CONSTRUCTIVE NONSTANDARD ANALYSIS WITHOUT ACTUAL INFINITY

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Academic dissertation

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1 Introduction

At least from the computational point of view we can only possess and process a finite amount of finitely precise information in a finitely long period of time. If we do not take into account any physical or other limitations to available space and time resources, then we may allow us to possess and process indefinitely large yet finite amount of indefinitely yet finitely precise information in an indefinitely yet finitely long period of time. But can we make any sense of this? Moreover, if we require that mathematical objects also be on a par with the above description, how can we then deal with infinite objects, like real numbers and the continuum, at all? It is clear that some nonstandard ideas are needed here.

Nonstandard analysis was invented by Robinson (1961, 1966). We refer to Albeverio et al. (1986) and Loeb and Wolff (2000) for a modern exposition. What interests us in nonstandard analysis is its use of hyperfinite sets, i.e. sets the elements of which can be enumerated from one up to an infinite, or hyperfinite, natural number. We might now consider a hyperfinite natural number as standing for an indefinitely large yet finite natural number. A real number could then be a hyperfinite sequence of rational numbers and the continuum a hyperfinite infinitesimal grid of real numbers. There is a problem, however. Tennenbaum's (1959) theorem says that even in a countable nonstandard model of Peano arithmetic addition and multiplication cannot be computable. For a proof of the theorem, see for instance Smoryński (1984), and for a discussion of its implications, see Kossak (1996).

The possibility of constructivization of nonstandard analysis has been studied thoroughly by Palmgren (1997, 1998, 2001). The model of constructive nonstandard analysis studied there is an extension of Moerdijk's (1995) model for constructive nonstandard arithmetic. The model satisfies many useful principles, e.g. the full transfer, overspill and underspill, idealization and countable saturation. But as in general in nonstandard analysis, the problem of interpretation of nonstandard results in terms of standard analysis remains due to the dichotomy between standard and nonstandard made in nonstandard analysis. In our opinion this approach do not manage to catch the idea of "indefinitely large yet finite" that we are looking for.

Schmieden and Laugwitz (1958) introduced a calculus of infinitesimals already before Robinson presented his nonstandard analysis. Their calculus has recently been reinvented and developed further by Henle (1999, 2003). The calculus has a distinct computational flavour. To show this, we present next the underlying idea of a slight generalization of the calculus by Laugwitz (1983). So, let T be a theory containing the theories of natural and rational numbers and perhaps some elementary set theory. Add a new number constant Ω to the language of T and define an extension $T(\Omega)$ of T by the following infinitary definition (called Basic Definition):

> (BD) Let $S(n)$ be a sentence in the language of T for all $n = 1, 2, 3, \ldots$. If $S(n)$ is a theorem of T from some point on, then $S(\Omega)$ belongs to $T(\Omega)$ by definition.

Now, if q_n is a rational number in T for all n, then q_Ω is called an Ω-rational number. It follows from (BD) that the Ω-rational numbers make up an ordered field. For instance, since

$$q_n < 0 \text{ or } q_n = 0 \text{ or } q_n > 0$$

is a theorem of T for all n,

$$q_\Omega < 0 \text{ or } q_\Omega = 0 \text{ or } q_\Omega > 0$$

belongs to $T(\Omega)$ by (BD). It is possible that none of the disjuncts belongs to $T(\Omega)$, so the interpretation of the disjunction in $T(\Omega)$ is far from being standard. But we have nevertheless, and more generally, the following:

> (I) If $\{S_1(\Omega), \ldots, S_p(\Omega)\} \subseteq T(\Omega)$ and $\{S_1(\Omega), \ldots, S_p(\Omega)\} \vdash S(\Omega)$, then $\{S_1(n), \ldots, S_p(n)\} \subseteq T$ and $\{S_1(n), \ldots, S_p(n)\} \vdash S(n)$ from some point on.

In this way meaning can be given to theorems of $T(\Omega)$, though this is not a compositional way. (In the 1958 article, (BD) is only applied to binary relations on the rational numbers, the interpretation of the logical operations is standard and hence the Ω-rational numbers just make up a partially ordered ring.) As above, if m_n is a natural number in T for all n, then m_Ω is called an Ω-natural number. It is said to be standard if there is a natural number N such that $m_n = N$ is a theorem of T from some point on. It is said to be infinite if for all natural numbers N, $m_n > N$ is a theorem of T from some point on. An Ω-rational number q_Ω is said to be infinitesimal if $|q_\Omega| < 1/N$ belongs to $T(\Omega)$ for all standard natural numbers N. Note that $\Omega > N$ belongs to $T(\Omega)$ for all standard natural numbers N, since $n > N$ is a theorem of T from some point on. Even though the flavour of this approach is clearly computational, classical logic is used throughout.

Martin-Löf (1990, 1999) worked out a constructive approach to nonstandard analysis along the above lines in his type theory (see Martin-Löf (1984)). Palmgren (1995, 1996) studied the idea in a constructive setting of Heyting arithmetic in all finite types $\mathbf{HA}^\omega$ using standard interpretation of logic. It turned out that then the transfer principle does not extend to all formulas of the language. Transfer principles in nonstandard extensions of Heyting arithmetic **HA** have been studied by Moerdijk and Palmgren (1997) and Avigad and Helzner (2002). Various weak theories of nonstandard arithmetic and analysis have been studied by Chuaqui and Suppes (1995), Sommer and Suppes (1996) and Avigad (2004). Zeilberger (2004) has recently suggested a finite foundation of mathematics. All these theories have problems in interpreting nonstandard results in terms of standard results.

We study in this thesis an approach invented by Mycielski (1980-81, 1981) that does not go beyond first-order arithmetic. He adds to the language of arithmetic a new constant symbol ∞_p for every rational number p and a new axiom $\infty_p \geq t$ for every term t containing no variables and no constant symbols ∞_q with $q \geq p$. Because of the new axioms, we can express with ∞_p's bounds, orders of magnitude and dependences. In particular, we can eliminate quantifiers with them. All the mathematical objects that we study will be finite, so there are no infinite numbers nor infinitesimals. Also, we do not model the new constant symbols and axioms in their totality. Instead, using a generalization of an idea of Martin-Löf (1990) similar to (I) above, we convert proofs of theorems with ∞_p's into proofs of theorems without ∞_p's. We believe this approach does manage to catch quite well the idea of "indefinitely large yet finite" that we have been looking for.

The content of this thesis is briefly as follows. In Chapter 2 we put up the framework. We present the formal system **HA** of Heyting arithmetic and its extension $\mathbf{HA}^*$ together with a conversion algorithm transforming proofs of theorems of the latter into proofs of theorems of the former. We also show how a theory of finite sets can be encoded in these systems. In Chapters 3, 4 and 5 we will work inside $\mathbf{HA}^*$. In Chapter 3 we go through some elementary analysis. In Chapter 4 we give a definition of an L^p-space and prove some results. In Chapter 5 we give definitions of some basic notions of probability theory, prove versions of weak and strong laws of large numbers and derive the Black-Scholes formula for the value of the European call option. Finally in Chapter 6 we give some simple examples of the conversion algorithm given in Chapter 2.

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2 Framework

2.1 Formal Systems HA and HA*

In this section we first describe shortly the intuitionistic first-order theory of arithmetic, the so-called Heyting arithmetic commonly denoted by **HA**, and then a certain extension of it which we denote by **HA***. The language of **HA** consists of a constant symbol 0, a unary function symbol S, a function symbol for every primitive recursive function and a binary relation symbol $=$. We assume for convenience that the binary relation symbol $\leq$ is also contained in the language. Terms and atomic formulas are built up as usual. Complex formulas are built up from atomic formulas by using the connectives $\vee$, $\wedge$ and $\rightarrow$ as well as the quantifiers $\exists$ and $\forall$. We define $\varphi \leftrightarrow \psi$ as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\neg\varphi$ as $\varphi \rightarrow 0 = S(0)$. The axioms of **HA** are the universal closures of the following formulas: the defining equation(s) for every primitive recursive function; the defining axiom

$$x \leq y \leftrightarrow (\exists z)(x + z = y);$$

the induction axiom

$$\varphi(0, \bar{z}) \wedge (\forall x)(\varphi(x, \bar{z}) \rightarrow \varphi(S(x), \bar{z})) \rightarrow (\forall x)\varphi(x, \bar{z})$$

for every formula $\varphi(u, \bar{z})$; the equality axioms

$$x = x,$$

$$x = y \wedge \varphi(x, \bar{z}) \rightarrow \varphi(y, \bar{z})$$

for every formula $\varphi(u, \bar{z})$. Heyting arithmetic **HA** is thus much like Peano arithmetic **PA** except that its underlying logic is intuitionistic predicate logic. Note that the "ex falso quodlibet" rule $0 = S(0) \Rightarrow \varphi$ is admissible in **HA** for every formula φ. For further information on intuitionistic logic and **HA**, see for instance Dragalin (1988), Troelstra (1973) or Troelstra and van Dalen (1988).

The extension **HA*** is the following. Its language consists of the language of **HA** together with a new constant symbol ∞_p for every rational number

p. Terms and formulas are built up as before. The axioms of $\mathbf{HA}^*$ are the axioms of $\mathbf{HA}$ together with a new axiom $\infty_p \geq t$ for every rational number p and for every closed term t of $\mathbf{HA}^*$ containing no constant symbols ∞_q with $q \geq p$. If a closed term t contains ∞_q with $q < p$, we say that ∞_p depends on ∞_q in the axiom $\infty_p \geq t$. Adding these new constant symbols and axioms to a theory of arithmetic is the main innovation of Mycielski (1980-81, 1981).

A constant symbol ∞_p should be thought of as an indefinitely large yet finite natural number instead of a potentially or otherwise infinite natural number, cf. the discussion in Lavine (1995, 1998). The indices run over the rational numbers instead of the natural numbers just for convenience: there is never need to renumber indices when writing proofs. Moreover, having the rational numbers as the index set gives a nice analogue to the dense linear order type $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ of any countable nonstandard model of Peano arithmetic. Note that here the "nonstandard" natural numbers are put not after the "standard" natural numbers but among them and, so to say, only "on demand".

We believe that with very few changes we could actually work in primitive recursive arithmetic **PRA** instead of Heyting arithmetic **HA**.

2.2 Conversion of Proofs of HA* into Proofs of HA

In this section we show some properties of our framework. Note first that the following transfer principle holds trivially since every axiom of **HA** is also an axiom of $\mathbf{HA}^*$:

Theorem 2.1. *Let φ be a formula of* **HA**. *If* $\mathbf{HA} \vdash \varphi$, *then* $\mathbf{HA}^* \vdash \varphi$.

The next theorem is quite important to this work. Its proof is a straightforward generalization of a proof of a corresponding theorem of Martin-Löf (1990), see also Palmgren (1993, 1995). It shows how proofs of theorems of $\mathbf{HA}^*$ can be converted into proofs of theorems of **HA** and thus gives computational content to them. It also (partly) motivates the way we formulate definitions in subsequent chapters. Some examples of the conversion will be given in Chapter 6.

Theorem 2.2. *Let $p_1 < p_2 < \cdots < p_n$ be rational numbers. If*

$$(\mathbf{HA}^* \vdash \varphi)[\infty_{p_1}, \infty_{p_2}, \ldots, \infty_{p_n}], \tag{1}$$

where our notation means that $\infty_{p_1}, \infty_{p_2}, \ldots, \infty_{p_n}$ are all the ∞_p's occurring in the proof, then

$$(\mathbf{HA} \vdash \varphi)[s_1/\infty_{p_1}, s_2/\infty_{p_2}, \ldots, s_n/\infty_{p_n}] \tag{2}$$

for some terms $s_1 = t_1 + x_1, s_2 = t_2(s_1) + x_2, \dots, s_n = t_n(s_1, s_2, \dots, s_{n-1}) + x_n$*, where* $t_1, t_2(y_1), \dots, t_n(y_1, y_2, \dots, y_{n-1})$ *are terms of* **HA** *having free variables among those indicated and* $x_1, x_2, \dots, x_n$ *are fresh variables, i.e. variables not occurring anywhere in the proof of* φ *in* $\mathbf{HA}^*$*. The converse holds, too.*

Proof. Suppose $\infty_{p_i} \geq u_{i,j}$, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m_i$, are the axioms used in the proof of φ in $\mathbf{HA}^*$. Let t_1 be the term

$$\max(0, u_{1,1}, \dots, u_{1,m_1});$$

let $t_2(y_1)$ be the term

$$\max(0, u_{2,1}(y_1), \dots, u_{2,m_2}(y_1)),$$

where each occurrence of ∞_{p_1} in $u_{2,j}$ has been replaced by y_1 for all $j = 1, 2, \dots, m_2$; and so on. Finally, let $t_n(y_1, y_2, \dots, y_{n-1})$ be the term

$$\max(0, u_{n,1}(y_1, y_2, \dots, y_{n-1}), \dots, u_{n,m_n}(y_1, y_2, \dots, y_{n-1})),$$

where each occurrence of ∞_{p_i} in $u_{n,j}$ has been replaced by y_i for all $i = 1, 2, \dots, n-1$ and $j = 1, 2, \dots, m_n$. Put then $s_1 = t_1 + x_1$, $s_2 = t_2(s_1) + x_2, \dots, s_n = t_n(s_1, s_2, \dots, s_{n-1}) + x_n$, where $x_1, x_2, \dots, x_n$ are fresh variables, and replace each occurrence of ∞_{p_i} by s_i in the proof of φ in $\mathbf{HA}^*$. Since each axiom $\infty_{p_i} \geq u_{i,j}$ becomes provable in **HA**, we have (2) as required.

For the converse, note that the given proof of φ in **HA** is also a proof of φ in $\mathbf{HA}^*$ by the transfer principle. We modify it as follows. We replace first each occurrence of x_1 by $\infty_{p_1} - t_1$ and use the axiom $\infty_{p_1} \geq t_1$ to get $s_1 = \infty_{p_1}$; we replace then each occurrence of x_2 by $\infty_{p_2} - t_2(\infty_{p_1})$ and use the axiom $\infty_{p_2} \geq t_2(\infty_{p_1})$ to get $s_2 = \infty_{p_2}$; and so on. Finally, we replace each occurrence of x_n by $\infty_{p_n} - t_n(\infty_{p_1}, \infty_{p_2}, \dots, \infty_{p_{n-1}})$ and use the axiom $\infty_{p_n} \geq t_n(\infty_{p_1}, \infty_{p_2}, \dots, \infty_{p_{n-1}})$ to get $s_n = \infty_{p_n}$. We thus get (1) as required. □

By reflecting on the above proof we notice that we can eliminate from a proof of φ in $\mathbf{HA}^*$ those ∞_p's that do not occur in φ itself. Moreover, it is only the order of the rational numbers $p_1 < p_2 < \cdots < p_n$ that matters in the conversion, not the rational numbers themselves. Namely, if we take any rational numbers $q_1 < q_2 < \cdots < q_n$ and replace each occurrence of ∞_{p_i} by ∞_{q_i} simultaneously for all $i = 1, 2, \dots, n$ in the given proof, then the proof thus obtained gets converted into exactly the same proof as the proof we started with. The following corollary summarizes these observations:

Corollary 2.3. *Let $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$ be rational numbers. If*

$$(\mathbf{HA}^* \vdash \varphi)[\infty_{p_1}, \infty_{p_2}, \ldots, \infty_{p_n}], \tag{3}$$

then

$$(\mathbf{HA}^* \vdash \varphi)[\infty_{q_1}/\infty_{p_1}, \infty_{q_2}/\infty_{p_2}, \ldots, \infty_{q_n}/\infty_{p_n}],$$

and both proofs get converted into the same proof. Moreover, we may assume that each $\infty_{p_1}, \infty_{p_2}, \ldots, \infty_{p_n}$ occurring in the proof of φ in (3) also occurs in φ itself.

