# Global behavior of quasiregular mappings and mappings of finite distortion under parabolic assumptions on spaces

**Pekka Pankka**

*Academic dissertation*

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Department of Mathematics and Statistics
Faculty of Science
University of Helsinki

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Pekka Pankka

## List of included articles

This dissertation consists of an introductory part and the following publications:

- [A] I. Holopainen and P. Pankka. *Mappings of finite distortion: global homeomorphism theorem. Ann. Acad. Sci. Fenn. Math.* 29(1):59-80, 2004.
- [B] P. Pankka. *Mappings of finite distortion and weighted parabolicity.* In *Future trends in geometric function theory*, volume 92 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pages 191-198. Univ. Jyväskylä, Jyväskylä 2003.
- [C] I. Holopainen and P. Pankka. *A big Picard theorem for quasiregular mappings into manifolds with many ends. Proc. Amer. Math. Soc.* 133 (2005), 1143-1150.
- [D] P. Pankka. *Quasiregular mappings from a punctured ball into compact manifolds.* Reports in Mathematics 399, Department of Mathematics and Statistics, University of Helsinki, Helsinki 2004.

In this introductory part these articles will be referred to [A], [B], [C], and [D] whereas other references will be numbered as [1], [2], ... .

## 1. On mappings, spaces, and questions

In this thesis, we study two classes of mappings between Riemannian manifolds distorting the local geometry in a controlled way. Under additional geometric and topological conditions on underlying manifolds, we obtain results on the global behavior of the mappings. To be more precise, we study quasiregular mappings and mappings of finite distortion between Riemannian manifolds, where the domain of the mappings is assumed to be a weighted parabolic manifold or having a conformally parabolic end and the range possesses a restricted topology. To describe the setting, let us first define the mapping classes and then consider the assumptions on manifolds.

A continuous mapping $f\colon M \to N$ between oriented Riemannian $n$-manifolds $M$ and $N$ is called $K$-*quasiregular*, $1 \leq K < \infty$, if $f$ is in the Sobolev class $W^{1,n}_{\text{loc}}(M,N)$ and satisfies an inequality

$$\|T_x f\|^n \leq K J(x,f)$$

for a.e $x \in M$, where $\|T_x f\|$ is the operator norm of the tangent map $T_x f\colon T_x M \to T_{f(x)} N$ and $J(x,f)$ is the Jacobian determinant of $f$ at $x$.

The theory of quasiregular mappings in dimension two differs drastically from the theory in higher dimensions. In dimension two the connection between quasiregular mappings and the Beltrami equation reduces the study of quasiregular mappings between planar Euclidean domains to the study of quasiregular homeomorphisms, i.e. quasiconformal mappings. Indeed, such a quasiregular mapping can be expressed as a composition of an analytic mapping and a quasiconformal mapping. Similarly, the global theory of quasiregular mappings between Riemannian surfaces reduces to the corresponding theory of analytic mappings. In higher dimensions the connection to methods arising from complex analysis is much weaker. We refer to [1] and [2] for details on the planar theory.

The theory of quasiregular mappings in dimensions $n \geq 3$ was initiated by Reshetnyak in the 1960's with a series of papers. By fundamental results of Reshetnyak, non-constant quasiregular mappings are sense-preserving, discrete, and open. Furthermore, these results include local Hölder continuity and almost everywhere differentiability of quasiregular mappings, see e.g. [31] or [35].

In 1969-1971, after Reshetnyak's foundational articles, Martio, Rickman, and Väisälä developed in a series of papers ([23], [24], and [25]) foundations of the metric and topological theory of quasiregular mappings. Geometric methods such as the capacity of a condenser and the modulus of a path family together with topological tools such as the local topological index of an open mapping led to metric and geometric definitions of quasiregular mappings, distortion theorems, and results on the properties of the branch set of non-constant quasiregular mappings. In Section 2 we discuss one of these results, an estimate for the injectivity radius of quasiregular local homeomorphisms, in more detail.

