R}^{nd}\to\mathbb{R}$ translationally invariant, if for every $a\in\mathbb{R}^d$ and $\mathbf{x}\in\mathbb{R}^{nd}$ we have $f(\mathbf{x})=f(\mathbf{x}+a)$.

In the limit $m\to 0$ the integral in (1.5.1) diverges, but in the formulas (1.4.6) and (1.4.7) $\mathcal{M}_n^{-1}$ acts on translationally invariant functions. If we set

$$d^{\alpha\beta}:=D_0-\mathcal{D}^{\alpha\beta} \tag{1.5.2}$$

we can write (1.3.49) as

$$\mathcal{M}_n = \sum_{1\le i<j\le n} d^{\alpha\beta}(x_i - x_j)\frac{\partial}{\partial x_i^\alpha}\frac{\partial}{\partial x_j^\beta} - \kappa\sum_{i=1}^n \Delta_i - \frac{1}{2}D_0\Big(\sum_{i=1}^n \frac{\partial}{\partial x_i^\beta}\Big)^2. \tag{1.5.3}$$

The last term here vanishes when acting on translationally invariant functions and for $m > 0$ we have

$$d^{\alpha\beta}(x_i - x_j) = \int_{\mathbb{R}^d} \frac{1-\cos(k\cdot x)}{(|k|^2+m^2)^{\frac{d+\xi}{2}}}\Big(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{|k|^2}\Big)\,dk. \tag{1.5.4}$$

At $m = 0$ this integral makes perfect sense and equals

$$d^{\alpha\beta}(x) = \frac{\Gamma((2-\xi)/2)}{(4\pi)^{d/2}2^\xi \xi\Gamma((d+\xi+2)/2)}|x|^\xi\Big((d-1+\xi)\delta^{\alpha\beta} - \xi\frac{x^\alpha x^\beta}{|x|^2}\Big). \tag{1.5.5}$$

A translationally invariant function on $\mathbb{R}^{nd}$ is really a function of $(n-1)d$ variables. So, let's throw away the last term from $\mathcal{M}_n$ and reduce the number of dimensions from nd to $(n-1)d$.

In other words, we set $x_i := y_i - y_{i+1}$ for $1 \le i \le n-1$, so

$$\frac{\partial}{\partial y_i^\alpha} = \begin{cases} \frac{\partial}{\partial x_1^\alpha} & \text{if } i = 1, \\ \frac{\partial}{\partial x_i^\alpha} - \frac{\partial}{\partial x_{i-1}^\alpha} & \text{if } 2 \le i \le n-1 \text{ and} \\ \frac{\partial}{\partial x_{n-1}^\alpha} & \text{if } i = n. \end{cases} \tag{1.5.6}$$

For an operator of the form $H := -\nabla\cdot A\nabla$ with A a matrix-valued function, denote $\sigma(H) := A$ and call A the symbol of H. Denote the symbol obtained in this way by $\sigma(M_n)$. At $\kappa = 0$ a simple calculation shows that $\sigma(M_n)$ equals

$$\sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\langle v_i, (d(\sum_{k=i}^{j} x_k) - d(\sum_{k=i}^{j-1} x_k) - d(\sum_{k=i+1}^{j} x_k) + d(\sum_{k=i+1}^{j-1} x_k))v_j\rangle \tag{1.5.7}$$

In particular,

$$\sigma(M_2) = \langle v_1, d(x_1)v_1\rangle, \tag{1.5.8}$$

$$\begin{aligned}\sigma(M_3) =& \langle v_1, d(x_1)v_1\rangle + \langle v_2, d(x_2)v_2\rangle + \\ & \langle v_1, (d(x_1+x_2) - d(x_1) - d(x_2))v_2\rangle\end{aligned} \tag{1.5.9}$$

and

$$\begin{aligned}\sigma(M_4) =& \langle v_1, d(x_1)v_1\rangle + \langle v_2, d(x_2)v_2\rangle + \langle v_3, d(x_3)v_3\rangle \\ & \langle v_1, (d(x_1+x_2) - d(x_1) - d(x_2))v_2\rangle \\ & \langle v_2, (d(x_2+x_3) - d(x_2) - d(x_3))v_3\rangle \\ & \langle v_1, (d(x_1+x_2+x_3) - d(x_1+x_2) - d(x_2+x_3) + d(x_2))v_3\rangle.\end{aligned} \tag{1.5.10}$$

The point of my paper ([14], Chapter 2) is that the following formula analogous to (1.4.14) holds directly at $\kappa = 0$:

$$\mathcal{F}_{2n}(\mathbf{x}) \leq C_n \sum_{\pi} \prod_{\{i,j\}\in\pi} (1 + |x_i - x_j|)^{2-d-\xi}. \tag{1.5.11}$$

To give an idea of the proof we investigate the easier question of the uniform local integrability of the Green's functions of the operators M_n in the following sense. We want to show that

$$\sup_{x\in\mathbb{R}^{(2n-1)d}} \int_{B(x,1)} d^{(2n-1)d}y\, G_{M_{2n}}(x,y) < \infty. \tag{1.5.12}$$

We prove (1.5.12) by analyzing the heat kernel $K_{M_{2n}}$ of the operator in detail and then using the fundamental relation

$$G_{M_{2n}}(x,y) = \int_0^\infty dt\, K(t,x,y) \tag{1.5.13}$$

to obtain estimates for the Green's functions.

The proof ([14], Chapter 2) has two phases. First of all, results of Davies and Varopoulos ([6], [5], [22]) are used to obtain $K(t,x,\cdot) \in L^\infty$ for $t > 0$, some coarse estimates for the tail of the heat kernel and to verify assumptions in the second phase. In the second phase the estimates are refined using the Harnack inequality ([13]) for a class of degenerate parabolic equations.

1.5.1 Results by Varopoulos and Davies

One may prove that the symbols of M_n satisfy the following estimates. The degeneration set F of $\sigma(M_n)$ is the set of $x \in \mathbb{R}^{(n-1)d}$ so that $\sigma(M_n)$ is not invertible. It is easily seen that F is a finite union of subspaces of codimension $d \geq 2$.

Let $w_1(x) := d(x,F)^\xi$ and let $w_2(x) := |x|^\xi$. One has a $C < \infty$ so that

$$C^{-1} d(x,F)^\xi |v||w| \leq \langle v, \sigma(M_n) w\rangle \leq C|x|^\xi |v||w|. \tag{1.5.14}$$

It follows by [4] (or by "direct" computation) that the following Sobolev estimate holds.

Proposition 1.5.1. *There is $C < \infty$ such that for all $f \in C_0^\infty(\mathbb{R}^{(n-1)d})$ we have*

$$||f||^2_{2\mu/(\mu-2)} \leq C\langle f, M_n f\rangle, \tag{1.5.15}$$

where $\mu := \frac{(n-1)d}{2-\xi/2}$ is called the dimension of the semigroup $e^{-M_n t}$.

Now the following result by Varopoulos [22] (the proof can also be found from [6], Theorem 2.4.2) yields that the heat kernel is pointwise bounded for $t > 0$.

Theorem 1.5.2. *Suppose $C_0^\infty(\mathbb{R}^n) \subseteq \mathrm{Dom}(H)$. Let e^{-Ht} be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$ and let $\mu > 2$ be given. Then there is $C_1 < \infty$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$||f||^2_{2\mu/(\mu-2)} \leq C_1 \langle f, Hf \rangle. \tag{1.5.16}$$

if and only if there is $C_2 < \infty$ such that for all $t > 0$ and $f \in L^2(\mathbb{R}^n)$ we have

$$||e^{-Ht} f||_\infty \leq C_2 t^{-\mu/4} ||f||_2. \tag{1.5.17}$$

Here the constants C_1 and C_2 depend only on each other and the dimension μ of the semigroup.

Since e^{-Ht} is self-adjoint (1.5.17) also implies that

$$||e^{-Ht} f||_2 \leq C_2 t^{-\mu/4} ||f||_1, \tag{1.5.18}$$

which in turn shows that

$$||e^{-Ht} f||_\infty \leq C_2 \left(\frac{t}{2}\right)^{-\mu/4} ||e^{-Ht/2} f||_2 \leq C_3 t^{-\mu/2} ||f||_1. \tag{1.5.19}$$

In particular $K(t, x, \cdot) \leq C_3 t^{-\mu/2}$.

The next part of phase one is to get Gaussian spatial estimate for the heat kernel. This is provided by the following Definition and Theorem.

Definition 1.5.3. Let A be a symbol on $\mathbb{R}^n$. The function

$$\begin{aligned} d_A(x,y) := \sup\big\{ &|\phi(x) - \phi(y)| : \phi \text{ is } C^\infty \text{ and bounded with} \\ &\langle \nabla\phi, A\nabla\phi \rangle \leq 1 \text{ on } \mathbb{R}^n \big\} \end{aligned} \tag{1.5.20}$$

is called the metric associated with A (or H, if $H := -\nabla \cdot A\nabla$ or e^{-tH} or the heat kernel of H).

Theorem 1.5.4. *Let μ be a positive real number. Suppose $H := -\nabla \cdot A\nabla \geq 0$ is a positive self-adjoint divergence form operator with e^{-Ht} a symmetric Markov semigroup of dimension μ. Then for each $\delta > 0$ there is $C_\delta < \infty$ such that the heat kernel K of e^{-Ht} satisfies*

$$0 \leq K(t, x, y) \leq C_\delta t^{-\mu/2} \exp\{-\frac{d_A(x,y)^2}{4(1+\delta)t}\} \tag{1.5.21}$$

for all $0 < t < \infty$ and $x, y \in \mathbb{R}^n$. Besides δ, C_δ depends only on μ and the constant C of (1.5.15).

Proof. See [5]. □

This estimate is useful for analysis of G_{M_n}, but using it alone fails to establish (1.5.12) as we shall shortly see. First of all we see that $\sigma(M_2) \sim |\cdot|^\xi$. Using Theorem 1.5.4 we get a $C < \infty$ such that

$$K_{M_2}(t,x,y) \le Ct^{-\frac{d}{2-\xi}} \exp\{-\frac{|x-y|^2}{Ct}\} \tag{1.5.22}$$

for $|x| = 1$ and $|x-y| \le \frac{1}{2}$. Integrating with respect to t from 0 to ∞ yields

$$G_{M_2}(x,y) \le C'|x-y|^{2-\frac{2d}{2-\xi}}. \tag{1.5.23}$$

Integration w.r.t. y over $|x-y| \le \frac{1}{2}$ yields a finite answer only if $2 - \frac{2d}{2-\xi} > -d$, that is $\xi < \frac{4}{d+2}$. This alone is not too bad, since we might be satisfied with (1.5.12) for small ξ, but we quickly run into problems. For each $n > 2$ there are points $x \in \mathbb{S}^{(n-1)d-1}$ so that $\sigma(M_n) \sim 1$ in a neighbourhood U of x. Similar arguments as above yield

$$G_{M_n}(x,y) \le C_n'|x-y|^{2-\frac{2(n-1)d}{2-\xi}}. \tag{1.5.24}$$

Now an integration w.r.t. y near x gives a finite answer only when $\xi < \frac{4}{(n-1)d+2}$. This means trouble: Given ξ with $0 < \xi < 2$, there will always be some N so that the above argument fails to give local integrability for M_n^{-1} with $n \ge N$.

The reason why this argument fails is that the estimate (1.5.22) is sub-optimal. Since M_2 is uniformly elliptic in a neighbourhood U of $x \in \mathbb{S}^{d-1}$, the heat kernel of M_2 should behave (by e.g. physical intuition) like the heat kernel of the Laplacian for small times and small distances from x.

So suppose we manage to verify that for $|x| = 1$, $|x-y| \le \epsilon \le \frac{1}{2}$ and $0 < t \le t_0$ we have

$$K_{M_2}(t,x,y) \le C_2 t^{-\frac{d}{2}} \exp\{-\frac{|x-y|^2}{C_2 t}\}. \tag{1.5.25}$$

Since there is some $C_3 < \infty$ so that $t^{-\frac{d}{2-\xi}} \le C_3 t^{-\frac{d}{2}}$ for $t \ge t_0$ we can combine (1.5.22) and (1.5.25) to get

$$K_{M_2}(t,x,y) \le C_4 t^{-\frac{d}{2}} \exp\{-\frac{|x-y|^2}{C_2 t}\} \tag{1.5.26}$$

for $|x| = 1$, $|x-y| \le \epsilon$ and $0 < t \le \infty$. Now an integration w.r.t. t from 0 to ∞ gives

$$G_{M_2}(x,y) \le C_5|x-y|^{2-d} \tag{1.5.27}$$

for $|x| = 1$ and $|x-y| \le \epsilon$. We get something similar for $n > 2$ for points $x \in \mathbb{S}^{(n-1)d-1}$ with $\sigma(M_n) \sim 1$ for some neighbourhood U of x.

Let us now briefly comment on how (1.5.25) can be proved. Let A be a symbol defined on some domain $U \subseteq \mathbb{R}^n$ with $B(0,2) \subseteq U$ and assume that A is uniformly elliptic on $B(0,2)$ and let λ and Λ be the corresponding lower and upper bounds for the symbol. Now Moser's (see [21]) parabolic Harnack inequality says that there is a $C < \infty$ depending on A only trough λ and Λ (and n) so that for any solution u of $u_t = \nabla \cdot A \nabla u$ on $(0,3) \times B(0,2)$ we have

$$\text{ess sup}_{x \in B(0,1)} u(1,x) \le C \,\text{ess inf}_{x \in B(0,1)} u(2,x). \tag{1.5.28}$$

Now we make a scaling argument. Let $\epsilon \in (0,1]$ be given. Define $u^\epsilon(t,x) := u(\epsilon t, \sqrt{\epsilon} x)$ and $A^\epsilon(x) := A(\sqrt{\epsilon} x)$. It is readily verified that u^ϵ satisfies $u^\epsilon_t = \nabla \cdot A^\epsilon \nabla u^\epsilon$ and since the C above depended only on λ and Λ (and n), we have

$$\text{ess sup}_{x \in B(0,1)} u^\epsilon(1,x) \le C \,\text{ess inf}_{x \in B(0,1)} u^\epsilon(2,x). \tag{1.5.29}$$

Scaling back to u this means that

$$\text{ess sup}_{x \in B(0,\sqrt{\epsilon})} u(\epsilon,x) \le C \,\text{ess inf}_{x \in B(0,\sqrt{\epsilon})} u(2\epsilon,x). \tag{1.5.30}$$

Now since for say fixed $x \in \mathbb{R}^d$ and $t > 0$ the heat kernel $K(\cdot, x, \cdot)$ is a solution, we can compute using the fact that the integral of a heat kernel is ≤ 1:

$$\begin{aligned} t^{\frac{n}{2}} K(t,x,0) &\le C|B(0,\sqrt{t})| \sup_{x' \in B(0,\sqrt{t})} K(t,x,x') \\ &\le C'|B(0,\sqrt{t})| \inf_{x' \in B(0,\sqrt{t})} K(2t,x,x') \\ &\le C' \int_{B(0,\sqrt{t})} K(2t,x,x')\, dx' \\ &\le C'. \end{aligned} \tag{1.5.31}$$

So this method gives the correct prefactor $t^{-\frac{d}{2}}$ of (1.5.25). Next we use a probabilistic method, i.e. killing probabilities to get the exponential tail of (1.5.25).

The following is Proposition 6.5 on page 179 of [1].

Proposition 1.5.5. *Suppose $A \sim^\lambda 1$ on $\mathbb{R}^n$. There is $C < \infty$ depending on A only through λ (and n) such that*

$$\mathbb{P}^y_A(\sup_{s \le t} |X_s - y| \ge \mu) \le C \exp\{-\frac{\mu^2}{Ct}\}. \tag{1.5.32}$$

Suppose then that $A \sim^\lambda \mathbb{1}$ on $B(0,2)$. Then obviously (1.5.32) holds for $y = 0$ and $\mu < 2$ with the same C, since obviously the killing probability of the diffusion can only depend on the symbol on the set on whose boundary the diffusion is killed at least if the sample paths of the diffusion are continuous almost surely.

To put it in another way, (1.5.32) says that the integral of the heat kernel $K(t,0,\cdot)$ over $|x| \geq \mu$ is less than $C\exp\{-\frac{\mu^2}{Ct}\}$. Therefore the argument of (1.5.31) yields for any $\epsilon > 0$ some $C(\epsilon) < \infty$ so that for any $x \in B(0, 2-\epsilon)$ we have

$$K(t,0,x) \leq C(\epsilon)t^{-\frac{n}{2}} \exp\{-\frac{|x|^2}{C(\epsilon)t}\}. \tag{1.5.33}$$

1.5.2 Gutiérrez-Wheeden results

For $n > 2$ we have degeneration points also outside of the origin. Large part of my article ([14], Chapter 2) is devoted to proving similar estimates as (1.5.33) also in this case. For this purpose one needs Harnack inequalities for degenerate parabolic equations. These are provided by the results of [13].

Definition 1.5.6. Let w be a nonnegative locally integrable function (a weight) defined on $\mathbb{R}^n$. We denote $w(A) := \int_A w(x)dx$. The function w is called a *doubling weight* (resp. an A_2-*weight*), if there is a constant C such that for every ball $B \subset \mathbb{R}^n$ we have $w(2B) \leq Cw(B)$ (resp. $\frac{1}{|B|^2} w(B)w^{-1}(B) \leq C$).

Since by Schwartz inequality $|B|^2 \leq w(B)w^{-1}(B)$, we have $|2B|^2 = 2^{2n}|B|^2 \leq 2^{2n}w(B)w^{-1}(B) \leq 2^{2n}w(B)w^{-1}(2B)$, so we can conclude that an A_2-weight is also a doubling weight.

Definition 1.5.7. Denote $u_B := |B|^{-1}\int_B u(x)\,dx$ and let w_1, w_2 be weights on $\mathbb{R}^n$ and let $q > 2$. We say that the *Poincaré inequality* holds for w_1, w_2 with q, if there is $C < \infty$ so that for every ball $B \subseteq \mathbb{R}^n$ and $u \in W^{1,2}(B)$ we have

$$\begin{aligned} &\left(w_2(B)^{-1} \int_B |u - u_B|^q w_2 \, dx \right)^{1/q} \\ &\qquad \leq C|B|^{1/n} \left(w_1(B)^{-1} \int_B |\nabla u|^2 w_1 \, dx \right)^{1/2}. \end{aligned} \tag{1.5.34}$$

Theorem 1.5.8. (Harnack inequality) *Suppose $H := -\nabla \cdot A\nabla$ is a divergence form operator with $w_1 \leq A \leq w_2$ and suppose that the weights w_1 and w_2 satisfy the following:*

1. *w_1 and w_2 are A_2,*

2. *The Poincaré inequality holds for w_1, w_2 with some $q > 2$ and*

3. The Poincaré inequality holds for w_1, 1 with some $q' > 2$.

Let $t_0, ..., t_4 \in \mathbb{R}$ with $t_0 < ... < t_4$, $\Omega \subseteq \mathbb{R}^n$ open and $K \subseteq \Omega$ compact and connected. Let u be a strictly positive solution to $u_t + Hu = 0$ in $\Omega \times (t_0, t_4)$. Then there is a constant $C < \infty$ depending on Ω, K and $t_0, ..., t_4$, but on A only through the bounds w_1 and w_2 so that

$$\text{ess sup}_{K\times(t_1,t_2)}\, u \leq C \,\text{ess inf}_{K\times(t_3,t_4)}\, u \tag{1.5.35}$$

Proof. This is just Theorem A of [13] supplemented with a covering argument from [21], pages 734-736. □

Remark 1.5.9. For the purposes of Theorem 1.5.8 the concept of u being a solution of $u_t + Hu = 0$ on $Q := \Omega \times (t_0, t_4)$ means exactly the following:

1. $u \in L^2(Q)$,
2. $u_t \in L^2(Q)$,
3. $|\nabla u|^2 w_2 \in L^1(Q)$ and
4. For all $\phi \in C_0^1(Q)$ we have

$$\int_Q u_t\phi + \langle A\nabla u, \nabla\phi\rangle \, dx\, dt = 0. \tag{1.5.36}$$

The assumptions above have been verified in my article ([14], Chapter 2).

Chapter 2

The Article

We prove estimates for the stationary state n-point functions at zero molecular diffusivity in the Kraichnan model [17]. This is done by proving upper bounds for the heat kernels and Green's functions of the degenerate elliptic operators M_n that occur in the Hopf equations for the n-point functions.

2.1 Introduction

The Kraichnan model of passive advection is an exactly solvable model that has a very similar phenomenology to the full Navier-Stokes turbulence, but is much simpler in many respects. I'll only give a very short reminder for the reader. More detailed introductions to the problem we are addressing can be found e.g. in [11] and [18]. See also [7], [19] and [20].

Let $T(t,x) \in \mathbb{R}, x \in \mathbb{R}^d$ be a scalar quantity satisfying

$$\partial_t T = \kappa \Delta T - v \cdot \nabla T + f. \tag{2.1.1}$$

In the Kraichnan model we take v and f random, decorrelated in time, independent and Gaussian with mean zero and covariances

$$\langle v^\alpha(t_1,x_1) v^\beta(t_2,x_2)\rangle = D^{\alpha\beta}(x_1-x_2)\delta(t_1-t_2) \text{ and} \tag{2.1.2}$$

$$\langle f(t_1,x_1) f(t_2,x_2)\rangle = C(x_1-x_2)\delta(t_1-t_2). \tag{2.1.3}$$

Here the $v \cdot \nabla T$ should be interpreted in the Stratonovich sense. The incompressibility of the velocity field v is guaranteed by taking

$$D^{\alpha\beta}(x) = \int e^{-ik\cdot x} D(|k|)\left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right) dk \tag{2.1.4}$$

where D is smooth, nonnegative and of compact support in $(0, \infty)$. A D that mimics turbulent velocities is

$$D(|k|) = |k|^{-(d+\xi)} \chi\left(|k|\eta + \frac{1}{|k|\ell} \right) \tag{2.1.5}$$

with χ smooth, $\chi = 1$ in a neighbourhood of the origin and $\chi(x) = 0$ for $x > 1$. The idea is that D behaves like $|x|^\xi$ in the so-called inertial range $\eta << |x| << \ell$. The number η is called the Kolmogorov scale and ℓ is called the inertial scale. We let $\tilde{C} \in C_0^\infty(\mathbb{R}^d)$ with a nonnegative Fourier transform and $C := \tilde{C}(\cdot/L)$, with $L > 0$.

One is interested in the statistics of $T(t,x)$ as $t \to \infty$. Let

$$\mathcal{F}_n(t, x_1, ..., x_n) := \langle T(t, x_1)...T(t, x_n) \rangle. \tag{2.1.6}$$

Given (2.1.2) and (2.1.3) the n-point functions $\mathcal{F}_n$ of the scalar T obey the so-called Hopf equations (see [20]):

$$\begin{aligned} \partial_t \mathcal{F}_n(t, x_1, ..., x_n) = -\mathcal{M}_n \mathcal{F}_n(t, x_1, ..., x_n) + \\ \sum_{1 \le i < j \le n} \mathcal{F}_{n-2}(t, x_1, \underset{\hat{i}\hat{j}}{...}, x_n) C(x_i - x_j), \end{aligned} \tag{2.1.7}$$

with

$$\mathcal{M}_n := -\sum_{1 \le i < j \le n} \sum_{1 \le \alpha, \beta \le d} D^{\alpha\beta}(x_i - x_j) \frac{\partial^2}{\partial x_i^\alpha \partial x_j^\beta} - \kappa \sum_{1 \le i \le n} \Delta_i. \tag{2.1.8}$$

The fact that the Hopf equation for $\mathcal{F}_n$ does not contain $\mathcal{F}_m$ with $m > n$ makes it easy to solve these equations inductively. The situation here differs drastically from full Navier-Stokes turbulence, where the Hopf equation for $\mathcal{F}_n$ contains also $\mathcal{F}_{n+1}$.

$\mathcal{M}_n$ is an elliptic operator and in terms of its heat kernel $\mathcal{F}_n$ (with zero initial condition for simplicity) is given by

$$\mathcal{F}_2(t, \mathbf{x}) = \int_{t_0}^t ds \int d\mathbf{y}\, e^{-(t-s)\mathcal{M}_2}(\mathbf{x}, \mathbf{y}) C(y_1 - y_2) \tag{2.1.9}$$

$$\begin{aligned} \mathcal{F}_{2n}(t, \mathbf{x}) = \sum_{1 \le i < j \le 2n} \int_{t_0}^t ds \int d\mathbf{y}\, e^{-(t-s)\mathcal{M}_{2n}}(\mathbf{x}, \mathbf{y}) \cdot \\ \cdot \mathcal{F}_{2n-2}(s, y_1, \underset{\hat{i}\hat{j}}{...}, y_{2n}) C(y_i - y_j)\, d\mathbf{y} \end{aligned} \tag{2.1.10}$$

with vanishing odd correlators.