It immediately follows from the preceding corollary that $\mathbf{HA}^*$ is a conservative extension of **HA** and therefore (together with the transfer principle) that $\mathbf{HA}^*$ is equiconsistent with **HA**:

Corollary 2.4. *Let φ be a formula of* **HA**. *Then*

$$\mathbf{HA} \vdash \varphi \textit{ if and only if } \mathbf{HA}^* \vdash \varphi.$$

Theorem 2.2 also has the consequence that intuitionistic logic gets a nonstandard interpretation in $\mathbf{HA}^*$ in the sense that both the disjunction and the explicit definability properties fail:

Corollary 2.5. *If $\mathbf{HA}^*$ is consistent, then it has neither the disjunction nor the explicit definability property.*

Proof. By the transfer principle, $\mathbf{HA}^*$ proves that every natural number is either even or odd. In particular, $\mathbf{HA}^*$ proves that ∞_p is either even or odd. But by Theorem 2.2, $\mathbf{HA}^*$ neither proves that ∞_p is even nor proves that ∞_p is odd, since otherwise **HA** and thus also $\mathbf{HA}^*$ would be inconsistent. For a proof of the failure of the explicit definability property, we refer to Palmgren (1993). □

We find the failure of the disjunction property in $\mathbf{HA}^*$ to accord well with our intuitive idea of an indefinitely large yet finite natural number. However, the failure of the explicit definability property in $\mathbf{HA}^*$ seems more like an accidental feature of $\mathbf{HA}^*$. Note that Martin-Löf's (1990) in some respects similar nonstandard extension of his type theory also lacks the disjunction property but has the explicit definability property since the latter is built-in into his type theory, see Martin-Löf (1984).

2.3 Finite Sets

In this section we outline briefly how the informal developments in subsequent chapters could be expressed formally inside $\mathbf{HA}^*$. To this end, we sketch a definitional extension of both $\mathbf{HA}$ and $\mathbf{HA}^*$. As Mycielski (1981), we use Ackermann's neat encoding of finite sets of natural numbers into natural numbers for providing us with a notion of a finite set. The encoding is as follows. Every natural number y has a unique (binary) representation

$$y = 2^{x_0} + 2^{x_1} + \cdots + 2^{x_{n-1}},$$

where n and $x_0 < x_1 < \cdots < x_{n-1}$ are natural numbers. Note that if $y = 0$, then $n = 0$. We take now y to be the code of the finite set $\{x_0, x_1, \ldots, x_{n-1}\}$. For further information, see the treatment of this topic (in the subsystem $\mathbf{I\Sigma_0}(exp)$ of $\mathbf{PA}$) in Hájek and Pudlák (1998).

The definitional extension is as follows. First we add to $\mathbf{HA}$ a relation symbol $\in$ for set membership and a relation symbol $\subseteq$ for a subset relation together with their defining axioms

$$x \in y \leftrightarrow (\exists v)(\exists w < 2^x)(y = v \cdot 2^{x+1} + 2^x + w),$$
$$x \subseteq y \leftrightarrow (\forall z)(z \in x \to z \in y).$$

It is not difficult to see that the resulting theory of finite sets is extensional, i.e. that

$$\mathbf{HA} \vdash (\forall y_1)(\forall y_2)((\forall x)(x \in y_1 \leftrightarrow x \in y_2) \leftrightarrow y_1 = y_2) \tag{4}$$

holds.

We sketch next a proof of a comprehension principle strong enough for our purposes:

Theorem 2.6. *Let φ be a formula of* $\mathbf{HA}$ *not having y free. If*

$$\mathbf{HA} \vdash (\forall x)(\varphi(x) \lor \neg\varphi(x)),$$

then

$$\mathbf{HA} \vdash (\forall z)(\exists! y)(\forall x)(x \in y \leftrightarrow (x \leq z \land \varphi(x))).$$

Proof. By induction on z. When $z = 0$, let $y = 2^0$ in case $\varphi(0)$ and $y = 0$ otherwise. Suppose now that y corresponding to z has already been constructed. Then $\bar{y}$ corresponding to $z + 1$ is constructed by letting $\bar{y} = 2^{z+1} + y$ in case $\varphi(z + 1)$ and $\bar{y} = y$ otherwise. Uniqueness of y follows from extensionality (4). □

Existence of the usual set-theoretic constructions follows now as a corollary:

Corollary 2.7. **HA** *proves the following: For every natural number y there are unique natural numbers $\{y\}$, $\bigcup y$ and $\mathsf{P}y$ such that*

(*a*) $(\forall x)(x \in \{y\} \leftrightarrow x = y)$,

(*b*) $(\forall x)(x \in \bigcup y \leftrightarrow (\exists w \in y)(x \in w))$,

(*c*) $(\forall x)(x \in \mathsf{P}y \leftrightarrow x \subseteq y)$.

For all natural numbers y_1 and y_2 there are unique natural numbers $y_1 \cup y_2$, $y_1 \cap y_2$, $y_1 \setminus y_2$ and $y_1 \times y_2$ such that

(*d*) $(\forall x)(x \in y_1 \cup y_2 \leftrightarrow x \in y_1 \vee x \in y_2)$,

(*e*) $(\forall x)(x \in y_1 \cap y_2 \leftrightarrow x \in y_1 \wedge x \in y_2)$,

(*f*) $(\forall x)(x \in y_1 \setminus y_2 \leftrightarrow x \in y_1 \wedge x \notin y_2)$,

(*g*) $(\forall x)(x \in y_1 \times y_2 \leftrightarrow (\exists x_1 \in y_1)(\exists x_2 \in y_2)(x = (x_1, x_2)))$,

where $(x_1, x_2) = \{\{x_1\}, \{x_1, x_2\}\} = \{\{x_1\}\} \cup \{\{x_1\} \cup \{x_2\}\}$ is the ordered pair of x_1 and x_2. Note that if $(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$, then $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$.

Proof. (*a*) Since

$$\mathbf{HA} \vdash (\forall x)(x = y \vee x \neq y) \text{ and } \mathbf{HA} \vdash (x = y \leftrightarrow (x \leq y \wedge x = y)),$$

the claim follows from the comprehension principle. The proofs of (*b*), (*c*), (*d*), (*e*), (*f*) and (*g*) are similar. □

Next we add to **HA** a predicate symbol rel for a binary relation predicate together with its defining axiom

$$\mathrm{rel}(y) \leftrightarrow (\exists y_1)(\exists y_2)(y \subseteq y_1 \times y_2).$$

It is easy to see that if $\mathrm{rel}(y)$, then the smallest y_1 and y_2 such that $y \subseteq y_1 \times y_2$ are unique. As usual, we write $\mathrm{dom}(y)$ for y_1 and $\mathrm{rng}(y)$ for y_2. We can now add to **HA** a predicate symbol fun for a function predicate together with its defining axiom

$$\mathrm{fun}(y) \leftrightarrow (\mathrm{rel}(y) \wedge (\forall x \in \mathrm{dom}(y))(\forall z_1, z_2 \in \mathrm{rng}(y)) \\ ((x, z_1) \in y \wedge (x, z_2) \in y \rightarrow z_1 = z_2)).$$

As usual, we write $y\colon u \to v$ when $\mathrm{fun}(y)$, $\mathrm{dom}(y) = u$ and $\mathrm{rng}(y) \subseteq v$.

We often need a notion of a finite sequence in the following chapters, so we add to **HA** predicate symbols $\mathbb{N}$ and seq for natural number and sequence predicates, respectively, and define them by the axioms

$$\mathbb{N}(y) \leftrightarrow (\exists x)(y = \mathrm{v_N}(x)),$$
$$\mathrm{seq}(y) \leftrightarrow (\mathrm{fun}(y) \wedge (\exists x)(\mathbb{N}(x) \wedge \mathrm{dom}(y) = x)),$$

where $\mathrm{v_N}$ is the primitive recursive function defined by the equations

$$\begin{cases} \mathrm{v_N}(0) = 0, \\ \mathrm{v_N}(x+1) = \mathrm{v_N}(x) + 2^x. \end{cases}$$

We define the length $\mathrm{lh}(y)$ of a sequence y to be its domain $\mathrm{dom}(y)$.

Finally, we add to **HA** predicate symbols $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Q}$ for integers, positive integers and rational numbers, respectively. Their defining axioms are

$$\mathbb{Z}(y) \leftrightarrow (\exists y_1)(\exists y_2)(\mathbb{N}(y_1) \wedge \mathbb{N}(y_2) \wedge y = (y_1, y_2)),$$
$$\mathbb{Z}^+(y) \leftrightarrow (\exists y_1)(\exists y_2)(\mathbb{N}(y_1) \wedge \mathbb{N}(y_2) \wedge y_1 > y_2 \wedge y = (y_1, y_2)),$$
$$\mathbb{Q}(y) \leftrightarrow (\exists y_1)(\exists y_2)(\mathbb{Z}(y_1) \wedge \mathbb{Z}^+(y_2) \wedge y = (y_1, y_2)).$$

We could go on defining equality and order relations and arithmetic operations and proving their properties, and so on. However, we leave all this to the reader and start instead developing some constructive nonstandard analysis inside $\mathbf{HA}^*$.

3 Finite Elementary Analysis in $\mathbf{HA}^*$

In this chapter we give definitions and prove some basic theorems of elementary analysis inside $\mathbf{HA}^*$.

3.1 On Notation

All definitions, theorems and proofs will be schematic in the rational number parameters of those ∞_p's that occur in them. Vector notation will be used for the parameters as follows. We write $\vec{s}$ for the ordered sequence $s_1 < s_2 < \cdots < s_l$ of rational number parameters. When $\vec{s}_1$ and $\vec{s}_2$ are two such sequences having the same length l, we write $\vec{s}_1 < \vec{s}_2$ for

$$s_{1,1} < s_{2,1} < s_{1,2} < s_{2,2} < \cdots < s_{1,l} < s_{2,l}.$$

Moreover, we write $\vec{n}$ for the sequence $n_1, n_2, \ldots, n_l$ of variables and $\vec{n} \leq \infty_{\vec{s}}$ for

$$n_1 \leq \infty_{s_1},\ n_2 \leq \infty_{s_2},\ \ldots,\ n_l \leq \infty_{s_l}.$$

Finally, when $\vec{s}_1 < \vec{s}_2$, we write $\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}$ for

$$\infty_{s_{1,1}} \leq n_1 \leq \infty_{s_{2,1}},\ \infty_{s_{1,2}} \leq n_2 \leq \infty_{s_{2,2}},\ \ldots,\ \infty_{s_{1,l}} \leq n_l \leq \infty_{s_{2,l}}.$$

Our main use of ∞_p's will be to express dependences with their help and thus to eliminate quantifiers. To emphasize some of the relevant dependences we list the rational number parameters in question at the beginning of definitions and theorems inside the brackets $\langle , \rangle$ using $\shortmid$ as a list separator. For instance, $\langle o, h < p < r \shortmid h, t < u \rangle$ says that (1) ∞_p may depend on ∞_o and/or ∞_h, (2) ∞_r may depend on ∞_o, ∞_h and/or ∞_p, (3) ∞_u may depend on ∞_h and/or ∞_t. Note that ∞_o may also depend on ∞_h or vice versa and ∞_h may also depend on ∞_t or vice versa even though $\langle o, h < p < r \shortmid h, t < u \rangle$ does not take a stand on that.

3.2 Real Numbers

We use predicate symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ (together with their defining axioms) also in $\mathbf{HA}^*$ even though their extensions are in a sense larger in $\mathbf{HA}^*$ than in

HA. But since these extensions are not finite sets, we cannot, for instance, take the extension of $\mathbb{Q}$ to play the role of the real line in $\mathbf{HA}^*$. Therefore, as Mycielski (1980-81, 1981), we split it into more and more saturated finite sets:

Definition 3.1. $\mathbb{Q}_t = \{z/\infty_t! : z \in \mathbb{Z} \text{ and } |z| \le (\infty_t!)^2\}$.

Note that $\mathbb{Q}_t$ contains any particular standard rational number (a rational number in the language of **HA**) once ∞_t is chosen big enough.

Lemma 3.2. $\langle t < u \rangle$ $\mathbb{Q}_t \subset \mathbb{Q}_u$.

Proof. The claim follows from the axiom $\infty_u \ge \infty_t + 1$. □

We define next three "approximative" relations on $\mathbb{Q}_t$. Each of them is decidable by transfer.

Definition 3.3. For each $x, y \in \mathbb{Q}_t$, put

(a) $x =_o y$ if and only if $|x - y| \le \lozenge_o$,

(b) $x <_o y$ if and only if $x + \lozenge_o < y$,

(c) $x \le_o y$ if and only if $x \le y + \lozenge_o$,

where $\lozenge_o = 1/\infty_o$.

We think of $\lozenge_o$'s as indefinitely small yet positive rational numbers. Note that $=_o$ is not an equivalence relation. It is reflexive and symmetric but it is not transitive since application of transitivity usually results in loss of known precision: If $o_1 < o_2$ and $x =_{o_2} y$ and $y =_{o_2} z$, then $x =_{o_1} z$ by the axiom $\infty_{o_2} \ge 2\infty_{o_1}$. Note also that if $x, y \in \mathbb{Q}_t$ and $o > t$, then it follows from the axiom $\infty_o \ge \infty_t! - 1$ that $=_o$, $<_o$ and $\le_o$ are the same relations as $=$, $<$ and $\le$, respectively. We leave it to the reader to list and prove other properties of these relations.

We take a real number to be a finite multisequence of rational numbers:

Definition 3.4. A finite multisequence $(x_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ of rational numbers belonging to $\mathbb{Q}_t$ is called a real number. We may write $(x_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ but usually we do not mention $\mathbb{Q}_t$ at all. We may also write just x instead of $(x)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ in case $x_{\vec{n}} = x$ for all $\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}$.