In the 1980's in a series of papers Rickman gave a description of the global behavior of entire quasiregular (or quasimeromorphic) mappings, i.e. quasiregular mappings from $\mathbb{R}^n$ into $\mathbb{S}^n$, by developing a value distribution theory for quasiregular mappings. These results include the Picard theorem and a version of Ahlfors's defect relation. Rickman's Picard theorem is discussed in more detail in Section 3. For a discussion on the defect relation and a comprehensive presentation on the geometric theory of quasiregular mappings in Euclidean spaces, we refer to Rickman's monograph [35].

Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 2$. A mapping $f\colon \Omega \to \mathbb{R}^n$ is called *a mapping of finite distortion*, with a measurable distortion function $K\colon \Omega \to [1,\infty]$, if $f$ is in the class $W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$, the Jacobian determinant of $f$ is locally integrable, and

$$\|Df(x)\|^n \leq K(x)J(x,f)$$

for a.e. $x \in \Omega$.

We obtain from the inequality that mappings of finite distortion, with $K \in L^\infty$, belong to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. Hence mappings of finite distortion extend the class of quasiregular mappings. Without any additional restrictions on the distortion, mappings of finite distortion do not possess analytical and topological properties similar to quasiregular mappings. Let us briefly discuss sharp additional assumptions on distortion, which guarantee these properties. Mappings of finite distortion are under rapid development and for an exposition on aspects of the theory we refer to the book of Iwaniec and Martin [16], see also e.g. [3], [15], [18], [A, Section 2], and references therein.

Following a recent work of Kauhanen, Koskela, Onninen, Malý, and Zhong [18] we say that a mapping $f$ of finite distortion $K$ satisfies *condition* $(A)$ if there exists an Orlicz-function $\mathcal{A}\colon [0,\infty) \to [0,\infty)$, see e.g. [A, Section 2] for the definition, satisfying

$$\exp(\mathcal{A}(K(\cdot))) \in L^1_{\mathrm{loc}}(M), \tag{A-0}$$

$$\int_1^\infty \frac{\mathcal{A}'(t)}{t}\,\mathrm{d}t = \infty, \text{ and} \tag{A-1}$$

$$t \mapsto t\mathcal{A}(t) \text{ increases to } \infty \text{ as } t \to \infty. \tag{A-2}$$

In [18], condition $(A)$ is shown to be the sharp Orlicz-condition which leads to a mapping class with properties similar to quasiregular mappings. That is, non-constant mappings of finite distortion satisfying condition $(A)$ are continuous, discrete, and open with almost everywhere positive Jacobian determinant. Furthermore, these mappings preserve sets of Lebesgue measure zero.

In [A] and [B], local homeomorphisms of finite distortion between Riemannian manifolds are studied. Since our considerations are restricted to local homeomorphisms, we assume *a priori continuity* in our definition of mappings of finite distortion between Riemannian manifolds; see [A, Section 2]. For a more general study of weakly differentiable mappings between manifolds, we refer to the recent article of Hajłasz, Iwaniec, Onninen, and Malý [10].

The geometric theory of mappings of finite distortion is under an extensive study. In [19], Koskela and Onninen showed that mappings of finite distortion satisfying condition $(A)$ enjoy capacity and modulus inequalities similar to those available for quasiregular mappings; see Section 2 for more details. By using these inequalities, Koskela, Onninen, and Rajala proved that local homeomorphisms of finite distortion have a uniform injectivity radius depending only on the dimension of the space and the distortion function of the mapping [20]. Furthermore, Rajala generalized a theorem of Dairbekov on removable sets for quasiregular local homeomorphisms to local homeomorphisms of finite distortion [30]. Especially we would like to note that also generalizations of the Picard theorem are under investigation, see [21] and [29].