As $t_0 \to -\infty$ these have the stationary limit

$$\mathcal{F}_2 = \int d\mathbf{y}\, (\mathcal{M}_2)^{-1}(\mathbf{x}, \mathbf{y}) C(y_1 - y_2) \tag{2.1.11}$$

$$\mathcal{F}_{2n} = \sum_{1 \le i < j \le 2n} \int (\mathcal{M}_{2n})^{-1}(\mathbf{x}, \mathbf{y}) \mathcal{F}_{2n-2}(y_1, \underset{\hat{i}\hat{j}}{\ldots}, y_{2n}) C(y_i - y_j)\, d\mathbf{y}. \tag{2.1.12}$$

One is interested in the study of $\mathcal{F}_{2n}$ for η small, ℓ large, κ small and L large. In this paper we prove bounds for these directly in the limit $\eta = 0$, $\ell = \infty$ and $\kappa = 0$ with fixed L, say $L = 1$. Our methods also allow the study of the limit $\eta \to 0$, $\ell \to \infty$ and $\kappa \to 0$ [15].

A comment on D is now in place. While sending $\eta \to 0$ and $\ell \to \infty$ in D, we get into trouble with ℓ, since D diverges as $\ell \to \infty$. Fortunately it doesn't matter: Let

$$d^{\alpha\beta}(x) := \int dk\, (1 - e^{ik\cdot x}) D(|k|) \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right). \tag{2.1.13}$$

Now (2.1.8) can be written in the following form:

$$\mathcal{M}_n := \sum_{1 \le i < j \le n} \sum_{1 \le \alpha, \beta \le d} d^{\alpha\beta}(x_i - x_j) \frac{\partial^2}{\partial x_i^\alpha \partial x_j^\beta} - \kappa \Delta - D_0 \left(\sum_{1 \le i \le n} \sum_{1 \le \alpha \le d} \frac{\partial}{\partial_{x_i^\alpha}} \right)^2 \tag{2.1.14}$$

In (2.1.7) $\mathcal{M}_n$ acts on translationally invariant functions, so the last term drops out and the rest has a limit as $\ell \to \infty$.

Finally, here's our main Theorem, proved directly at $\eta = 0$, $\ell = \infty$ and $\kappa = 0$:

Theorem 2.1.1.

$$\mathcal{F}_{2n}(\mathbf{x}) \le C_n \sum_{\pi} \prod_{\{i,j\} \in \pi} (1 + |x_i - x_j|)^{2-\xi-d}, \tag{2.1.15}$$

where the sum is over pairings of $\{1, ..., 2n\}$.

2.2 Preliminaries

This section fixes the notation and discusses the results from other papers ([4], [6], [13], [22]) used in this paper. There is an overview of this paper in §2.3, so the reader might want to start there.

2.2.1 Degenerate elliptic operators in divergence form

Let $\Omega \subset \mathbb{R}^n$ be a domain. We shall be interested in second order differential operators in divergence form, i.e. in operators H of the form $H = -\nabla \cdot A\nabla$, where A is a locally square integrable function from Ω to real symmetric positive $n \times n$ matrices with locally square integrable distributional derivative, i.e. $A \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{M}^n)$. One can make sense of more general operators, but this is not relevant to the results presented in this paper.

Definition 2.2.1. Let H and A be as above. The matrix A is called the *symbol* of H, and we denote $\sigma(H) := A$. The function $w_1^H(x) := \inf_{v\in\mathbb{S}^{n-1}}\langle v, \sigma(H)(x)v\rangle$ (resp. $w_2^H(x) := \sup_{v\in\mathbb{S}^{n-1}}\langle v, \sigma(H)(x)v\rangle$) is called the greatest lower bound (resp. least upper bound) of the symbol.

We shall also use $\sigma(H)$ to denote the quadratic form $\langle v, A(x)v\rangle$. The usage will be clear from the context. We often speak loosely and forget the attributes "greatest" and "lowest" from the bounds.

If A and B are two symbols and $U \subseteq \mathbb{R}^m$, we shall denote $A \sim^\lambda B$ on U, if $\lambda A \le B \le \lambda^{-1}A$ a.e. on U. If there is $\lambda > 0$ so that $A \sim^\lambda B$ on U we also say $A \sim B$ on U. If "on U" is dropped, we refer to whole $\mathbb{R}^m$.

We shall use $\mathbb{1}$ to denote the identity matrix. Thus a symbol A is uniformly elliptic iff $A \sim \mathbb{1}$. Moreover, if A and B are symbols on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, then $A \oplus B$ is just the natural symbol on $\mathbb{R}^{n_1+n_2}$.

Definition 2.2.2. Let w be a nonnegative locally integrable function (a weight) defined on $\mathbb{R}^n$. We denote $w(A) := \int_A w(x)dx$. The function w is called a *doubling weight* (resp. an A_2*-weight*), if there is a constant C such that for every ball $B \subset \mathbb{R}^n$ we have $w(2B) \le Cw(B)$ (resp. $\frac{1}{|B|^2}w(B)w^{-1}(B) \le C$).

Since by Schwartz inequality $|B|^2 \le w(B)w^{-1}(B)$, we have $|2B|^2 = 2^{2n}|B|^2 \le 2^{2n}w(B)w^{-1}(B) \le 2^{2n}w(B)w^{-1}(2B)$, so we can conclude that an A_2-weight is also a doubling weight.

Definition 2.2.3. Denote $u_B := |B|^{-1}\int_B u(x)\,dx$ and let w_1, w_2 be weights on $\mathbb{R}^n$ and let $q > 2$. We say that the *Poincaré inequality* (resp. *Sobolev inequality*) holds for w_1, w_2 with q, if there is $C < \infty$ so that for every ball $B \subseteq \mathbb{R}^n$ and $u \in W^{1,2}(B)$ (resp. $u \in W^{1,2}_0(B)$) we have

$$\left(w_2(B)^{-1}\int_B |u-u_B|^q w_2\,dx\right)^{1/q} \le C|B|^{1/n}\left(w_1(B)^{-1}\int_B |\nabla u|^2 w_1\,dx\right)^{1/2} \tag{2.2.1}$$

(resp.

$$\left(w_2(B)^{-1}\int_B |u|^q w_2\,dx\right)^{1/q} \le C|B|^{1/n}\left(w_1(B)^{-1}\int_B |\nabla u|^2 w_1\,dx\right)^{1/2} \tag{2.2.2}$$

).

Theorem 2.2.4. (Harnack inequality) *Suppose $H := -\nabla \cdot A\nabla$ is a divergence form operator with $w_1 \le A \le w_2$ and suppose that the weights w_1 and w_2 satisfy the following:*

1. *w_1 and w_2 are in A_2,*
2. *The Poincaré inequality holds for w_1, w_2 with some $q > 2$ and*
3. *The Poincaré inequality holds for w_1, 1 with some $q' > 2$.*

Let $t_0, ..., t_4 \in \mathbb{R}$ with $t_0 < ... < t_4$, $\Omega \subseteq \mathbb{R}^n$ open and $K \subseteq \Omega$ compact and connected. Let u be a strictly positive solution to $u_t + Hu = 0$ in $\Omega \times (t_0, t_4)$. Then there is a constant $C < \infty$ depending on Ω, K and $t_0, ..., t_4$, but on A only through the bounds w_1 and w_2 so that

$$\text{ess sup}_{K\times(t_1,t_2)}\, u \le C\,\text{ess inf}_{K\times(t_3,t_4)}\, u \tag{2.2.3}$$

Proof. This is just Theorem A of [13] supplemented with a covering argument from [21], pages 734-736. □

Remark 2.2.5. For the purposes of Theorem 2.2.4 the concept of u being a solution of $u_t + Hu = 0$ on $Q := \Omega \times (t_0, t_4)$ means exactly the following:

1. $u \in L^2(Q)$,
2. $u_t \in L^2(Q)$,
3. $|\nabla u|^2 w_2 \in L^1(Q)$ and
4. For all $\phi \in C_0^1(Q)$ we have

$$\int_Q u_t\phi + \langle A\nabla u, \nabla\phi\rangle\,dx\,dt = 0 \tag{2.2.4}$$

We are going to apply to apply the Harnack inequality only to heat kernels of some degenerate elliptic operators. In particular as long as $t_0 > 0$ all the above items will hold.

Since the heat kernel is a positive distribution, it is a measure and (4) follows from the fact that the heat kernel is a distributional solution of the corresponding degenerate heat equation.

First of all (1) holds because for $t_0 > 0$ the heat kernel is a bounded function on $\Omega \times (t_0, t_4)$ (by Corollary 2.4.22).

Secondly (2) holds because of the following computation which is justified by Remark 2.2.8:

$$\begin{aligned}(\partial_t K)(s, \cdot, y) &= -HK(s, \cdot, y) \\ &= -e^{-(s-t_0)H} H e^{-t_0 H/2} K(\frac{t_0}{2}, \cdot, y).\end{aligned} \tag{2.2.5}$$

Now since by Remark 2.2.7 e^{-tH} is a contraction on L^2,

$$\sup_{s \in (t_0, t_4)} ||\partial_t K(s, \cdot, y)||_2 < \infty. \tag{2.2.6}$$

Let A be the symbol of H. To prove (3) it suffices to show that $|\nabla K|$ is locally in L^2, since w_2 is locally bounded. Since

$$\int_Q |\nabla K|^2 \, dx \, dt \le \int_Q w_1^{-1} \langle A \nabla K, \nabla K \rangle \, dx \, dt. \tag{2.2.7}$$

Since w_1 is in A_2 (by Lemma A.1.2), w_1^{-1} is locally integrable, so it suffices to prove that $\langle A\nabla K, \nabla K\rangle$ is essentially bounded on Q. We show that for any $0 \le \phi \in C_0^\infty(Q)$ we have

$$\int_Q \phi \langle A \nabla K, \nabla K \rangle \le C \int_Q \phi, \tag{2.2.8}$$

with C not depending on ϕ.

So we compute using the facts that K and $\nabla \cdot A \nabla K$ are locally bounded:

$$\begin{aligned}\int_Q \phi \langle A \nabla K, \nabla K \rangle &= \left| \int_Q K \nabla \cdot \phi A \nabla K \right| \\ &\le C \left| \int_Q \langle A \nabla \phi, \nabla K \rangle \right| + C \left| \int_Q \phi \nabla \cdot A \nabla K \right| \\ &\le 2C \left| \int_Q \phi \nabla \cdot A \nabla K \right| \le C' \int_Q \phi.\end{aligned} \tag{2.2.9}$$

It follows from the results in §2.4.2 and Appendix A.1 that this Harnack inequality holds for the operators M_n, which will be our main interest and will be defined in §2.2.3.

2.2.2 Gaussian upper bounds for heat kernels

The material in this section is mostly taken from [6]. For more information, see sections 1.3, 2.4 and 3.2 there. See also [5] and [22].

Definition 2.2.6. Let $H \geq 0$ be a real self-adjoint operator on $L^2(\mathbb{R}^n)$. We call the semigroup e^{-Ht} a *symmetric Markov semigroup*, if it is positivity-preserving and a contraction on $L^\infty(\mathbb{R}^n)$.

Remark 2.2.7. By saying that e^{-Ht} is a contraction on L^p with $p \neq 2$ we mean that e^{-Ht} is a contraction on $L^p \cap L^2$ and can be extended to a unique contraction on L^p. In the case of L^∞ we have to impose the extra condition of weak* continuity to achieve uniqueness since $L^\infty \cap L^2$ is not norm dense in L^∞.

Remark 2.2.8. A symmetric Markov semigroup is strongly continuous on L^p with $1 \leq p < \infty$, see Theorem 1.4.1 of [6]. This in particular implies that the generator H commutes with the semigroup e^{-Ht} (see [5]).

By Theorem 1.3.5 of [6], any self-adjoint divergence form operator with non-negative symbol and core $C_0^\infty(\mathbb{R}^n)$ gives rise to a symmetric Markov semigroup. The Theorem there is stated for "elliptic" operators, but the proof works for any non-negative symbol. The keywords here are self-adjointness and core C_0^∞. Both follow for M_n from the fact that $\sigma(M_n) \in W^{1,2}_{\text{loc}}(\mathbb{R}^{(n-1)d})$ (Proposition 2.4.3). See Theorem 1.2.5 of [6].

Definition 2.2.9. Let e^{-Ht} be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$. We say that e^{-Ht} is *ultracontractive* if the map e^{-Ht} is bounded from L^2 to L^∞ for every $t > 0$.

Definition 2.2.10. Suppose that $C_0^\infty(\mathbb{R}^n) \subseteq \text{Dom}(H)$. Let e^{-Ht} be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$. We say that e^{-Ht} (or H or $\sigma(H)$) is of *dimension* μ if there is $C_2 < \infty$ such that for all $t > 0$ and $f \in L^2(\mathbb{R}^n)$ we have

$$||e^{-Ht}f||_\infty \leq C_2 t^{-\mu/4} ||f||_2. \tag{2.2.10}$$

Note that the dimension of a semigroup need not be unique.

There is a standard method for obtaining global Gaussian upper bounds for heat kernels of divergence form operators with nonnegative symbols using global space-independent bounds. A good reference for this is [5].

Definition 2.2.11. Let A be a symbol on $\mathbb{R}^n$. The function

$$\begin{aligned} d_A(x,y) := \sup\big\{ |\phi(x) - \phi(y)| : \phi \text{ is } C^\infty \text{ and bounded with} \\ \langle \nabla\phi, A\nabla\phi\rangle \leq 1 \text{ on } \mathbb{R}^n \big\} \end{aligned} \tag{2.2.11}$$

is called the metric associated with A (or H, if $H := -\nabla \cdot A\nabla$ or e^{-tH} or the heat kernel of H).

The following Theorem was proved by Varopoulos [22].

Theorem 2.2.12. *Let μ be a positive real number. Suppose $H := -\nabla \cdot A\nabla \geq 0$ is a positive self-adjoint divergence form operator with e^{-Ht} a symmetric Markov semigroup of dimension μ. Then for each $\delta > 0$ there is $C_\delta < \infty$ such that the heat kernel K of e^{-Ht} satisfies*

$$0 \leq K(t,x,y) \leq C_\delta t^{-\mu/2} \exp\{-\frac{d_A(x,y)^2}{4(1+\delta)t}\} \tag{2.2.12}$$

for all $0 < t < \infty$ and $x, y \in \mathbb{R}^n$. Besides δ, C_δ depends only on μ and the constant C_2 of Definition 2.2.10.

Proof. See [22] or [5]. □

We shall use the following Theorem later to get the dimension of M_n in Corollary 2.4.22.

Theorem 2.2.13. *Suppose $C_0^\infty(\mathbb{R}^n) \subseteq \mathrm{Dom}(H)$. Let e^{-Ht} be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$ and let $\mu > 2$ be given. Then there is $C_1 < \infty$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$||f||^2_{2\mu/(\mu-2)} \leq C_1 \langle f, Hf\rangle. \tag{2.2.13}$$

if and only if there is $C_2 < \infty$ such that for for all $t > 0$ and $f \in L^2(\mathbb{R}^n)$ we have

$$||e^{-Ht}f||_\infty \leq C_2 t^{-\mu/4}||f||_2. \tag{2.2.14}$$

Here the constants C_1 and C_2 depend only on each other and the number μ.

Proof. This is just Theorem 2.4.2 of [6]. □

Remark 2.2.14. One can show using the Schwartz Kernel and Radon-Nikodym Theorems that a bounded linear map $L : L_1 \to L_\infty$ has a integral kernel that is a function in L_∞ whose L_∞-norm equals the operator norm of L. Since our e^{-Ht} is self-adjoint, boundedness of $e^{-Ht} : L_2 \to L_\infty$ implies boundedness of $e^{-Ht} : L_1 \to L_2$, so in this case we have a heat kernel that is a genuine function.

Finally, we give a nice way to estimate heat kernels of operators H that have symbols satisfying $\sigma(H) \sim A_1 \oplus A_2$.

Theorem 2.2.15. *Suppose that for $i = 1, 2$, A_i is a symbol on $\mathbb{R}^{n_i}$ such that $e^{t\nabla\cdot A_i\nabla}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^{n_i})$ and $B \sim^\lambda A_1 \oplus A_2$. Suppose also that the heat kernels of A_i's satisfy*

$$K_{A_i}(t,x,y) \leq C_i t^{-\frac{\mu_i}{2}} \exp\{-\frac{d_{A_i}(x,y)^2}{C_i t}\}. \tag{2.2.15}$$

Then there is $C < \infty$ depending only on C_1, C_2, μ_1, μ_2 and λ so that the heat kernel of B satisfies

$$K_B(t,x,y) \le Ct^{-\frac{\mu_1+\mu_2}{2}} \exp\{-\frac{d_{A_1}(x,y)^2 + d_{A_2}(x,y)^2}{Ct}\}. \tag{2.2.16}$$

Proof. Since $K_{A_1 \oplus A_2}(t,(x_1,x_2),(y_1,y_2)) = K_{A_1}(t,x_1,y_1)K_{A_2}(t,x_2,y_2)$, we can conclude that

$$K_{A_1 \oplus A_2} \le C_1 C_2 t^{-\frac{\mu_1+\mu_2}{2}}, \tag{2.2.17}$$

which by Riesz-Thorin interpolation theorem and the fact that $e^{t\nabla \cdot A_1 \oplus A_2 \nabla}$ is a contraction L^∞ imply (2.2.14) for $H = -\nabla \cdot A_1 \oplus A_2 \nabla$. Therefore by Theorem 2.2.13

$$||f||^2_{2\mu/(\mu-2)} \le C_3 \langle \nabla f, (A_1 \oplus A_2)\nabla f \rangle \tag{2.2.18}$$

for any $f \in C_0^\infty(\mathbb{R}^{n_1+n_2})$ with C_3 depending only on C_1C_2 and $\mu_1 + \mu_2$. Since $A_1 \oplus A_2 \le \lambda^{-1}B$, we have

$$K_B \le C_4 t^{-\frac{\mu_1+\mu_2}{2}}, \tag{2.2.19}$$

with C_4 depending only on C_1C_2, $\mu_1 + \mu_2$ and λ. We now apply Theorem 2.2.12 to conclude the claim. □

2.2.3 The definition of the operators M_n

For the rest of the paper, we fix a constant ξ, $0 < \xi < 2$ and an integer $d \ge 2$. Here d is the dimension of the "physical" space.

Next, we overload the symbol d immediately and let d be the map from R^d to $d \times d$ matrices defined by

$$d(x) := C \int_{\mathbb{R}^d} \frac{1 - \cos(k \cdot x)}{|k|^{d+\xi}} (1 - \hat{k} \otimes \hat{k})\, dk, \tag{2.2.20}$$

with

$$C := \frac{(4\pi)^{d/2} 2^\xi \xi \Gamma((d+\xi+2)/2)}{(d-1)\Gamma((2-\xi)/2)}. \tag{2.2.21}$$

A computation (see e.g. [9]) shows that

$$d(x) = |x|^\xi \left((1 + \frac{\xi}{d-1})\mathbb{1} - \frac{\xi}{d-1} \hat{x} \otimes \hat{x} \right). \tag{2.2.22}$$

In the following definition, we denote vectors in $\mathbb{R}^{nd}$ by $\{v_i\}_{i=1}^n$, where each v_i is a vector in $\mathbb{R}^d$.

Definition 2.2.16. Let $n \geq 2$. The operator $\mathcal{M}_n^{sc} := -\nabla \cdot \sigma(\mathcal{M}_n^{sc})\nabla$ is the one with the symbol

$$\sigma(\mathcal{M}_n^{sc}) := -\sum_{1\leq i<j\leq n} \langle v_i, d(x_i - x_j)v_j\rangle \tag{2.2.23}$$

If $a \in \mathbb{R}^d$, we denote the vector $(x_i+a)_{i=1}^n$ by $\mathbf{x}+a$. We call a function $f : \mathbb{R}^{nd} \to \mathbb{R}$ translationally invariant, if for every $a \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^{nd}$ we have $f(\mathbf{x}) = f(\mathbf{x}+a)$.

We shall be interested in $\mathcal{M}_n^{sc}$ acting on translationally invariant functions, so we need to reduce the number of total space dimensions to $(n-1)d$.

In other words, we set $x_i := y_i - y_{i+1}$ for $1 \leq i \leq n-1$, so

$$\frac{\partial}{\partial y_i^\alpha} = \begin{cases} \frac{\partial}{\partial x_1^\alpha} & \text{if } i = 1, \\ \frac{\partial}{\partial x_i^\alpha} - \frac{\partial}{\partial x_{i-1}^\alpha} & \text{if } 2 \leq i \leq n-1 \text{ and} \\ \frac{\partial}{\partial x_{n-1}^\alpha} & \text{if } i = n. \end{cases} \tag{2.2.24}$$

Denote the symbol obtained in this way by $\sigma(M_n)$. A simple calculation shows that $\sigma(M_n)$ equals

$$\sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\langle v_i, (d(\sum_{k=i}^{j} x_k) - d(\sum_{k=i}^{j-1} x_k) - d(\sum_{k=i+1}^{j} x_k) + d(\sum_{k=i+1}^{j-1} x_k))v_j\rangle \tag{2.2.25}$$

In particular,

$$\sigma(M_2) = \langle v_1, d(x_1)v_1\rangle, \tag{2.2.26}$$

$$\begin{aligned} \sigma(M_3) =& \langle v_1, d(x_1)v_1\rangle + \langle v_2, d(x_2)v_2\rangle + \\ & \langle v_1, (d(x_1+x_2) - d(x_1) - d(x_2))v_2\rangle \end{aligned} \tag{2.2.27}$$

and

$$\begin{aligned} \sigma(M_4) =& \langle v_1, d(x_1)v_1\rangle + \langle v_2, d(x_2)v_2\rangle + \langle v_3, d(x_3)v_3\rangle \\ & \langle v_1, (d(x_1+x_2) - d(x_1) - d(x_2))v_2\rangle \\ & \langle v_2, (d(x_2+x_3) - d(x_2) - d(x_3))v_3\rangle \\ & \langle v_1, (d(x_1+x_2+x_3) - d(x_1+x_2) - d(x_2+x_3) + d(x_2))v_3\rangle \end{aligned} \tag{2.2.28}$$

2.3 Overview

Our intent here is to give some intuition on the arguments of this paper and how they lead to the proof of Theorem 2.1.1. What is obvious at first sight, is that if Theorem 2.1.1 is to hold, the Green's functions of the operators M_{2n} should be locally integrable in the sense that for every $n \geq 2$ there is $C < \infty$ so that for every $x \in \mathbb{R}^{(2n-1)d}$ we have

$$\int_{B(x,1)} d^{(2n-1)d}y \, G_{M_{2n}}(x,y) < C. \tag{2.3.1}$$

One might hope to get (2.3.1) to hold using the heat kernel estimate of Theorem 2.2.12, but unfortunately this direct approach fails. First of all we see that $\sigma(M_2) \sim |\cdot|^\xi$. Applying Definition 2.2.11, Corollary 2.4.22 and Theorem 2.2.12 to this, we find a $C < \infty$ such that

$$K_{M_2}(t,x,y) \leq Ct^{-\frac{d}{2-\xi}} \exp\{-\frac{|x-y|^2}{Ct}\} \tag{2.3.2}$$

for $|x| = 1$ and $|x-y| \leq \frac{1}{2}$. Integrating with respect to t from 0 to ∞ we get

$$G_{M_2}(x,y) \leq C'|x-y|^{2-\frac{2d}{2-\xi}}. \tag{2.3.3}$$

This estimate yields (2.3.1) only when $2 - \frac{2d}{2-\xi} > -d$, that is $\xi < \frac{4}{d+2}$. We might be satisfied with the fact that (2.3.1) holds only for small ξ, but there is worse to come: For each $\sigma(M_n)$ will have points $x \in \mathbb{S}^{(n-1)d-1}$ so that $\sigma(M_n) \sim \mathbb{1}$ in a neighbourhood U of x. A similar argument as above now yields

$$G_{M_{2n}}(x,y) \leq C'|x-y|^{2-\frac{2(n-1)d}{2-\xi}} \tag{2.3.4}$$

for $y \in U$. This yields (2.3.1) for M_n only when $\xi < \frac{4}{(n-1)d+2}$, which means trouble: Given ξ with $0 < \xi < 2$, there will always some be N so that our argument above fails to give local integrability for M_n with $n \geq N$.

On the other hand, since M_2 is uniformly elliptic in a neighbourhood U of x, the heat kernel of M_2 should behave like the heat kernel of the Laplacian for small times and small distances from x.

Turning this analysis into formulas let's suppose

$$K_{M_2}(t,x,y) \leq C_2 t^{-\frac{d}{2}} \exp\{-\frac{|x-y|^2}{C_2 t}\} \tag{2.3.5}$$

for $|x| = 1$, $|x-y| \leq \epsilon \leq \frac{1}{2}$ and $0 < t \leq t_0$. Since there is $C_3 < \infty$ so that $t^{-\frac{d}{2-\xi}} \leq C_3 t^{-\frac{d}{2}}$ for $t \geq t_0$, we can combine (2.3.2) with (2.3.5) and conclude that

$$K_{M_2}(t,x,y) \leq C_4 t^{-\frac{d}{2}} \exp\{-\frac{|x-y|^2}{C_4 t}\} \tag{2.3.6}$$

for $|x| = 1$, $|x-y| \leq \epsilon$ and $0 < t < \infty$. Now an integration w.r.t. t from 0 to ∞ yields

$$G_{M_2}(x,y) \leq C_5|x-y|^{2-d} \tag{2.3.7}$$

for $|x| = 1$ and $|x - y| \le \epsilon$. The same holds for M_n with $n > 2$. This leads us to a further twist: for $n > 2$, $\sigma(M_n)$ has degeneracies also outside of the origin, but fortunately in the end these turn out not to be problematic.