The reason why we consider multisequences of rational numbers instead of sequences of rational numbers is given by Theorem 3.19 below. The reason why we consider multisequences that are not only bounded from above but also from below is that the lower bounds internalize the idea of "from some point on" and thus simplify arguments a lot by eliminating some quantifiers. To lighten notation and ease reading we usually mention the bounds only once and leave other occurrences away. Thus, if we state at the beginning of a definition or a theorem that $\vec{n}$ has bounds $\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}$, then it has these bounds throughout the definition or the theorem (together with its proof) unless stated otherwise.

Equality of real numbers is defined pointwise as follows:

Definition 3.5. $\langle o < \vec{s}_1 < \vec{s}_2 \rangle$ We say that two real numbers $(x_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ and $(y_{\vec{n}})$ are o-equal if $x_{\vec{n}} =_o y_{\vec{n}}$ for all $\vec{n}$. We write then $(x_{\vec{n}}) =_o (y_{\vec{n}})$.

The order relations $<_o$ and $\leq_o$ can be extended to real numbers in a similar way. Note that since $=_o$, $<_o$ and $\leq_o$ are decidable on rational numbers, so are the corresponding relations on real numbers. In fact, all objects we define in the sequel will be finite and all properties we define on them will be decidable. Hence, if we want to prove a claim C, it is enough to prove $\neg\neg C$.

We take Cauchyness to be a property of real numbers like any other:

Definition 3.6. $\langle o < \vec{s}_1 < \vec{s}_2 \rangle$ We say that a real number $(x_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ is o-Cauchy if $x_{\vec{n}_1} =_o x_{\vec{n}_2}$ for all $\vec{n}_1, \vec{n}_2$.

Note that if we defined a real number to be just an element of $\mathbb{Q}_t$ instead of a finite multisequence of elements of $\mathbb{Q}_t$, then we would not be able to express inside $\mathbf{HA}^*$ whether it is Cauchy or not.

The property of being Cauchy respects equality:

Lemma 3.7. $\langle o_1 < o_2 \rangle$ *If* $(x_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is an* o_2*-Cauchy real number and* o_2*-equal to* $(y_{\vec{n}})$*, then the latter is an* o_1*-Cauchy real number.*

Proof. For every $\vec{n}_1, \vec{n}_2$, since $y_{\vec{n}_1} =_{o_2} x_{\vec{n}_1} =_{o_2} x_{\vec{n}_2} =_{o_2} y_{\vec{n}_2}$ by the assumption, $y_{\vec{n}_1} =_{o_1} y_{\vec{n}_2}$ follows by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. □

We have not introduced any apartness relation on real numbers, since we do not need such a relation. This can be seen as follows. Let $o_1 < o_2 < \vec{s}_1 < \vec{s}_2$ and suppose $(x_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ is an o_2-Cauchy real number. We have either $(x_{\vec{n}}) =_{o_1} 0$ or $(x_{\vec{n}}) \neq_{o_1} 0$. If we have $(x_{\vec{n}}) \neq_{o_1} 0$, then there is $\vec{m}$ such that $|x_{\vec{m}}| >_{o_1} 0$. Now, since for every $\vec{n}$,

$$|x_{\vec{n}}| \geq |x_{\vec{m}}| - |x_{\vec{m}} - x_{\vec{n}}| > \Diamond_{o_1} - \Diamond_{o_2} \geq \Diamond_{o_2}$$

by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$, we must have either $(x_{\vec{n}}) <_{o_2} 0$ or $(x_{\vec{n}}) >_{o_2} 0$. Yet, the reader should keep in mind that $\mathbf{HA}^*$ does not have the disjunction property.

The arithmetic operations, maximum, minimum and absolute value are defined on real numbers pointwise in the obvious way. We prove next that they preserve equality. In case of the addition, multiplication, maximum and minimum we do this for N operands instead of just two since the usual inductive argument from two to N operands does not work due to the fact that $=_o$ is not transitive. But first we introduce some terminology:

Definition 3.8. Let $(x_{\vec{n}})_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ and $(M_{\vec{n}}) > 0$ be real numbers. We say that $(x_{\vec{n}})$ is $(M_{\vec{n}})$-bounded in case $|x_{\vec{n}}| \le M_{\vec{n}}$ for all $\vec{n}$. We say that $(x_{\vec{n}})$ is $(1/M_{\vec{n}})$-appreciable in case $(1/x_{\vec{n}})$ is $(M_{\vec{n}})$-bounded.

Lemma 3.9. $\langle h, k, o_1 < o_2\rangle$ *Let $N \le \infty_k$ be a natural number. If for each $i \le N$, $(x_{i,\vec{n}})_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ and $(y_{i,\vec{n}})$ are real numbers such that $(x_{i,\vec{n}}) =_{o_2} (y_{i,\vec{n}})$, then*

- *(a)* $(\sum_{i=1}^N x_{i,\vec{n}}) =_{o_1} (\sum_{i=1}^N y_{i,\vec{n}})$,
- *(b)* $(\prod_{i=1}^N x_{i,\vec{n}}) =_{o_1} (\prod_{i=1}^N y_{i,\vec{n}})$,
- *(c)* $(x_{1,\vec{n}}^{-1}) =_{o_1} (y_{1,\vec{n}}^{-1})$,
- *(d)* $(\max\{x_{i,\vec{n}} : i \le N\}) =_{o_2} (\max\{y_{i,\vec{n}} : i \le N\})$,
- *(e)* $(\min\{x_{i,\vec{n}} : i \le N\}) =_{o_2} (\min\{y_{i,\vec{n}} : i \le N\})$,
- *(f)* $(|x_{1,\vec{n}}|) =_{o_2} (|y_{1,\vec{n}}|)$.

In (b) we assume that $(x_{i+1,\vec{n}})$ and $(y_{i,\vec{n}})$ are ∞_h-bounded for all $i < N$. In (c) we assume that $(x_{1,\vec{n}})$ and $(y_{1,\vec{n}})$ are $\Diamond_h$-appreciable.

Proof. Take any $\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}$.

(*a*) We have

$$\left|\sum_{i=1}^N x_{i,\vec{n}} - \sum_{i=1}^N y_{i,\vec{n}}\right| \le \sum_{i=1}^N |x_{i,\vec{n}} - y_{i,\vec{n}}| \le \infty_k \Diamond_{o_2} \le \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \ge \infty_{o_1}\infty_k$.

(*b*) We have

$$\left|\prod_{i=1}^N x_{i,\vec{n}} - \prod_{i=1}^N y_{i,\vec{n}}\right| \le \sum_{i=1}^N \left(|x_{i,\vec{n}} - y_{i,\vec{n}}| \prod_{j=i+1}^N |x_{j,\vec{n}}| \prod_{k=1}^{i-1} |y_{k,\vec{n}}| \right)$$
$$\le \infty_k \infty_h^{\infty_k - 1} \Diamond_{o_2} \le \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \ge \infty_{o_1}\infty_k\infty_h^{\infty_k-1}$.

(c) We have

$$|x_{1,\vec{n}}^{-1} - y_{1,\vec{n}}^{-1}| = \left| \frac{x_{1,\vec{n}} - y_{1,\vec{n}}}{x_{1,\vec{n}} y_{1,\vec{n}}} \right| \leq \infty_h^2 \lozenge_{o_2} \leq \lozenge_{o_1}$$

by the axiom $\infty_{o_2} \geq \infty_{o_1} \infty_h^2$.

(d) Suppose $x_{i_x,\vec{n}} = \max\{x_{i,\vec{n}} : i \leq N\}$ and $y_{i_y,\vec{n}} = \max\{y_{i,\vec{n}} : i \leq N\}$. If $x_{i_x,\vec{n}} <_{o_2} y_{i_y,\vec{n}}$, then $i_x \neq i_y$ and $x_{i_x,\vec{n}} < x_{i_y,\vec{n}}$, which is not possible. Similarly if $x_{i_x,\vec{n}} >_{o_2} y_{i_y,\vec{n}}$. Hence we must have $x_{i_x,\vec{n}} =_{o_2} y_{i_y,\vec{n}}$. The proof of (e) goes in a similar way. Finally, the proof of (f) is obvious. □

A similar proof shows that the arithmetic operations, maximum, minimum and absolute value preserve Cauchyness:

Lemma 3.10. $\langle h, k, o_1 < o_2 \rangle$ *Let* $N \leq \infty_k$ *be a natural number. If for each* $i \leq N$, $(x_{i,\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is an* o_2*-Cauchy real number, then*

(a) $(\sum_{i=1}^{N} x_{i,\vec{n}})$,

(b) $(\prod_{i=1}^{N} x_{i,\vec{n}})$,

(c) $(x_{1,\vec{n}}^{-1})$,

(d) $(\max\{x_{i,\vec{n}} : i \leq N\})$,

(e) $(\min\{x_{i,\vec{n}} : i \leq N\})$,

(f) $(|x_{1,\vec{n}}|)$

are o_1*-Cauchy real numbers. In (b) we assume that* $(x_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $i \leq N$*. In (c) we assume that* $(x_{1,\vec{n}})$ *is* $\lozenge_h$*-appreciable.*

Note that, because of being a finite set, $\mathbb{Q}_t$ is in general not closed under addition, multiplication or taking the inverse. Nevertheless, it is easy to see that if $k < t < u$ and $(x_{i,\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ for all $i \leq N \leq \infty_k$, then $(\sum_{i=1}^{N} x_{i,\vec{n}}), (\prod_{i=1}^{N} x_{i,\vec{n}}) \in \mathbb{Q}_u$. Moreover, $(x_{1,\vec{n}}^{-1}) \in \mathbb{Q}_u$ in case $(x_{1,\vec{n}}) \neq 0$.

3.3 Finite Sequences and Series of Real Numbers

We take a sequence of real numbers to be a "double" sequence of rational numbers:

Definition 3.11. $\langle r < \vec{s}_1 < \vec{s}_2 \rangle$ A sequence of real numbers is a finite sequence $(x_{m,\vec{n}})_{m \leq \infty_r, \infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ of rational numbers.

Equality of sequences of real numbers is defined obviously as follows:

Definition 3.12. $\langle o, r < \vec{s}_1 < \vec{s}_2 \rangle$ We say that two sequences of real numbers $(x_{m,\vec{n}})_{m \le \infty_r, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ and $(y_{m,\vec{n}})_{m \le \infty_{r_2}}$ are o-equal if $x_{m,\vec{n}} =_o y_{m,\vec{n}}$ for all $m \le \infty_r$ and $\vec{n}$. We write then $(x_{m,\vec{n}})_{m \le \infty_r} =_o (y_{m,\vec{n}})_{m \le \infty_r}$.

We state the following definition for clarity:

Definition 3.13. $\langle o, r < \vec{s}_1 < \vec{s}_2 \rangle$ We say that $(x_{m,\vec{n}})_{m \le \infty_r, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ is a sequence of o-Cauchy real numbers if $(x_{m,\vec{n}})$ is an o-Cauchy real number for all $m \le \infty_r$.

The property of being a sequence of Cauchy real numbers respects equality:

Lemma 3.14. $\langle o_1 < o_2 \rangle$ *If* $(x_{m,\vec{n}})_{m \le \infty_r, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *is a sequence of* o_2*-Cauchy real numbers and is* o_2*-equal to* $(y_{m,\vec{n}})_{m \le \infty_r}$*, then the latter is a sequence of* o_1*-Cauchy real numbers.*

Proof. Take any $m \le \infty_r$ and $\vec{n}_1, \vec{n}_2$. Since $y_{m,\vec{n}_1} =_{o_2} x_{m,\vec{n}_1} =_{o_2} x_{m,\vec{n}_2} =_{o_2} y_{m,\vec{n}_2}$ by the assumption, we have $y_{m,\vec{n}_1} =_{o_1} y_{m,\vec{n}_2}$ by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$. □

We say next what we mean by a Cauchy sequence of real numbers:

Definition 3.15. $\langle o < r_1 < r_2 < \vec{s}_1 < \vec{s}_2 \rangle$ We say that a sequence of real numbers $(x_{m,\vec{n}})_{m \le \infty_{r_2}, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ is $r_1 o$-Cauchy in case $x_{m_1,\vec{n}} =_o x_{m_2,\vec{n}}$ for all $\infty_{r_1} \le m_1, m_2 \le \infty_{r_2}$ and $\vec{n}$.

We sometimes say that a sequence of real numbers converges instead of that it is Cauchy. The property of being a Cauchy sequence of real numbers respects equality:

Lemma 3.16. $\langle o_1 < o_2 \rangle$ *If* $(x_{m,\vec{n}})_{m \le \infty_{r_2}, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *is an* $r_1 o_2$*-Cauchy sequence of real numbers and* o_2*-equal to* $(y_{m,\vec{n}})_{m \le \infty_{r_2}}$*, then the latter is an* $r_1 o_1$*-Cauchy sequence of real numbers.*

Proof. For each $\infty_{r_1} \le m_1, m_2 \le \infty_{r_2}$ and $\vec{n}$, since $y_{m_1,\vec{n}} =_{o_2} x_{m_1,\vec{n}} =_{o_2} x_{m_2,\vec{n}} =_{o_2} y_{m_2,\vec{n}}$ by the assumption, $y_{m_1,\vec{n}} =_{o_1} y_{m_2,\vec{n}}$ follows by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$. □

The arithmetic operations on sequences of real numbers are defined pointwise in the obvious way. They preserve equality:

Lemma 3.17. $\langle h, k, o_1 < o_2 \rangle$ *Let* $N \le \infty_k$ *be a natural number. If for each* $i \le N$, $(x_{i,m,\vec{n}})_{m \le \infty_r, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *and* $(y_{i,m,\vec{n}})_{m \le \infty_r}$ *are sequences of real numbers such that* $(x_{i,m,\vec{n}})_{m \le \infty_r} =_{o_2} (y_{i,m,\vec{n}})_{m \le \infty_r}$*, then*

(a) $(\sum_{i=1}^{N} x_{i,m,\vec{n}})_{m\leq\infty_r} =_{o_1} (\sum_{i=1}^{N} y_{i,m,\vec{n}})_{m\leq\infty_r}$,

(b) $(\prod_{i=1}^{N} x_{i,m,\vec{n}})_{m\leq\infty_r} =_{o_1} (\prod_{i=1}^{N} y_{i,m,\vec{n}})_{m\leq\infty_r}$,

(c) $(x^{-1}_{1,m,\vec{n}})_{m\leq\infty_r} =_{o_1} (y^{-1}_{1,m,\vec{n}})_{m\leq\infty_r}$.

In (b) we assume that $(x_{i,m,\vec{n}})$ and $(y_{i,m,\vec{n}})$ are ∞_h-bounded real numbers for all $i \leq N$ and $m \leq \infty_r$. In (c) we assume that $(x_{1,m,\vec{n}})$ and $(y_{1,m,\vec{n}})$ are $\Diamond_h$-appreciable real numbers for all $m \leq \infty_r$.