In order to describe the geometric assumptions on the manifolds in more detail, let us introduce the notion of a capacity. A set $F \subset M$ is said to have *zero $p$-capacity* for $p \in [1, \infty)$, if for every compact subset $C \subset F$ and every open set $\Omega \subset M$ containing $C$ we have

$$\mathsf{cap}_p(\Omega, C) := \inf_u \int_M |\nabla u|^p \, \mathrm{d}m = 0,$$

where the infimum is taken over all functions $u \in C_0^\infty(M)$ such that $u|C \geq 1$. Furthermore, $M$ is said to be $p$-parabolic, if every compact subset of $M$ has zero $p$-capacity with respect to $M$, i.e. $\mathsf{cap}_p(M, C) = 0$ for every compact set $C \subset M$. Otherwise, we say that $M$ is $p$-hyperbolic. Due to the conformal invariance of $n$-parabolicity and $n$-hyperbolicity on Riemannian $n$-manifolds, we also refer to these properties as *conformal parabolicity* and *conformal hyperbolicity*, respectively. For other characterizations of parabolicity, see e.g. [11, Section 5] and [13].

Let us consider the presence of this classification of Riemannian manifolds in the Euclidean theory of quasiregular mappings. Given a closed set $E \subset \mathbb{R}^n$, simple arguments show that $\mathbb{R}^n \setminus E$ is conformally parabolic if and only if $E$ has zero $n$-capacity in $\mathbb{R}^n$. Especially $\mathbb{R}^n$ is conformally parabolic. A fundamental result in the Euclidean theory is that a non-constant entire quasiregular mapping cannot omit a set of positive $n$-capacity, see e.g. [35, III.2.12]. In the setting of Riemannian manifolds this result takes a form that non-constant quasiregular mappings preserve conformal parabolicity in the sense that the image of a conformally parabolic manifold under a non-constant quasiregular mapping is a conformally parabolic submanifold of the target manifold. Since a conformally hyperbolic manifold does not contain any conformally parabolic open submanifolds, we may state this result in another form that every quasiregular mapping from a conformally parabolic manifold into a conformally hyperbolic manifold is constant. See e.g. [5, 5.1] for a detailed discussion.

We also study quasiregular mappings defined on $B^n \setminus \{0\}$, where $B^n$ is the unit ball of $\mathbb{R}^n$. In this case our domain is not conformally parabolic, but it has a parabolic end at the origin in the sense that the boundary component $\{0\}$ of $B^n \setminus \{0\}$ in $\mathbb{R}^n$ is a set of zero $n$-capacity. For our purposes it is not necessary to give a more general definition of a parabolic end of a Riemannian manifold, but such a definition can be acquired from the alternative definition of $n$-parabolicity given in Section 2 and the definition of an end of a manifold in Section 3. In general, the parabolic end at the origin does not impose any additional restrictions to quasiregular mappings defined in $B^n \setminus \{0\}$. However, if the origin is an essential singularity of a quasiregular mapping defined in a neighborhood of the origin, then the image of the mapping is always conformally parabolic, see e.g. [C, Lemma 3.1].

For mappings of finite distortion, capacity inequalities indicate that a natural counterpart for conformally parabolic manifolds are weighted conformally parabolic manifolds. We discuss this subject further in Section 2.

We are now ready to formulate two questions which are discussed in the forthcoming sections.

*First question:* Given a local homeomorphism of finite distortion $f\colon M \to N$ and a weighted conformally parabolic manifold $M$ with a weight comparable to the distortion of $f$, what is the relation between multiplicity of the mapping and the fundamental group of the manifold $N$?

*Second question:* Given a quasiregular mapping $f\colon B^n \setminus \{0\} \to N$, which additional topological conditions on $N$ guarantee that $f$ has a limit at the origin?

In the following section we show that generalizations of Zorich's theorem for mappings of finite distortion give an answer to the first question. With respect to the second question, we consider separately the cases when $N$ is compact or non-compact. For non-compact target manifolds, we give an answer to this question in terms of the number of ends of $N$ in Section 3. In Section 4 we show that for compact target manifolds, the limit exists when the dimension of the de Rham cohomology ring of $N$ exceeds a bound given in terms of the dimension of $N$ and the dilatation $K$ of the mapping $f$.