A few words on the structure of the rest of the paper. In §2.4 the symbols of M_n are analyzed in detail. The local analysis of the heat kernels is done in §2.5. Theorem 2.1.1 is proved in §2.6 and §2.7 is devoted to proving a technicality needed in §2.6. Finally, there are three appendices containing technicalities.

2.4 The operators M_n

From now on, we live in $\mathbb{R}^{(n-1)d}$ and denote vectors of $\mathbb{R}^{(n-1)d}$ with $\mathbf{v} = (v_i)_{i=1}^{n-1}$ and $\mathbf{x} = (x_i)_{i=1}^{n-1}$, where $v_i, x_i \in \mathbb{R}^d$.

The symbol of M_n has a bunch of useful symmetries, inherited from $\mathcal{M}_n^{sc}$. For $L : \mathbb{R}^k \to \mathbb{R}^l$ a surjective linear mapping and A a symbol on $\mathbb{R}^k$ which for all $x \in \mathbb{R}^k$ is constant on $\{x\} + \ker L$ denote $A^L(x) := LA(L^{-1}x)L^T$, where L^{-1} is some right-inverse of L. Let $L_n : \mathbb{R}^{nd} \to \mathbb{R}^{(n-1)d}$ be given by the matrix $(L_n)_{ij} := \delta_{ij} - \delta_{i+d,j}$, so that $\sigma(M_n) = \sigma(\mathcal{M}^{sc})^{L_n}$

We let

$$\mathcal{L}_n = \{L_n L L_n^{-1} : L \text{ is a permutation of the coordinate axes of } \mathbb{R}^{nd}\}. \tag{2.4.1}$$

Now $\sigma(M_n)^L = \sigma(M_n)$ for every $L \in \mathcal{L}_n$

Remark 2.4.1. Let A_1 and A_2 be two symbols on $\mathbb{R}^k$ and let $G_1, G_2 \subseteq \mathrm{GL}(\mathbb{R}^k)$ be their respective symmmetry groups, i.e

$$G_i := \{L \in \mathrm{GL}(\mathbb{R}^k) : A_i^L = A_i\}, \tag{2.4.2}$$

for $i \in \{1, 2\}$. Now if $A_1 \sim A_2$ on U, then $A_1 \sim A_2$ on LU for any $L \in G_1 \cap G_2$.

Remark 2.4.2. A simple calculation shows that M_n is degenerate, whenever $\sum_{i=a}^{b} x_i = 0$, where $1 \le a \le b \le n - 1$. In fact these are the only points where M_n degenerates, as we see in Theorem 2.4.7. To avoid lengthy statements in the rest of the paper, we denote $\{\mathbf{x} \in \mathbb{R}^{(n-1)d} : x_i = 0\}$ by $\{x_i = 0\}$ and similarly for the other sets.

Proposition 2.4.3.

$$\sigma(M_n) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{(n-1)d}) \tag{2.4.3}$$

Proof. The case $1 < \xi < 2$ is an easy computation, since then $\sigma(M_n)$ is continuously differentiable.

In case $0 < \xi \leq 1$, we let

$$F := \bigcup_{1\leq a\leq b<n} \{\sum_{i=a}^{b} x_i = 0\}. \tag{2.4.4}$$

A relatively simple calculation shows that there is $C < \infty$ such that

$$|\nabla(\sigma(M_n))(\mathbf{x})| \leq C d(\mathbf{x}, F)^{\xi-1}. \tag{2.4.5}$$

Since F is a finite union of vector subspaces of codimension $d \geq 2$, we can conclude that $d(\mathbf{x}, F)^{\xi-1}$ is a locally square integrable function. □

Remark 2.4.4. It is trivial to get an upper bound for M_n:

$$\sigma(M_n) \leq (\sup_{|\mathbf{y}|=|\mathbf{w}|=1} \langle \mathbf{w}, \sigma(M_n)(\mathbf{y})\mathbf{w}\rangle)|\mathbf{x}|^{\xi}|\mathbf{v}|^2. \tag{2.4.6}$$

We obtain a better upper bound in section §2.4.2.

Proposition 2.4.5. *For any $\epsilon \in (0,1)$ there is $C < \infty$ such that*

$$d_{\sigma(M_n)}(x, y) \leq C|x-y|^{1-\xi/2}, \tag{2.4.7}$$

when $|x - y| \geq \epsilon|x|$.

Proof. By Definition 2.2.11 and Remark 2.4.4 it suffices to show that there is $C < \infty$ such that $d_{|\cdot|^\xi}(x, y)^2 \leq C|x-y|^{2-\xi}$, when $|x - y| \geq \epsilon|x|$. Trivial dimensional analysis gives $d_{|\cdot|^\xi}(x, y) = |x|^{1-\xi/2} d_{|\cdot|^\xi}(\hat{x}, \frac{y}{|x|})$. Therefore we may assume $|x| = 1$. By rotational symmetry, we may fix x. By scaling, there is $C' < \infty$ so that $C'|y|^{1-\xi/2} = d_{|\cdot|^\xi}(0, y)$. Since now

$$\frac{d_{|\cdot|^\xi}(x, y)^2}{|x-y|^{2-\xi}} = C' \frac{d_{|\cdot|^\xi}(x, y)^2}{d_{|\cdot|^\xi}(0, x-y)^2}, \tag{2.4.8}$$

it suffices to show that $f(R) := \sup_{|x-y|=R} d_{|\cdot|^\xi}(x, y)/d_{|\cdot|^\xi}(0, x-y)$ is a bounded function of R for $R \in [\epsilon, \infty)$. Obviously f is continuous. By continuity of $d_{|\cdot|^\xi}$ we have

$$\frac{d_{|\cdot|^\xi}(x, y)^2}{d_{|\cdot|^\xi}(0, x-y)^2} = \frac{d_{|\cdot|^\xi}(\frac{x}{|x-y|}, \frac{y}{|x-y|})^2}{d_{|\cdot|^\xi}(0, \widehat{x-y})^2} \to \frac{d_{|\cdot|^\xi}(0, \hat{y})^2}{d_{|\cdot|^\xi}(0, \hat{y})^2} = 1 \tag{2.4.9}$$

as $|x - y| \to \infty$. □

2.4.1 Fourier integral representation and the degeneration set

Definition 2.4.6. Let A be a symbol. We call the set

$$\mathrm{Dgn}(A) := \{x \in R^n : A(x) \text{ is not invertible}\} \tag{2.4.10}$$

the degeneration set of A.

The following Fourier integral representation of the symbol is crucial for the computation of the degeneration sets of $\mathcal{M}_n$ (which then implies corresponding properties for the operators M_n to be introduced later).

Theorem 2.4.7. *The degeneration set of M_n is*

$$\mathrm{Dgn}(M_n) = \bigcup_{1\le i\le j<n} \{\mathbf{x} \in R^{(n-1)d} : |x_i + ... + x_j| = 0\}. \tag{2.4.11}$$

Proof. By Remark 2.4.2, it suffices to show that for every $\mathbf{v} \in \mathbb{R}^{nd}$ with $\sum_{i=1}^n v_i = 0$ we have $-\sum_{1\le i<j\le n}\langle v_i, d(x_i - x_j)v_j\rangle > 0$ whenever $x_i \neq x_j$ for all $1 \le i < j \le n$. We have

$$\begin{aligned} -\sum_{1\le i<j\le n} \langle v_i, d(x_i - x_j)v_j\rangle &= -\frac{1}{2}\sum_{1\le i,j\le n} \langle v_i, d(x_i - x_j)v_j\rangle \\ &= -\frac{C}{2}\int_{\mathbb{R}^d} \mathrm{Re}(\sum_{1\le i,j\le n} \frac{1 - e^{ik\cdot(x_i - x_j)}}{|k|^{d+\xi}} \langle v_i, (\mathbb{1} - k\otimes k)v_j\rangle)\, dk \\ &= \frac{C}{2}\int_{\mathbb{R}^d} \mathrm{Re}\langle\sum_{i=1}^n v_i e^{ik\cdot x_i}, \frac{\mathbb{1} - k\otimes k}{|k|^{d+\xi}} \sum_{i=1}^n v_i e^{ik\cdot x_i}\rangle\, dk \end{aligned} \tag{2.4.12}$$

The rest goes as in Proposition 1 of [9]: For the integral to be zero, we have to have

$$\sum_{i=1}^n v_i e^{ik\cdot x_i} = \alpha(k)k \tag{2.4.13}$$

almost everywhere for some scalar function α. Taking the exterior product (i.e. the antisymmetric part of the tensor product) with respect to k and Fourier transforming in the sense of distributions we arrive at

$$\sum_{i=1}^n v_i \wedge \nabla\delta(x - x_n) = 0. \tag{2.4.14}$$

Thus for any smooth test function ϕ

$$\sum_{i=1}^{n} v_i \wedge \nabla\phi(x_n) = 0. \tag{2.4.15}$$

This is a contradiction since the values of $\nabla\phi$ can be arbitrarily specified on a discrete set and the x_n's are all distinct. □

2.4.2 Estimates for the symbol of M_n

We shall now show that the symbol of M_n can be estimated using the symbols of M_m, $m \in \{2, ..., n-1\}$.

Definition 2.4.8. Let $\mathbf{x} \in \mathbb{R}^{(n-1)d}$. The dimension of the zero eigenspace of $\sigma(M_n)$ at $\mathbf{x}$ divided by d is called the *rank* of the point $\mathbf{x}$ and denoted $\mathrm{rk}(\mathbf{x})$. In particular $\mathbf{x}$ is a degeneration point of $\sigma(M_n)$ iff $\mathrm{rk}(\mathbf{x}) > 0$.

Below, for a symbol A and invertible linear transformation L we define the symbol A^L by the formula $A^L(x) := LA(L^{-1}x)L^T$.

Theorem 2.4.9. *Let $n \geq 2$ and $\mathbf{x} \in \mathbb{S}^{(n-1)d-1}$. Then either M_n is uniformly elliptic in some neighbourhood of $\mathbf{x}$ or there is a invertible linear transformation L of $\mathbb{R}^{(n-1)d}$, a neighbourhood U of $L\mathbf{x}$ so that $\sigma(M_n)^L \sim \bigoplus_{i=1}^k \sigma(M_{n_i}) \oplus \mathbb{1}$ on U with $k \geq 1$, each $n_k \geq 2$, $\mathrm{rk}(\mathbf{x}) = \sum_{i=1}^k (n_k - 1) < n - 1$ and $(L\mathbf{x})_i = 0$ for $1 \leq i \leq \sum_{j=1}^k (n_k - 1)$.*

Let's introduce some convenient notation at this point. First of all $[i, j] := \{i, ..., j\}$. Let $A \subseteq [1, n]$. Then we write

$$\begin{aligned} x_A &:= \sum_{i \in A} x_i \\ \gamma_A &:= \langle v_{\min A}, (d(x_A) - d(x_{A\setminus\{\min A\}}) \\ &\qquad - d(x_{A\setminus\{\max A\}}) + d(x_{A\setminus\{\min A, \max A\}}))v_{\max A}\rangle \\ \sigma_A &:= \sum_{i,j \in A; i \leq j} \gamma_{A\cap[i,j]}. \end{aligned} \tag{2.4.16}$$

Moreover $x_{i,j} := x_{[i,j]}$, $\sigma_{i,j} := \sigma_{[i,j]}$, $\gamma_{i,j} := \gamma_{[i,j]}$ and $\sigma_i := \gamma_i := \gamma_{\{i\}}$.

2.4.3 Two propositions for the proof of Theorem 2.4.9

Our purpose here is to prove Proposition 2.4.10 and Proposition 2.4.17. Let us illustrate what we're going to do by studying $\sigma(M_3)$ in some detail.

Let $\mathbf{x} \in \mathbb{S}^{2d-1}$ be such that $x_1 = 0$, i.e. $\mathbf{x} = (x_1, x_2)$ with $x_2 \in \mathbb{S}^{d-1}$. We'll show that there is a neighbourhood U of $\mathbf{x}$ and $C < \infty$ so that for every $\mathbf{y} \in U$ we have

$$\frac{1}{C}\big(|y_1|^\xi |v_1|^2 + |y_2|^\xi |v_2|^2\big) \le \sigma(M_3)(\mathbf{y}) \le C\big(|y_1|^\xi |v_1|^2 + |y_2|^\xi |v_2|^2\big). \tag{2.4.17}$$

Let E be given by Lemma 2.4.11 and let $\epsilon \in (0, \frac{1}{4})$ be such that

$$E\big((2\epsilon)^{1-\xi/2} + (2\epsilon)^{\xi/2}\big) \le \frac{1}{2} \tag{2.4.18}$$

and let

$$U := B(0, \epsilon) \times \{\frac{1}{2} < |y_2| < \frac{3}{2}\}. \tag{2.4.19}$$

By our choice of ϵ we have

$$|\gamma_{1,2}| \le \frac{1}{2}(|y_1|^\xi |v_1|^2 + |y_2|^\xi |v_2|^2) \tag{2.4.20}$$

in U. In other words (2.4.17) holds and thus $\sigma(M_3) \sim \sigma(M_2) \oplus \mathbb{1}$ on U.

Proposition 2.4.10 will be used when we have several (or all) coordinates away from the degeneration set. As might be guessed from our calculation with $\sigma(M_3)$, the point of Lemmata 2.4.11-2.4.14 is that in the proof of Theorem 2.4.9 we need to have estimates for the crossterms with the flavor

$$|\gamma_{i,j}| \le \text{ something } \cdot (|x_i|^\xi |v_i|^2 + |x_j|^\xi |v_j|^2). \tag{2.4.21}$$

We have neatly blackboxed all this mess into Proposition 2.4.17; the Lemmata of this section are not directly used in the proof of Theorem 2.4.9. The proofs can be found in Appendix A.2.

Proposition 2.4.10. *Suppose $n \ge 1$, $\epsilon \in (0, 1)$ and let*

$$A := \{\mathbf{x} \in \mathbb{R}^{nd} : \epsilon \max\{|x_{i,j}| : 1 \le i \le j \le n\} \le \min\{|x_{i,j}| : 1 \le i \le j \le n\}\}. \tag{2.4.22}$$

Then there is $C < \infty$ so that for every $\mathbf{x} \in A$ we have

$$\frac{1}{C}\sum_{i=1}^{n} |x_i|^\xi |v_i|^2 \le \sigma(M_{n+1}) \le C \sum_{i=1}^{n} |x_i|^\xi |v_i|^2 \tag{2.4.23}$$

Lemma 2.4.11. *There is a constant $E < \infty$ such that if $1 \le i < n$ and $|x_i| < \frac{1}{2}|x_{i+1}|$, then*

$$\begin{aligned} &|\langle v_i, (d(x_i + x_{i+1}) - d(x_i) - d(x_{i+1}))v_{i+1}\rangle| \\ &\qquad \le E\bigg(\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{1-\xi/2} + \Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{\xi/2}\bigg)(|x_i|^\xi |v_i|^2 + |x_{i+1}|^\xi |v_{i+1}|^2). \end{aligned} \tag{2.4.24}$$

Lemma 2.4.12. *There is a constant $E < \infty$ such that if $1 \le i < i+1 < j \le n$, $|x_i| < \frac{1}{2}\min\{|x_{i+1,j}|, |x_{i+1,j-1}|\}$ and $|x_j| > 0$, then*

$$\begin{aligned} &|\langle v_i,(d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j\rangle| \\ &\quad \le E\Big((\frac{|x_i|}{|x_{i+1,j}|})^{1-\xi/2} (\frac{|x_{i+1,j}|}{|x_j|})^{\xi/2} + (\frac{|x_i|}{|x_{i+1,j-1}|})^{1-\xi/2} (\frac{|x_{i+1,j-1}|}{|x_j|})^{\xi/2} \Big) \\ &\quad \cdot (|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2). \end{aligned} \tag{2.4.25}$$

Lemma 2.4.13. *There is a constant $E < \infty$ such that if $1 \le i < i+1 < j \le n$, $\frac{1}{2}|x_{i+1,j-1}| \le |x_i| < \frac{1}{2}|x_{i+1,j}|$ and $|x_j| > 0$, then*

$$\begin{aligned} &|\langle v_i,(d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j\rangle| \\ &\quad \le E\Big((\frac{|x_i|}{|x_{i+1,j}|})^{1-\xi/2} (\frac{|x_{i+1,j}|}{|x_j|})^{\xi/2} + (\frac{|x_i|}{|x_j|})^{\xi/2} \Big) (|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2). \end{aligned} \tag{2.4.26}$$

Lemma 2.4.14. *There is $E < \infty$ so that if $1 \le i < i+1 < j \le n$ and $\max\{|x_i|, |x_j|\} < \frac{1}{3}\{|x_{i+1,j-1}|\}$, we have*

$$\begin{aligned} &|\langle v_i,(d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j\rangle| \\ &\quad \le E(\frac{|x_i|}{|x_{i+1,j-1}|})^{1-\xi/2} (\frac{|x_i|}{|x_{i+1,j-1}|})^{1-\xi/2} (|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2). \end{aligned} \tag{2.4.27}$$

We still have one more Lemma to go before we can start proving Proposition 2.4.17. We'll illustrate it with $\sigma(M_6)$. Let $\mathbf{x} \in \mathbb{S}^{5d-1}$ with $|x_1| = |x_3| = |x_5| = 0$ and $|x_2|, |x_4|, |x_{2,4}| > 0$. By Proposition 2.4.10 $\sigma_{\{2,4\}}(y_2, y_4)$ behaves like $|y_2|^{\xi}|v_2|^2 + |y_4|^{\xi}|v_4|^2$ in a neighbourhood of (x_2, x_4). Unfortunately the relevant part of $\sigma(M_6)$ is $\gamma_2 + \gamma_4 + \gamma_{2,4}$, but at least we would have some hope, if we could get an estimate of the form

$$|\gamma_{2,4} - \gamma_{\{2,4\}}| \le \text{ something } \cdot (|y_2|^{\xi}|v_2|^2 + |y_4|^{\xi}|v_4|^2) \tag{2.4.28}$$

for $\mathbf{y}$ in a neighbourhood of $\mathbf{x}$.

This is the point of Lemma 2.4.15. More precisely, let

$$\mu := \min\{|y_2|, |y_4|, |y_{2,4}|\} \le \max\{|y_2|, |y_4|, |y_{2,4}|\} =: \nu \tag{2.4.29}$$

and let $C < \infty$ be such that if $\frac{\mu}{2} < |y_2|, |y_4|, |y_2 + y_4| < 2\nu$ we have

$$\frac{1}{C}(|y_2|^{\xi}|v_2|^2 + |y_4|^{\xi}|v_4|^2) \le \sigma_{\{2,4\}} \le C(|y_2|^{\xi}|v_2|^2 + |y_4|^{\xi}|v_4|^2). \tag{2.4.30}$$

Let $\epsilon \in (0, \frac{\mu}{6})$ be such that

$$E\Big((\frac{2\epsilon}{\mu})^{1-\xi/2} (\frac{4\nu}{\mu})^{\xi/2} + (\frac{2\epsilon}{\mu})^{\xi/2} \Big) \le \frac{1}{2C}. \tag{2.4.31}$$

Let

$$U := \{|y_3| < \epsilon \text{ and } \frac{\mu}{2} < |y_2|, |y_4|, |y_{2,4}|, |y_2 + y_4| < 2\nu\}. \tag{2.4.32}$$

By Lemma 2.4.15 for $\mathbf{y} \in U$ we have

$$|\gamma_{2,4} - \gamma_{\{2,4\}}| \le \frac{1}{2C}(|y_2|^\xi |v_2|^2 + |y_4|^\xi |v_4|^2). \tag{2.4.33}$$

Combining (2.4.33) with (2.4.30) we conclude that $\gamma_2 + \gamma_4 + \gamma_{2,4}$ behaves like $|y_2|^\xi |v_2|^2 + |y_4|^\xi |v_4|^2$ in U.

Again, the proof of the following Lemma can be found in Appendix A.2.

Lemma 2.4.15. *There is $E < \infty$ such that if $1 \le i < j \le n$ and $\{i,j\} \subseteq A \subseteq [i,j]$ and if $\sum_{k \in [i,j] \setminus A} |x_k| \le \frac{1}{2} \min\{|x_{k,l}| : k, l \in A, k \le l\}$ Then*

$$\begin{aligned} |\gamma_{i,j} - \gamma_A| \le E\Big(&(\frac{\sum_{k \in [i,j] \setminus A} |x_k|}{\min\{|x_{k,l}| : k, l \in A, k \le l\}})^{1-\xi/2} (\frac{\sum_{k \in A} |x_k|}{|x_j|})^{\xi/2} \\ &+ (\frac{\sum_{k \in [i,j] \setminus A} |x_k|}{\min\{|x_{k,l}| : k, l \in A, k \le l\}})^{\xi/2} \Big) (|x_i|^\xi |v_i|^2 + |x_j|^\xi |v_j|^2). \end{aligned} \tag{2.4.34}$$

If $L \in \mathrm{GL}(\mathbb{R}^{(n-1)d})$, we shall use the following somewhat weird notation: If $\mathbf{x} \in \mathbb{R}^{(n-1)d}$, we let $Lx_i := (L\mathbf{x})_i$ for $1 \le i \le n-1$. Similarly, we let $Lx_{i,j} := (L\mathbf{x})_{i,j}$ for $1 \le i \le j \le n-1$.

Remark 2.4.16. Let $\mathbf{x}$ be a degeneration point of $\sigma(M_n)$. We claim that there is a symmetry $L \in \mathcal{L}_n$ and $A \subsetneq \{1, ..., n-1\}$ so that $|Lx_i| = 0$ if $i \in A$ and $Lx_{i,j} > 0$ if $\{i, ..., j\} \not\subseteq A$. This is easy to see, if we look at the original symbol $\sigma(\mathcal{M}_n^{sc})$. Then the claim above simply says that if we have points $y_1, ..., y_n \in \mathbb{R}^d$, then there is a permutation $\pi \in S_n$ so that if $y_{\pi(i)} = y_{\pi(j)}$ with $\pi(i) \le \pi(j)$, then $y_{\pi(i)} = y_k$ with every k with $\pi(i) \le k \le \pi(j)$. Still in other words: if we pick n possibly coinciding points from $\mathbb{R}^d$, we can label them with numbers $1, ..., n$ so that the coinciding points get consecutive numbers as labels.

Given x and A as above, write A as

$$\{i_1, ..., j_1\} \cup ... \cup \{i_m, ..., j_m\} \tag{2.4.35}$$

with $i_1 \le j_1 < j_1 + 1 < i_2 \le ... < i_m \le j_m$ and write $\sigma(M_n)$ as

$$\sigma(M_n) = \sum_{l=1}^{m} \sigma_{i_l, j_l} + \sigma_{A^c} + \sum_{i,j \in A^c} \gamma_{i,j} - \sigma_{A^c} + \text{ the rest.} \tag{2.4.36}$$

Let $\mu := \min\{|x_{i,j}| : \{i, ..., j\} \not\subseteq A\}$ and $\nu := \max\{|x_{i,j}| : \{i, ..., j\} \not\subseteq A\}$.

Proposition 2.4.17. *For any $C > 0$ there is a neighbourhood U of $\mathbf{x}$ so that*

$$|\sum_{i,j\in A^c} \gamma_{i,j} - \sigma_{A^c} + \text{ the rest } | \leq \frac{1}{2C}\sum_{i=1}^{n} |y_i|^\xi |v_i|^2 \tag{2.4.37}$$

for any $\mathbf{y} \in U$.

Proof. For $\epsilon > 0$ let
(2.4.38)
$$U^\epsilon := \{\mathbf{y} \in \mathbb{R}^{nd} : |y_{i,j}| < \epsilon \text{ if } \{i, ..., j\} \subseteq A \text{ and } \mu/2 < |y_{i,j}| < 2\nu \text{ otherwise}\}.$$

Let $N := \frac{n(n-1)}{2}$ be the number of terms in $\sigma(M_n)$. We'll find $\epsilon > 0$ so that each term in (2.4.37) is $\leq \frac{1}{2NC}\sum_{i=1}^{n} |x_i|^\xi |v_i|^2$ where we count each $\gamma_{i,j} - \gamma_{A^c\cap[i,j]}$ with $i, j \in A^c$ as one term.

A (long) moment's look at Lemmata 2.4.11-2.4.15 reveals us that this is possible. Here's a list of the requirements for ϵ.

1. Lemma 2.4.11: $\epsilon < \frac{\mu}{4}$ and $E((\frac{2\epsilon}{\mu})^{1-\xi/2} + (\frac{2\epsilon}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$
2. Lemma 2.4.12: $\epsilon < \frac{\mu}{4}$ and $2E(\frac{2\epsilon}{\mu})^{1-\xi/2}(\frac{4\nu}{\mu})^{\xi/2} \leq \frac{1}{2NC}$.
3. Lemma 2.4.13: $\epsilon < \frac{\mu}{4}$ and $E((\frac{2\epsilon}{\mu})^{1-\xi/2}(\frac{4\nu}{\mu})^{\xi/2} + (\frac{2\epsilon}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$
4. Lemma 2.4.14: $\epsilon < \frac{\mu}{6}$ and $E(\frac{2\epsilon}{\mu})^{2-\xi} \leq \frac{1}{2NC}$
5. Lemma 2.4.15: $n\epsilon < \frac{\mu}{4}$ and $E((\frac{2n\epsilon}{\mu})^{1-\xi/2}(\frac{4n\nu}{\mu})^{\xi/2} + (\frac{2n\epsilon}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$.