The arithmetic operations also preserve Cauchyness of sequences of real numbers:

Lemma 3.18. $\langle h, k, o_1 < o_2\rangle$ *Let $N \leq \infty_k$ be a natural number. If for each $i \leq N$, $(x_{i,m,\vec{n}})_{m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ is an r_1o_2-Cauchy sequence of real numbers, then*

(a) $(\sum_{i=1}^{N} x_{i,m,\vec{n}})_{m\leq\infty_{r_2}}$,

(b) $(\prod_{i=1}^{N} x_{i,m,\vec{n}})_{m\leq\infty_{r_2}}$,

(c) $(x^{-1}_{1,m,\vec{n}})_{m\leq\infty_{r_2}}$

are r_1o_1-Cauchy sequences of real numbers. In (b) we assume that $(x_{i,m,\vec{n}})$ are ∞_h-bounded real numbers for all $i \leq N$ and $m \leq \infty_{r_2}$. In (c) we assume that $(x_{1,m,\vec{n}})$ is a $\Diamond_h$-appreciable real number for all $m \leq \infty_{r_2}$.

The next theorem gives our version of the Cauchy completeness property of the real numbers.

Theorem 3.19. $\langle o_1 < o_2\rangle$ *If $(x_{m,\vec{n}})_{m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ is an r_1o_2-Cauchy sequence of o_2-Cauchy real numbers, then $(x_{m,\vec{n}})_{\infty_{r_1}\leq m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ is an o_1-Cauchy real number.*

Proof. For all $\infty_{r_1} \leq m_1, m_2 \leq \infty_{r_2}$ and $\vec{n}_1, \vec{n}_2$, since $x_{m_1,\vec{n}_1} =_{o_2} x_{m_1,\vec{n}_2} =_{o_2} x_{m_2,\vec{n}_2}$ by the assumption, we get $x_{m_1,\vec{n}_1} =_{o_1} x_{m_2,\vec{n}_2}$ by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. □

There may not be any way to construct a Cauchy limit of a Cauchy sequence of Cauchy real numbers that would avoid the introduction of multisequences. The usual construction making use of the countable axiom of choice (see Bishop and Bridges, 1985) seems to be out of question. Also, if we just substitute ∞_{r_2} for m in $(x_{m,\vec{n}})$ to get the Cauchy real number $(x_{\infty_{r_2},\vec{n}})$ as a limit, then we lose the piece of information saying that $(x_{m,\vec{n}})_{m\leq\infty_{r_2}}$ is a Cauchy sequence of real numbers.

We illustrate the above notions by the following little example needed in the proof of Theorem 5.11.

Lemma 3.20. $\langle o_1, h, r_1 < r_2 \mid o_1 < o_2\rangle$ *If* $(x_{m,\vec{n}})_{m\leq\infty_{r_3},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ *is an* r_1o_2*-Cauchy sequence of* ∞_h*-bounded real numbers, then the real number*

$$\left(\frac{x_{1,\vec{n}}+\cdots+x_{m,\vec{n}}}{m}\right)_{\infty_{r_2}\leq m\leq\infty_{r_3}}$$

is o_1*-equal to the real number* $(x_{m,\vec{n}})_{\infty_{r_2}\leq m\leq\infty_{r_3}}$.

Proof. For each $\infty_{r_2}\leq m\leq\infty_{r_3}$ and $\vec{n}$,

$$\begin{aligned}\left|\frac{x_{1,\vec{n}}+\cdots+x_{m,\vec{n}}}{m}-x_{m,\vec{n}}\right| &\leq \frac{|x_{1,\vec{n}}-x_{m,\vec{n}}|}{m}+\cdots+\frac{|x_{\infty_{r_1},\vec{n}}-x_{m,\vec{n}}|}{m}\\ &\quad+\frac{|x_{\infty_{r_1}+1,\vec{n}}-x_{m,\vec{n}}|}{m}+\cdots+\frac{|x_{m,\vec{n}}-x_{m,\vec{n}}|}{m}\\ &\leq \infty_{r_1}\frac{2\infty_h}{m}+(m-\infty_{r_1})\frac{\Diamond_{o_2}}{m}\\ &\leq 2\infty_{r_1}\Diamond_{r_2}\infty_h+\Diamond_{o_2}\leq\Diamond_{o_1}\end{aligned}$$

by the axioms $\infty_{r_2}\geq 4\infty_{r_1}\infty_h\infty_{o_1}$ and $\infty_{o_2}\geq 2\infty_{o_1}$. □

We define divergence of a sequence of real numbers as follows:

Definition 3.21. $\langle o < r_1 < r_2 < \vec{s}_1 < \vec{s}_2\rangle$ We say that a sequence of real numbers $(x_{m,\vec{n}})_{m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ r_1o-diverges if there are $\infty_{r_1}\leq m_1, m_2\leq\infty_{r_2}$ and $\vec{n}$ such that $x_{m_1,\vec{n}}\neq_o x_{m_2,\vec{n}}$.

Note that each sequence of real numbers either converges or diverges. However, we cannot in general prove that an increasing sequence of real numbers with an upper bound converges even though it has a least upper bound, namely the last element of the sequence.

The property of being a divergent sequence of real numbers respects equality:

Lemma 3.22. $\langle o_1 < o_2\rangle$ *Let* $(x_{m,\vec{n}})_{m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ *be a sequence of real numbers* r_1o_1*-diverging and being* o_2*-equal to a sequence of real numbers* $(y_{m,\vec{n}})_{m\leq\infty_{r_2}}$*. Then the latter* r_1o_2*-diverges.*

Proof. By the assumption, $x_{m_1,\vec{n}}\neq_{o_1} x_{m_2,\vec{n}}$ for some $\infty_{r_1}\leq m_1, m_2\leq\infty_{r_2}$ and $\vec{n}$. Moreover, $x_{m_1,\vec{n}}=_{o_2} y_{m_1,\vec{n}}$ and $x_{m_2,\vec{n}}=_{o_2} y_{m_2,\vec{n}}$. If we had $y_{m_1,\vec{n}}=_{o_2} y_{m_2,\vec{n}}$, then $x_{m_1,\vec{n}}=_{o_1} x_{m_2,\vec{n}}$ by the axiom $\infty_{o_2}\geq 3\infty_{o_1}$, but this is not possible. So we must have $y_{m_1,\vec{n}}\neq_{o_2} y_{m_2,\vec{n}}$. □

The following lemma shows that to assume that a sequence of Cauchy real numbers diverges it is enough to assume a bit less:

Lemma 3.23. $\langle o_1 < o_2 \rangle$ *If* $(x_{m,\vec{n}})_{m \le \infty_{r_2}, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *is a sequence of* o_2*-Cauchy real numbers such that* $x_{m_1,\vec{n}_1} \neq_{o_1} x_{m_2,\vec{n}_2}$ *for some* $\infty_{r_1} \le m_1, m_2 \le \infty_{r_2}$ *and* $\vec{n}_1, \vec{n}_2$*, then* $x_{m_1,\vec{n}_1} \neq_{o_2} x_{m_2,\vec{n}_1}$*.*

Proof. By the assumption, $x_{m_1,\vec{n}_1} \neq_{o_1} x_{m_2,\vec{n}_2}$ and $x_{m_2,\vec{n}_1} =_{o_2} x_{m_2,\vec{n}_2}$. Suppose $x_{m_1,\vec{n}_1} =_{o_2} x_{m_2,\vec{n}_1}$. Then $x_{m_1,\vec{n}_1} =_{o_1} x_{m_2,\vec{n}_2}$ by the axiom $\infty_{o_2} \ge 2\infty_{o_1}$, which is impossible. So we must have $x_{m_1,\vec{n}_1} \neq_{o_2} x_{m_2,\vec{n}_1}$. □

We turn now to the notion of a series, i.e. an indefinitely long yet finite sequence of real numbers the terms of which are to be summed.

Definition 3.24. $\langle o_1 < o_2 \mid r_2, o_3 < o_4 \rangle$ Let $(x_{m,\vec{n}})_{m \le \infty_{r_2}, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ be a sequence of real numbers and let $(S_{m,\vec{n}})_{m \le \infty_{r_2}}$ be the sequence of real numbers defined by putting

$$S_{m,\vec{n}} = \sum_{i=0}^{m} x_{i,\vec{n}}.$$

We call $(S_{m,\vec{n}})_{m \le \infty_{r_2}}$ the sequence of partial sums of the series. In case $(S_{m,\vec{n}})_{\infty_{r_1} \le m \le \infty_{r_2}}$ is an o_1-Cauchy real number, we call it the $r_1 o_1$-sum of the series. Note that if $(x_{m,\vec{n}})_{m \le \infty_{r_2}}$ is a sequence of o_4-Cauchy real numbers, then $(S_{m,\vec{n}})_{m \le \infty_{r_2}}$ is a sequence of o_3-Cauchy real numbers. Also, if $(S_{m,\vec{n}})_{m \le \infty_{r_2}}$ is an $r_1 o_2$-Cauchy sequence of o_2-Cauchy real numbers, then $(S_{m,\vec{n}})_{\infty_{r_1} \le m \le \infty_{r_2}}$ is an o_1-Cauchy real number by Theorem 3.19.

The property of being a series respects equality:

Lemma 3.25. $\langle r, o_1 < o_2 \rangle$ *If* $(x_{m,\vec{n}})_{m \le \infty_r, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$*,* $(y_{m,\vec{n}})_{m \le \infty_r}$ *are* o_2*-equal sequences of real numbers, then* $(\sum_{i=0}^{m} x_{i,\vec{n}})_{m \le \infty_r}$*,* $(\sum_{i=0}^{m} y_{i,\vec{n}})_{m \le \infty_r}$ *are* o_1*-equal sequences of real numbers.*

Proof. Let $m \le \infty_r$ and $\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}$. We get

$$\left| \sum_{i=0}^{m} x_{i,\vec{n}} - \sum_{i=0}^{m} y_{i,\vec{n}} \right| \le \sum_{i=0}^{m} |x_{i,\vec{n}} - y_{i,\vec{n}}| \le (\infty_r + 1)\lozenge_{o_2} \le \lozenge_{o_1}$$

by the axiom $\infty_{o_2} \ge \infty_{o_1}(\infty_r + 1)$. □

We deal next with the geometric series.

Lemma 3.26. $\langle h, r, o_1 < o_2 \rangle$ *Suppose* $(x_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *is an* ∞_h*-bounded* o_2*-Cauchy real number. Then* $(x_{\vec{n}}^m)_{m \le \infty_r}$ *is a sequence of* o_1*-Cauchy real numbers.*

Proof. For each $m \le \infty_r$ and $\infty_{\vec{s}_1} \le \vec{n}_1, \vec{n}_2 \le \infty_{\vec{s}_2}$,

$$|x_{\vec{n}_1}^m - x_{\vec{n}_2}^m| \le |x_{\vec{n}_1} - x_{\vec{n}_2}| \sum_{i=1}^{m} |x_{\vec{n}_1}|^{m-i}|x_{\vec{n}_2}|^{i-1} \le \Diamond_{o_2}\infty_r\infty_h^{\infty_r - 1} \le \Diamond_{o_1}$$

holds by the axiom $\infty_{o_2} \ge \infty_{o_1}\infty_r\infty_h^{\infty_r - 1}$. □

Lemma 3.27. *$\langle o < r_1 \rangle$ If $(x_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ is a real number such that $|x_{\vec{n}}| <_o 1$ for all $\vec{n}$, then $(x_{\vec{n}}^m)_{\infty_{r_1} \le m \le \infty_{r_2}} =_o 0$.*

Proof. Take any $\infty_{r_1} \le m \le \infty_{r_2}$ and $\vec{n}$. Since $|x_{\vec{n}}| < 1 - \Diamond_o$ and $1 - \Diamond_o = (1+M)^{-1}$, where $M = (\infty_o - 1)^{-1}$, it is enough to show that $(1+M)^{-m} =_o 0$. Now, since

$$(1+M)^m = \sum_{i=0}^{m} \binom{m}{i} M^i > \binom{m}{1} M = mM \ge \infty_{r_1} M, \qquad (5)$$

we get $(1+M)^{-m} < \Diamond_{r_1}(\infty_o - 1) \le \Diamond_o$ by the axiom $\infty_{r_1} \ge \infty_o(\infty_o - 1)$. □

The usual condition for convergence and divergence of the geometric series holds:

Lemma 3.28. *$\langle o_1 < o_2 < r_1 \shortmid r_2, o_1 < o_3 \rangle$ Let $(x_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ be an o_2-Cauchy real number.*

(a) If $|x_{\vec{n}}| <_{o_2} 1$ for all $\vec{n}$, then the series $(\sum_{i=0}^m x_{\vec{n}}^i)_{m \le \infty_{r_2}}$ $r_1 o_1$-converges and the real number $(\sum_{i=0}^m x_{\vec{n}}^i)_{\infty_{r_1} \le m \le \infty_{r_2}}$ is o_1-equal to $((1 - x_{\vec{n}})^{-1})$.

(b) If $|x_{\vec{n}}| \ge_{o_3} 1$ for all $\vec{n}$, then the series $(\sum_{i=0}^m x_{\vec{n}}^i)_{m \le \infty_{r_2}}$ $r_1 o_1$-diverges.

Proof. (a) Since $\sum_{i=0}^m x_{\vec{n}}^i = (1 - x_{\vec{n}}\, x_{\vec{n}}^m)(1 - x_{\vec{n}})^{-1}$ for all $\infty_{r_1} \le m \le \infty_{r_2}$ and $\vec{n}$, it is enough to show that

$$((1 - x_{\vec{n}}\, x_{\vec{n}}^m)(1 - x_{\vec{n}})^{-1})_{m \le \infty_{r_2}}$$

$r_1 o_1$-converges and is o_1-equal to $((1 - x_{\vec{n}})^{-1})$. This follows from Lemmas 3.17 and 3.18, since $x_{\vec{n}}^m =_{o_2} 0$ for all $\infty_{r_1} \le m \le \infty_{r_2}$ by Lemma 3.27.