## 2. Zorich's theorem for mappings of finite distortion between Riemannian manifolds

Zorich's theorem for entire quasiregular mappings exhibits the striking difference between planar and spatial theory of quasiregular mapping. Zorich's theorem reads as follows.

**Theorem 1** ([37])**.** *For $n \geq 3$ every quasiregular local homeomorphism from $\mathbb{R}^n$ into itself is a homeomorphism.*

For quasiregular mappings from $\mathbb{R}^2$ into itself this theorem fails as the exponential mapping $z \mapsto \exp(z)$ reveals. For $n \geq 3$, a result of Martio, Rickman, and Väisälä on the injectivity radius of quasiregular local homeomorphisms yields Zorich's theorem as a corollary.

**Theorem 2** ([25, 2.3])**.** *If $n \geq 3$ and if $f\colon B^n \to \mathbb{R}^n$ is a $K$-quasiregular local homeomorphism, then $f$ is injective in $B^n(\psi(n,K))$, where $\psi(n,K)$ is a positive constant depending only on $n$ and $K$.*

Indeed, given a quasiregular local homeomorphism from $\mathbb{R}^n$ into itself we have, by scaling and Theorem 2, that $f$ is injective in $B^n(R)$ for every $R > 0$. Thus Zorich's theorem follows.

In [20], Koskela, Onninen, and Rajala showed that local homeomorphisms of finite distortion satisfying condition $(A)$ possess a similar local injectivity property. Hence local homeomorphisms of finite distortion from $\mathbb{R}^n$ into itself are homeomorphisms. We refer to [20] for the precise statement.

When studying quasiregular mappings or mappings of finite distortion between Riemannian manifolds we readily note that this kind of scaling argument is not available. However, for quasiregular mappings, methods of the proof of Theorem 1 can be extended to the setting of Riemannian manifolds.

**Theorem 3** ([38])**.** *Let $n \geq 3$, $M$ a conformally parabolic Riemannian $n$-manifold, and $N$ a simply connected Riemannian $n$-manifold. Then every quasiregular local homeomorphism from $M$ into $N$ is an embedding. Moreover, the set $N \setminus fM$ has zero $n$-capacity.*

In the literature, this version of Zorich's theorem is attributed to Gromov and known as the geometric version of the global homeomorphism theorem. Indeed, in [8], Gromov states that the proof of Theorem 1 has a wider range containing Theorem 3, see also [9, pp. 336]. To author's knowledge, the proof of Theorem 3 was discussed first time in detail in [38]. For quasiregular local homeomorphisms from conformally parabolic submanifolds of $\mathbb{S}^n$ into $\mathbb{S}^n$, this result follows from theorems of Dairbekov [6] and Martio and Srebro [26].

In [A], Holopainen and the author showed that Theorem 3 generalizes to mappings of finite distortion, when parabolicity of the domain is interpreted properly.

**Theorem 4** ([A, Theorem 1])**.** *Let $n \geq 3$. Given a $K^{n-1}$-parabolic $n$-manifold $M$, where $K\colon M \to [1,\infty]$ is a measurable function satisfying condition $(A)$, and a simply connected manifold $N$, then every local homeomorphism $f\colon M \to N$ of finite distortion $K$ is an embedding. Furthermore, $N \setminus fM$ has zero $n$-capacity.*

In order to define the weighted parabolicity, let us first define the $w$-weighted $p$-modulus of a path family. Let $w\colon M \to [0,\infty]$ be a Borel function and $p \in [1,\infty)$. We define the $w$-weighted $p$-modulus $\mathsf{M}_{p,w}(\Gamma)$ of a path family $\Gamma$ by

$$\mathsf{M}_{p,w}(\Gamma) = \inf_{\rho} \int_M \rho^p w \,\mathrm{d}x,$$

where the infimum is taken over all Borel functions $\rho\colon M \to [0,\infty]$ satisfying

$$\int_\gamma \rho \,\mathrm{d}s \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$.