□

2.4.4 The proof of Theorem 2.4.9

Proof. (of Theorem 2.4.9) We shall prove this Theorem by induction on n and we shall accomplish this by proving in parallel that there is a constant $C < \infty$ so that for any $\mathbf{x} \in \mathbb{R}^{(n-1)d}$ there is $K \in \mathcal{L}_n$ so that

$$\frac{1}{C}\sum_{i=1}^{n-1} |Kx_i|^\xi |v_i|^2 \leq \sigma(M_n) \leq C\sum_{i=1}^{n-1} |Kx_i|^\xi |v_i|^2. \tag{2.4.39}$$

This is trivial for $\sigma(M_2)$. We assume now that the claim above is true for $\sigma(M_m)$, $2 \leq m < n$ and prove it for $\sigma(M_n)$. This is done as follows. For every $\mathbf{x} \in \mathbb{S}^{nd-1}$ we find a neighbourhood $U_{\mathbf{x}}$ of $\mathbf{x}$ so that the claim above holds on $U_{\mathbf{x}}$ with a constant $C(\mathbf{x})$ depending on $\mathbf{x}$. Since $\mathbb{S}^{nd-1}$ is compact, there is a finite set

$\{\mathbf{x}_1, ..., \mathbf{x}_k\}$ so that $\mathbb{S}^{nd-1} \subseteq \bigcup_{i=1}^k U_{\mathbf{x}_k}$, so the claim above will then hold with $C = \max_{1\leq i\leq k} C(\mathbf{x}_i)$.

If $\mathbf{x}$ is not a degeneration point of M_{n+1}, then by Proposition 2.4.10 the estimate above can be satisfied in a neighbourhood of $\mathbf{x}$ with $K = \mathbb{1}$, so we assume $\mathbf{x}$ is a degeneration point.

We now apply the symmetry discussed in Remark 2.4.16, so we can assume there is nonempty $A \subsetneq \{1, ..., n\}$ so that $|x_i| = 0$ if $i \in A$ and $|x_{i,j}| > 0$ if $\{i, ..., j\} \not\subseteq A$. Write A as $\{i_1, ..., j_1\} \cup ... \cup \{i_m, ..., j_m\}$ with $i_1 \leq j_1 < j_1 + 1 < i_2 < ... < i_m \leq j_m$. Denote $A^c := \{1, ..., n\} \setminus A$. We may even assume that $i_1 = 1$ and if $m > 1$, we have $j_m = n$. Note that $\mathrm{rk}(\mathbf{x}) = \#(A)$. Let U' be the neighbourhood of $\mathbf{x}$ given by Proposition 2.4.17.

Recall that μ and ν were defined as $\mu := \min\{|x_{i,j}| : \{i, ..., j\} \not\subseteq A\}$ and $\nu := \max\{|x_{i,j}| : \{i, ..., j\} \not\subseteq A\}$. Let

$$U := U' \cap \{\frac{\mu}{2} < |y_B| < 2\nu : B \not\subseteq A\}. \tag{2.4.40}$$

First of all, let $C < \infty$ be such that our induction hypothesis is satisfied with it for $2 \leq m < n$ and also that C is so large that the conclusion of Proposition 2.4.10 holds with $\epsilon := \frac{\mu}{4\nu}$. Also we require that

$$\frac{1}{C} \max_{B \not\subseteq A} |y_B|^\xi \leq 1 \leq C \min_{B \not\subseteq A} |y_B|^\xi \tag{2.4.41}$$

holds whenever $\mathbf{y} \in U$.

We claim that on U we have $\sigma(M_n) \sim \sigma(M_{j_1+1}) \oplus \mathbb{1}$ if $m = 1$ and $\sigma(M_n) \sim \sigma(M_{j_1+1}) \oplus 1 \oplus \sigma(M_{j_2-i_2+2}) \oplus ... \oplus \mathbb{1} \oplus \sigma(M_{n-i_m+2})$ otherwise. Denote the right-hand sides of these expressions collectively as Σ.

By our induction hypotheses, for any $\mathbf{y}' \in U$ and any $k \in \{1, ..., m\}$ there is a symmetry $K \in \mathcal{L}_n$ so that for $1 \leq k \leq m$ we have

$$\frac{1}{C} \sum_{i=i_k}^{j_k} |Ky_i'|^\xi |v_i|^2 \leq \sigma(M_{j_k-i_k+2})(Ky_{i_k}', ..., Ky_{j_k}') \leq C \sum_{i=i_k}^{j_k} |Ky_i'|^\xi |v_i|^2 \tag{2.4.42}$$

with C not depending on $\mathbf{y}'$: Just pick such a symmetry $K_k \in \mathcal{L}_{j_k-i_k+2}$ for $k \in \{1, ..., m\}$ and take any $K \in \mathcal{L}_n$ such that the restriction to the $y_{i_k}, ..., y_{j_k}$ coordinates is K_k. Here we have been abusing notation with the K_k's so that K_k above operates on coordinates $y_{i_k}, ..., y_{j_k}$ and not $y_1, ..., y_{j_k-i_k+1}$. Extend K_k now naturally to whole of $\mathbb{R}^{(n-1)d}$. We can now take K to be say $K = K_1 K_2 ... K_{m-1} K_m$.

Now for every $\mathbf{y}' \in U$ fix such a transformation $K_{\mathbf{y}'}$ and denote $\mathbf{y} := K_{\mathbf{y}'}\mathbf{y}'$.

By (2.4.41) and (2.4.42) we have

$$\frac{1}{C}\sum_{i=1}^{n-1}|y_i|^{\xi}|v_i|^2 \leq \Sigma(\mathbf{y}) \leq C\sum_{i=1}^{n-1}|y_i|^{\xi}|v_i|^2. \tag{2.4.43}$$

As before, we write

$$\sigma(M_n) = \sum_{l=1}^{m}\sigma_{i_l,j_l} + \sigma_{A^c} + \sum_{i,j\in A^c}\gamma_{i,j} - \sigma_{A^c} + \text{ the rest.} \tag{2.4.44}$$

The first two terms satisfy

$$\frac{1}{C}\sum_{i=1}^{n-1}|y_i|^{\xi}|v_i|^2 \leq \sum_{l=1}^{m}\sigma_{i_l,j_l} + \sigma_{A^c} \leq C\sum_{i=1}^{n-1}|y_i|^{\xi}|v_i|^2, \tag{2.4.45}$$

and by Proposition 2.4.17 we have

$$|\sum_{i,j\in A^c}\gamma_{i,j} - \sigma_{A^c} + \text{ the rest}| \leq \frac{1}{2C}\sum_{i=1}^{n}|y_i|^{\xi}|v_i|^2. \tag{2.4.46}$$

So we have

$$\frac{1}{2C}\sum_{i=1}^{n-1}|y_i|^{\xi}|v_i|^2 \leq \sigma(M_n)(\mathbf{y}) \leq (C+\frac{1}{2C})\sum_{i=1}^{n-1}|y_i|^{\xi}|v_i|^2. \tag{2.4.47}$$

Let $U^K := \{\mathbf{y}' \in U : K_{\mathbf{y}'} = K\}$. Clearly $U = \bigcup\{U^K : K \in \mathcal{K}_n\}$. We just proved that for any $\mathbf{y}' \in U$ we have $\sigma(M_n) \sim \Sigma$ in $K_{\mathbf{y}'}U^{K_{\mathbf{y}'}}$. Since both Σ and $\sigma(M_n)$ are invariant under $K_{\mathbf{y}'}^{-1}$ for any $\mathbf{y}' \in U$, we can conclude by Remark 2.4.1 that $\sigma(M_n) \sim \Sigma$ on $U^{K_{\mathbf{y}'}}$. Since $\mathcal{L}_n$ is finite we can conclude that $\sigma(M_n) \sim \Sigma$ on U. ☐

Let

$$\begin{aligned}\mathcal{L}'_n := \{L \in \mathrm{GL}(\mathbb{R}^{(n-1)d}) : \exists i_1, j_1, ..., i_{n-1}, j_{n-1} : \forall x_1, ..., x_{n-1} :\\ L((x_1, ..., x_{n-1})) = (x_{i_1,j_1}, ..., x_{i_{n-1},j_{n-1}})\}.\end{aligned} \tag{2.4.48}$$

Obviously, $\mathcal{L}'_n$ is a finite set. Note that the L as constructed in Theorem 2.4.9 belongs to $\mathcal{L}'_n$.

Remark 2.4.18. The following Proposition simply says the following: Suppose we have a symbol of the form

$$\bigoplus_{i=1}^{k}\sigma(M_{n_i+1}) \oplus \mathbb{1}. \tag{2.4.49}$$

This corresponds to a splitting $\mathbb{R}^{nd} = \mathbb{R}^{ld} \oplus \mathbb{R}^{(n-l)d}$ with $l = n_1 + ... + n_k$. Then we can replace $\mathbb{R}^{(n-l)d}$ with any complementary subspace to $\mathbb{R}^{ld}$ and the symbol looks the same in these new coordinates as looks the symbol in an neighbourhood of 0 which is bounded in the $\mathbb{R}^{ld}$-direction.

Proposition 2.4.19. *Let $\sigma \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus \mathbb{1}$ on a set $U \subseteq B \times \mathbb{R}^{(n-l)d}$ with B bounded and $l := \mathrm{rk}(0) = \sum_{i=1}^k n_i$. Let $L \in \mathrm{GL}(\mathbb{R}^{nd})$ be such that*

1. *$L : \{0\} \times \mathbb{R}^{(n-l)d} = \{0\} \times \mathbb{R}^{(n-l)d}$ and*
2. *Let $P : \mathbb{R}^{nd} \to \mathbb{R}^{ld}$ be the natural projection onto the first ld coordinates and let $L' := L \restriction \mathbb{R}^{ld} \times \{0\}$. Then*

$$(\bigoplus_{i=1}^k \sigma(M_{n_i+1}))^{L'} \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}). \tag{2.4.50}$$

With these assumptions

$$\sigma^L \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.4.51}$$

and LU.

Proof. Without loss of generality we may assume that

$$L := \begin{pmatrix} \mathbb{1} & 0 \\ M & \mathbb{1} \end{pmatrix}, \tag{2.4.52}$$

with M an $\mathbb{R}^{(n-l)d} \times \mathbb{R}^{ld}$-matrix.

Also without loss of generality we may assume $U = B(0,1) \times \mathbb{R}^{(n-l)d}$.

Let $A := \bigoplus_{i=1}^k \sigma(M_{n_i+1})$. Denote $v := (v_1, v_2)$ and $x := (x_1, x_2)$ where $v_1, x_1 \in \mathbb{R}^{ld}$ and $v_2, x_2 \in \mathbb{R}^{(n-l)d}$. Then

$$\begin{aligned} \langle v, (A \oplus \mathbb{1})^L(x) v\rangle = \langle v_1, A((L^{-1}x)_1) v_1\rangle + \langle v_1, A((L^{-1}x)_1) M^T v_2\rangle + \\ + \langle M^T v_2, A((L^{-1}x)_1) v_1\rangle + |v_2|^2 =: (*). \end{aligned} \tag{2.4.53}$$

Since $A(x)$ is a symmetric matrix for every x the two middle terms are equal. Moreover, $(L^{-1}x)_1 = x_1$ and thus

$$(*) = \langle v_1, A(x_1) v_1\rangle + 2\langle v_1, A(x_1) M^T v_2\rangle + |v_2|^2 =: (**) \tag{2.4.54}$$

Next, we use induction on $\mathrm{rk}\, 0 = n_1 + ... + n_k$. If $\mathrm{rk}\, 0 = 1$, i.e. $A = \sigma(M_2)$ we have

$$\frac{1}{C}(|v_1|^2 + |v_2|^2) \le (**) \le C(|v_1|^2 + |v_2|^2) \tag{2.4.55}$$

for some $C < \infty$ when $(x_1, x_2) \in \mathbb{S}^{d-1} \times \mathbb{R}^{(n-1)d}$. Adding $(|x_1|^{-\xi} - 1)|v_2|^2$ and multiplying by $|x_1|^\xi$ yields

$$\frac{1}{C}(|x_1|^\xi |v_1|^2 + |v_2|^2) \leq (**) \leq C(|x_1|^\xi |v_1|^2 + |v_2|^2) \tag{2.4.56}$$

when $(x_1, x_2) \in B(0,1) \times \mathbb{R}^{(n-1)d}$. Since $\sigma(M_2) \sim |\cdot|^\xi$ we can conclude our claim.

Next, suppose our Proposition is true for configurations of rank $< l$ and we prove our claim when $\operatorname{rk} 0 = l$. Now cover $\mathbb{S}^{ld-1}$ by finitely many open sets $B_1, ..., B_m$ so that

$$\Big(\bigoplus_{i=1}^{k} \sigma(M_{n_i+1})\Big)^{L_j} \sim \bigoplus_{i=1}^{k_j} \sigma(M_{n_{j,i}+1}) \oplus \mathbb{1} \tag{2.4.57}$$

on B_j with some linear transformation L_j and with $\sum_{i=1}^{k_j} n_{j,i} < l$.

Letting $L'_j := L(L_j \oplus \mathbb{1})$, and applying this Theorem on $U_j := B_j \times \mathbb{R}^{(n-l)d}$ we see that

$$\sigma(M_{n+1})^{L'_j} \sim \bigoplus_{i=1}^{k_j} \sigma(M_{n_{j,i}+1}) \oplus \mathbb{1} \tag{2.4.58}$$

on U_j.

Now a similar argument as above for rank 0 yields the desired conclusion. The reader may fill in the details. □

The following is an immediate corollary to this proposition.

Corollary 2.4.20. *Let $L \in \mathcal{L}$ be such that for some neighbourhood U of x we have*

$$\sigma(M_{n+1})^{L} \sim \bigoplus_{i=1}^{k} \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.4.59}$$

on LU. Then for every $L' \in \mathcal{L}$ such that

$$L^{-1} = L'^{-1} \text{ on } \{|x_i| = 0 : 1 \leq i \leq \operatorname{rk} x\} \tag{2.4.60}$$

we have

$$\sigma(M_{n+1})^{L'} \sim \bigoplus_{i=1}^{k} \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.4.61}$$

on $L'U$.

2.4.5 Some Corollaries

Corollary 2.4.21. *For every $n \geq 2$ there is $C > 0$ such that*

$$Cd(\mathbf{x}, \mathrm{Dgn}(M_n))^{\xi} \leq \sigma(M_n). \tag{2.4.62}$$

The proof of this fact is easy and thus omitted. The assumptions of Theorem 2.2.4 are now satisfied (by Corollary 2.4.21, Theorem A.1.1 and Proposition A.1.3) for M_n. Moreover, we can directly calculate the dimension of M_n:

Corollary 2.4.22. *There is $C < \infty$ such that for any $f \in L^2(\mathbb{R}^{(n-1)d})$ we have*

$$||e^{-M_n t} f||_\infty \leq Ct^{-\frac{(n-1)d}{4-2\xi}} ||f||_2. \tag{2.4.63}$$

Moreover, C depends only on the lower bound for $\sigma(M_n)$.

Proof. By Proposition A.1.3 there is $C < \infty$ so that

$$||f||_q \leq C||d(\mathbf{x}, \mathrm{Dgn}(M_n))^{\xi/2} \nabla f||_2 =: (*) \tag{2.4.64}$$

for any $f \in C_0^\infty(\mathbb{R}^{(n-1)d})$ with $q := \frac{2n}{n+\xi-2}$.
By Corollary 2.4.21 we have

$$(*) \leq C'\langle f, M_n f\rangle. \tag{2.4.65}$$

Finally, by Theorem 2.2.13 we can conclude that (2.4.63) holds. □

Corollary 2.4.23. *For any $\rho \in (0, 1)$ there is $C < \infty$ such that for any $\mathbf{x} \in \mathbb{R}^{(n-1)d}$ and any $\mathbf{y} \notin B(\mathbf{x}, \rho|\mathbf{x}|)$ we have*

$$K_{M_n}(t, \mathbf{x}, \mathbf{y}) \leq Ct^{-\frac{(n-1)d}{2-\xi}} \exp\{-\frac{|\mathbf{x} - \mathbf{y}|^{2-\xi}}{Ct}\} \tag{2.4.66}$$

and

$$G_{M_n}(\mathbf{x}, \mathbf{y}) \leq C|\mathbf{x} - \mathbf{y}|^{2-\xi-(n-1)d}. \tag{2.4.67}$$

Proof. This is a direct consequence of Proposition 2.4.5, Theorem 2.2.12 and Corollary 2.4.22. □

Corollary 2.4.24. *Suppose $A \sim^\lambda \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1}$ on $\mathbb{R}^{ld} \times \mathbb{R}^{(n-l)d}$ with $l := n_1 + ... + n_k < n$ and let $\epsilon > 0$ be given. Then there is $C < \infty$ such that if $z \notin B(y_1, \epsilon|y_1|) \times B(y_2, \epsilon|y_1|^{1-\xi/2})$ (here $y := (y_1, y_2) \in \mathbb{R}^{ld} \times \mathbb{R}^{(n-l)d}$), we have*

$$K_A(t, y, z) \leq Ct^{-\frac{ld}{2-\xi}-\frac{n-l}{2}} \exp\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{Ct}\}. \tag{2.4.68}$$

Moreover C depends on A only through λ, $n_1, ..., n_k$ and n.

Proof. The proof is straightforward using Theorem 2.2.15, Proposition 2.4.5 and Corollary 2.4.22 and we leave the details for the reader. The only finesse is the appearance of $B(y_2, \epsilon|y_1|^{1-\xi/2})$ above. This is due to the fact that if $z_1 \in B(y_1, \epsilon|y_1|)$ and $z_2 \notin B(y_2, \epsilon|y_1|^{1-\xi/2})$, we have

$$|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2 \leq (\epsilon|y_1|)^{2-\xi} + |y_2 - z_2|^2 \\ \leq \epsilon^{-\xi}|y_2 - z_2|^2 + |y_2 - z_2|^2. \tag{2.4.69}$$

□

2.5 Local estimates for the heat kernel

The main result in this section is Theorem 2.5.12. Superficially it is very similar to Corollary 2.4.24, but there is a very important difference: In Corollary 2.4.24 one assumes that

$$A \sim \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1} \tag{2.5.1}$$

in $\mathbb{R}^{nd}$ but in Theorem 2.5.12 $A = \sigma(M_{n+1})$ and (2.5.1) holds only in a relatively compact neighbourhood of a point x. The point of this section is to close the gap between these two results. We start with some technicalities and prove a uniform version of the Harnack inequality adapted to our case.

Remark 2.5.1. In a few places we use the somewhat terse assumption "A has a heat kernel". In these places we assume that A has a heat kernel K such that both $K(\cdot, x, \cdot)$ and $K(\cdot, \cdot, x)$ are solutions to $u_t + Au = 0$ in the sense of Remark 2.2.5 and that for every t and x we have both

$$\int dy\, K(t, x, y) \leq 1 \text{ and } \int dy\, K(t, y, x) \leq 1. \tag{2.5.2}$$

In the cases that are of interest to us (see Remark 2.2.14) this is the case and moreover our heat kernels are symmetric in the spatial coordinates.

A well-known argument (see for example [23], section I.3, page 5) yields the following: Suppose A is a divergence-form operator on $\mathbb{R}^n$ with a nonnegative symbol. Suppose also that A is uniformly elliptic on some ball B and that A has a heat kernel. Then for any ball $B' \subset\subset B$ there is $C < \infty$ such that we have

$$K(t, x, y) \leq Ct^{-n/2} \tag{2.5.3}$$

whenever $t \in (0, 1]$, $x \in B'$ and $y \in \mathbb{R}^d$. We shall now make a generalization (Corollary 2.5.5) of this result.

So for the rest of the section we fix a symbol A on $\mathbb{R}^{nd}$ and suppose that

$$A \sim^{\lambda} \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1} \tag{2.5.4}$$

on $B(0,2) \times B(0,2) \subseteq \mathbb{R}^{ld} \times \mathbb{R}^{(n-l)d}$, where $l := n_1 + ... + n_k$. Let's denote

$$Q := \overline{B}(0,1) \times \overline{B}(0,1) \text{ and } D := \mathbb{S}^{nd-1} \times \overline{B}(0,1). \tag{2.5.5}$$

Proposition 2.5.2. *For each $t \in (0,1]$ there is an open covering $\{U_y^t\}_{y \in Q}$ of Q with the following properties:*

1. *$y \in U_y^t$ for every $y \in Q$ and $t \in (0,1]$.*
2. *There is $\epsilon > 0$ not depending on t such that $B(y_1, \epsilon t^{1/2-\xi}) \times B(y_2, \epsilon\sqrt{t}) \subseteq U_y^t$*
3. *For every $t \in (0,1]$, every $y \in Q$ and every positive solution u of $u_t = \nabla \cdot A\nabla u$ on $(0,3) \times U_y^t$ we have*

$$\sup_{y' \in U_y^t} u(t, y') \leq C \inf_{y' \in U_y^t} u(2t, y'). \tag{2.5.6}$$

Moreover, C depends on A only through λ, $n_1, ..., n_k$ and n.

Remark 2.5.3. Strictly speaking in (3) we only assume u is a solution of $u_t = \nabla \cdot A\nabla u$ in the sense of Remark 2.2.5 on $(\epsilon, 3) \times U_y^t$ for every $\epsilon \in (0,3)$.

Corollary 2.5.4. *Proposition 2.5.2 holds with obvious modifications for any affine transform A^K of A with possibly different ϵ and C.*

To give some intuition to the reader we first give a Corollary to this Proposition.

Corollary 2.5.5. *There is $C < \infty$ such that*

$$K_A(t, y, y') \leq C t^{-\frac{ld}{2-\xi} - \frac{(n-l)d}{2}} \tag{2.5.7}$$

for any $y \in Q$, $y' \in \mathbb{R}^{nd}$ and $t \in (0,1]$.

Proof. By Proposition 2.5.2 for any $y \in Q$ and $y' \in \mathbb{R}^{nd}$ we have

$$\begin{aligned} t^{\frac{ld}{2-\xi} + \frac{(n-l)d}{2}} K_A(t, y, y') &\leq C' |U_y^t| \sup_{y'' \in U_y^t} K_A(t, y'', y') \\ &\leq CC' |U_y^t| \inf_{y'' \in U_y^t} K_A(2t, y'', y') \\ &\leq CC' \int_{U_y^t} K_A(2t, y'', y')\, dy'' \\ &\leq CC'. \end{aligned} \tag{2.5.8}$$

□

Next we prove a small Lemma used in the proof of Proposition 2.5.2. The setup here is the following. Let $y \in D$. In our proof of Proposition 2.5.2 we use induction on rank. By Theorem 2.4.9 there is an invertible affine transformation K_y of $\mathbb{R}^{nd}$ sending y to 0 so that

$$A^{K_y} \sim \sigma(M_{n'_1+1}) \oplus ... \oplus \sigma(M_{n'_k+1}) \oplus \mathbb{1} \tag{2.5.9}$$

on $B(0,2) \times B(0,2)$ with $l' := n'_1 + ... + n'_k < l$. Now Lemma 2.5.6 allows us to conclude that if (2) of Proposition 2.5.2 holds for the covering associated with y in K_y-coordinates with some ϵ (for convenience, we have put this ϵ equal to 1 in the statement of Lemma 2.5.6), then it holds in the usual coordinates of $\mathbb{R}^{nd}$ with some other ϵ.

Here is our choice of the subspaces for Lemma 2.5.6:

1. $S_1 := K_y^{-1}[\mathbb{R}^{l'd} \times \{0\}] - \{y\}$ and
2. $S_2 := K_y^{-1}[\{0\} \times \mathbb{R}^{(n-l'))d}] - \{y\}$.

In other words S_2 is the degeneration subspace associated with y. The fact that $y \in Q$ guarantees that $\{0\} \times \mathbb{R}^{(n-l)d} \subseteq S_2$. Note that the $-\{y\}$ in the definition of S_2 is redundant, since $y \in S_2$, but we didn't want to confuse the reader a few lines ago, did we?

Lemma 2.5.6. *Let S_1, S_2 be a splitting of $\mathbb{R}^{nd}$ into complementary subspaces so that $\{0\} \times \mathbb{R}^{(n-l)d} \subseteq S_2$. Assume also that each of them is equipped with a norm and denote the balls with respect to these norms with $B_i(x,r)$ with $i = 1,2$. Then there is $\epsilon > 0$ so that*

$$B(0, \epsilon t^{1/(2-\xi)}) \times B(0, \epsilon\sqrt{t}) \subseteq B_1(0, t^{1/(2-\xi)}) \times B_2(0, \sqrt{t}) \tag{2.5.10}$$

for any $t \in (0,1]$.

Proof. Obviously there is $\epsilon > 0$ so that

$$B(0,\epsilon) \times B(0,\epsilon) \subseteq B_1(0,1) \times B_2(0,1) \tag{2.5.11}$$

Let us write $B(0, \epsilon t^{1/(2-\xi)}) \times B(0, \epsilon\sqrt{t})$ as

$$B(0, \epsilon t^{\frac{1}{2-\xi}}) \times \mathbb{R}^{(n-l)d} \cap B(0, \epsilon\sqrt{t}) \times B(0, \epsilon\sqrt{t}) \tag{2.5.12}$$

and similarly for $B_1(0, t^{1/(2-\xi)}) \times B_2(0, \sqrt{t})$ (we used the fact that $t^{1/(2-\xi)} \leq \sqrt{t}$ for $t \in (0,1]$).