(b) It is enough to show that

$$\left| \sum_{i=0}^{\infty_{r_2}} x_{\infty_{\vec{s}_2}}^i - \sum_{i=0}^{\infty_{r_2} - 1} x_{\infty_{\vec{s}_2}}^i \right| = |x_{\infty_{\vec{s}_2}}^{\infty_{r_2}}| \ge_{o_1} 1.$$

Now, since $|x_{\infty_{\vec{s}_2}}| \geq 1-\lozenge_{o_3}$ and $1-\lozenge_{o_3} = (1+M)^{-1}$, where $M = (\infty_{o_3}-1)^{-1}$, it is enough to show that $(1+M)^{-\infty_{r_2}} \geq (1+L)^{-1}$, where $L = (\infty_{o_1}-1)^{-1}$. But we have

$$\begin{aligned}(1+M)^{\infty_{r_2}} &= \sum_{i=0}^{\infty_{r_2}} \binom{\infty_{r_2}}{i} M^i \leq 1 + M \sum_{i=1}^{\infty_{r_2}} \binom{\infty_{r_2}}{i} \\ &= 1 + M(2^{\infty_{r_2}}-1) \leq 1+L,\end{aligned}$$

where the last inequality holds by the axiom $\infty_{o_3} \geq (\infty_{o_1}-1)(2^{\infty_{r_2}}-1)+1$. Thus $|x_{\infty_{\vec{s}_2}}^{\infty_{r_2}}| \geq_{o_1} 1$, so $(\sum_{i=0}^{m} x_{\vec{n}}^{i})_{m\leq\infty_{r_2}}$ r_1o_1-diverges. □

We prove next some convergence and divergence tests. The first one is the comparison test:

Theorem 3.29. *Let $(x_{m,\vec{n}})_{m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ and $(y_{m,\vec{n}})_{m\leq\infty_{r_2}}$ be sequences of real numbers.*

- *(a) If $(\sum_{i=0}^{m} y_{i,\vec{n}})_{m\leq\infty_{r_2}}$ r_1o-converges and $|x_{i,\vec{n}}| \leq y_{i,\vec{n}}$ for all $\infty_{r_1} \leq i \leq \infty_{r_2}$ and $\vec{n}$, then $(\sum_{i=0}^{m} x_{i,\vec{n}})_{m\leq\infty_{r_2}}$ also r_1o-converges.*
- *(b) If $(\sum_{i=0}^{m} y_{i,\vec{n}})_{m\leq\infty_{r_2}}$ r_1o-diverges and $x_{i,\vec{n}} \geq |y_{i,\vec{n}}|$ for all $\infty_{r_1} \leq i \leq \infty_{r_2}$ and $\vec{n}$, then $(\sum_{i=0}^{m} x_{i,\vec{n}})_{m\leq\infty_{r_2}}$ also r_1o-diverges.*

Proof. (a) Let $\infty_{r_1} \leq m_1 \leq m_2 \leq \infty_{r_2}$ and $\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}$. Then

$$\left|\sum_{i=0}^{m_2} x_{i,\vec{n}} - \sum_{i=0}^{m_1} x_{i,\vec{n}}\right| \leq \sum_{i=m_1+1}^{m_2} |x_{i,\vec{n}}| \leq \sum_{i=m_1+1}^{m_2} y_{i,\vec{n}} \leq \lozenge_o,$$

so $(\sum_{i=0}^{m} x_{i,\vec{n}})_{m\leq\infty_{r_2}}$ r_1o-converges.

(b) There are $\infty_{r_1} \leq m_1 < m_2 \leq \infty_{r_2}$ and $\vec{n}$ such that

$$\sum_{i=m_1+1}^{m_2} |y_{i,\vec{n}}| \geq \left|\sum_{i=m_1+1}^{m_2} y_{i,\vec{n}}\right| = \left|\sum_{i=0}^{m_2} y_{i,\vec{n}} - \sum_{i=0}^{m_1} y_{i,\vec{n}}\right| \geq \lozenge_o.$$

Now

$$\left|\sum_{i=0}^{m_2} x_{i,\vec{n}} - \sum_{i=0}^{m_1} x_{i,\vec{n}}\right| = \sum_{i=m_1+1}^{m_2} x_{i,\vec{n}} \geq \sum_{i=m_1+1}^{m_2} |y_{i,\vec{n}}| \geq \lozenge_o,$$

so $(\sum_{i=0}^{m} x_{i,\vec{n}})_{m\leq\infty_{r_2}}$ r_1o-diverges. □

The following one is the ratio test:

Theorem 3.30. $\langle o_1, h, k, r_1 < o_2 < o_3 < r_2 < r_3\rangle$ *Let $(x_{m,\vec{n}})_{m\leq\infty_{r_3},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ be a sequence of real numbers and let $(c_{\vec{n}})$ be an o_3-Cauchy real number.*

(a) *If* $0 <_k c_{\vec{n}} <_{o_3} 1$ *and* $|x_{\infty_{r_1},\vec{n}}| \leq \Diamond_h$ *and* $|x_{m+1,\vec{n}}/x_{m,\vec{n}}| \leq c_{\vec{n}}$ *for all* $\infty_{r_1} \leq m < \infty_{r_3}$ *and* $\vec{n}$*, then* $(\sum_{i=0}^m x_{i,\vec{n}})_{m\leq\infty_{r_3}}$ r_2o_1*-converges.*

(b) *If* $c_{\vec{n}} >_{o_1} 1$ *and* $|x_{\infty_h,\vec{n}}| \geq \Diamond_h$ *and* $|x_{m+1,\vec{n}}/x_{m,\vec{n}}| \geq c_{\vec{n}}$ *for all* $\infty_{r_1} \leq m < \infty_{r_3}$ *and* $\vec{n}$*, then* $(\sum_{i=0}^m x_{i,\vec{n}})_{m\leq\infty_{r_3}}$ r_2o_1*-diverges.*

Proof. (a) We have

$$|x_{m,\vec{n}}| \leq |x_{\infty_{r_1},\vec{n}}| c_{\vec{n}}^{m-\infty_{r_1}} \leq \Diamond_h \infty_k^{\infty_{r_1}} c_{\vec{n}}^m$$

for all $\infty_{r_2} \leq m \leq \infty_{r_3}$ and $\vec{n}$. Then, since $(\sum_{i=0}^m c_{\vec{n}}^i)_{m\leq\infty_{r_3}}$ r_2o_2-converges as a geometric series, $(\Diamond_h \infty_k^{\infty_{r_1}} \sum_{i=0}^m c_{\vec{n}}^i)_{m\leq\infty_{r_3}}$ r_2o_1-converges by the axiom $\infty_{o_2} \geq \lceil \infty_{o_1} \Diamond_h \infty_k^{\infty_{r_1}} \rceil$, so $(\sum_{i=0}^m x_{i,\vec{n}})_{m\leq\infty_{r_3}}$ r_2o_1-converges by the comparison test.

(b) It is enough to show that

$$\left| \sum_{i=0}^{\infty_{r_3}} x_{i,\infty_{\vec{s}_2}} - \sum_{i=0}^{\infty_{r_3}-1} x_{i,\infty_{\vec{s}_2}} \right| = |x_{\infty_{r_3},\infty_{\vec{s}_2}}| >_{o_1} 0.$$

But we have

$$\begin{aligned} |x_{\infty_{r_3},\infty_{\vec{s}_2}}| &\geq c_{\infty_{\vec{s}_2}}^{\infty_{r_3}-\infty_{r_1}} |x_{\infty_{r_1},\infty_{\vec{s}_2}}| \\ &\geq (1+\Diamond_{o_1})^{\infty_{r_3}-\infty_{r_1}} \Diamond_h \geq (1+\Diamond_{o_1})^{\infty_h} \Diamond_h > \Diamond_{o_1}, \end{aligned}$$

where the last inequality holds by a calculation similar to (5) in the proof of Lemma 3.27. Thus $(\sum_{i=0}^m x_{i,\vec{n}})_{m\leq\infty_{r_3}}$ r_2o_1-diverges. □

The last test we prove is known as Kummer's criterion. We only show the part dealing with convergence.

Lemma 3.31. $\langle o_1 < o_2 \rangle$ *Suppose* $(x_m)_{\infty_{r_1}\leq m\leq\infty_{r_2}}$ *is a positive real number and* $(y_{m,\vec{n}})_{m\leq\infty_{r_2},\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ *is a sequence of positive real numbers. If* $x_m y_{m,\vec{n}} =_{o_2} 0$ *and*

$$\frac{x_m y_{m,\vec{n}}}{y_{m+1,\vec{n}}} - x_{m+1} \geq \Diamond_{o_1}$$

for all $\infty_{r_1} \leq m \leq \infty_{r_2}$ *and* $\vec{n}$*, then the series* $(\sum_{i=0}^m y_{i,\vec{n}})_{m\leq\infty_{r_2}}$ r_1o_1*-converges.*

Proof. For each $\infty_{r_1} \leq m_1 < m_2 \leq \infty_{r_2}$ and $\vec{n}$, since $x_{m_1} y_{m_1,\vec{n}} =_{o_2} x_{m_2} y_{m_2,\vec{n}}$ holds by the assumption,

$$\begin{aligned} \sum_{i=m_1+1}^{m_2} y_{i,\vec{n}} &\leq \infty_{o_1} \sum_{i=m_1+1}^{m_2} y_{i,\vec{n}} \Big(\frac{x_{i-1} y_{i-1,\vec{n}}}{y_{i,\vec{n}}} - x_i \Big) \\ &= \infty_{o_1} (x_{m_1} y_{m_1,\vec{n}} - x_{m_2} y_{m_2,\vec{n}}) \leq \infty_{o_1} \Diamond_{o_2} \leq \Diamond_{o_1} \end{aligned}$$

holds by the axiom $\infty_{o_2} \geq \infty_{o_1}^2$. □

As an example we show the following little result needed in the proof of Theorem 5.11. Let $o_1 < o_2 < r_1$. Now, $(m)_{\infty_{r_1} \le m \le \infty_{r_2}}$ is a positive real number and $(m^{-2})_{m \le \infty_{r_2}}$ is a sequence of positive real numbers such that for each $\infty_{r_1} \le m \le \infty_{r_2}$ we have $m \cdot m^{-2} = m^{-1} \le \Diamond_{r_1} \le \Diamond_{o_2}$ by the axiom $\infty_{r_1} \ge \infty_{o_2}$ and

$$\frac{m \cdot m^{-2}}{(m+1)^{-2}} - (m+1) = 1 + \frac{1}{m} \ge 1 + \Diamond_{r_2} \ge \Diamond_{o_1}.$$

Hence the series $(\sum_{i=1}^{m} i^{-2})_{m \le \infty_{r_2}}$ $r_1 o_1$-converges by Kummer's criterion.

The following theorem says the familiar fact that the product of two converging series converges to the product of the sums of the series if at least one of the series converges absolutely.

Theorem 3.32. *$\langle h, o_1 < o_2 \mid r_1 < r_2 \rangle$ Let $(\sum_{i=0}^{m} x_{i,\vec{n}})_{m \le \infty_{r_3}, \infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$, $(\sum_{i=0}^{m} |x_{i,\vec{n}}|)_{m \le \infty_{r_3}}$ and $(\sum_{i=0}^{m} y_{i,\vec{n}})_{m \le \infty_{r_3}}$ be $r_1 o_2$-converging series such that the latter two are ∞_h-bounded. If we put*

$$z_{m,\vec{n}} = \sum_{j=0}^{m} x_{j,\vec{n}}\, y_{m-j,\vec{n}}$$

for all $m \le \infty_{r_3}$ and $\vec{n}$, then we have

$$\sum_{i=0}^{m} z_{i,\vec{n}} =_{o_1} \sum_{i=0}^{m} x_{i,\vec{n}} \cdot \sum_{i=0}^{m} y_{i,\vec{n}}$$

for all $\infty_{r_2} \le m \le \infty_{r_3}$.

Proof. Take any $\infty_{r_2} \le m \le \infty_{r_3}$ and $\vec{n}$. If we write $X = \sum_{i=0}^{m} x_{i,\vec{n}}$ and $Y_j = \sum_{i=0}^{j} y_{i,\vec{n}}$, then

$$\begin{aligned}\sum\nolimits_{i=0}^{m} z_{i,\vec{n}} &= x_{0,\vec{n}} Y_m + x_{1,\vec{n}} Y_{m-1} + \cdots + x_{m,\vec{n}} Y_0 \\ &= x_{0,\vec{n}} Y_m + x_{1,\vec{n}}(Y_m - y_{m,\vec{n}}) + \cdots + x_{m,\vec{n}}(Y_m - (y_{1,\vec{n}} + \cdots + y_{m,\vec{n}})) \\ &= X_m Y_m - (x_{1,\vec{n}} y_{m,\vec{n}} + \cdots + x_{m,\vec{n}}(y_{1,\vec{n}} + \cdots + y_{m,\vec{n}})).\end{aligned}$$

By the assumption and the axiom $\infty_{r_2} \ge 2\infty_{r_1}$, $|y_{l,\vec{n}} + \cdots + y_{m,\vec{n}}| \le \Diamond_{o_2}$ for all $l = m - \infty_{r_1} + 1, \ldots, m$. Moreover, since $|x_{\infty_{r_1}+1,\vec{n}}| + \cdots + |x_{m,\vec{n}}| \le \Diamond_{o_2}$, we get

$$\begin{aligned}&|x_{1,\vec{n}}\, y_{m,\vec{n}} + \cdots + x_{m,\vec{n}}\,(y_{1,\vec{n}} + \cdots + y_{m,\vec{n}})| \\ &\quad \le |x_{1,\vec{n}}\, y_{m,\vec{n}} + \cdots + x_{\infty_{r_1},\vec{n}}\,(y_{m-\infty_{r_1}+1,\vec{n}} + \cdots + y_{m,\vec{n}})| \\ &\qquad + |x_{\infty_{r_1}+1,\vec{n}}\,(y_{m-\infty_{r_1},\vec{n}} + \cdots + y_{m,\vec{n}}) + \cdots + x_{m,\vec{n}}\,(y_{1,\vec{n}} + \cdots + y_{m,\vec{n}})| \\ &\quad \le \infty_h \Diamond_{o_2} + \infty_h \Diamond_{o_2} = 2\infty_h \Diamond_{o_2} \le \Diamond_{o_1}\end{aligned}$$

by the axiom $\infty_{o_2} \ge 2\infty_h \infty_{o_1}$. □

3.4 Subsets and Functions

We took a real number to be a finite multisequence of elements of $\mathbb{Q}_t$ instead of just an element of $\mathbb{Q}_t$ because we wanted to talk about its convergence. Correspondingly, we take a subset of real numbers to be a finite multisequence of subsets of $\mathbb{Q}_t$:

Definition 3.33. A finite multisequence $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ of subsets of $\mathbb{Q}_t$ is called a subset of real numbers. We write then $(S_{\vec{n}}) \subseteq \mathbb{Q}_t$.