We say that an $n$-manifold $M$ is $w$-parabolic for given measurable function $w\colon M \to [0,\infty]$ if $\mathsf{M}_{n,w}(\Gamma_M^\infty) = 0$, where $\Gamma_M^\infty$ is the family of all paths $\gamma$ in $M$ for which the locus of $\gamma$ is not relatively compact in $M$, see [A, Section 3].

To show that the definition of $w$-parabolicity extends the definition of $n$-parabolicity given in Section 1 it is sufficient to note the following. For $w \equiv 1$ we recover the $p$-modulus $\mathsf{M}_p(\Gamma)$ of a path family $\Gamma$. Furthermore, given a domain $\Omega \subset M$ and a compact set $C \subset \Omega$ then

$$\mathsf{cap}_p(\Omega, C) = \mathsf{M}_p(\Gamma_\Omega^\infty \cap \Gamma_C),$$

where $\Gamma_C$ is the family of paths intersecting $C$. See [35, II.10.2] for details. Hence the class of $w$-parabolic manifolds with $w \equiv 1$ is exactly the class of $n$-parabolic manifolds.

The most fundamental tool in the proof of Theorem 4 is the weighted version of Väisälä's inequality for mappings of finite distortion. For mappings of finite distortion between Euclidean domains this inequality is due to Koskela and Onninen [19]. In [A], we gave a modification of this proof in the case of Riemannian manifolds. For notations, see [A, Section 3].

**Theorem 5** ([A, Theorem 7])**.** *Let $f\colon M \to N$ be a continuous non-constant mapping of finite distortion $K$ satisfying condition $(A)$. Let $\Gamma$ be a path family in $M$, $\Gamma'$ a path family in $N$, and $m$ a positive integer such that the following is true: For every path $\beta\colon I \to N$ in $\Gamma'$ there are paths $\alpha_1, \ldots, \alpha_m$ in $\Gamma$ such that $f \circ \alpha_j \subset \beta$ for all $j$ and such that for every $x \in M$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x,f)$ indices $j$. Then*

$$\mathsf{M}_n(\Gamma') \leq \frac{\mathsf{M}_{n,K_I(\cdot,f)}(\Gamma)}{m}.$$

Simple examples, like the mapping $\mathbb{S}^{n-1} \times \mathbb{S}^1 \to \mathbb{S}^{n-1} \times \mathbb{S}^1$, $(x,z) \mapsto (x,z^2)$, show that simply connectedness of the target manifold is essential in obtaining injectivity of the mapping $f$ in Theorem 4. This raises a question, what can be said about the covering properties of local homeomorphisms of finite distortion from a weighted parabolic manifold into another manifold, if the assumption on the simply connectedness of the range is relaxed. In [B], we show that, apart from a small exceptional set, these mappings behave like covering mappings in the following sense. We refer to [B] for terminology.

**Theorem 6** ([B, Theorem 3])**.** *Let $n \geq 3$, $M$ a $K^{n-1}$-parabolic $n$-manifold, where $K\colon M \to [1,\infty]$ is a measurable function satisfying condition (A), $N$ a Riemannian $n$-manifold, and $f\colon M \to N$ a local homeomorphism of finite distortion $K$. Then there exists a set $E \subset N$ of zero $n$-capacity such that $f$ is an $m$-to-1 mapping on $M \setminus f^{-1}E$, where*

$$m = \text{card } (\pi_1(N)/f_*\pi_1(M)) \in \mathbb{Z}_+ \cup \{\infty\},$$

*and* card $f^{-1}(y) \leq m$ *for every $y \in E$. If $m < \infty$, then* card $f^{-1}(y) < m$ *for every $y \in E$. Moreover, $N \setminus fM \subset E$ and $E$ is the set of asymptotic limits of $f$.*

As theorems 4 and 6 show, parabolicity of the domain, when suitably interpreted, transforms the local rigidity of a mapping of finite distortion into a global rigidity.