Now since $\{0\} \times \mathbb{R}^{(n-l)d} \subseteq S_2$, we conclude by scaling that

$$B(0, \epsilon t^{\frac{1}{2-\xi}}) \times \mathbb{R}^{(n-l)d} \subseteq B_1(0, t^{\frac{1}{2-\xi}}) \times S_2. \tag{2.5.13}$$

for any $t > 0$.

Also by scaling we get

$$B(0, \epsilon\sqrt{t}) \times B(0, \epsilon\sqrt{t}) \subseteq B_1(0, \sqrt{t}) \times B_2(0, \sqrt{t}). \tag{2.5.14}$$

for any $t > 0$. □

Proof. (of Proposition 2.5.2)

If $l = 0$, then we just choose $U_y^t := B(y, \sqrt{t})$. Obviously, these sets satisfy (2) above and by classical results (see again [23], section I.3, page 5) they satisfy (3) too.

Next we assume that the cases $< l$ have been handled and prove the Proposition for l. This is done in three phases:

1. Phase 1: Use our induction hypothesis (i.e. that the cases $< l$ have been handled) to handle points in D.
2. Phase 2: Use scaling to handle points $z \in Q$ with $0 < |z_1| < 1$ and times $t \in (0, |z_1|^{2-\xi}]$. And finally
3. Phase 3: Do something creative for points $z \in Q$ and times $t \in (|z_1|^{2-\xi}, 1]$. Note that this includes defining the sets U_z^t when $|z_1| = 0$.

First, phase 1: By compactness, there is $\{y_1, ..., y_k\} \subseteq D$ so that $\{K_{y_i}^{-1}[B(0,1) \times B(0,1)]\}_{i=1}^k$ cover D. Obviously each y_i is of rank $< l$. For each $t \in (0, 1]$ and $z \in D$ pick U_z^t to be one of the U_z^t's associated with some of the $y_1, ..., y_k$ (this is possible by induction hypothesis and Corollary 2.5.4). Now these U_z^t's satisfy (2) and (3), where (3) satisfied by induction and (2) is satisfied by Lemma 2.5.6 (and the discussion before it) and finiteness of the set $\{y_1, ..., y_k\}$.

Next, phase 2: We define the sets U_z^t for z's with $0 < |z_1| < 1$ and $t \in (0, |z_1|^{2-\xi}]$. This is achieved by scaling A outwards so that in this scaling z travels to D. Then the symbol A^z obtained this way has the same upper and lower bounds as A on $B(0,2) \times B(0,2)$, so we can use our sets U_y^t defined above for $y \in D$. After this we just scale things back.

So, let $z \in Q$ with $0 < |z_1| < 1$ and let

$$y^z := (y_1/|z_1|, z_2 + (y_2 - z_2)/|z_1|^{1-\xi/2}). \tag{2.5.15}$$

Let A^z be defined by

$$A_{ij}^z(y) := \begin{cases} |z_1|^{\xi} A_{ij}(y^z) & \text{if } 1 \le i, j \le ld \\ |z_1|^{\xi/2} A_{ij}(y^z) & \text{if } 1 \le i \le ld < j \le nd \text{ or} \\ & 1 \le j \le ld < i \le nd \\ \sigma(A_{ij}(y^z) & \text{if } ld < i, j \le nd \end{cases} \tag{2.5.16}$$

Similarly define u^z by $u^z(t,y) := u(|z_1|^{\xi-2}t, y^z)$. Now if u satisfies $u_t = \nabla A \cdot \nabla u$ on $(0,3) \times B(0,2) \times B(0,2)$, then u^z satisfies $u^z_t = \nabla \cdot A^z \nabla u^z$ on this same set. Since now if $A \sim^\lambda \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1}$ on $B(0,2) \times B(0,2)$, then the same is true of A^z we can conclude that (2) and (3) hold for A^z with the same constants as for A. So if we scale back and let

$$U^t_z = \{(|z_1|y_1, z_2 + |z_1|^{1-\xi/2}(y_2 - z_2)) : (y_1, y_2) \in U_{\hat{z}}^{|z_1|^{\xi-2}t}\} \tag{2.5.17}$$

then (2) and (3) hold for these whenever defined.

Finally, phase 3: To finish the argument, we set for $t \geq |z_1|^{2-\xi}$

$$U^t_z = B(0, \frac{3}{2}t^{1/(2-\xi)}) \times B(z_2, \frac{1}{2}\sqrt{t}). \tag{2.5.18}$$

Now (2) holds for these sets. To prove (3) we may assume without loss of generality that $z_2 = 0$ and let A^t be defined as follows:

$$A^t_{ij}(y_1, y_2) := \begin{cases} t^{-\frac{\xi}{2-\xi}} A_{ij}(y_1 t^{1/(2-\xi)}, y_2\sqrt{t}) & \text{if } 1 \leq i,j \leq ld \\ t^{-\frac{\xi}{4-2\xi}} A_{ij}(y_1 t^{1/(2-\xi)}, y_2\sqrt{t}) & \text{if } 1 \leq i \leq ld < j \leq nd \text{ or} \\ & 1 \leq j \leq ld < i \leq nd \\ A_{ij}(y_1 t^{1/(2-\xi)}, y_2\sqrt{t}) & \text{if } ld < i,j \leq nd \end{cases} \tag{2.5.19}$$

As before, for $t \in (0,1]$ the substitution $A \mapsto A^t$ preserves the constant in the Harnack inequality (Theorem 2.2.4) and thus we can conclude that (3) holds. □

Remark 2.5.7. It is not hard to modify the previous proof so that for given $\epsilon' > 0$ there is $\epsilon > 0$ so that

1. $B(y_1, \epsilon t^{1/(2-\xi)}) \times B(y_2, \epsilon\sqrt{t}) \subseteq U^t_y$ for every $t \in (0,1]$ and
2. $U^t_y \subseteq B(y_1, \epsilon' t^{1/(2-\xi)}) \times B(y_2, \epsilon'\sqrt{t})$, when $|y_1|^{2-\xi} \leq t \leq 1$.
3. $U^t_y \subseteq B(y_1, \epsilon'|y_1|) \times B(y_2, \epsilon'|y_1|^{(2-\xi)/2})$, when $0 < t \leq |y_1|^{2-\xi}$.

We need (2) and (3) in the proof of Theorem 2.5.12. There we need to find $\epsilon' > 0$ so that U^t_z and $B(y_1, \epsilon' t^{1/(2-\xi)}) \times B(y_2, \epsilon'\sqrt{t})$ are disjoint whenever $z \notin B(y_1, t^{1/(2-\xi)}) \times B(y_2, \sqrt{t})$ and this is hard to arrange if we don't have any kind of control over the U^t_z's from outside. This required control is provided by (2) and (3) above. The actual choice of $\epsilon' > 0$ is done in Lemma 2.5.10.

Anyway, it is quite easy to make (2) and (3) hold. First of all, it is easy to see that (2) and (3) hold with some $\epsilon'_0 > 0$ when U^t_y's are defined as in the proof of Proposition 2.5.2. By letting $V^t_y := U^{t/T}_y$ with $T := (\epsilon'_0/\epsilon')^2$ we see that V^t_y's for $t \in (0,1]$ satisfy (1)-(3) above together with the claims of Proposition 2.5.2. The details are left to the reader. We will use Proposition 2.5.2 in this form in the proofs below.

We now have to estimate the tails of the heat kernel. We use a common probabilistic argument for this (killing probabilities). Denote

$$d(x,y)^2 := \max\{|x_1 - y_1|^{2-\xi}, |x_2 - y_2|^2\}. \tag{2.5.20}$$

Obviously there is $C < \infty$ so that

$$C^{-1}d(x,y) \leq \sqrt{|x_1 - y_1|^{2-\xi} + |x_2 - y_2|^2} \leq Cd(x,y) \tag{2.5.21}$$

Below, $P_A^y(\sup_{s\leq t} d(X_s, y) \geq \mu)$ denotes the probability of the diffusion X associated with A starting from y at time 0 hitting the set $\{z : d(y,z) = \mu\}$ before time t.

The following is Proposition 6.5 on page 179 of [1].

Proposition 2.5.8. *Suppose $A \sim^\lambda \mathbb{1}$ on $\mathbb{R}^l$. There is $C < \infty$ depending on A only through λ such that*

$$\mathbb{P}_A^y(\sup_{s\leq t} |X_s - y| \geq \mu) \leq C \exp\{-\frac{\mu^2}{Ct}\}. \tag{2.5.22}$$

Corollary 2.5.9. *Suppose $A \sim^\lambda \mathbb{1}$ on $B(0,2) \subseteq \mathbb{R}^{nd}$. Then there is $C < \infty$ depending on A only through λ such that for every $y \in B(0,1)$, $z \in B(y, \frac{1}{2})$ and $0 < t \leq 1$ we have*

$$K_A(t,y,z) \leq Ct^{-\frac{nd}{2}} \exp\{-\frac{|y-z|^2}{Ct}\} \tag{2.5.23}$$

The proof of this Corollary is quite simple and well-known (folklore) and we shall not prove it here, but the interested reader can reconstruct the argument from the proof of Theorem 2.5.12 which is a generalization of Corollary 2.5.9.

Unfortunately we need the following technicality in the proofs of Proposition 2.5.11 and Theorem 2.5.12.

Lemma 2.5.10. *Suppose $\epsilon'' > 0$ is given. Then there is $\epsilon' > 0$ so that if $d(y,z) \geq \epsilon''|y_1|^{1-\xi/2}$, we have*

$$\{z' : d(z,z') \leq \epsilon'|z_1|^{1-\xi/2}\} \subseteq \{z' : d(z,z') \leq \frac{d(y,z)}{2}\} \tag{2.5.24}$$

and

$$B(y_1, \epsilon'|y_1|) \times B(y_2, \epsilon'|y_1|^{1-\xi/2}) \cap B(z_1, \epsilon'|z_1|) \times B(z_2, \epsilon'|z_1|^{1-\xi/2}) = \emptyset. \tag{2.5.25}$$

Proof. Let

$$\alpha := \frac{d(y,z)^{2/(2-\xi)}}{|y_1|}, \tag{2.5.26}$$

Then we have

$$|z_1| \leq |y_1| + |y_1 - z_1| \leq |y_1| + d(y,z)^{2/(2-\xi)} \leq (1+\alpha)|y_1|. \tag{2.5.27}$$

So to prove (2.5.24), we just have to find $\epsilon' > 0$ so that

$$\epsilon'((1+\alpha)|y_1|)^{1-\xi/2} \leq \frac{1}{2}(\alpha|y_1|)^{1-\xi/2}, \tag{2.5.28}$$

whenever $\alpha \geq (\epsilon'')^{2/(2-\xi)}$. By elementary calculus, we see that this is possible. Using similar reasoning, we see that to prove (2.5.25) we have to find $\epsilon' > 0$ so that

1. $\epsilon'|y_1| + \epsilon'(1+\alpha)|y_1| \leq \alpha|y_1|$ and
2. $\epsilon'|y_1|^{1-\xi/2} + \epsilon'((1+\alpha)|y_1|)^{1-\xi/2} \leq (\alpha|y_1|)^{1-\xi/2}$,

when $\alpha \geq (\epsilon'')^{2/(2-\xi)}$. Again, this is possible. □

Proposition 2.5.11. *Suppose $A \sim^\lambda \sigma(M_{n_1}) \oplus ... \oplus \sigma(M_{n_k}) \oplus \mathbb{1}$ on $\mathbb{R}^{ld+(n-l)d}$ with $\sum_{i=1}^k (n_i - 1) = l$ and let $\epsilon'' > 0$ be given. Then there is $C < \infty$ such that for $\mu \geq \epsilon''|y_1|^{1-\xi/2}$ we have*

$$\mathbb{P}_A^y(\sup_{s\leq t} d(X_s, y) \geq \mu) \leq C \exp\{-\frac{\mu^2}{Ct}\}. \tag{2.5.29}$$

Proof. Let $\epsilon' > 0$ be given by Lemma 2.5.10. By Corollary 2.4.24, there is $C_1 < \infty$ so that if $d(y,z) \geq \epsilon'|y_1|^{1-\xi/2}$ we have

$$K_A(t,y,z) \leq C_1 t^{-\frac{ld}{2-\xi} - \frac{(n-l)d}{2}} \exp\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{C_1 t}\}. \tag{2.5.30}$$

Now a direct computation gives

$$\begin{aligned}
\mathbb{P}_A^y(\sup_{s\leq t} d(X_s, y) \geq \mu) &\leq \mathbb{P}_A^y(d(X_t,y) \geq \mu/2) \\
&\quad + \mathbb{P}_A^y(d(X_t,y) \leq \mu/2 \text{ and } \exists s < t : d(X_s,y) = \mu) \\
&\leq \mathbb{P}_A^y(d(X_t,y) \geq \mu/2) \\
&\quad + \mathbb{P}_A^y(\exists s < t : d(X_s,s) = \mu \text{ and } d(X_s,X_t) \geq \mu/2) \\
&\leq \mathbb{P}_A^y(d(X_t,y) \geq \mu/2) + \sup_{d(y,z)=\mu, s\leq t} \mathbb{P}_A^z(d(X_s,z) \geq \mu/2) \\
&= (*).
\end{aligned} \tag{2.5.31}$$

By (2.5.24) of Lemma 2.5.10, for every $z \in \mathbb{R}^{nd}$ with $d(y,z) = \mu$ we have

$$\{z' : d(z,z') \leq \epsilon' |z_1|^{1-\xi/2}\} \subseteq \{z' : d(z,z') \leq \frac{\mu}{2}\}. \tag{2.5.32}$$

A fortiori we also have

$$\{z' : d(y,z') \leq \epsilon' |y_1|^{1-\xi/2}\} \subseteq \{z' : d(y,z') \leq \frac{\mu}{2}\}, \tag{2.5.33}$$

since there are points $z \in \mathbb{R}^d$ with $d(y,z) = \mu$ and $|z_1| \geq |y_1|$.

Thus by (2.5.30) we can conclude that

$$\begin{aligned}
(*) &\leq C_2 \int_{d(y,z)\geq \mu/2} t^{-\frac{ld}{2-\xi}-\frac{(n-l)d}{2}} \exp\{-\frac{|y_1-z_1|^{2-\xi}+|y_2-z_2|^2}{C_1 t}\}\, dy \\
&\leq C_3 \int_{|y_1-z_1|^{2-\xi}\geq \mu^2} t^{-\frac{ld}{2-\xi}} \exp\{-\frac{|y_1-z_1|^{2-\xi}}{C_1 t}\}\, dy_1 \\
&+ C_3 \int_{|y_2-z_2|\geq \mu} t^{-\frac{(n-l)d}{2}} \exp\{-\frac{|y_2-z_2|^2}{C_1 t}\}\, dy_2 \\
&\leq C \exp\{-\frac{\mu^2}{Ct}\}.
\end{aligned} \tag{2.5.34}$$

□

Now we can finish with the local estimates.

Theorem 2.5.12. *Suppose that $A \sim^{\lambda} \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1}$ on $B(0,2) \times B(0,2)$ with $l := n_1 + ... + n_k < n$ and that A has a heat kernel. For any $\epsilon'' \in (0,1]$ there is $C < \infty$ so that if $y \in Q$, $0 < t \leq 1$ and $\epsilon'' |y_1|^{1-\xi/2} \leq d(z,y) \leq \frac{1}{2}$ we have*

$$K_{M_{n+1}}(t,y,z) \leq C t^{-ld/(2-\xi)-(n-l)d/2} \exp\{\frac{|y_1-z_1|^{2-\xi}+|y_2-z_2|^2}{Ct}\}. \tag{2.5.35}$$

Moreover, this estimate depends on A only through λ, $n_1, ..., n_k$ and n.

Proof. If $0 < d(z,y)^2 \leq t$, then there is $C < \infty$ so that

$$1 \leq C \exp\{-\frac{|y_1-z_1|^{2-\xi}+|y_2-z_2|^2}{Ct}\}. \tag{2.5.36}$$

Thus in view of Corollary 2.5.5 we only need to prove the claim for $t \leq d(z,y)^2 \leq 1$.

Let $\epsilon' > 0$ be given by Lemma 2.5.10 and let $\{U_t^y\}$ be a collection of open coverings given by Proposition 2.5.2 and Remark 2.5.7 associated with this ϵ'. We may assume $\epsilon' \leq \min\{\frac{1}{2}, \frac{1}{2^{\xi}}\}$.

We want to show that U_z^t and $B(y_1, \epsilon' t^{1/(2-\xi)}) \times B(y_2, \epsilon'\sqrt{t})$ are disjoint whenever $z \notin B(y_1, t^{1/(2-\xi)}) \times B(y_2, \sqrt{t})$. The case $|y_1|^{2-\xi} \leq t \leq 1$ follows easily, since we assumed $\epsilon' \leq \min\{\frac{1}{2}, \frac{1}{2^\xi}\}$. In case $0 < t \leq |y_1|^{2-\xi}$ we just use Lemma 2.5.10 to conclude that

$$B(y_1, \epsilon' t^{1/(2-\xi)}) \times B(y_2, \epsilon'\sqrt{t}) \cap B(z_1, \epsilon'|z_1|) \times B(z_2, \epsilon'|z_1|^{(2-\xi)/2}) = \emptyset, \tag{2.5.37}$$

whenever $d(y,z) \geq \epsilon''|y_1|^{1-\xi/2}$.

By the proof of Corollary 2.5.5 we have

$$\begin{aligned} &t^{\frac{ld}{2-\xi}+\frac{(n-l)d}{2}} \sup_{z'\in U_z^t} K_{M_{n+1}}(t,y,z') \\ &\leq C_2 \int_{U_z^t} dy'\, K_{M_{n+1}}(2t,x,y'). \end{aligned} \tag{2.5.38}$$

By Proposition 2.5.11 we have

$$\int_{U_z^t} K_{M_{n+1}}(2t,y,z) \leq C_3 \exp\{-\frac{|y_1-z_1|^{2-\xi}+|y_2-z_2|^2}{C_3 t}\}, \tag{2.5.39}$$

so we are done. □

2.6 Construction of the stationary state

In this section, we shall prove Theorem 2.1.1 modulo some technicalities whose proofs are postponed until Appendix A.3 and §2.7. To this end, we shall inductively show the following

Theorem 2.6.1. *Let $\chi : \mathbb{R}^d \to \mathbb{R}$ be compactly supported and nonnegative. Then for some $C_n < \infty$ we have*

$$M_{2n}^{-1}(M_{2n-2}^{-1}(\ldots(M_2^{-1}\chi \otimes \chi)\ldots) \otimes \chi) \leq C_n \prod_{i=1}^n (1+|x_{2i-1}|)^{2-\xi-d}. \tag{2.6.1}$$

Obviously Theorem 2.1.1 follows directly from this.

The following formula is a central tool in this section.

Proposition 2.6.2. *Let $1 \leq l \in \mathbb{N}$. Then*

$$\int_{\mathbb{R}^{ld}} d^{ld}y\, |x-y|^{2-\xi-ld} \prod_{i=1}^{l-1}(1+|y_i|)^{2-\xi-d}\chi(y_l) \leq C \prod_{i=1}^{l}(1+|x_i|)^{2-\xi-d}. \tag{2.6.2}$$

The proof of this Proposition can be found in Appendix A.3.

Let $\mathcal{S}$ be the smallest collection of subspaces of $\mathbb{R}^{nd}$ that contains

$$\{\{y_{i,j} = 0\} \subseteq \mathbb{R}^{nd} : 1 \le i \le j \le l\} \tag{2.6.3}$$

and is closed under nonempty pairwise intersections.

The following Proposition whose proof is postponed until §2.7 is the key to proving Theorem 2.6.1. Unfortunately the statement of Proposition 2.6.3 is somewhat messy, but hopefully the following comments will help the reader to comprehend the point.

The sets A_j^x below in the statement of Proposition 2.6.3 correspond to a splitting of the domain of integration into parts, i.e. since we want to show that

$$\int_{\mathbb{R}^{(2n-1)d}} G_{M_{2n}}(x,y) \prod_{i=1}^{n-1} (1+|y_{2i-1}|)^{2-\xi-d} \chi(y_{2n-1})\, dy \le C \prod_{i=1}^{n} (1+|x_i|)^{2-\xi-d}, \tag{2.6.4}$$

we shall write the above integral as

$$\int_{\mathbb{R}^{(2n-1)d}} = \int_{|x-y|\ge\rho|x|} + \sum_{j=0}^{k(x)} \int_{A_j^x \setminus A_{j-1}^x} \tag{2.6.5}$$

and then prove the desired estimate of (2.6.4) separately for each term of the right-hand side. During our construction of the sets A_j^x we will take care that $k(x) \le n$ so that the number of terms in the RHS of the equation above will not blow up.

The first integral can be handled in a straightworward manner using Corollary 2.4.23. We would like to take $A_j^x := B(x, r_j^x)$ with suitably chosen r_j^x's. This is not possible, but if it were, then we could handle A_j^x's with $r_j^x \le 1$ easily: By (3) of Proposition 2.6.3 below we would have

$$\begin{aligned} &\int_{A_j^x \setminus A_{j-1}^x} G_{M_{2n}}(x,y) \prod_{i=1}^{n-1} (1+|y_{2i-1}|)^{2-\xi-d} \chi(y_{2n-1})\, dy \\ &\quad \le C \sup_{y\in B(x,1)} \prod_{i=1}^{n} (1+|y_{2i-1}|)^{2-\xi-d} \int_{A_j^x \setminus A_{j-1}^x} G_{M_{2n}}(x,y)\, dy \\ &\quad \le C' \prod_{i=1}^{n} (1+|x_{2i-1}|)^{2-\xi-d}. \end{aligned} \tag{2.6.6}$$

This would leave us with the problem of handling A_j^x with $r_j^x \in (1, \rho|x|)$. The estimate of (2) in Proposition 2.6.3 is sufficient for this purpose as will be seen in the proof of Theorem 2.6.1. By the way, our argument for $r_j^x \in (1, \rho|x|)$ does not work down to 0 (see (2.6.21)), so we have to do something different for $r_j^x \in (0,1)$ anyway.

Unfortunately life is not so simple and our sets A_j^x will not be balls, but instead they can look like very flat ellipsoids, so our simple-minded argument in (2.6.6) does not work directly. The reason for this will become obvious when we define A_j^x's in §2.7, but one manifestation of this problem can be seen from the statement of Theorem 2.5.12: The estimate there holds in the set $\{z : \epsilon''|y_1|^{1-\xi/2} \le d(z,y) \le \frac{1}{2}\}$ (d was defined in (2.5.20)), which might be our $A_k^y \setminus A_{k-1}^y$. Morally then A_{k-1}^y would be $\{z : d(z,y) \le \epsilon''|y_1|^{1-\xi/2}\} = B(z_1, (\epsilon'')^{2/(2-\xi)}|y_1|) \times B(z_2, \epsilon''|y_1|^{1-\xi/2})$. If we divide the latter radius by former, then this ratio diverges as $|y_1| \to 0$. Although spiced with lots of handwaving, this could be seen as the heart of the matter.

Fortunately the orientation of these sets is such that a somewhat similar argument can be carried through; this is the point of (4) and (5) below.

Proposition 2.6.3. *There is $\rho \in (0,1)$ and $C < \infty$ such that for every $x \in \mathbb{R}^{nd} \setminus \{0\}$ there is a finite sequence $x \in A_1^x \subset \ldots \subset A_{k(x)}^x = B(x, \rho|x|)$ of sets with $k(x) \le n$ satisfying the following*

1. *Each of these sets A_j^x has an associated size r_j^x, non-trivial (possibly improper) subspace $S_j^x \in \mathcal{S}$ and a linear transformation $L_j^x \in \mathcal{L}'_{n+1}$ ($\mathcal{L}'_{n+1}$ was defined in (2.4.48)). The relation between S_j^x and L_j^x is the following:*

$$S_j^x = (L_j^x)^{-1}\{y_i = 0 : 1 \le i \le \frac{\operatorname{codim}(S_j^x)}{d}\} \tag{2.6.7}$$

2. *For every $x \in \mathbb{R}^{nd}$ and $j \in \{1, ..., k(x)\}$, we have*

$$\begin{aligned} G_{M_{n+1}}(x,y) \le & C\Big(\prod_{i=l+1}^{n} a_i^{-\frac{\xi d}{2}}\Big)\Big(\sum_{i=1}^{l} |(L_j^x x)_i - (L_j^x y)_i|^{2-\xi} + \\ & + \sum_{i=l+1}^{n} a_i^{-\xi}|(L_j^x x)_i - (L_j^x y)_i|^2\Big)^{1-\frac{ld}{2-\xi}-\frac{(n-l)d}{2}} \end{aligned} \tag{2.6.8}$$

whenever $y \in A_j^x \setminus A_{j-1}^x$ with some positive numbers a_i depending on x and j and

$$l := \frac{\operatorname{codim}(S_j^x)}{d}. \tag{2.6.9}$$

3. *We have*

$$\int_{A_j^x \setminus A_{j-1}^x} d^{nd}y\, G_{M_{n+1}}(x,y) \le C(r_j^x)^{2-\xi} \tag{2.6.10}$$

4. *If* $S_j^x \subseteq \{y_i = 0\}$, *then* $|x_i| \leq r_j^x$.