The most important example of a subset of real numbers is the "closed" interval:

Definition 3.34. Let $(x_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (y_{\vec{n}}) \in \mathbb{Q}_t$. We define an interval $([x_{\vec{n}}, y_{\vec{n}}]) \subseteq \mathbb{Q}_t$ by putting

$$[x_{\vec{n}}, y_{\vec{n}}] = \{z \in \mathbb{Q}_t : x_{\vec{n}} \leq z \leq y_{\vec{n}}\}$$

for all $\vec{n}$. If we write $(I_{\vec{n}})$ for such an interval, then we write $(I_{L,\vec{n}})$ and $(I_{R,\vec{n}})$ for its left and right end points, respectively. The length of $(I_{\vec{n}})$ is the real number $(|I_{\vec{n}}|)$ defined by putting $|I_{\vec{n}}| = I_{R,\vec{n}} - I_{L,\vec{n}}$ for all $\vec{n}$. We say that $(I_{\vec{n}})$ is $(M_{\vec{n}})$-proper in case $(|I_{\vec{n}}|)$ is $(M_{\vec{n}})$-appreciable, where $(M_{\vec{n}})$ is a positive real number.

Note that intervals like $([-\infty_h, \infty_h])_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$, where $h \leq t$, are also covered by the preceding definition. This particular interval contains any "standard" real number $(x_{\vec{n}})$ once ∞_h is chosen big enough.

Equality between subsets is defined as follows:

Definition 3.35. $\langle o < \vec{s}_1 < \vec{s}_2 \rangle$ We say that $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (T_{\vec{n}}) \subseteq \mathbb{Q}_t$ are o-equal if for each $\vec{n}$ and $x \in S_{\vec{n}}$ there is $y \in T_{\vec{n}}$ such that $x =_o y$ and vice versa. We write then $(S_{\vec{n}}) =_o (T_{\vec{n}})$.

Once again $=_o$ is not an equivalence relation. It is reflexive and symmetric but it is not transitive. However, if $o_1 < o_2$ and $(S_{\vec{n}}) =_{o_2} (T_{\vec{n}})$ and $(T_{\vec{n}}) =_{o_2} (U_{\vec{n}})$, then $(S_{\vec{n}}) =_{o_1} (U_{\vec{n}})$ by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$.

We say next what we mean by a Cauchy subset:

Definition 3.36. $\langle o < \vec{s}_1 < \vec{s}_2 \rangle$ We say that $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ is o-Cauchy if for each $\vec{n}_1, \vec{n}_2$ and $x \in S_{\vec{n}_1}$ there is $y \in S_{\vec{n}_2}$ such that $x =_o y$.

Note that an interval $(I_{\vec{n}})$ is o-Cauchy if and only if the end points $(I_{L,\vec{n}})$, $(I_{R,\vec{n}})$ are o-Cauchy real numbers.

The property of being a Cauchy interval respects equality:

Lemma 3.37. $\langle o_1 < o_2 \rangle$ *If* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *is* o_2*-Cauchy and* o_2*-equal to* $(T_{\vec{n}}) \subseteq \mathbb{Q}_t$*, then the latter is* o_1*-Cauchy.*

Proof. Take any $\vec{n}_1, \vec{n}_2$ and $x \in T_{\vec{n}_1}$. By the assumption, there are $u \in S_{\vec{n}_1}$, $v \in S_{\vec{n}_2}$ and $y \in T_{\vec{n}_2}$ such that $x =_{o_2} u =_{o_2} v =_{o_2} y$. Now $x =_{o_1} y$ follows by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. □

A real valued function of one variable is defined now obviously as follows:

Definition 3.38. Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. A real valued function of one variable defined on $(S_{\vec{n}})$ and taking values in $\mathbb{Q}_u$ is a finite multisequence $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ of functions from $S_{\vec{n}}$ to $\mathbb{Q}_u$.

Note that for each function $(f_{\vec{n}})$ in the sense of the above definition there is the unique function $(\bar{f}_{\vec{n}})$ in the customary sense of taking real numbers to real numbers and defined by putting $(\bar{f}_{\vec{n}})((x_{\vec{n}})) = (f_{\vec{n}}(x_{\vec{n}}))$, but the converse does not hold. We have chosen the smaller class of functions since it is rich enough and corresponds better with the hyperfinite techniques of nonstandard analysis.

We define next equality of functions. Recall that our definition of equality of subsets was not pointwise but approximative, so the definition of equality of functions cannot be pointwise either but must also be approximative. Accordingly, there is a close connection between definitions of equality and continuity of functions: a function is equal to itself if and only if it is continuous, see Definition 3.44 below.

Definition 3.39. $\langle o < p_1 \leq p_2 \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$, $(T_{\vec{n}}) \subseteq \mathbb{Q}_t$. We say that $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ and $(g_{\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)$ are p_2p_1o-equal if $(S_{\vec{n}}) =_{p_2} (T_{\vec{n}})$ and $f_{\vec{n}}(x) =_o g_{\vec{n}}(y)$ for all $\vec{n}$ and $x \in S_{\vec{n}}$, $y \in T_{\vec{n}}$ such that $x =_{p_1} y$. We write then $(f_{\vec{n}}) =_{p_2p_1o} (g_{\vec{n}})$.

Evidently, $=_{p_2p_1o}$ is not an equivalence relation. As we mentioned above, it is reflexive if and only if the function in question is continuous. Similarly, it is symmetric if and only if both the functions in question are continuous. Moreover, it is not transitive but we have the following:

Lemma 3.40. $\langle o_1 < o_2 \mid p_1 < p_2 \rangle$ *Let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$, $(T_{\vec{n}})$, $(U_{\vec{n}}) \subseteq \mathbb{Q}_t$ *and let* $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$, $(g_{\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)$, $(h_{\vec{n}} \colon U_{\vec{n}} \to \mathbb{Q}_u)$ *be functions. If* $(f_{\vec{n}})$ *is* $p_3p_2o_2$*-equal to* $(g_{\vec{n}})$ *and* $(g_{\vec{n}})$ *is* $p_3p_1o_2$*-equal to* $(h_{\vec{n}})$*, then* $(f_{\vec{n}})$ *is* $p_2p_2o_1$*-equal to* $(h_{\vec{n}})$*.*

Proof. Take any $\vec{n}$ and suppose $x \in S_{\vec{n}}$, $z \in U_{\vec{n}}$ are such that $x =_{p_2} z$. Since $x =_{p_2} y$ for some $y \in T_{\vec{n}}$, we get $y =_{p_1} z$ by the axiom $\infty_{p_2} \geq 2\infty_{p_1}$. Now $f_{\vec{n}}(x) =_{o_2} g_{\vec{n}}(y) =_{o_2} h_{\vec{n}}(z)$ by the assumption, so $f_{\vec{n}}(x) =_{o_1} h_{\vec{n}}(z)$ follows by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. □

Pointwise equality makes sense for functions defined on the same set:

Definition 3.41. Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. We say that $(f_{\vec{n}}, g_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ are pointwise o-equal if $f_{\vec{n}}(x) =_o g_{\vec{n}}(x)$ for all $\vec{n}$ and $x \in S_{\vec{n}}$. We write then $(f_{\vec{n}}) =_o (g_{\vec{n}})$.

The following definition is similar to Definition 3.8 above:

Definition 3.42. Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ and let $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ be a function and $(M_{\vec{n}})$ be a positive real number. We say that $(f_{\vec{n}})$ is $(M_{\vec{n}})$-bounded in case $|f_{\vec{n}}(x)| \leq M_{\vec{n}}$ for all $\vec{n}$ and $x \in S_{\vec{n}}$. We say that $(f_{\vec{n}})$ is $(1/M_{\vec{n}})$-appreciable in case $(1/f_{\vec{n}})$ is $(M_{\vec{n}})$-bounded.

The arithmetic operations, maximum, minimum and absolute value are defined on functions pointwise in the obvious way. They preserve equality:

Lemma 3.43. $\langle h, k, o_1 < o_2 \rangle$ *Let* $N \leq \infty_k$ *be a natural number and let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (T_{\vec{n}}) \subseteq \mathbb{Q}_t$. *If* $(f_{i,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is* $p_2 p_1 o_2$*-equal to* $(g_{i,\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)$ *for all* $i \leq N$, *then*

(*a*) $(\sum_{i=1}^{N} f_{i,\vec{n}}) =_{p_2 p_1 o_1} (\sum_{i=1}^{N} g_{i,\vec{n}})$,

(*b*) $(\prod_{i=1}^{N} f_{i,\vec{n}}) =_{p_2 p_1 o_1} (\prod_{i=1}^{N} g_{i,\vec{n}})$,

(*c*) $(f_{1,\vec{n}}^{-1}) =_{p_2 p_1 o_1} (g_{1,\vec{n}}^{-1})$,

(*d*) $(\max\{f_{i,\vec{n}} : i \leq N\}) =_{p_2 p_1 o_1} (\max\{g_{i,\vec{n}} : i \leq N\})$,

(*e*) $(\min\{f_{i,\vec{n}} : i \leq N\}) =_{p_2 p_1 o_1} (\min\{g_{i,\vec{n}} : i \leq N\})$,

(*f*) $(|f_{1,\vec{n}}|) =_{p_2 p_1 o_1} (|g_{1,\vec{n}}|)$.

In (*b*) *we assume that* $(f_{i+1,\vec{n}})$ *and* $(g_{i,\vec{n}})$ *are* ∞_h*-bounded for all* $i < N$. *In* (*c*) *we assume that* $(f_{1,\vec{n}})$ *and* $(g_{1,\vec{n}})$ *are* $\Diamond_h$*-appreciable.*

We define now continuity as follows:

Definition 3.44. $\langle o < p \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. We say that $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ is po-continuous if $f_{\vec{n}}(x) =_o f_{\vec{n}}(y)$ for all $\vec{n}$ and $x, y \in S_{\vec{n}}$ such that $x =_p y$.

The property of being a continuous function respects equality:

Lemma 3.45. $\langle o_1 < o_2 \mid p_1 < p_2 \rangle$ *Let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$, $(T_{\vec{n}}) \subseteq \mathbb{Q}_t$. *If* $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is* $p_1 o_2$*-continuous and* $p_2 p_2 o_2$*-equal to* $(g_{\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)$, *then the latter is* $p_2 o_1$*-continuous.*

Proof. Take any $\vec{n}$ and let $x, y \in T_{\vec{n}}$ be such that $x =_{p_2} y$. Since $u =_{p_2} x$ and $y =_{p_2} v$ for some $u, v \in S_{\vec{n}}$, we get $u =_{p_1} v$ by the axiom $\infty_{p_2} \geq 3\infty_{p_1}$. Then $g_{\vec{n}}(x) =_{o_2} f_{\vec{n}}(u) =_{o_2} f_{\vec{n}}(v) =_{o_2} g_{\vec{n}}(y)$ by the assumption, so $g_{\vec{n}}(x) =_{o_1} g_{\vec{n}}(y)$ follows by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. □

The arithmetic operations, maximum, minimum and absolute value preserve continuity:

Lemma 3.46. $\langle h, k, o_1 < o_2 \rangle$ *Let* $N \leq \infty_k$ *be a natural number and let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. *If* $(f_{i,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is* po_2*-continuous for all* $i \leq N$, *then*

(a) $(\sum_{i=1}^{N} f_{i,\vec{n}})$,

(b) $(\prod_{i=1}^{N} f_{i,\vec{n}})$,

(c) $(f_{1,\vec{n}}^{-1})$,

(d) $(\max\{f_{i,\vec{n}} : i \leq N\})$,

(e) $(\min\{f_{i,\vec{n}} : i \leq N\})$,

(f) $(|f_{1,\vec{n}}|)$

are po_1*-continuous. In (b) we assume that* $(f_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $i \leq N$. *In (c) we assume that* $(f_{1,\vec{n}})$ *is* $\lozenge_h$*-appreciable.*

The notion of continuity is quite weak. For instance, $(f_n \colon \mathbb{Q}_t \to \mathbb{Q}_t)$ with $f_n(x) = (-1)^n$ is continuous but has $|f_{n+1}(x) - f_n(x)| = 2$ for all n and $x \in \mathbb{Q}_t$. So we define next a notion stronger than continuity:

Definition 3.47. $\langle o < p < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. We say that $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ is *po*-Cauchy if $f_{\vec{n}_1}(x) =_o f_{\vec{n}_2}(y)$ for all $\vec{n}_1, \vec{n}_2$ and $x \in S_{\vec{n}_1}$, $y \in S_{\vec{n}_2}$ such that $x =_p y$.

The property of being a Cauchy function respects equality:

Lemma 3.48. $\langle o_1 < o_2 \mid p_1 < p_2 \rangle$ *Let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (T_{\vec{n}}) \subseteq \mathbb{Q}_t$. *If* $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is* $p_1 o_2$*-Cauchy and* $p_2 p_2 o_2$*-equal to* $(g_{\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)$, *then the latter is* $p_2 o_1$*-Cauchy.*

A Cauchy function takes a Cauchy argument to a Cauchy value:

Theorem 3.49. *Suppose* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. *If* $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is po-Cauchy and* $(x_{\vec{n}}) \in (S_{\vec{n}})$ *is p-Cauchy, then* $(f_{\vec{n}}(x_{\vec{n}}))$ *is o-Cauchy.*

Proof. This follows from Definition 3.47. □

The arithmetic operations, maximum, minimum and absolute value preserve Cauchyness:

Lemma 3.50. $\langle h, k, o_1 < o_2 \rangle$ *Let* $N \leq \infty_k$ *be a natural number and let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$*. If* $(f_{i,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is* po_2*-Cauchy for all* $i \leq N$*, then*

(a) $(\sum_{i=1}^{N} f_{i,\vec{n}})$,

(b) $(\prod_{i=1}^{N} f_{i,\vec{n}})$,

(c) $(f_{1,\vec{n}}^{-1})$,

(d) $(\max\{f_{i,\vec{n}} : i \leq N\})$,

(e) $(\min\{f_{i,\vec{n}} : i \leq N\})$,

(f) $(|f_{1,\vec{n}}|)$

are po_1*-Cauchy. In (b) we assume that* $(f_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $i \leq N$*. In (c) we assume that* $(f_{1,\vec{n}})$ *is* $\Diamond_h$*-appreciable.*

The intermediate value theorem:

Theorem 3.51. *Let* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *be an interval and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *be a po-continuous function such that* $f_{\vec{n}}(I_{L,\vec{n}}) < 0 < f_{\vec{n}}(I_{R,\vec{n}})$ *for all* $\vec{n}$*. Then* $(f_{\vec{n}}(c_{\vec{n}})) =_o 0$ *for some* $(c_{\vec{n}}) \in (I_{\vec{n}})$*.*

Proof. Take any $\vec{n}$ and let N be the smallest natural number such that $|I_{\vec{n}}| \leq N\Diamond_p$. Write $x_i = I_{L,\vec{n}} + i\Diamond_p$ for all $i < N$ and $x_N = I_{R,\vec{n}}$. Define $c_{\vec{n}}$ to be that x_i, where $i < N$ is the smallest natural number such that $f_{\vec{n}}(x_i) \leq 0$ and $f_{\vec{n}}(x_{i+1}) > 0$. Then $f_{\vec{n}}(c_{\vec{n}}) =_o 0$. □

The reason why we partitioned the interval $(I_{\vec{n}})$ anew in the above proof instead of using the partition of $\mathbb{Q}_t$ is that we want $(c_{\vec{n}})$ to depend only on the data given in the assumptions. Note that $(c_{\vec{n}})$ may not be Cauchy. To find a Cauchy root that depends only on the data given in the assumptions, we need a stronger assumption, for instance the assumption that $(f_{\vec{n}})$ is strictly increasing:

Definition 3.52. $\langle o < p \rangle$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval. We say that $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ is *op*-strictly increasing if $f_{\vec{n}}(x) <_p f_{\vec{n}}(y)$ for all $\vec{n}$ and $x, y \in I_{\vec{n}}$ such that $x <_o y$.