In [B], we also consider characterizations of parabolic and weighted parabolic manifolds. The main result, which is a modest extension of a result of Keselman and Zorich [39], reads as follows.

**Theorem 7** ([B, Theorem 4])**.** *Let $(M,g)$ be a Riemannian $n$-manifold and $w\colon M \to [0,\infty]$ be in $L^1_{\text{loc}}(M)$. Then $(M,g)$ is $w$-parabolic if and only if there exists a $C^\infty$-function $\lambda\colon M \to (0,\infty)$ such that the manifold $(M,\lambda g)$ is complete and*

$$\|w\|_{L^1(M,\lambda g)} = \int_M w\lambda^{n/2}\, \mathrm{d}m_g < \infty.$$

*Here $m_g$ is the measure given by the Riemannian metric $g$.*

## 3. Picard type theorems for quasiregular mappings

At a very early stage of the theory of quasiregular mappings it was discovered that an entire non-constant quasiregular mapping cannot omit a set of positive capacity. The proofs of this result did not impose any finite bound on the cardinality of the omitted set and it was conjectured for a long time that an entire quasiregular mapping into $\mathbb{S}^n$ could not omit more than two points also in dimensions $n \geq 3$. In [32], Rickman proved the Picard theorem for quasiregular mappings, that is, for non-constant entire quasiregular mappings there exists a finite bound on the cardinality of the omitted set. This bound is given in terms of the dimension of the space and the distortion of the mapping. In dimension three, this result is known to be qualitatively best possible by Rickman's construction in [33]. Although it is possible to construct, by elementary methods, an entire quasiregular mapping into $\mathbb{R}^n$ omitting one point, no other method apart from Rickman's construction in [33] is known how to construct entire quasiregular mappings omitting more than one point. It is also an open question whether Rickman's Picard theorem is sharp in dimensions above three.

Rickman's proof for the Picard theorem for quasiregular mappings is based on a careful analysis of the behavior of the averaged counting function of the mapping and sharp estimates on the moduli of path families. In these estimates, the geometry of the Euclidean space, or actually $\mathbb{S}^n$, is involved, making this approach difficult to generalize. In [7], Eremenko and Lewis gave a potential theoretic method to prove Rickman's theorem. This method was further simplified by Lewis in [22]. Lewis's approach, with Harnack functions, to the Picard-Rickman theorem is not restricted to Euclidean spaces but allow generalizations into Riemannian manifolds. In [12] and [14], Holopainen and Rickman used Lewis's method to generalize the Picard theorem first for quasiregular mappings from $\mathbb{R}^n$ into manifolds with many ends and then for quasiregular mappings from manifolds with controlled geometry to manifolds with many ends.

The original method of Rickman in [32] gives also a proof for the corresponding big Picard type theorem. That is, there exists a number $q$ depending only on the dimension and the dilatation $K$ such that if a $K$-quasiregular mapping from $B^n \setminus \{0\}$ to $\mathbb{R}^n$ omits more than $q$ points, then there exists a limit at the origin. This limit can also be infinity. In [C], Holopainen and the author showed that Lewis's method, as further developed by Holopainen and Rickman, can be used to obtain a big Picard type theorem for quasiregular mappings from $B^n \setminus \{0\}$ into manifolds with many ends.

**Theorem 8** ([C, Theorem 1.3])**.** *Let $N$ be a Riemannian $n$-manifold. For every $K \geq 1$ there exists $q = q(K,n)$ such that every $K$-quasiregular mapping $f \colon B^n \setminus \{0\} \to N$ has a removable singularity at the origin if $N$ has at least $q$ ends.*

We say that a manifold $N$ has at least $q$ ends, if there exists a compact set $E \subset N$ such that $N \setminus E$ has at least $q$ components that are not relatively compact. For the definition of a removable singularity in this setting, see [C, Section 1].