5. *If* $S_j^x \not\subseteq \{y_i = 0\}$, *then* $A_j^x \subseteq \{|y_i| \geq C^{-1}|x_i|\}$.

Proof. (of Theorem 2.6.1 using Proposition 2.6.3). The proof is by induction, i.e. we show that

$$M_{2n}^{-1}(\prod_{i=1}^{n-1}(1+|x_{2i-1}|)^{2-\xi-d}\chi(x_{2n-1})) \leq C\prod_{i=1}^{n}(1+|x_{2i-1}|)^{2-\xi-d} \tag{2.6.11}$$

Without loss of generality, we may assume that the support of χ is contained in the ball $B(0, C^{-1})$, where C is from Proposition 2.6.3.

As discussed briefly before the statement of Proposition 2.6.3, our proof goes as follows: First we split the domain of integration into parts as in (2.6.5) and then we proceed in three phases:

1. Phase 1: Handle the integral $\int_{|x-y|\geq\rho|x|}$.
2. Phase 2: Handle the integrals $\int_{A_j^x \setminus A_{j-1}^x}$ with $r_j^x \leq 1$ and
3. Phase 3: Handle the integrals $\int_{A_j^x \setminus A_{j-1}^x}$ with $r_j^x \geq 1$.

First, phase 1: Let $\rho \in (0,1)$ be given by Proposition 2.6.3. We know by Corollary 2.4.23 that for $|x-y| \geq \rho|x|$ we have

$$G_{M_{2n}}(x,y) \leq C_1(\sum_{i=1}^{2n-1}|x_i - y_i|)^{2-\xi-2(n-1)d}, \tag{2.6.12}$$

so we can conclude that

$$\begin{aligned} &\int_{|x-y|\geq\rho|x|} d^{(2n-1)d}y\, G_{M_{2n}}(x,y)\prod_{i=1}^{n-1}(1+|y_{2i-1}|)^{2-\xi-d}\chi(|y_{2n-1}|) \\ &\leq C_1\int_{\mathbb{R}^{(2n-1)d}} d^{(2n-1)d}y\,(\sum_{i=1}^{2n-1}|x_i-y_i|)^{2-\xi-(2n-1)d}. \\ &\cdot\prod_{i=1}^{n-1}(1+|y_{2i-1}|)^{2-\xi-d}\chi(|y_{2n-1}|) =: (*) \end{aligned} \tag{2.6.13}$$

By a change of variables we see that

$$\begin{aligned}
&\int_{\mathbb{R}^{(n-1)d}} \prod_{i=1}^{n-1} d^d y_{2i} \, (\sum_{i=1}^{2n-1} |x_i - y_i|)^{2-\xi-(2n-1)d} \\
&= (\sum_{i=1}^{n} |x_{2i-1} - y_{2i-1}|)^{2-\xi-nd} \int_{\mathbb{R}^{(n-1)d}} \prod_{i=1}^{n-1} d^d z_{2i} \cdot \\
&\cdot (1 + \sum_{i=1}^{n-1} |z_{2_i}|)^{2-\xi-(2n-1)d},
\end{aligned} \tag{2.6.14}$$

so we can conclude that

$$\begin{aligned}
(*) &\leq C_1 \int_{\mathbb{R}^{nd}} \prod_{i=1}^{n} d^d y_{2i-1} \, (\sum_{i=1}^{n} |x_{2i-1} - y_{2i-1}|)^{2-\xi-nd} \cdot \\
&\cdot \prod_{i=1}^{n-1} (1 + |y_{2i-1}|)^{2-\xi-d} \chi(|y_{2n-1}|) =: (*^2).
\end{aligned} \tag{2.6.15}$$

By Proposition 2.6.2, we have

$$(*^2) \leq C_2 \prod_{i=1}^{n} (1 + |x_{2i-1}|)^{2-\xi-d}. \tag{2.6.16}$$

Next, some initial preparation for phases 2 and 3: Let x and $j \leq k(x)$ be given and let $U := \{1, 3, ..., 2n-1\}$, $U_1 := \{i \in U : S_j^x \subseteq \{y_i = 0\}\}$ and $U_2 = U \setminus U_1$. Then, phase 2: so suppose $r_j^x \leq 1$. Then by (3) of Proposition 2.6.3 we have

$$\begin{aligned}
&\int_{y \in A_j^x \setminus A_{j-1}^x} d^{(n-1)d} y \, G_{M_{2n}}(x, y) \prod_{i=1}^{n-1} (1 + |y_{2i-1}|)^{2-\xi-d} \chi(|y_{2n-1}|) \\
&\leq C_3 \sup_{y \in A_j^x \setminus A_{j-1}^x} \prod_{i=1}^{n} (1 + |y_{2n-i}|)^{2-\xi-d} =: (*^3)
\end{aligned} \tag{2.6.17}$$

Since $(1 + |y_i|)^{2-\xi-d} \leq 1$ for any $i \in U$, we can conclude that

$$(*^3) \leq C_3 \sup_{y \in A_j^x \setminus A_{j-1}^x} \prod_{i \in U_2} (1 + |y_i|)^{2-\xi-d} := (*^4) \tag{2.6.18}$$

By (5) of Proposition 2.6.3 we have

$$(*^4) \leq C_4 \prod_{i \in U_2} (1 + |x_i|)^{2-\xi-d} =: (*^5) \tag{2.6.19}$$

Since by (4) of Proposition 2.6.3 we have $|x_i| \leq 1$ for $i \in U_1$ we can finally conclude that

$$(*^5) \leq C_5 \prod_{i=1}^{n} (1+|x_{2i-1}|)^{2-\xi-d}. \tag{2.6.20}$$

Finally, phase 3: If $r_j^x \geq 1$ and $2n-1 \in U_2$, then

$$(\mathbb{R}^{(2n-2)d} \times \text{supp } \chi) \cap A_j^x = \emptyset \tag{2.6.21}$$

whenever $|x_i| \geq 1$ by (5) of Proposition 2.6.3 and the fact that supp $\chi \subseteq B(0, C^{-1})$ and thus in this case we have

$$\int_{y \in A_j^x} d^{(n-1)d} y \, G_{M_{2n}}(x,y) \prod_{i=1}^{n-1} (1+|y_{2i-1}|)^{2-\xi-d} \chi(|y_{2n-1}|) = 0. \tag{2.6.22}$$

So we may assume $2n-1 \in U_1$. By (5) of Proposition 2.6.3 we have

$$(1+|y_i|)^{2-\xi-d} \leq C_6 (1+|x_i|)^{2-\xi-d} \tag{2.6.23}$$

for $i \in U_2$. Therefore

$$\begin{aligned} &\int_{y \in A_j^x \setminus A_{j-1}^x} d^{(2n-1)d} y \, G_{M_{2n}}(x,y) \prod_{i=1}^{n} (1+|y_{2i-1}|)^{2-\xi-d} \\ &\leq C_7 \prod_{i \in U_2} (1+|x_i|)^{2-\xi-d} \int_{y \in A_j^x \setminus A_{j-1}^x} d^{(2n-1)d} y \, G_{M_{2n}}(x-y) \cdot \\ &\cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1+|y_i|)^{2-\xi-d} \chi(y_{2n-1}) = (*^6). \end{aligned} \tag{2.6.24}$$

Writing $x' := L_j^x x$, $y' := L_j^x y$ and $l' := 2n-1-l$ we get

$$\begin{aligned} &\int_{y \in A_j^x \setminus A_{j-1}^x} d^{(2n-1)d} y \, G_{M_{2n}}(x-y) \prod_{i \in U_1 \setminus \{2n-1\}} (1+|y_i|)^{2-\xi-d} \chi(y_{2n-1}) \\ &\leq C_8 \int_{R^{(2n-1)d}} d^{(2n-1)d} y' \, \Big(\prod_{i=l+1}^{n} a_i^{-\frac{\xi d}{2}} \Big) \Big(\sum_{i=1}^{l} |x_i' - y_i'|^{2-\xi} + \\ &+ \sum_{i=l+1}^{n} a_i^{-\xi} |x_i' - y_i'|^2 \Big)^{1-\frac{ld}{2-\xi}-\frac{l'd}{2}} \cdot \\ &\cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1+|y_i|)^{2-\xi-d} \chi(y_{2n-1}) = (*^7), \end{aligned} \tag{2.6.25}$$

Note that since for U_1 is the set of those i such that $S_j^x \subseteq \{y_i = 0\}$, we have that in the y_i'-coordinates the expression

$$\prod_{i \in U_1 \setminus \{2n-1\}} (1 + |y_i|)^{2-\xi-d} \chi(y_{2n-1}) \tag{2.6.26}$$

depends only on the variables $y_1', ..., y_l'$.

Using a similar change of variables as in (2.6.14), we get

$$\begin{aligned}(*^7) \leq C_8 \int_{\mathbb{R}^{ld}} \prod_{i=1}^{l} d^d y_i' \, (\sum_{i=1}^{l} |x_i' - y_i'|^{2-\xi})^{1-\frac{ld}{2-\xi}} \cdot \\ \cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1 + |y_i|)^{2-\xi-d} \chi(y_{2n-1}) \int_{\mathbb{R}^{l'd}} \prod_{i=l+1}^{2n-1} d^d y_i' \cdot \\ \cdot \, (\prod_{i=l+1}^{2n-1} a_i^{-\frac{\xi d}{2}})(1 + \sum_{i=l+1}^{2n-1} a_i^{-\xi} |x_i' - y_i'|^2)^{1-\frac{ld}{2-\xi}-\frac{l'd}{2}} = (*^8).\end{aligned} \tag{2.6.27}$$

By substituting $y_i'' = a_i^{-\xi/2} y_i'$ we see that the last integral is $\leq C_9$.

Therefore

$$\begin{aligned}(*^8) \leq C_{10} \int_{\mathbb{R}^{ld}} \prod_{i=1}^{l} d^d y_i' \, (\sum_{i=1}^{l} |x_i' - y_i'|^{2-\xi})^{1-\frac{ld}{2-\xi}} \cdot \\ \cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1 + |y_i|)^{2-\xi-d} \chi(y_{2n-1}) =: (*^9).\end{aligned} \tag{2.6.28}$$

Noticing that $\sum_{i=1}^{l} |x_i' - y_i'|^{2-\xi}$ is essentially just $|(x_1' - y_1', ..., x_l' - y_l')|^{2-\xi}$ for estimation purposes, using a similar change-of-variables argument as before, we can conclude that

$$(*^9) \leq C_{11} \prod_{i \in U_1} (1 + |x_i|)^{2-\xi-d}, \tag{2.6.29}$$

so

$$(*^6) \leq C_{12} \prod_{i=1}^{n} (1 + |x_{2i-1}|)^{2-\xi-d}. \tag{2.6.30}$$

□

2.7 Uniform local integrability

To give the reader an idea what the sets A_j^x in the statement of Proposition 2.6.3 are all about, we'll analyze the operator associated with $\sigma(M_2)^{\oplus n}$ in some detail. The analysis has less moving parts than the corresponding analysis for $\sigma(M_{n+1})$, so we hope this serves as a soft landing for the reader. (Actually $\sigma(M_2)^{\oplus n}$ is easy to handle, since it is a product of $\sigma(M_2)$'s and the heat kernel factorizes nicely, but this isn't true for $\sigma(M_{n+1})$ so we'll need to do something different here).

So let $\mathbf{x} \in \mathbb{R}^{nd}$ be such that $\max_{1\le i\le n} |x_i| = 1$ and by symmetry we may assume $|x_1| \le ... \le |x_n| = 1$. Let $k(\mathbf{x}) := \#(\{|x_1|, ..., |x_n|\} \setminus \{0\})$ (i.e. the number of distinct strictly positive numbers) and let ℓ be defined by

$$\begin{aligned} 0 < |x_{\ell(1)}| = ... = |x_{\ell(2)-1}| < |x_{\ell(2)}| = ... = |x_{\ell(3)-1}| < ... \\ < |x_{\ell(k(x))}| = ... = |x_n| = 1 \end{aligned} \tag{2.7.1}$$

with $\ell(1)$ being the smallest integer so that $|x_{\ell(1)}| > 0$.

We'll first give the sets $A_j^{\mathbf{x}}$ explicitly and then an inductive construction which is relevant for the proof of Proposition 2.6.3. So for each $\mathbf{x}$ the number of sets $A_j^{\mathbf{x}}$ is $k(\mathbf{x})$ and

$$\begin{aligned} A_j^{\mathbf{x}} := \{\mathbf{y} \in \mathbb{R}^{nd} : &\forall i < \ell(j+1) : |x_i - y_i| \le \frac{1}{2}|x_{\ell(j)}| \text{ and} \\ &\forall l \in \{j+1, ..., k(\mathbf{x})-1\} \forall i \in \{\ell(l), ..., \ell(l+1)-1\} : \\ &|x_i - y_i| \le \frac{1}{2}|x_{\ell(j)}||x_{\ell(l+1)}|^{-\xi/2}\} \end{aligned} \tag{2.7.2}$$

Inductively, this is done as follows. If $\mathbf{x}$ is such that $k(\mathbf{x}) > 1$, let $\tilde{\mathbf{x}}$ be defined as follows:

1. $\tilde{x}_i = x_i / |x_{\ell(k(\mathbf{x}))-1}|$ for $1 \le i \le \ell(k(\mathbf{x})) - 1$ and
2. $\tilde{x}_i = x_i$ for $\ell(k(\mathbf{x})) \le i \le n$.

Obviously $k(\tilde{\mathbf{x}}) = k(\mathbf{x}) - 1$.

If $k(\mathbf{x}) = 1$, i.e. $|x_{\ell(1)}| = ... = |x_n| = 1$, we have

$$A_1^{\mathbf{x}} = \{\mathbf{y} \in \mathbb{R}^{nd} : |y_i - x_i| \le \frac{1}{2} \text{ for every } i \in \{1, ..., n\}\}. \tag{2.7.3}$$

Next suppose $k(\mathbf{x}) > 1$ and let $r := |x_{\ell(k(\mathbf{x}))-1}|$. For $j \in \{1, ..., k(\mathbf{x}) - 1\}$ we have

$$\begin{aligned} A_j^{\mathbf{x}} := \{(ry_1, ..., ry_{\ell(k(\mathbf{x}))-1}, x_{\ell(k(\mathbf{x}))} + r^{1-\xi/2}(y_{\ell(k(\mathbf{x}))} - x_{\ell(k(\mathbf{x}))}), \\ ..., x_n + r^{1-\xi/2}(y_n - x_n)) : \mathbf{y} \in A_j^{\tilde{\mathbf{x}}}\} \end{aligned} \tag{2.7.4}$$

and

$$A^{\mathbf{x}}_{k(\mathbf{x})} = \{\mathbf{y} : |y_i - x_i| \leq \frac{1}{2} \text{ for every } i \in \{1, ..., n\}\}. \tag{2.7.5}$$

Below, denote $M_2^{\oplus n} := -\nabla \cdot \sigma(M_2)^{\oplus n}\nabla$. The philosophy for showing for example (2) of Proposition 2.6.3 is as follows: First find a $C < \infty$ so that for every $\mathbf{x}$ with $\max_{1\leq i\leq n} |x_i| = 1$ and $\mathbf{y} \in A^{\mathbf{x}}_{k(\mathbf{x})} \setminus A^{\mathbf{x}}_{k(\mathbf{x})-1}$ we have

$$\begin{aligned} G_{M_2^{\oplus n}}(\mathbf{x}, \mathbf{y}) \leq C(&\sum_{i=1}^{\ell(k(\mathbf{x}))-1} |x_i - y_i|^{2-\xi} + \\ &+ \sum_{i=\ell(k(\mathbf{x}))}^{n} |x_i - y_i|^2)^{1-\frac{(\ell(k(\mathbf{x}))-1)d}{2-\xi} - \frac{(n-\ell(k(\mathbf{x}))+1)d}{2}}. \end{aligned} \tag{2.7.6}$$

This holds because of Theorem 2.5.12, Theorem 2.2.12 and Corollary 2.4.22 and an integration from 0 to ∞ with respect to t.

The rest is an inductive argument. One needs to use (2.7.4), some dimensional analysis and witchcraft to show that if for $j \in \{1, ..., k(\tilde{\mathbf{x}})\}$ and $\mathbf{y} \in A^{\tilde{\mathbf{x}}}_j \setminus A^{\tilde{\mathbf{x}}}_{j-1}$ we have

$$\begin{aligned} G_{M_2^{\oplus n}}(\tilde{\mathbf{x}}, \mathbf{y}) \leq C\Big(\prod_{i=\ell(j)}^{n} a_i^{-\frac{\xi d}{2}}\Big)(&\sum_{i=1}^{\ell(j)-1} |\tilde{x}_i - y_i|^{2-\xi} + \\ &+ \sum_{i=\ell(j)}^{n} a_i^{-\xi}|\tilde{x}_i - y_i|^2)^{1-\frac{(\ell(j)-1)d}{2-\xi} - \frac{(n-\ell(j)+1)d}{2}}, \end{aligned} \tag{2.7.7}$$

with some positive constants a_i, then for $\mathbf{y} \in A^{\mathbf{x}}_j \setminus A^{\mathbf{x}}_{j-1}$ and with $r := |x_{\ell(k(\mathbf{x}))-1}|$ we get

$$\begin{aligned} G_{M_2^{\oplus n}}(\mathbf{x}, \mathbf{y}) \leq C\Big(\prod_{i=\ell(j)}^{\ell(k(\mathbf{x})-1} (ra_i)^{-\frac{\xi d}{2}} \prod_{i=\ell(k(\mathbf{x}))}^{n} a_i^{-\frac{\xi d}{2}}\Big)(&\sum_{i=1}^{\ell(j)-1} |x_i - y_i|^{2-\xi} + \\ &+ \sum_{i=\ell(j)}^{\ell(k(\mathbf{x}))-1} (ra_i)^{-\xi}|x_i - y_i|^2 + \\ &+ \sum_{i=\ell(k(\mathbf{x}))}^{n} (a_i)^{-\xi}|x_i - y_i|^2)^{1-\frac{(\ell(j)-1)d}{2-\xi} - \frac{(n-\ell(j)+1)d}{2}}. \end{aligned} \tag{2.7.8}$$

This is immediately seen to be of the required form. The constants a_i above can also be given explicitly.

For $\mathbf{y} \in A^{\mathbf{x}}_j \setminus A^{\mathbf{x}}_{j-1}$ we have

$$G_{M_2^{\oplus n}}(\mathbf{x},\mathbf{y}) \leq C\Big(\prod_{i=\ell(j)}^{n} |x_i|^{-\frac{\xi d}{2}}\Big)\Big(\sum_{i=1}^{\ell(j)-1} |x_i - y_i|^{2-\xi} + \\ + \sum_{i=\ell(j)}^{n} |x_i|^{-\xi}|x_i - y_i|^2\Big)^{1-\frac{(\ell(j)-1)d}{2-\xi}-\frac{(n-\ell(j)+1)d}{2}}. \tag{2.7.9}$$

We want to modify this construction to apply to $\sigma(M_n)$.

Remark 2.7.1. Note that if $x, y \in \mathbb{R}^{nd}$, $\operatorname{rk} x = \operatorname{rk} y$ and $S^x \neq S^y$, then for every $z \in S^x \cap S^y$ we have $\operatorname{rk} z > \operatorname{rk} x$. In particular, if $\operatorname{rk} x = n-1$, then $S^x \cap S^y = \{0\}$.

Lemma 2.7.2. *There is a finite set $P \subseteq \mathbb{R}^{nd} \setminus \{0\}$, a partial ordering $\prec$ on P and for each $x \in P$ an associated linear transformation $L_x \in \mathcal{L}'_{n+1}$ ($\mathcal{L}'_{n+1}$ was defined in (2.4.48)) having the following properties:*

1. *There is $C \in (0,\infty)$ such that if K_x is the affine transformation sending x to 0 whose linear part is CL_x, there are $n_1, ..., n_k$ so that*

$$\sigma(M_{n+1})^{K_x} \sim \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1} \tag{2.7.10}$$

   on $B(0,2) \times B(0,2) \subseteq \mathbb{R}^{(\operatorname{rk} x)d} \times \mathbb{R}^{(n-\operatorname{rk} x)d}$.

2. *Let $Q_x := K_x^{-1}[B(0,1) \times B(0,1)]$, where the first $B(0,1)$ lies in $\mathbb{R}^{(\operatorname{rk} x)d}$ and the second one in $\mathbb{R}^{(n-\operatorname{rk} x)d}$. Then $\{Q_x : x \in P \cap \mathbb{S}^{nd-1}\}$ covers $\mathbb{S}^{nd-1}$.*

3. *Let $D_x := K_x^{-1}[\mathbb{S}^{(\operatorname{rk} x)d-1} \times \overline{B}(0,1)]$. Then for every $x \in P$ of rank > 0, $\{C_{x'}(1) : x' \in P \cap D_x \text{ and } x' \prec x\}$ covers D_x.*

4. *Suppose $x' \prec x$. Then*

$$L_x L_{x'}^{-1} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2.7.11}$$

   with A (resp. B) a $(\operatorname{rk} x)d \times (\operatorname{rk} x)d$ (resp. an $(n-\operatorname{rk} x)d \times (n-\operatorname{rk} x)d$) matrix. More precisely, we prove the following; If for every $y \in \mathbb{R}^{nd}$ we have $L_x y = (y_{i_1,j_1}, ..., y_{i_n,j_n})$ and $L_{x'} y = (y_{i'_1,j'_1}, ..., y_{i'_n,j'_n})$, then

$$i_m = i'_m \text{ and } j_m = j'_m \text{ for } \operatorname{rk} x + 1 \leq m \leq n \tag{2.7.12}$$

   and

$$\{y \in \mathbb{R}^{nd} : y_{i_1,j_1} = ... = y_{i_{\operatorname{rk} x},j_{\operatorname{rk} x}} = 0\} \\ = \{y \in \mathbb{R}^{nd} : y_{i'_1,j'_1} = ... = y_{i'_{\operatorname{rk} x},j'_{\operatorname{rk} x}} = 0\}. \tag{2.7.13}$$

Proof. The proof is by downward induction on rank. Let $F_0 := \mathbb{S}^{nd-1}$ and for every $x \in F_0$ of rank $n-1$, fix a linear transformation $L_x^0 \in \mathcal{L}'_{n+1}$ and a neighbourhood U_x^0 of x such that

$$\sigma(M_{n+1})^{L_x^0} \sim \bigoplus_{i=1}^{k} \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.7.14}$$

on $L_x^0 U_x^0$. (Actually here $1 \le k \le 2$.)

Next, choose $C_x^0 \in (0,\infty)$ so that if K_x^0 is the affine transformation sending x to 0 with $C_x^0 L_x^0$ as the linear part, we have

$$\sigma(M_{n+1})^{K_x} \sim \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1} \tag{2.7.15}$$

on $B(0,2) \times B(0,2)$.

Denote $Q_x^0 := K_x^{-1}[B(0,1) \times B(0,1)]$ and $D_x^0 := K_x^{-1}[\mathbb{S}^{(\operatorname{rk} x)d-1} \times \overline{B}(0,1)]$.

Let P_0 be a finite set such that

$$F_0 \cap \{x : \operatorname{rk} x = n-1\} \subseteq \bigcup_{x \in P_0} Q_x^0. \tag{2.7.16}$$

Next, suppose $k \le n-1$ is given and we have defined F_m and P_m for every $m < k$ and for given $m < k$ we have defined U_x^m, L_x^m, C_x^m, K_x^m, Q_x^m and D_x^m. for every $x \in P_m$.

Let

$$F_k := (F_{k-1} \cup \bigcup_{x \in P_{k-1}} D_x^{k-1}) \setminus \bigcup_{\substack{x \in P_{k-1} \\ \operatorname{rk} x = n-k}} Q_x^{k-1}. \tag{2.7.17}$$

We generate the set P_k and for every $x \in P_k$ of rank $n-k-1$ the linear transformation L_x^k as follows. List the points of P_{k-1} as $x_1, ..., x_\alpha$. Then we inductively define $P_{k,\beta}$ for $0 \le \beta \le \alpha$ and finally set $P_k = P_{k,\alpha}$. The partial order $\prec$ is generated so that for every β with $1 \le \beta \le \alpha$ and for every $x' \in P_{k,\alpha} \setminus P_{k,\alpha-1}$ we put $x' \prec x_\alpha$ and in the end of the induction we take a transitive closure (i.e. force $\prec$ to be a partial order).