The property of being a strictly increasing function respects equality:

Lemma 3.53. $\langle o_1 < o_2, p_3 \mid p_1 < p_2 \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (J_{\vec{n}}) \subseteq \mathbb{Q}_t$ *are intervals. If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *is* o_2p_1*-strictly increasing and* $p_3p_3p_2$*-equal to* $(g_{\vec{n}} \colon J_{\vec{n}} \to \mathbb{Q}_u)$*, then the latter is* o_1p_2*-strictly increasing.*

Proof. Take any $\vec{n}$ and let $x, y \in J_{\vec{n}}$ be such that $x <_{o_1} y$. Since $u =_{p_3} x$ and $y =_{p_3} v$ for some $u, v \in I_{\vec{n}}$, we have $u <_{o_2} v$ by the axioms $\infty_{o_2} \geq 2\infty_{o_1}$ and $\infty_{p_3} \geq 4\infty_{o_1}$. Now $g_{\vec{n}}(x) =_{p_2} f_{\vec{n}}(u) <_{p_1} f_{\vec{n}}(v) =_{p_2} g_{\vec{n}}(y)$ by the assumption, so $g_{\vec{n}}(x) <_{p_2} g_{\vec{n}}(y)$ follows by the axiom $\infty_{p_2} \geq 3\infty_{p_1}$. □

A stronger assumption yields now a stronger conclusion:

Theorem 3.54. $\langle o_1 < o_2, p_2 \mid p_1 < o_3 \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *is a* p_2*-Cauchy interval and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *is a* p_2o_3*-Cauchy and* o_2p_1*-strictly increasing function such that* $(f_{\vec{n}}(I_{L,\vec{n}})) < 0 < (f_{\vec{n}}(I_{R,\vec{n}}))$*. Then* $(f_{\vec{n}}(c_{\vec{n}})) =_{o_3} 0$ *for some* o_1*-Cauchy real number* $(c_{\vec{n}}) \in (I_{\vec{n}})$*.*

Proof. Take any $\vec{n}_1, \vec{n}_2$. Let $c_{\vec{n}_1}, c_{\vec{n}_2}$ be as in the proof of the intermediate value theorem and let $z \in I_{\vec{n}_1}$ be such that $z =_{p_2} c_{\vec{n}_2}$. If $c_{\vec{n}_1} <_{o_1} c_{\vec{n}_2}$, then $c_{\vec{n}_1} <_{o_2} z$ by the axioms $\infty_{p_2} \geq 2\infty_{o_1}$ and $\infty_{o_2} \geq 2\infty_{o_1}$. Now by the assumption,

$$0 =_{o_3} f_{\vec{n}_1}(c_{\vec{n}_1}) <_{p_1} f_{\vec{n}_1}(z) =_{o_3} f_{\vec{n}_2}(c_{\vec{n}_2}) =_{o_3} 0,$$

which is impossible because of the axiom $\infty_{o_3} \geq 4\infty_{p_1}$. Hence $c_{\vec{n}_1} \geq_{o_1} c_{\vec{n}_2}$. A similar argument shows $c_{\vec{n}_1} \leq_{o_1} c_{\vec{n}_2}$, so $c_{\vec{n}_1} =_{o_1} c_{\vec{n}_2}$. □

The extreme value theorem:

Theorem 3.55. *Let* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *be an interval. If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *is* po*-continuous, then there are* $(c_{\vec{n}}), (d_{\vec{n}}) \in (I_{\vec{n}})$ *such that* $f_{\vec{n}}(c_{\vec{n}}) \leq_o f_{\vec{n}}(x) \leq_o f_{\vec{n}}(d_{\vec{n}})$ *for all* $\vec{n}$ *and* $x \in I_{\vec{n}}$*. Moreover, if* $(I_{\vec{n}})$ *is* p*-Cauchy and* $(f_{\vec{n}})$ *is* po*-Cauchy, then* $(f_{\vec{n}}(c_{\vec{n}})), (f_{\vec{n}}(d_{\vec{n}}))$ *are* o*-Cauchy.*

Proof. Take any $\vec{n}$ and let $N_{\vec{n}}$ be the smallest natural number such that $|I_{\vec{n}}| \leq N_{\vec{n}} \Diamond_p$. Write $x_{\vec{n},i} = I_{L,\vec{n}} + i\Diamond_p$ for all $i < N_{\vec{n}}$ and $x_{\vec{n},N_{\vec{n}}} = I_{R,\vec{n}}$. Define $c_{\vec{n}}$ to be that $x_{\vec{n},i}$, where $i \leq N_{\vec{n}}$ is the smallest natural number such that $f_{\vec{n}}(x_{\vec{n},i}) \leq f_{\vec{n}}(x_{\vec{n},j})$ for all $j \leq N_{\vec{n}}$. Take now any $x \in I_{\vec{n}}$. Since $x_{\vec{n},j} \leq x \leq x_{\vec{n},j+1}$ for some $j < N_{\vec{n}}$, we have $f_{\vec{n}}(c_{\vec{n}}) \leq f_{\vec{n}}(x_{\vec{n},j}) =_o f_{\vec{n}}(x)$ by the assumption. For the additional claim, take any $\vec{n}_1, \vec{n}_2$. Since there are $j_1 \leq N_{\vec{n}_1}$ and $j_2 \leq N_{\vec{n}_2}$ such that $x_{\vec{n}_1,j_1} =_p c_{\vec{n}_2}$ and $x_{\vec{n}_2,j_2} =_p c_{\vec{n}_1}$, we have

$$f_{\vec{n}_1}(c_{\vec{n}_1}) \leq f_{\vec{n}_1}(x_{\vec{n}_1,j_1}) =_o f_{\vec{n}_2}(c_{\vec{n}_2}) \leq f_{\vec{n}_2}(x_{\vec{n}_2,j_2}) =_o f_{\vec{n}_1}(c_{\vec{n}_1})$$

by the assumption. Therefore $f_{\vec{n}_1}(c_{\vec{n}_1}) =_o f_{\vec{n}_2}(c_{\vec{n}_2})$, so $(f_{\vec{n}}(c_{\vec{n}}))$ is o-Cauchy. A similar argument works for $(d_{\vec{n}})$. □

Composition of functions is defined in the obvious way:

Definition 3.56. Let $(S_{\vec{n}})_{\infty_{\bar{s}_1} \leq \vec{n} \leq \infty_{\bar{s}_2}} \subseteq \mathbb{Q}_t$, $(T_{\vec{n}}) \subseteq \mathbb{Q}_u$ and let $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$, $(g_{\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_v)$ be such that $(f_{\vec{n}}(S_{\vec{n}})) \subseteq (T_{\vec{n}})$. The composition of $(g_{\vec{n}})$ with $(f_{\vec{n}})$ is the function $(g_{\vec{n}} \circ f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_v)$ defined by putting $(g_{\vec{n}} \circ f_{\vec{n}})(x) = g_{\vec{n}}(f_{\vec{n}}(x))$ for all $\vec{n}$ and $x \in S_{\vec{n}}$.

Equality is respected:

Lemma 3.57. *Suppose* $(S_{\vec{n}})_{\infty_{\bar{s}_1} \leq \vec{n} \leq \infty_{\bar{s}_2}}, (U_{\vec{n}}) \subseteq \mathbb{Q}_t$ *and* $(T_{\vec{n}}), (V_{\vec{n}}) \subseteq \mathbb{Q}_u$. *If* $(f_{\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)$ *is* $p_2p_1o_2$*-equal to* $(h_{\vec{n}} \colon U_{\vec{n}} \to \mathbb{Q}_u)$ *and* $(g_{\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_v)$ *is* $p_1o_2o_1$*-equal to* $(i_{\vec{n}} \colon V_{\vec{n}} \to \mathbb{Q}_v)$ *and* $(f_{\vec{n}}(S_{\vec{n}})) \subseteq (T_{\vec{n}})$, $(g_{\vec{n}}(U_{\vec{n}})) \subseteq (T_{\vec{n}})$, *then* $(g_{\vec{n}} \circ f_{\vec{n}})$ *is* $p_2p_1o_1$*-equal to* $(i_{\vec{n}} \circ h_{\vec{n}})$.

3.5 Differentiation

We define the derivative of a function in the usual way:

Definition 3.58. $\langle o < p < q \rangle$ Let $(I_{\vec{n}})_{\infty_{\bar{s}_1} \leq \vec{n} \leq \infty_{\bar{s}_2}} \subseteq \mathbb{Q}_t$ be an interval and $(f_{\vec{n}}, f'_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ be functions. We say that $(f'_{\vec{n}})$ is a *qpo*-derivative of $(f_{\vec{n}})$ on $(I_{\vec{n}})$ if

$$\frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x - y} =_o f'_{\vec{n}}(x) \tag{6}$$

for all $\vec{n}$ and $x, y \in I_{\vec{n}}$ with $x \neq_q y$ and $x =_p y$. We write then $(f_{\vec{n}})' =_{qpo} (f'_{\vec{n}})$.

The assumption $x \neq_q y$ in the above definition is needed for showing that the notion of derivative respects equality:

Lemma 3.59. $\langle o_1 < o_2 \mid p_1 < p_2, q_3 \mid o_1, h, q_1 < q_2 < q_3 \rangle$ *Let* $(I_{\vec{n}})_{\infty_{\bar{s}_1} \leq \vec{n} \leq \infty_{\bar{s}_2}}$, $(J_{\vec{n}}) \subseteq \mathbb{Q}_t$ *be intervals and let* $(f_{\vec{n}}, f'_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$, $(g_{\vec{n}}, g'_{\vec{n}} \colon J_{\vec{n}} \to \mathbb{Q}_u)$ *be functions. If* $(f'_{\vec{n}})$ *is* ∞_h*-bounded,* $(g'_{\vec{n}})$ *and* $(f'_{\vec{n}})$ *are* $q_3q_3o_2$*-equal,* $(f'_{\vec{n}})$ *is a* $q_3p_1o_2$*-derivative of* $(f_{\vec{n}})$ *and* $(f_{\vec{n}})$ *and* $(g_{\vec{n}})$ *are* $q_3q_3q_2$*-equal, then* $(g'_{\vec{n}})$ *is a* $q_1p_2o_1$*-derivative of* $(g_{\vec{n}})$.

Proof. Take any $\vec{n}$ and $x, y \in J_{\vec{n}}$ with $x \neq_{q_1} y$ and $x =_{p_2} y$. Choose $u, v \in I_{\vec{n}}$ so that $u =_{q_3} x$ and $y =_{q_3} v$. Then on the one hand,

$$|u - v| \geq |x - y| - |x - u| - |y - v| > \Diamond_{q_1} - 2\Diamond_{q_3} \geq \Diamond_{q_3}$$

by the axiom $\infty_{q_3} \geq 3\infty_{q_1}$; on the other hand,

$$|u - v| \leq |x - y| + |x - u| + |y - v| \leq \Diamond_{p_2} + 2\Diamond_{q_3} \leq \Diamond_{p_1}$$

by the axioms $\infty_{p_2} \geq 2\infty_{p_1}$ and $\infty_{q_3} \geq 4\infty_{p_1}$. Now

$$\left|\frac{g_{\vec{n}}(x) - g_{\vec{n}}(y)}{x - y} - g'_{\vec{n}}(x)\right| \leq \left|\frac{g_{\vec{n}}(x) - g_{\vec{n}}(y)}{x - y} - \frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v}\right| + \left|\frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v} - f'_{\vec{n}}(u)\right| + |f'_{\vec{n}}(u) - g'_{\vec{n}}(x)|,$$

where the sum of the last two terms is $\leq \Diamond_{o_2} + \Diamond_{o_2} = 2\Diamond_{o_2}$ and

$$\begin{aligned}
&\left|\frac{g_{\vec{n}}(x) - g_{\vec{n}}(y)}{x - y} - \frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v}\right| \\
&\leq \left|\frac{g_{\vec{n}}(x) - f_{\vec{n}}(u)}{x - y}\right| + \left|\frac{f_{\vec{n}}(u) - g_{\vec{n}}(y)}{x - y} - \frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v}\right| \\
&\leq \infty_{q_1}\Diamond_{q_2} + \left|\frac{f_{\vec{n}}(v) - g_{\vec{n}}(y)}{x - y}\right| + \left|\frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{x - y} - \frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v}\right| \\
&\leq 2\infty_{q_1}\Diamond_{q_2} + \left|\frac{(u - x) - (v - y)}{x - y}\right| \cdot \left|\frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v}\right| \\
&\leq 2\infty_{q_1}\Diamond_{q_2} + 2\infty_{q_1}\Diamond_{q_3}\left(\left|\frac{f_{\vec{n}}(u) - f_{\vec{n}}(v)}{u - v} - f'_{\vec{n}}(u)\right| + |f'_{\vec{n}}(u)|\right) \\
&\leq 2\infty_{q_1}\Diamond_{q_2} + 2\infty_{q_1}\Diamond_{q_2}(\Diamond_{o_2} + \infty_h) \leq 4\infty_{q_1}\Diamond_{q_2} + 2\infty_h\infty_{q_1}\Diamond_{q_2},
\end{aligned}$$

where we used the axiom $\infty_{q_3} \geq \infty_{q_2}$. So we get altogether

$$\left|\frac{g_{\vec{n}}(x) - g_{\vec{n}}(y)}{x - y} - g'_{\vec{n}}(x)\right| \leq 4\infty_{q_1}\Diamond_{q_2} + 2\infty_h\infty_{q_1}\Diamond_{q_2} + 2\Diamond_{o_2} \leq \Diamond_{o_1}$$

by the axioms $\infty_{q_2} \geq 8\infty_{o_1}\infty_h\infty_{q_1}$ and $\infty_{o_2} \geq 8\infty_{o_1}$. □

It follows from the definition that if a function has a derivative, then both the function and its derivative are continuous:

Lemma 3.60. $\langle o_1, h < p < q \shortmid o_1 < o_2\rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has a* $\Diamond_p$*-appreciable length and* $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ *has a* qpo_2*-derivative* $(f'_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$. *Then* $(f'_{\vec{n}})$ *is* po_1*-continuous. If in addition* $(f'_{\vec{n}})$ *is* ∞_h*-bounded, then* $(f_{\vec{n}})$ *is also* po_1*-continuous.*