## 4. de Rham cohomology and quasiregular mappings

A highly interesting open problem in the theory of quasiregular mappings is the classification of *quasiregularly elliptic manifolds*, that is, manifolds admitting non-constant quasiregular mappings from $\mathbb{R}^n$.

To be more precise, we say that a connected oriented Riemannian $n$-manifold $N$ is *$K$-quasiregularly elliptic* if there exists a non-constant $K$-quasiregular mapping from $\mathbb{R}^n$ into $N$. A manifold is *quasiregularly elliptic* if it is $K$-quasiregularly elliptic for some $K \geq 1$.

In dimension two, this question can be fully answered. By quasiregular Liouville theorem, the only connected and oriented Riemannian 2-manifolds receiving non-constant quasiregular mappings from $\mathbb{R}^2$ are $\mathbb{R}^2$ itself, the 2-sphere $\mathbb{S}^2$, and the 2-torus $T^2$. See e.g. [4] for details.

In higher dimensions, no such characterization is known. As described in the introduction, standard capacity estimates show that quasiregularly elliptic manifolds are conformally parabolic. Furthermore, the Picard type theorem of Holopainen and Rickman shows that $K$-quasiregularly elliptic manifolds have a bounded number of ends in terms of the dimension and $K$. Since closed manifolds, that is, compact manifolds without boundary, meet these two requirements trivially, we may ask which closed manifolds are quasiregularly elliptic. This question was originally posed by Gromov and Rickman, see e.g [8], [9], and [34]

In [17, 1.3] Jormakka showed, by assuming the Geometrization Conjecture, that all quasiregularly elliptic 3-manifolds are quotients of $\mathbb{S}^3$, $\mathbb{R}^3$, or $\mathbb{S}^2 \times \mathbb{R}$. In general, it is easy to see that $\mathbb{S}^n$ and $T^n$ are quasiregularly elliptic for every $n \geq 2$. Furthermore, one can show that also manifolds $\mathbb{S}^k \times \mathbb{S}^{n-k}$ and $\mathbb{S}^k \times T^{n-k}$ are quasiregularly elliptic. On the other hand, Peltonen showed in [28] that given a closed $n$-manifold $N$ the manifold $T^n \# N$ is not quasiregularly elliptic, if $H^m(N) \neq 0$ for some $m \in \{1, \ldots, n-1\}$. Here $H^m(N)$ is the $m$-th de Rham cohomology group of $N$. It was an open question until very recently whether the connected sum of $\mathbb{S}^2 \times \mathbb{S}^2$ with itself, i.e. $\mathbb{S}^2 \times \mathbb{S}^2 \# \mathbb{S}^2 \times \mathbb{S}^2$, is quasiregularly elliptic. Rickman gave an affirmative answer to this question in [36].

In [4], Bonk and Heinonen showed that quasiregularly elliptic manifolds have bounded cohomology in the following sense.

**Theorem 9** ([4, Theorem 1.1])**.** *If $N$ is a closed $K$-quasiregularly elliptic $n$-manifold, $n \geq 2$, then*

$$\dim H^*(N) \leq C(n, K),$$

*where* $\dim H^*(N)$ *is the dimension of the de Rham cohomology ring $H^*(N)$ of $N$ and $C(n, K)$ is a constant depending only on $n$ and $K$.*

In [D], we consider quasiregular mappings from a punctured $n$-ball into closed manifolds. Picard type theorems for quasiregular mappings suggest that the theorem of Bonk and Heinonen has a local counterpart for this class of mappings. This is indeed the case.