First, let $P_{k,0} := P_{k-1}$ and let $L_x^k := L_x^{k-1}$ for $x \in P_{k,0}$. Next, suppose $\beta > 0$ is given with $\beta \le \alpha$ and that we have defined $P_{k,\beta-1}$ and L_x^k for $x \in P_{k,\beta-1}$. Suppose

$$z \in D_{x_\beta}^{k-1} \cap F_k \cap \{x : \operatorname{rk} x = n-k-1\} \tag{2.7.18}$$

and that for every $y \in \mathbb{R}^{nd}$ we have

$$L_{x_1}^{k-1}(y_1, ..., y_n) = (y_{i_1,j_1}, ..., y_{i_n,j_n}). \tag{2.7.19}$$

Let $L \in \mathcal{L}'_{n+1}$ be a linear transformation (given by Theorem 2.4.9) such that

$$\sigma(M_{n+1})^L \sim \bigoplus_{i=1}^{\gamma} \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.7.20}$$

on LU, with U being a neighbourhood of z. Suppose for every $y \in \mathbb{R}^{nd}$ we have

$$L(y_1, ..., y_n) = (y_{i'_1,j'_1}, ..., y_{i'_n,j'_n}). \tag{2.7.21}$$

Since $z \in D^{k-1}_{x_1}$, we have

$$\{y_{i_1,j_1} = ... = y_{i'_{\operatorname{rk} x_\beta},j'_{\operatorname{rk} x_\beta}} = 0\} \subseteq \{y_{i'_1,j'_1} = ... = y_{i'_{\operatorname{rk} z},j'_{\operatorname{rk} z}} = 0\}. \tag{2.7.22}$$

Let

$$i''_i = \begin{cases} i'_i & \text{if } 1 \le i \le \operatorname{rk} z \text{ and} \\ i_i & \text{if } \operatorname{rk} x_\beta + 1 \le i \le n, \end{cases} \tag{2.7.23}$$

and similarly for j''_i.

By (2.7.22) we may choose $i''_{\operatorname{rk} z+1}, ..., i''_{\operatorname{rk} z_\beta}$ and $j''_{\operatorname{rk} z+1}, ..., j''_{\operatorname{rk} z+1}$ so that if we let

$$L'_z(y_1, ..., y_n) := (y_{i''_1,j''_1}, ..., y_{i''_n,j''_n}) \tag{2.7.24}$$

for all $y \in \mathbb{R}^{nd}$ we have $L'_z \in \mathcal{L}'_{n+1}$ and by Proposition 2.4.19 we have

$$\sigma(M_{n+1})^{L'_z} \sim \bigoplus_{i=1}^{\beta} \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.7.25}$$

on $L'_z U'$ with U' a neighbourhood of z.

Next we choose $C'_x \in (0, \infty)$ for $x \in D := D^{k-1}_{x_\beta} \cap F_k \cap \{x : \operatorname{rk} x = n - k - 1\}$ so that if K'_x is the linear transformation sending x to 0 having $C'_x L'_x$ as its linear part we have

$$\sigma(M_{n+1})^{K'_x} \sim \bigoplus_{i=1}^{\beta} \sigma(M_{n_i+1}) \oplus \mathbb{1} \tag{2.7.26}$$

on $K'^{-1}_x[B(0,2) \times B(0,2)]$.

We have a small twist here. We want to cover D with finitely many sets $Q'_x := K'^{-1}_x[B(0,1) \times B(0,1)]$. However, we do not want to use Q'_x's associated with $x \in P^k_{\beta-1}$. This can be avoided by taking C'_x's to be such that there is $\epsilon > 0$ so that $B(x, \epsilon) \subseteq Q'_x$. This is easy to arrange. The point is that D is a compact set consisting of points of equal rank. D splits into finitely many connected components $D_1, ..., D_\delta$. Thus, actually (2.7.25) holds in a neighbourhood of each D_i.

So now that the Q'_x's are such that $B(x, \epsilon) \subseteq Q'_x$ and $\{Q'_x : x \in D \setminus P^k_{\beta-1}$ covers D, pick a finite subcover $\{Q'_x : x \in D'\}$ and let $P^k_\beta := P^k_{\beta-1} \cup D'$. For $x \in D'$, let $L^k_x := L'_x$ and $C^k_x := C'_x$ and set K^k_x, Q^k_x and D^k_x accordingly.

Finally, let $P := P_{n-1}$ and for every $x \in P$ let $L_x := L^{n-1}_x$ and $C_x = C^{n-1}_x$. □

For every $x \in P$, let $\mathcal{L}_x = \{L_{x'} : x \preceq x'\}$.

From now on, we'll use a rather schizophrenic convention of coordinates. Pairs like (y_1, y_2) are in K_x-coordinates, where x is clear from the context (See before Proposition 2.5.2). Sequences like $(y_1, ..., y_n)$ are in the original $\mathbb{R}^{nd}$-coordinates.

For $x \in P$, let $C_x(r) := K_x^{-1}[B(0, r) \times B(0, r)]$. (if $\operatorname{rk} x = 0$, we just set $C_x(r) := K_x^{-1}[B(0, r)]$).

For each $x \in P$ we find a set R_x as follows. If $\operatorname{rk}(x) = 0$ we set $R_x := C_x(\frac{1}{2})$. Obviously now $C_x(1) + R_x \subseteq C_x(\frac{3}{2})$ and moreover there is $C < \infty$ so that for every $x' \in C_x(1)$ and every $y \in x' + R_x$ we have

1. For $0 < t \leq 1$ we have $K_{M_{n+1}}(t, x, y) \leq Ct^{-\frac{(n+1)d}{2}} \exp\{-\frac{|x-y|^2}{Ct}\}$ (by Corollary 2.5.9) and
2. For $0 < t < \infty$ we have $K_{M_{n+1}}(t, x, y) \leq Ct^{-\frac{(n+1)d}{2-\xi}} \exp\{-\frac{|x-y|^2}{Ct}\}$ (by Theorem 2.2.12 and Corollary 2.4.22)

Since for $1 \leq t < \infty$ we have $t^{-(n+1)d/2} \leq t^{-(n+1)d/(2-\xi)}$ we can replace $0 < t \leq 1$ above with $0 < t < \infty$ so by integrating w.r.t. time we conclude that for every $x \in P$ with $\operatorname{rk}(x) = 0$ there is $C(x) < \infty$ so that for every $x' \in C_x(1)$, every $y \in x' + R_x$ and every $L \in \mathcal{L}'_{n+1}$ we have

$$G_{M_{n+1}}(x', y) \leq C(x)|Lx' - Ly|^{2-\xi-nd}. \tag{2.7.27}$$

Moreover we want that $C(x)$ depends on $\sigma(M_{n+1})$ only through upper and lower bounds on $C_x(2)$.

Remark 2.7.3. When we say that C depends on $\sigma(M_{n+1})$ only through the upper and lower bounds for $\sigma(M_{n+1})$ on $C_x(2)$ we really mean that if in $C_x(2)$ we have

$$\sigma(M_{n+1}) \sim^\lambda \sigma(M_{n_1+1}) \oplus ... \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1}, \tag{2.7.28}$$

with $\lambda > 0$ minimal, then C can be chosen so that the given claim holds for any symbol A that has a heat kernel and for which (2.7.28) holds with $\sigma(M_{n+1})$ replaced by A.

For $x \neq 0$ let

$$S^x := \bigcap_{\substack{1\leqslant i\leqslant j\leqslant n \\ |x_{i,j}|=0}} \{y \in \mathbb{R}^{nd} : |y_{i,j}| = 0\}. \tag{2.7.29}$$

Note that $\dim(S^x) + \mathrm{rk}(x)d = nd$.

If $l := \mathrm{rk}(x) > 0$, we set $R_x := K_x^{-1}\{y : d(x,y) < \frac{1}{2}\}$ (d was defined between Remark 2.5.7 and Proposition 2.5.8). Now a similar argument as above (but this time using Theorem 2.5.12 instead of Corollary 2.5.9) yields a $C(x,\epsilon'') < \infty$ ($\epsilon'' \in (0, \frac{1}{2})$) so that

1. $C_x(1) + R_x \subseteq C_x(\frac{3}{2})$ and
2. There is $C < \infty$ so that for every $x' \in C_x(1)$, for every y that satisfies $\epsilon''|x'_1|^{1-\xi/2} \leq d(x,y) \leq \frac{1}{2}$ and for every $L \in \mathcal{L}_x$ we have

$$\begin{aligned} G_{M_{n+1}}(x', y) \leq C(x,\epsilon'')\big(&\sum_{i=0}^{l} |(Lx)_i - (Ly)_i|^{2-\xi} + \\ &+ \sum_{i=l+1}^{n} |(Lx)_i - (Ly)_i|^2\big)^{1-\frac{ld}{2-\xi}-\frac{(n-l)d}{2}}. \end{aligned} \tag{2.7.30}$$

Again we want that $C(x)$ depends on $\sigma(M_{n+1})$ only through the upper and lower bounds on $C_x(2)$. Let

$$C(\epsilon'') := \max_{x\in P} C(x,\epsilon''). \tag{2.7.31}$$

We will fix our $\epsilon'' \in (0, \frac{1}{2})$ after we've defined our sets A_j^x.

Now it's time for the

2.7.1 Proof of Proposition 2.6.3.

Proof. (of Proposition 2.6.3). We start by defining the sets A_j^x and then prove that these sets have the desired properties.

First of all we handle the points $x \in \mathbb{S}^{nd-1}$ and then use scaling: If $A_1^{\hat{x}}, ..., A_k^{\hat{x}}$ is the sequence associated with $\hat{x}$, we let $A_j^x = |x|A_j^{\hat{x}}$. Moreover we let $r_j^x := |x|r_j^{\hat{x}}$.

This sequence is generated as follows: First, to each point $x \in P$ and $y \in C_x(1)$ we associate inductively a finite sequence $A_1^{x,y}, ..., A_{k(x,y)}^{x,y}$ (with $k(x,y)$ depending on x and y but not exceeding $\mathrm{rk}(x) + 1$). We shall denote $k(x,y)$ by k below when confusion is not possible. We also define associated sizes $r_1^{x,y}, ..., r_k^{x,y}$ and linear transformations $L_1^{x,y}, ..., L_k^{x,y}$.

Having done this we choose for every $y \in \mathbb{S}^{nd-1}$ a $x \in P_0$ so that $y \in C_x(1)$ and set $A_j^y := A_j^{x,y}$, $r_j^y := r_j^{x,y}$ and $L_j^y := L_j^{x,y}$ for $1 \leq j \leq k(y)$, where $k(y) := k(x,y)$. To confuse the reader even more (just kidding), for every $x \in P$ of rank > 0 we'll also define $\epsilon''(x) \in (0, \frac{1}{2})$ so that in K_x-coordinates we have

$$A_k^{x,y} \setminus A_{k-1}^{x,y} \subseteq \{y' : \epsilon''|y_1|^{1-\xi/2} \leq d(y,y') \leq \frac{1}{2}\} \tag{2.7.32}$$

for every $y \in C_1(x)$. Then we let

$$\epsilon'' = \min_{\substack{x\in P \\ \mathrm{rk}(x)>0}} \epsilon''(x) \tag{2.7.33}$$

Suppose $x \in P$ is of rank 0. Then for every $y \in C_x(1)$, we let $k(x,y) := 1$, $A_1^{x,y} := (y - x) + R_x$, $r_j^{x,y} := 1$ and let $L_1^{x,y} := L_x$.

Next, suppose all the points in P of rank $< l$ have been handled and let $x \in P$ be of rank l. Set $k(x,x) := 1$, $A_1^{x,x} := R_x$, $r_1^{x,x} := 1$ and let $L_1^{x,x} := L_x$.

For $y \in D_x$ we choose $x' \in P \cap D_x$ so that $y \in C_{x'}(1)$. Then we let $k(x,y) := k(x',y) + 1$, $A_i^{x,y} := A_i^{x',y} \cap [(y-x) + R_x]$ for $1 \leq i \leq k(x',y)$ and set $A_{k(x,y)}^{x,y} := [(y-x) + R_x]$. We let $r_i^{x,y} := r_i^{x',y}$ and set $r_{k(x,y)}^{x,y} := 1$. Finally, we let $L_i^{x,y} = L_i^{x',y}$ and set $L_{k(x,y)}^{x,y} := L_x$.

Since now the sets $A_{k(x',y)}^{x',y} = A_{k(x,y)-1}^{x,y}$ are just translates of $R_{x'}$, we see that $\epsilon''(x) > 0$ can be chosen so that our condition is satisfied for these sets.

Finally, if $y \in r\mathbb{S}^{\mathrm{rk}(x)d-1} \times \overline{B}(0,1)$ with $r \in (0,1)$, let $y' := (\hat{y}_1, y_2)$. Then we set $k(x,y) := k(x,y')$, and for $1 \leq i < k$ we let

$$A_i^{x,y} := \{(rz_1, y_2 + r^{1-\xi/2}(z_2 - y_2)) : (z_1, z_2) \in A_i^{x,y'}\}, \tag{2.7.34}$$

$r_i^{x,y} := r r_i^{x,y'}$ and $L_i^{x,y} := L_i^{x,y'}$. Finally, as above, we set $A_{k(x,y)}^{x,y} := [(y-x) + R_x]$, $r_k^{x,y} := 1$ and $L_i^{x,y} := L_x$. Note that our scaling was such that our condition for $\epsilon''(x)$ is still satisfied.

Now we have to prove that this choice of the sets $A_i^{x,y}$ satisfies all the claims of Proposition 2.6.3. Obviously, (1) is trivial.

By scaling, it suffices to prove (2)-(5) for $x \in \mathbb{S}^{nd-1}$.

We prove all these claims by chasing through the definition of the $A_j^{x,y}$'s. That is, we do a double induction by going through every $x \in P$ by induction on $\mathrm{rk}(x)$ and then for each $x \in P$ we go through $A_j^{x,y}$'s by downward induction on j.

So we start with (2). We prove that (2.6.8) holds with C given in (2.7.31).

By (2.7.27) and (2.7.30) our claim is trivial for $A^{x,y}_{k(x,y)}$'s, so we have to handle the case where $l := \mathrm{rk}(x) > 0$ and $j < k(x, y)$.

Our induction hypothesis now is that we have proven (2) for all sets $A^{x',y'}_{j'}$ and all linear transformations $L^{x',y'}_{j'}$ with $\mathrm{rk}(x') < l$, $y' \in C_{x'}(1)$ and $1 \leq j' \leq k(x', y')$ and also for all symbols A having the same upper and lower bounds as $\sigma(M_{n+1})$ on $A^{x',y'}_{j'}$.

So, in K^x-coordinates, let

$$y := (y_1, y_2) \in r\mathbb{S}^{ld-1} \times \mathbb{R}^{(n-l)d} \tag{2.7.35}$$

with $r \in (0, 1)$ ($r = 0, 1$ have been handled by our induction hypotheses). Without loss of generality, we may assume that $y_2 = 0$ (this eases our notation a bit).

Let A be some symbol having the same upper and lower bounds as $\sigma(M_{n+1})$ on $B(0, 2) \times B(0, 2)$.

Denote $z^y := (z_1/|y_1|, z_2/|y_1|^{1-\xi/2})$ and let the symbol B^y be defined by

$$B^y_{ij}(z) := \begin{cases} |y_1|^\xi \sigma(M_{n+1})_{ij}(z^y) & \text{if } 1 \leq i, j \leq ld \\ |y_1|^{\xi/2} \sigma(M_{n+1})_{ij}(z^y) & \text{if } 1 \leq i \leq ld < j \leq nd \text{ or} \\ & 1 \leq j \leq ld < i \leq nd \\ \sigma(M_{n+1})_{ij}(z^y) & \text{if } ld < i, j \leq nd. \end{cases} \tag{2.7.36}$$

Now B^y and A have the same upper and lower bounds on $B(0, 2) \times B(0, 2)$. Let $l' = \mathrm{codim}(S^{x,y^y}_j)/d < l$. Then we can conclude by our induction hypothesis (since $A^{x,y^y}_j = A^{x',y^y}_j$ for some $x' \in P$ with $\mathrm{rk}(x') < \mathrm{rk}(x)$) that we have

$$\begin{aligned} G_{B^y}(y^y, z^y) \leq C\Big(\prod_{i=l'+1}^{n} a_i^{-\frac{\xi d}{2}} \Big) \Big(\sum_{i=1}^{l'} |(Ly^y)_i - (Lz^y)_i|^{2-\xi} + \\ + \sum_{i=l'+1}^{n} a_i^{-\xi} |(Ly^y)_i - (Lz^y)_i|^2 \Big)^{1-\frac{ld}{2-\xi}-\frac{(n-l)d}{2}} \end{aligned} \tag{2.7.37}$$

whenever $z \in A^{x,y^y}_j \setminus A^{x,y^y}_{j-1}$ and $L = L^{x,y^y}_j$.

A little pondering in dimensional analysis implies

$$G_A(y, z) = |y_1|^{2-\xi-ld-(1-\xi/2)(n-l)d} G_{B^y}(y^y, z^y). \tag{2.7.38}$$

whenever $z \in A^{x,y}_j$.

Note that $L = L_{x'}$ for some $x' \prec x$. Therefore by (4) of Lemma 2.7.2 we have both

$$L_x'^{-1}\{w_i = 0 : 1 \le i \le l\} = K_x^{-1}\{w_i = 0 : 1 \le i \le l\} \tag{2.7.39}$$

and

$$L_x'^{-1}\{w_i = 0 : l+1 \le i \le n\} = K_x^{-1}\{w_i = 0 : l+1 \le i \le d\}. \tag{2.7.40}$$

Thus

$$(Ly^y)_i - (Lz^y)_i = \begin{cases} |y_1|^{-1}((Ly)_i - (Lz)_i) & \text{if } 1 \le i \le l \text{ and} \\ |y_1|^{\xi/2-1}((Ly)_i - (Lz)_i) & \text{if } l+1 \le i \le n. \end{cases} \tag{2.7.41}$$

Combining this with (2.7.37) and (2.7.38) we get

$$\begin{aligned} G_A(y,z) \le C\Big(\prod_{i=l'+1}^{l} (|y_1| a_i)^{-\frac{\xi d}{2}} \prod_{i=l+1}^{n} a_i^{-\frac{\xi d}{2}} \Big) \Big(\sum_{i=1}^{l} |(Ly)_i - (Lz)_i|^{2-\xi} + \\ + \sum_{i=l'+1}^{l} (|y_1| a_i)^{-\xi} |(Ly)_i - (Lz)_i|^2 \\ + \sum_{i=l+1}^{n} a_i^{-\xi} |(Ly)_i - (Lz)_i|^2 \Big)^{1 - \frac{ld}{2-\xi} - \frac{(n-l)d}{2}} \end{aligned} \tag{2.7.42}$$

for any $z \in A_j^{x,y} \setminus A_{j-1}^{x,y}$. This is seen to be of the form required, so we're done with (2).

Next we handle (3). First of all it is easy to see that it suffices to find $C(x,j) < \infty$ such that

$$\int_{A_j^x \setminus A_{j-1}^x} G_{M_{n+1}}(x,y)\, dy \le C(x,j)(r_j^x)^{2-\xi}. \tag{2.7.43}$$

For every $x \in P$ with $\mathrm{rk}(x) = 0$, there is $C(x,1) < \infty$ so that for $y \in C_x(1)$ we have

$$\int_{A_1^{x,y}} G_{M_{n+1}}(y,z)\, dy \le C(x,1)(r_1^{x,y})^2 \le C(r_1^{x,y})^{2-\xi}. \tag{2.7.44}$$

So suppose all points $x \in P$ of rank $< l$ have been handled and x is of rank l.

For $A_k^{x,y}$ the claim follows from (2.7.30) and the fact that $r_k^{x,y} = 1$:

$$
\begin{aligned}
&\int_{A_k^{x,y}\setminus A_{k-1}^{x,y}} G_{M_{n+1}}(y,y')\,dy' \\
&\quad\le C\int_{A_k^{x,y}\setminus A_{k-1}^{x,y}} dy'\,\Big(\sum_{i=1}^{l}|(Ly)_i-(Ly')_i|^{2-\xi}+ \\
&\qquad + \sum_{i=l+1}^{n}|(Ly)_i-(Ly')_i|^2\Big)^{1-\frac{ld}{2-\xi}-\frac{n-l}{d}} \\
&\quad\le C\int_{\{d(y,y')\le\frac{1}{2}\}} \le C' = C'(r_k^{x,y})^{2-\xi}.
\end{aligned}
\tag{2.7.45}
$$

For $j<k$, by (2.7.34), (2.7.37) and (2.7.38) we have

$$
\begin{aligned}
\int_{A_j^{x,y}\setminus A_{j-1}^{x,y}} G_{M_{n+1}}(y,y')\,dy' &= |y_1|^{2-\xi}\int_{A_j^{x,y^y}\setminus A_{j-1}^{x,y^y}} G_{B^y}(y^y,y')\,dy' \\
&\le C(x,j)|y_1|^{2-\xi}(r_j^{x,y^y})^{2-\xi}
\end{aligned}
\tag{2.7.46}
$$

Next in line is (4). Again we chase through the definition of the $A_j^{x,y}$'s. If $\mathrm{rk}(x)=0$, then $r_0^{x,y}=1$ and the claim is trivial, as $|y_i|\le|y|=1$. Suppose the points in P of rank $<l$ have been handled and $x\in P$ is of rank l. Next, for $A_k^{x,y}$ it holds, since again $r_k^{x,y}=1$ and $|y_i|\le|y|=1$.

So suppose again that $j<k(x,y)$ is given and the claim has been proven for every $A_{j'}^{x,y}$ with $j<j'\le k(x,y)$. Let again $y:=(y_1,y_2)\in r\mathbb{S}^{\mathrm{rk}(x)d-1}\times\overline{B}(0,1)$ with $r\in(0,1]$ and let z^y be defined as before. Since $S_j^{x,y}\subseteq\{z_i=0\}$ we have $|y_j|=r|y_j^y|$ and since $A_j^{x,y^y}=A_j^{x',y^y}$ for some $x'\in P$ with $\mathrm{rk}(x')<\mathrm{rk}(x)$ we conclude that

$$
|y_i| = r|y_j^y| \le r r_j^{x,y^y} = r_j^{x,y}.
\tag{2.7.47}
$$

Finally, we handle (5). Again, the basic steps in the induction are quite trivial, so we just show the nontrivial step. So the setup is that we have shown (5) for all points of rank $<l$, x is of rank l and the claim has also been shown for every $A_{j'}^{x,y}$ with $j<j'\le k(x,y)$. Now if $S_j^x\subseteq\{y_i=0\}$, then if

$$
A_j^{x,y^y}\subseteq\{|z_i|\ge C\},
\tag{2.7.48}
$$

we can conclude that

$$
A_j^{x,y}\subseteq\{|z_i|\ge C|y_i|\}.
\tag{2.7.49}
$$

On the other hand, if $S_j^x \not\subseteq \{y_i = 0\}$, then

$$A_j^{x,y^y} \subseteq \{|z_i| \geq C|y_j^y|\}, \tag{2.7.50}$$

implies that

$$A_j^{x,y} \subseteq \{|z_i| \geq C|y_j^y|\} \subseteq \{|z_i| \geq C|y_j|\}. \tag{2.7.51}$$

□

Appendix A

Some Technicalities

A.1 Poincaré and Sobolev inequalities

The following Theorem was proved in [4].

Theorem A.1.1. *Let $q > 2$ and let w_1 and w_2 be two weights on $\mathbb{R}^n$ and suppose that w_1 is A_2 and that w_2 is doubling. Suppose also that for all balls B' and B with $B' \subseteq 2B$*

$$\left(\frac{|B'|}{|B|}\right)^{1/n}\left(\frac{w_2(B')}{w_2(B)}\right)^{1/q} \leq c\left(\frac{w_1(B')}{w_1(B)}\right)^{1/2} \tag{A.1.1}$$

with c independent of the balls.

Then the Poincaré and Sobolev inequalities hold for w_1, w_2 with q.

So in order to conclude that the Harnack inequality holds for M_n, it suffices to check the assumptions of above Theorem with $w_1 = d(x,F)^\xi$ with F be a finite union of vector subspaces of $\mathbb{R}^n$ or just $\{0\}$ and w_2 either $|x|^\xi$ or 1.

Lemma A.1.2. *Let F be a finite union of vector subspaces of $\mathbb{R}^n$ or just $\{0\}$. Suppose $\xi > -n$. Then $w_\xi(x) := d(x,F)^\xi$ satisfies the following: There is a constant $C < \infty$ such that for every $x \in \mathbb{R}^n$ we have*

1. *If $0 < r < \frac{d(x,F)}{2}$, then $\frac{1}{C}d(x,F)^\xi r^n \leq w_\xi(B(x,r)) \leq Cd(x,F)^\xi r^n$ and*

2. *If $r \geq \frac{d(x,F)}{2}$, then $\frac{1}{C}r^{n+\xi} \leq w_\xi(B(x,r)) \leq Cr^{n+\xi}$.*

Proof. Since for $y \in B(x,r) \subseteq B(x, \frac{d(x,F)}{2})$ we have $\left(\frac{d(x,F)}{2}\right)^\xi \leq w_\xi(y) \leq \left(\frac{3d(x,F)}{2}\right)^\xi$, the first estimate follows.

For the second estimate, since $w_\xi(x,r) = |x|^{n+\xi} w_\xi(\hat{x}, \frac{r}{|x|})$ we see that by scaling it suffices to prove the inequality for $x \in \mathbb{S}^{n-1}$. To conclude the proof, it suffices to prove that

$$0 < \lim_{r\to\infty} \sup_{x\in\mathbb{S}^{n-1}} \frac{1}{r^{n+\xi}} w_\xi(B(x,r)) = \lim_{r\to\infty} \inf_{x\in\mathbb{S}^{n-1}} \frac{1}{r^{n+\xi}} w_\xi(B(x,r)) < \infty. \tag{A.1.2}$$

The computation is omitted. □

Naturally, the choice of the borderline at $\frac{d(x,F)}{2}$ was arbitrary. We can and will put the borderline at $\epsilon d(x,F)$ with $\epsilon \in (0,1)$ depending on the situation.

Proposition A.1.3. *Let $0 < \xi < 2$ and let F be a finite union of vector subspaces of $\mathbb{R}^n$ or just $\{0\}$. Let $w_1 = C_1 d(x,F)^\xi$ and $w_2 = C_2|x|^\xi$ with $0 < C_1, C_2 < \infty$. Then the Poincaré and Sobolev inequalities hold for w_1, w_2 with $q := \frac{2n}{n+\xi-2}$ and for w_1, 1 with q.*

Proof. By Lemma A.1.2 both w_1 and w_2 are A_2, so it suffices to prove the scaling assumption in Theorem A.1.1 with q. Now Lemma A.1.2 implies that there is a constant $C < \infty$ such that for every $x \in \mathbb{R}^d$ and $r > 0$ and every ball $B' := B(x', r') \subseteq B(x, 2r)$ we have

1. If $0 < r < \frac{d(x,F)}{4}$, then $C^{-1}|x|^\xi r'^n \le w(B') \le C|x|^\xi r'^n$.
2. If $\frac{d(x,F)}{4} < r$, then $C^{-1} r'^{n+\xi} \le w(B') \le C r^\xi r'^n$.

Here w stood for either w_1 or w_2. Thus we have for some $C < \infty$ the following:

1. If $0 < r < \frac{d(x,F)}{4}$, then
$$C^{-1}\left(\frac{r'^n}{r^n}\right) \le \left(\frac{w(B')}{w(B)}\right) \le C\left(\frac{r'^n}{r^n}\right). \tag{A.1.3}$$
2. If $\frac{d(x,F)}{4} < r$, then
$$C^{-1}\left(\frac{r'^{n+\xi}}{r^{n+\xi}}\right) \le \left(\frac{w(B')}{w(B)}\right) \le C\left(\frac{r'^n}{r^n}\right) \tag{A.1.4}$$

Therefore, the claim reduces to finding $C < \infty$ such that for every r and r' with $r' \le 2r$ we have:

1. If $0 < r < \frac{d(x,F)}{4}$, then

$$\left(\frac{r'}{r}\right)\left(\frac{r'^n}{r^n}\right)^{\frac{n-2+\xi}{2n}} \leq C\left(\frac{r'^n}{r^n}\right)^{1/2} \tag{A.1.5}$$

and

$$\left(\frac{r'}{r}\right)^{\frac{n+\xi}{2}} \leq C\left(\frac{r'}{r}\right)^{\frac{n}{2}} \tag{A.1.6}$$

2. If $\frac{d(x,F)}{4} < r$, then

$$\left(\frac{r'}{r}\right)\left(\frac{r'^n}{r^n}\right)^{\frac{n-2+\xi}{2n}} \leq C\left(\frac{r'^{n+\xi}}{r^{n+\xi}}\right)^{1/2} \tag{A.1.7}$$

and

$$\left(\frac{r'}{r}\right)^{\frac{n+\xi}{2}} \leq C\left(\frac{r'}{r}\right)^{\frac{n+\xi}{2}} \tag{A.1.8}$$

Obviously, such a C exists, so our claim has been proved. □

A.2 Proofs for §2.4.3

Proof. (of Proposition 2.4.10) Let $C_1 := \inf\{\langle \mathbf{v}, \sigma(M_{n+1})(\mathbf{x})\mathbf{v}\rangle : |\mathbf{x}| = |\mathbf{v}| = 1 \text{ and } \mathbf{x} \in A\}$ and $C_2 := \sup\{\langle \mathbf{v}, \sigma(M_{n+1})(\mathbf{x})\mathbf{v}\rangle : |\mathbf{x}| = |\mathbf{v}| = 1 \text{ and } \mathbf{x} \in A\}$. Since A is conical with $A \cap S^{nd-1}$ compact and disjoint from the degeneration set, we have $C_1 > 0$.

For $\mathbf{x} \in A$ we have

$$\begin{aligned}
C_1 \sum_{i=1}^{n} |x_i|^\xi |v_i|^2 &\leq C_1 \sum_{i=1}^{n} |\mathbf{x}|^\xi |v_i|^2 \\
&= C_1 |\mathbf{x}|^\xi |\mathbf{v}|^2 \\
&\leq \sigma(M_n) \\
&\leq C_2 |\mathbf{x}|^\xi |\mathbf{v}|^2 \\
&= C_2 \sum_{i=1}^{n} |\mathbf{x}|^\xi |v_i|^2 \\
&\leq C_2 \left(\frac{\sqrt{n}}{\epsilon}\right)^\xi \sum_{i=1}^{n} |x_i|^\xi |v_i|^2,
\end{aligned} \tag{A.2.1}$$

where the last inequality follows from the fact that

$$
\begin{aligned}
|\mathbf{x}|^\xi &= (\sum_{i=1}^n |x_i|^2)^{\xi/2} \\
&\le n^{\xi/2} \max\{|x_i|^\xi : 1 \le i \le n\}) \\
&\le (\frac{\sqrt{n}}{\epsilon})^\xi \min\{|x_i|^\xi : 1 \le i \le n\} \\
&\le (\frac{\sqrt{n}}{\epsilon})^\xi |x_i|^\xi.
\end{aligned}
\tag{A.2.2}
$$

□

Proof. (of Lemma 2.4.11) We write $|\langle v_i, (d(x_i + x_{i+1}) - d(x_i) - d(x_{i+1}))v_{i+1}\rangle| \le |\langle v_i, (d(x_i + x_{i+1}) - d(x_{i+1}))v_{i+1}\rangle| + |\langle v_i, d(x_i)v_{i+1}\rangle|$ and estimate the two terms separately.

Since d is differentiable in the ball $\overline{B}(x_{i+1}, \frac{1}{2}|x_{i+1}|)$ a simple application of the mean value theorem of elementary calculus gives

$$
\begin{aligned}
|\langle v_i, (d(x_i + x_{i+1}) &- d(x_{i+1}))v_{i+1}\rangle| \\
&\le \sup_{0\le r\le 1} \langle v_i, (x_i \cdot \nabla)d(x_{i+1} + r x_i)v_{i+1}\rangle \\
&\le \sup_{\frac{1}{2}\le |y| \le \frac{3}{2}} |\langle \hat{v}_i, (\hat{x_i} \cdot \nabla)d(y)\hat{v}_{i+1}\rangle||x_i||x_{i+1}|^{\xi-1}|v_i||v_{i+1}| \\
&:= C|x_i||x_{i+1}|^{\xi-1}|v_i||v_{i+1}| \\
&= C\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{1-\xi/2}|x_i|^{\xi/2}|x_{i+1}|^{\xi/2}|v_i||v_{i+1}| \\
&\le \frac{C}{2}\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{1-\xi/2}(|x_i|^\xi|v_i|^2 + |x_{i+1}|^\xi|v_{i+1}|^2).
\end{aligned}
\tag{A.2.3}
$$

Similarly,

$$
\begin{aligned}
|\langle v_i, d(x_i)v_{i+1}\rangle| &\le (1 + \frac{\xi}{d-1})|x_i|^\xi|v_i||v_{i+1}| \\
&\le (1 + \frac{\xi}{d-1})\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{\xi/2}|x_i|^{\xi/2}|x_{i+1}|^{\xi/2}|v_i||v_{i+1}| \\
&\le (\frac{1}{2} + \frac{\xi}{2d-2})\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{\xi/2}(|x_i|^\xi|v_i|^2 + |x_{i+1}|^\xi|v_{i+1}|^2).
\end{aligned}
\tag{A.2.4}
$$

Therefore, by setting $E := \max\{\frac{C}{2}, \frac{1}{2} + \frac{\xi}{2d-2}\}$, we can conclude our claim. □

Proof. (of Lemma 2.4.12) We just estimate $|\langle v_i, (d(v_{i,j}) - d(v_{i+1,j}))v_j\rangle|$ and the other part is estimated similarly.

Again an application of mean value theorem gives us

$$
\begin{aligned}
|\langle v_i, (d(x_{i,j}) - &d(x_{i+1,j}))v_j\rangle| \\
&\leq \sup_{0\leq r\leq 1} \langle v_i, (x_i\cdot\nabla)d(x_{i+1,j}+rx_i)v_j\rangle \\
&\leq \sup_{\frac{1}{2}\leq|y|\leq\frac{3}{2}} |\langle \hat{v_i}, (\hat{x_i}\cdot\nabla)d(y))\hat{v_j}\rangle||x_i||x_{i+1,j}|^{\xi-1}|v_i||v_j| \\
&:= C|x_i||x_{i+1,j}|^{\xi-1}|v_i||v_j| \\
&= C\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{1-\xi/2}\Big(\frac{|x_{i+1,j}|}{|x_j|}\Big)^{\xi/2}|x_i|^{\xi/2}|x_j|^{\xi/2}|v_i||v_j| \\
&\leq \frac{C}{2}\Big(\frac{|x_i|}{|x_{i+1}|}\Big)^{1-\xi/2}\Big(\frac{|x_{i+1,j}|}{|x_j|}\Big)^{\xi/2}(|x_i|^{\xi}|v_i|^2+|x_{i+1}|^{\xi}|v_{i+1}|^2).
\end{aligned}
\tag{A.2.5}
$$

□

Proof. (of Lemma 2.4.13) First we make a split:

$$
\begin{aligned}
&|\langle v_i, (d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j\rangle| \\
&\leq |\langle v_i, (d(x_{i,j}) - d(x_{i+1,j})v_j\rangle| + |\langle v_i, d(x_{i,j-1})v_j\rangle| + |\langle v_i, d(x_{i+1,j-1})v_j\rangle|.
\end{aligned}
\tag{A.2.6}
$$

Now the first term is estimated exactly as in the previous Lemma and the latter as follows. (Actually we only estimate the second one, the third one is handled identically).

$$
\begin{aligned}
|\langle v_i, &d(x_{i,j-1})v_j\rangle| \\
&\leq \Big(1+\frac{\xi}{d-1}\Big)|x_{i,j-1}|^{\xi}|v_i||v_j| \\
&\leq \Big(1+\frac{\xi}{d-1}\Big)\Big(\frac{|x_{i,j-1}|}{|x_i|}\Big)^{\xi}\Big(\frac{|x_i|}{|x_j|}\Big)^{\xi/2}|x_i|^{\xi/2}|x_j|^{\xi/2}|v_i||v_j| \\
&\leq \Big(\frac{1}{2}+\frac{\xi}{2d-2}\Big)\Big(\frac{|x_i|+|x_{i+1,j-1}|}{|x_i|}\Big)^{\xi}\Big(\frac{|x_i|}{|x_j|}\Big)^{\xi/2}(|x_i|^{\xi}|v_i|^2+|x_j|^{\xi}|v_j|^2) \\
&\leq \Big(\frac{1}{2}+\frac{\xi}{2d-2}\Big)3^{\xi}\Big(\frac{|x_i|}{|x_j|}\Big)^{\xi/2}(|x_i|^{\xi}|v_i|^2+|x_j|^{\xi}|v_j|^2).
\end{aligned}
\tag{A.2.7}
$$

□

Proof. (of Lemma 2.4.14) Two applications of the mean value theorem give us

$$
\begin{aligned}
&|\langle v_i,(d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j\rangle| \\
&\quad \leq \sup_{0\leq r\leq 1} \langle v_i, ((x_i \cdot \nabla)d(x_{i+1,j} + rx_i) - (x_i \cdot \nabla)d(x_{i+1,j-1} + rx_i)v_j\rangle \\
&\quad \leq \sup_{0\leq r,r'\leq 1} \langle v_i, (x_i \cdot \nabla)(x_j \cdot \nabla)d(x_{i+1,j-1} + rx_i + r'x_j)v_j\rangle \\
&\quad \leq \sup_{\frac{1}{3}\leq|y|\leq\frac{4}{3}} |\langle \hat{v_i}, (\hat{x_i} \cdot \nabla)(\hat{x_j} \cdot \nabla)d(y)\hat{v_j}\rangle|x_i||x_j||x_{i+1,j-1}|^{\xi-2}|v_i||v_j| \\
&\quad := 2E|x_i||x_j||x_{i+1,j-1}|^{\xi-2}|v_i||v_j| \\
&\quad \leq E\Big(\frac{|x_i|}{|x_{i+1,j-1}|}\Big)^{1-\xi/2}\Big(\frac{|x_i|}{|x_{i+1,j-1}|}\Big)^{1-\xi/2}(|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2)
\end{aligned}
\tag{A.2.8}
$$

□

Proof. (of Lemma 2.4.15) We estimate the terms individually. The mean value theorem gives us

$$
\begin{aligned}
&|\langle v_i,(d(x_{i,j}) - d(x_A))v_j\rangle| \\
&\quad \leq 2^{\xi/2}C\Big(\sum_{k\in[i,j]\setminus A} |x_k|\Big)|x_A|^{\xi-1}|v_i||v_j| \\
&\quad \leq 2^{\xi/2}C\Big(\frac{\sum_{k\in[i,j]\setminus A}|x_k|}{|x_A|}\Big)^{1-\xi/2}\Big(\frac{|x_A|}{|x_j|}\Big)^{\xi/2}.
\end{aligned}
\tag{A.2.9}
$$

Since we assumed that $\sum_{k\in[i,j]\setminus A}|x_k| \leq \frac{1}{2}\min\{|x_{k,l}| : k,l \in A, k \leq l\}$, we have

$$
\begin{aligned}
&2^{\xi/2}C\Big(\frac{\sum_{k\in[i,j]\setminus A}|x_k|}{|x_A|}\Big)^{1-\xi/2}\Big(\frac{|x_A|}{|x_j|}\Big)^{\xi/2} \\
&\quad \leq 2^{\xi/2-1}C\Big(\frac{\sum_{k\in[i,j]\setminus A}|x_k|}{\min\{|x_{k,l}| : k,l \in A, k \leq l\} - \sum_{k\in[i,j]\setminus A}|x_k|}\Big)^{1-\xi/2}\Big(\frac{|x_A|}{|x_j|}\Big)^{\xi/2} \\
&\quad\quad \cdot (|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2) \\
&\quad \leq C\Big(\frac{\sum_{k\in[i,j]\setminus A}|x_k|}{\min\{x_{k,l} : k,l \in A, k \leq l\}}\Big)^{1-\xi/2}\Big(\frac{\sum_{k\in A}|x_k|}{|x_j|}\Big)^{\xi/2} \\
&\quad\quad \cdot (|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2)
\end{aligned}
\tag{A.2.10}
$$

Similar estimates hold for the other terms, except when $A = \{i, j\}$, which causes modifications to the last pair of terms. Then

$$\begin{aligned}
&|\langle v_i, d(x_{i+1,j-1})v_j\rangle| \\
&\quad\le (1+\frac{\xi}{d-1})|x_{i+1,j-1}|^{\xi}|v_i||v_j| \\
&\quad\le (\frac{1}{2}+\frac{\xi}{2d-2})\Big(\frac{\sum_{k\in[i,j]\setminus A}|x_k|}{|x_j|}\Big)^{\xi/2}(|x_i|^{\xi}|v_i|^2+|x_j|^{\xi}|v_j|^2).
\end{aligned} \tag{A.2.11}$$

□

A.3 Proof of Proposition 2.6.2

In order to prove Proposition 2.6.2 we first need a Lemma.

Lemma A.3.1. *Let $2 \le l \in \mathbb{N}$. Then there is $C < \infty$ such that*

$$\int_{\mathbb{R}^d} d^d y\,(k+|x-y|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} \le Ck^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}. \tag{A.3.1}$$

Proof. We split the domain of integration into three parts and estimate these separately:

$$\begin{aligned}
&\int_{\mathbb{R}^d} d^d y\,(k+|x-y|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} \\
&\qquad = \int_{|x-y|\le|x|/2} + \int_{|y|\le|x|/2} + \int_{|y|,|x-y|\ge|x|/2} =: (*^1)+(*^2)+(*^3)
\end{aligned} \tag{A.3.2}$$

To estimate $(*^1)$ we note that in $|x-y| \le |x|/2$ we have $|x|/2 \le |y|$ which implies that in $|x-y| \le |x|/2$ we have

$$(1+|y|)^{2-\xi-d} \le (1+|x|/2)^{2-\xi-d} \le C_1(1+|x|)^{2-\xi-d}. \tag{A.3.3}$$

Therefore

$$\begin{aligned}
(*^1) &\le C_1\int_{|x-y|\le|x|/2} d^d y\,(k+|x-y|)^{2-\xi-ld}(1+|x|)^{2-\xi-d} \\
&= C_1 k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}\int_{|x-y|\le|x|/(2k)} d^d y\,(1+|x-y|)^{2-\xi-ld} \\
&\le C_1 k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}\int_{\mathbb{R}^d} d^d y\,(1+|x-y|)^{2-\xi-ld} \\
&\le C_2 k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}.
\end{aligned} \tag{A.3.4}$$

We make in a similar estimate in $(*^2)$: in $|y| \leq |x|/2$ we have $|x|/2 \leq |x-y|$ which implies that in $|y| \leq |x|/2$ we have

$$(k+|x-y|)^{2-\xi-ld} \leq (k+|x|/2)^{2-\xi-ld} \leq C_3(k+|x|)^{2-\xi-ld}. \tag{A.3.5}$$

Now we can compute:

$$\begin{aligned}(*^2) &\leq C_3 \int_{|y|\leq|x|/2} d^d y\,(k+|x|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} \\ &= C_4(k+|x|)^{2-\xi-ld} \cdot \begin{cases} |x|^d & \text{if } |x| \leq 1 \text{ and} \\ |x|^{2-\xi} & \text{if } |x| \geq 1. \end{cases}\end{aligned} \tag{A.3.6}$$

To treat the case $|x| \leq 1$, we compute:

$$\begin{aligned}C_4(k+|x|)^{2-\xi-ld}|x|^d &\leq C_4 k^{2-\xi-(l-1)d}|x|^{-d}|x|^d \\ &\leq C_5 k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}.\end{aligned} \tag{A.3.7}$$

If $|x| \geq 1$ we have

$$\begin{aligned}C_4(k+|x|)^{2-\xi-ld}|x|^{2-\xi} &\leq C_4 k^{2-\xi-(l-1)d}|x|^{-d}|x|^{2-\xi} \\ &= C_4 k^{2-\xi-(l-1)d}|x|^{2-\xi-d}.\end{aligned} \tag{A.3.8}$$

Finally, we handle $(*^3)$. When $|y|, |x-y| \geq |x|/2$, we have $|y|/3 \leq |x-y|$. Since this might not be obvious, we compute: Since $B(x,|x|/2) \subseteq B(0,3|x|/2)$, we have

$$\begin{aligned}|y|/3 &= |x|/2 + \frac{1}{3}d(y, B(0,3|x|/2)) \\ &\leq |x|/2 + d(y, B(0,3|x|/2) \\ &\leq |x|/2 + d(y, B(x,|x|/2) \\ &= |x-y|.\end{aligned} \tag{A.3.9}$$

Therefore, when $|y|, |x-y| \geq |x|/2$, we have

$$(k+|x-y|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} \leq C_6(k+|y|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} \tag{A.3.10}$$

and thus

$$(*^3) \leq C_6 \int_{|y|\geq|x|/2} d^d y\,(k+|y|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} =: (*^4) \tag{A.3.11}$$

We split the analysis of $(*^4)$ into two subcases: $|x| \geq 2$ and $|x| \leq 2$.

If $|x| \geq 2$, then we have

$$\begin{aligned}(*^4) &\leq C_6 \int_{|y|\geq|x|/2} (k+|y|)^{2-\xi-ld}|y|^{2-\xi-d} \\ &= C_7 k^{2(2-\xi)-ld} \int_{|y|\geq|x|/(2k)} d^d y\,(1+|y|)^{2-\xi-ld}|y|^{2-\xi-d} \\ &= C_8 k^{2(2-\xi)-ld}(1+|x|/k)^{2(2-\xi)-ld} =: (*^5)\end{aligned} \tag{A.3.12}$$

If $|x| \leq k$, then

$$(*^5) = C_8 k^{2(2-\xi)-ld} \leq C_9 k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}. \tag{A.3.13}$$

On the other hand, if $k \leq |x|$, then

$$(*^5) = C_8 |x|^{2(2-\xi)-ld} \leq C_{10} k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}. \tag{A.3.14}$$

If instead of $|x| \geq 2$ we have $|x| \leq 2$ in $(*^4)$, we compute

$$\begin{aligned}(*^4) &\leq C_6 \int_{\mathbb{R}^d} d^d y\,(k+|y|)^{2-\xi-ld}(1+|y|)^{2-\xi-d} \\ &\leq C_{11} \int_{|y|\leq 1} d^d y\,(k+|y|)^{2-\xi-ld} + C_{11}\int_{|y|\geq 1} (k+|y|)^{2-\xi-ld}|y|^{2-\xi-d} \\ &\leq C_{12} k^{2-\xi-(l-1)d} \int_{\mathbb{R}^d} d^d y\,(1+|y|)^{2-\xi-ld} + C_{12} k^{2-\xi-(l-1)d} \\ &\leq C_{13} k^{2-\xi-(l-1)d}(1+|x|)^{2-\xi-d}.\end{aligned} \tag{A.3.15}$$

□

Proof. (of Proposition 2.6.2.) Without loss of generality, we may assume that χ is the characteristic function of the unit ball. First we integrate y_l out:
Write $k := \sum_{i=1}^{l-1} |x_i - y_i|$. Now we have

$$\begin{aligned}&\int_{y_l\in B(0,1)} d^d y_l\,(k+|x_l-y_l|)^{2-\xi-ld} \\ &\qquad \leq C_1 \begin{cases} (k+|x_l|)^{2-\xi-ld} & \text{if } |x_l|\geq 2, \\ k^{2-\xi-(l-1)d} & \text{if } |x_l|\leq 2 \text{ and } k\leq 1 \text{ and} \\ k^{2-\xi-ld} & \text{if } |x_l|\leq 2 \text{ and } k\geq 1. \end{cases}\end{aligned} \tag{A.3.16}$$

The first case, i.e. $|x_l| \geq 2$, is computed by a repeated application of Lemma A.3.1:

$$
\begin{aligned}
&\int_{\mathbb{R}^{(l-1)d}} \prod_{i=1}^{l-1} d^d y_i \, (|x_l| + \sum_{i=1}^{l-1} |x_i - y_i|)^{2-\xi-ld} \prod_{i=1}^{l-1} (1+|y_i|)^{2-\xi-d} \\
&\qquad \leq C_2 (1+|x_{l-1}|)^{2-\xi-d} \int_{\mathbb{R}^{(l-1)d}} \prod_{i=1}^{l-1} d^d y_i \cdot \\
&\qquad \cdot (|x_l| + \sum_{i=1}^{l-2} |x_i - y_i|)^{2-\xi-(l-1)d} \prod_{i=1}^{l-2} (1+|y_i|)^{2-\xi-d} \\
&\qquad \leq \ldots \\
&\qquad \leq C_l \prod_{i=2}^{l-1} (1+|x_i|)^{2-\xi-d} \int_{\mathbb{R}^d} d^d y_1 \cdot \\
&\qquad \cdot (|x_l| + |x_1 - y_1|)^{2-\xi-2d} (1+|y_1|)^{2-\xi-d} \\
&\qquad \leq C_{l+1} \prod_{i=1}^{l} (1+|x_i|)^{2-\xi-d}.
\end{aligned} \tag{A.3.17}
$$

In the second case, i.e. $|x_l| \leq 2$ and $k \leq 1$, we get

$$
\begin{aligned}
&\int_{k\leq 1} \prod_{i=1}^{l-1} d^d y_i \, k^{2-\xi-(l-1)d} \prod_{i=1}^{l-1} (1+|y_i|)^{2-\xi-d} \\
&\qquad \leq \sup_{k\leq 1} \prod_{i=1}^{l-1} (1+|y_i|)^{2-\xi-d} \int_{k\leq 1} \prod_{i=1}^{l-1} d^d y_i \, k^{2-\xi-(l-1)d} \\
&\qquad \leq C' \prod_{i=1}^{l-1} (1+|x_i|)^{2-\xi-d}.
\end{aligned} \tag{A.3.18}
$$

The third case, i.e. $|x_l| \leq 2$ and $k \geq 1$, uses the following trick:

$$
\begin{aligned}
&\int_{k\geq 1} \prod_{i=1}^{l-1} d^d y_1 \, k^{2-\xi-ld} \prod_{i=1}^{l-1} (1+|y_i|)^{2-\xi-d} \\
&\qquad \leq C_2' \int_{\mathbb{R}^{(l-1)d}} \prod_{i=1}^{l-1} d^d y_i \, (1+k)^{2-\xi-ld} \prod_{i=1}^{l-1} (1+|y_i|)^{2-\xi-d} = (*)
\end{aligned} \tag{A.3.19}
$$

Now repeating the computation of the first case, we get:

$$
(*) \leq C_3' \prod_{i=1}^{l-1} (1+|x_i|)^{2-\xi-d} \leq C_4' \prod_{i=1}^{l} (1+|x_i|)^{2-\xi-d}. \tag{A.3.20}
$$

□

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