**Theorem 10** ([D, Theorem 2])**.** *Let $n \geq 2$ and $K \geq 1$. There exists a constant $C'(n, K)$ depending only on $n$ and $K$ such that whenever*

*$f\colon B^n\setminus\{0\}\to N$ is a $K$-quasiregular mapping into a closed, connected, and oriented Riemannian $n$-manifold $N$ with* $\dim H^*(N) > C'(n,K)$, *the limit of $f$ at the origin exists.*

The $n$-torus $T^n$ shows that both constants $C(n,K)$ and $C'(n,K)$ are at least $\dim H^*(T^n) = 2^n$ for every $K \geq 1$. The exact values of these constants are not known.

Theorem 10 together with a short argument give the theorem of Bonk and Heinonen. Indeed, let $K \geq 1$ and $N$ be a closed, connected, and oriented Riemannian $n$-manifold such $\dim H^*(N) \geq C'(n,K)$. By Theorem 10, every $K$-quasiregular mapping $f\colon \mathbb{R}^n \to N$ has a limit at the infinity. Hence the averaged counting function of $f$ is bounded. By [4, Theorem 1.11], for every non-constant quasiregular mapping from $\mathbb{R}^n$ into $N$ the averaged counting function is unbounded, since $H^\ell(N) \neq 0$ for some $\ell \in \{1,\ldots,n-1\}$. Hence $f$ is constant.

The proof of Theorem 10 employs the method of pull-backing $p$-harmonic forms under quasiregular mappings as in the proof of Theorem 9, but instead of a scaling argument that was used in [4], we organize the proof around a ball decomposition method due to Rickman. Both proofs use the value distribution result [27, 5.11] of Mattila and Rickman to relate the information on the cohomology of the target manifold, via $p$-harmonic forms, to the averaged counting function of $f$. For the ball decomposition method we require an estimate for the lower growth of the averaged counting function $A(\cdot;f)$ of $f$. For the definition of $A(\cdot;f)$, see e.g. [D, Section 3].

**Theorem 11** ([D, Theorem 14])**.** *Let $N$ be a closed, connected, and oriented Riemannian $n$-manifold such that $H^\ell(N) \neq 0$ for some $\ell \in \{2,\ldots,n-2\}$, and let $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ be a $K$-quasiregular mapping having an essential singularity at the infinity. Then there exists constants $C_0 > 1$ and $\lambda > 1$ depending only on $n$ and $K$ such that*

$$\liminf_{t\to\infty} \frac{A(\lambda t;f)}{A(t;f)} \geq C_0.$$

*Furthermore, there exists $\alpha > 0$ depending only on $n$ and $K$ such that*

$$\liminf_{t\to\infty} \frac{A(t;f)}{t^\alpha} > 0.$$

This theorem corresponds to [4, Theorem 1.11] which states that the averaged counting function of a non-constant quasiregular mapping from Euclidean $n$-space into a closed manifold $N$ grows like a power, if $\dim H^\ell(N) \neq 0$ for some $\ell \in \{1,\ldots,n-1\}$. We show by an example in [D, Section 5] that the stricter assumption on the cohomology of $N$ is necessary in Theorem 11.

## 5. Errata

Article [B] contains following errors known to the author.

1. Page 192: line 3 should be

$$m = \text{card}\,(\pi_1(N)/f_*\pi_1(M)) \in \mathbb{Z}_+ \cup \{\infty\},$$

2. Page 194: formula (3) should be

$$\|w\|_{L^1(M,\lambda g)} = \int_M w\lambda^{n/2}\,\mathrm{d}m_g < \infty. \tag{3}$$

3. Page 195: line 5 should be

$$\lambda = \left(\varphi + \sum_{i=1}^{\infty}\|\nabla u_i\|^n\right)^{2/n}$$

4. Page 197: The sentence "For class $c \in \pi_1(N)/f_*\pi_1(M)$ choose a loop $\alpha_c$ starting from $y$" in line 3 should be "For a class $c \in \pi_1(N)/f_*\pi_1(M)$ choose a loop $\alpha_c$ starting from $y$ such that the homotopy class of $\alpha_c$ belongs to $c$".

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