Proof. Take any $\vec{n}$ and let $x, y \in I_{\vec{n}}$ be such that $x =_p y$. If $x \neq_q y$, then

$$\begin{aligned}
|f'_{\vec{n}}(x) - f'_{\vec{n}}(y)| &\leq \left|f'_{\vec{n}}(x) - \frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x - y}\right| + \left|\frac{f_{\vec{n}}(y) - f_{\vec{n}}(x)}{y - x} - f'_{\vec{n}}(y)\right| \\
&\leq 2\Diamond_{o_2} \leq \Diamond_{o_1}
\end{aligned}$$

by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. If $x =_q y$, then $x \neq_q z \neq_q y$ and $x =_p z =_p y$ for some $z \in I_{\vec{n}}$ because of the assumption $|I_{\vec{n}}| \geq \Diamond_p$ and the axiom $\infty_q \geq 3\infty_p$, so

$$|f'_{\vec{n}}(x) - f'_{\vec{n}}(y)| \leq |f'_{\vec{n}}(x) - f'_{\vec{n}}(z)| + |f'_{\vec{n}}(z) - f'_{\vec{n}}(y)| \leq 4\Diamond_{o_2} \leq \Diamond_{o_1}$$

by the first case and the axiom $\infty_{o_2} \geq 4\infty_{o_1}$. Hence $(f'_{\vec{n}})$ is po_1-continuous. The proof of the other claim is similar. □

If a function has a derivative, then the derivative is unique in the following sense:

Lemma 3.61. $\langle o_1 < o_2 \mid p < q\rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has a* $\Diamond_p$*-appreciable length. If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *has* qpo_2*-derivatives* $(g_{\vec{n}}, h_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$*, then* $(g_{\vec{n}})$ *and* $(h_{\vec{n}})$ *are* o_1*-equal.*

Proof. Take any $\vec{n}$ and $x \in I_{\vec{n}}$. Since $|I_{\vec{n}}| \geq \Diamond_p$ by the assumption, we get $x \neq_q y$ and $x =_p y$ for some $y \in I_{\vec{n}}$ by the axiom $\infty_q \geq 3\infty_p$. Now

$$g_{\vec{n}}(x) =_{o_2} \frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x - y} =_{o_2} h_{\vec{n}}(x),$$

so $(g_{\vec{n}}) =_{o_1} (h_{\vec{n}})$ holds by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. □

The usual rules of derivation hold under some extra conditions:

Lemma 3.62. $\langle h, k, o_1 < o_2 < o_3 < o_4 \mid p < q\rangle$ *Let* $N \leq \infty_k$ *be a natural number and* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *be an interval having a* $\Diamond_p$*-appreciable length. If* $(f_{i,\vec{n}}, f'_{i,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *are such that* $(f_{i,\vec{n}})' =_{qpo_4} (f'_{i,\vec{n}})$ *for all* $i \leq N$*, then*

(*a*) $(\sum_{i=1}^N f_{i,\vec{n}})' =_{qpo_1} (\sum_{i=1}^N f'_{i,\vec{n}})$,

(*b*) $(\prod_{i=1}^N f_{i,\vec{n}})' =_{qpo_1} (\sum_{i=1}^N (f'_{i,\vec{n}} \prod_{j=1, j\neq i}^N f_{j,\vec{n}}))$,

(*c*) $(f_{1,\vec{n}})' =_{qpo_1} (-f'_{1,\vec{n}}/f_{1,\vec{n}}^2)$.

In (b) we assume that $(f_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $i \leq N$ *and* $(f'_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $1 < i \leq N$*. In (c) we assume that* $(f_{1,\vec{n}})$ *is* ∞_h*-bounded and* $\Diamond_h$*-appreciable.*

Proof. Take any $\vec{n}$ and let $x, y \in I_{\vec{n}}$ be such that $x \neq_q y$ and $x =_p y$.

(a) We have

$$\left|\frac{\sum_{i=1}^N f_{i,\vec{n}}(x)-\sum_{i=1}^N f_{i,\vec{n}}(y)}{x-y}-\sum_{i=1}^N f'_{i,\vec{n}}(x)\right|$$
$$\leq \sum_{i=1}^N \left|\frac{f_{i,\vec{n}}(x)-f_{i,\vec{n}}(y)}{x-y}-f'_{i,\vec{n}}(x)\right| \leq \infty_k \Diamond_{o_4} \leq \Diamond_{o_1}$$

by the axiom $\infty_{o_4} \geq \infty_{o_1}\infty_k$.

(b) Each $(f_{i,\vec{n}})$ is po_3-continuous by Lemma 3.60, so each $(\prod_{k=1}^{i-1} f_{k,\vec{n}})$ is po_2-continuous by Lemma 3.46. If we put $z_{i,j}=x$ in case $j \geq i$ and $z_{i,j}=y$ otherwise, we get

$$\left|\frac{\prod_{i=1}^N f_{i,\vec{n}}(x)-\prod_{i=1}^N f_{i,\vec{n}}(y)}{x-y}-\sum_{i=1}^N\left(f'_{i,\vec{n}}(x)\prod_{j=1,j\neq i}^N f_{j,\vec{n}}(x)\right)\right|$$
$$\leq \sum_{i=1}^N \prod_{j=1,j\neq i}^N |f_{j,\vec{n}}(z_{i,j})|\left|\frac{f_{i,\vec{n}}(x)-f_{i,\vec{n}}(y)}{x-y}-f'_{i,\vec{n}}(x)\right|$$
$$+\sum_{i=2}^N\left(|f'_{i,\vec{n}}(x)|\prod_{j=i+1}^N |f_{j,\vec{n}}(x)|\left|\prod_{k=1}^{i-1} f_{k,\vec{n}}(x)-\prod_{k=1}^{i-1} f_{k,\vec{n}}(y)\right|\right)$$
$$\leq \infty_k \infty_h^{\infty_k-1}\Diamond_{o_4}+\infty_h\left(\sum_{i=0}^{\infty_k-2}\infty_h^i\right)\Diamond_{o_2} \leq \Diamond_{o_1}$$

by the axioms $\infty_{o_4} \geq 2\infty_{o_1}\infty_k\infty_h^{\infty_k-1}$ and $\infty_{o_2} \geq 2\infty_{o_1}\infty_h(\sum_{i=0}^{\infty_k-2}\infty_h^i)$.

(c) Since $(f_{1,\vec{n}})$ is po_3-continuous by Lemma 3.60, we have

$$\left|\frac{f_{1,\vec{n}}^{-1}(x)-f_{1,\vec{n}}^{-1}(y)}{x-y}+\frac{f'_{1,\vec{n}}(x)}{f_{1,\vec{n}}^2(x)}\right|$$
$$=|f_{1,\vec{n}}^{-1}(x)|\cdot|f_{1,\vec{n}}^{-1}(y)|\cdot\left|\frac{f_{1,\vec{n}}(x)-f_{1,\vec{n}}(y)}{x-y}-f_{1,\vec{n}}(y)f_{1,\vec{n}}^{-1}(x)f'_{1,\vec{n}}(x)\right|$$
$$\leq |f_{1,\vec{n}}^{-1}(x)|\cdot|f_{1,\vec{n}}^{-1}(y)|\cdot\left|\frac{f_{1,\vec{n}}(x)-f_{1,\vec{n}}(y)}{x-y}-f'_{1,\vec{n}}(x)\right|$$
$$+|f_{1,\vec{n}}^{-2}(x)|\cdot|f_{1,\vec{n}}^{-1}(y)|\cdot|f'_{1,\vec{n}}(x)|\cdot|f_{1,\vec{n}}(x)-f_{1,\vec{n}}(y)|$$
$$\leq \infty_h^2\Diamond_{o_4}+\infty_h^3\Diamond_{o_3} \leq \Diamond_{o_1}$$

by the axioms $\infty_{o_3} \geq 2\infty_h^3\infty_{o_1}$ and $\infty_{o_4} \geq 2\infty_h^2\infty_{o_1}$. □

The chain rule:

Theorem 3.63. $\langle h, o_1 < o_2 \mid q_1, o_1 < q_2, q_3 \rangle$ *Let* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *and* $(J_{\vec{n}}) \subseteq \mathbb{Q}_u$ *be intervals and* $(f_{\vec{n}}, f'_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *and* $(g_{\vec{n}}, g'_{\vec{n}} \colon J_{\vec{n}} \to \mathbb{Q}_v)$ *be functions such that* $(f'_{\vec{n}})$ *is a* $q_1p_2o_2$*-derivative of* $(f_{\vec{n}})$ *and* $(g'_{\vec{n}})$ *is a* $q_3p_1o_2$*-derivative of* $(g_{\vec{n}})$*. If* $(f_{\vec{n}})$ *is* p_2p_1*-continuous,* $(g_{\vec{n}})$ *is* q_3q_2*-continuous, both* $(f'_{\vec{n}})$ *and* $(g'_{\vec{n}})$ *are* ∞_h*-bounded and* $(f_{\vec{n}}(I_{\vec{n}})) \subseteq (J_{\vec{n}})$*, then*

$$(g_{\vec{n}} \circ f_{\vec{n}})' =_{q_1p_2o_1} ((g'_{\vec{n}} \circ f_{\vec{n}}) \cdot f'_{\vec{n}}).$$

Proof. Take any $\vec{n}$ and let $x, y \in I_{\vec{n}}$ be such that $x \neq_{q_1} y$ and $x =_{p_2} y$. Then $f_{\vec{n}}(x) =_{p_1} f_{\vec{n}}(y)$ by p_2p_1-continuity. We now have two cases. If $f_{\vec{n}}(x) \neq_{q_3} f_{\vec{n}}(y)$, then

$$\begin{aligned}
&\left| \frac{g_{\vec{n}}(f_{\vec{n}}(x)) - g_{\vec{n}}(f_{\vec{n}}(y))}{x-y} - g'_{\vec{n}}(f_{\vec{n}}(x)) f'_{\vec{n}}(x) \right| \\
&\quad \leq \left| \frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x-y} \right| \cdot \left| \frac{g_{\vec{n}}(f_{\vec{n}}(x)) - g_{\vec{n}}(f_{\vec{n}}(y))}{f_{\vec{n}}(x) - f_{\vec{n}}(y)} - g'_{\vec{n}}(f_{\vec{n}}(x)) \right| \\
&\qquad + |g'_{\vec{n}}(f_{\vec{n}}(x))| \cdot \left| \frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x-y} - f'_{\vec{n}}(x) \right| \\
&\quad \leq (|f'_{\vec{n}}(x)| + 1) \Diamond_{o_2} + |g'_{\vec{n}}(f_{\vec{n}}(x))| \Diamond_{o_2} \\
&\quad \leq (\infty_h + 1) \Diamond_{o_2} + \infty_h \Diamond_{o_2} \leq \Diamond_{o_1}
\end{aligned}$$

by the axiom $\infty_{o_2} \geq \infty_{o_1}(2\infty_h + 1)$. If $f_{\vec{n}}(x) =_{q_3} f_{\vec{n}}(y)$, then

$$\begin{aligned}
&\left| \frac{g_{\vec{n}}(f_{\vec{n}}(x)) - g_{\vec{n}}(f_{\vec{n}}(y))}{x-y} - g'_{\vec{n}}(f_{\vec{n}}(x)) f'_{\vec{n}}(x) \right| \\
&\quad \leq \left| \frac{g_{\vec{n}}(f_{\vec{n}}(x)) - g_{\vec{n}}(f_{\vec{n}}(y))}{x-y} \right| + |g'_{\vec{n}}(f_{\vec{n}}(x))| \cdot |f'_{\vec{n}}(x)| \\
&\quad \leq \infty_{q_1} \Diamond_{q_2} + |g'_{\vec{n}}(f_{\vec{n}}(x))| \left(\left| f'_{\vec{n}}(x) - \frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x-y} \right| + \left| \frac{f_{\vec{n}}(x) - f_{\vec{n}}(y)}{x-y} \right| \right) \\
&\quad \leq \infty_{q_1} \Diamond_{q_2} + \infty_h (\Diamond_{o_2} + \infty_{q_1} \Diamond_{q_3}) \leq \Diamond_{o_1}
\end{aligned}$$

by the axioms $\infty_{q_2} \geq 3\infty_{q_1}\infty_{o_1}$ and $\infty_{o_2} \geq 3\infty_{o_1}\infty_h$ and $\infty_{q_3} \geq 3\infty_{q_1}\infty_{o_1}$. □

Rolle's theorem:

Theorem 3.64. $\langle o_1 < o_2 < o_3 \mid p < q \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *is a* $\Diamond_p$*-proper interval and* $(f_{\vec{n}}, f'_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *are such that* $(f_{\vec{n}})' =_{qpo_3} (f'_{\vec{n}})$ *and* $(f_{\vec{n}}(I_{L,\vec{n}})) =_{o_2} (f_{\vec{n}}(I_{R,\vec{n}}))$*. Then* $(f'_{\vec{n}}(c_{\vec{n}})) =_{o_1} 0$ *for some* $(c_{\vec{n}}) \in (I_{\vec{n}})$*.*

Proof. Since $(f'_{\vec{n}})$ is po_2-continuous by Lemma 3.60, it follows from the extreme value theorem that there is $(c_{\vec{n}}) \in (I_{\vec{n}})$ such that $|f'_{\vec{n}}(c_{\vec{n}})| \leq_{o_2} |f'_{\vec{n}}(x)|$ for all $\vec{n}$ and $x \in I_{\vec{n}}$. Take now any $\vec{n}$. Suppose $f'_{\vec{n}}(c_{\vec{n}}) >_{o_1} 0$ and choose points $I_{L,\vec{n}} = d_0 < d_1 < \ldots < d_N < d_{N+1} = I_{R,\vec{n}}$ so that $d_i \neq_q d_{i+1}$ and $d_i =_p d_{i+1}$ for all $i = 0, \ldots, N$. Then contrary to the assumption $f_{\vec{n}}(I_{L,\vec{n}}) =_{o_2} f_{\vec{n}}(I_{R,\vec{n}})$,

$$\begin{aligned} & f_{\vec{n}}(I_{L,\vec{n}}) - f_{\vec{n}}(I_{R,\vec{n}}) \\ &\quad = \sum_{i=0}^{N} f'_{\vec{n}}(d_i)(d_{i+1} - d_i) + \sum_{i=0}^{N} [f_{\vec{n}}(d_{i+1}) - f_{\vec{n}}(d_i) - f'_{\vec{n}}(d_i)(d_{i+1} - d_i)] \\ &\quad \geq (f'_{\vec{n}}(c_{\vec{n}}) - \lozenge_{o_2})(I_{R,\vec{n}} - I_{L,\vec{n}}) - \lozenge_{o_3}(I_{R,\vec{n}} - I_{L,\vec{n}}) \\ &\quad > (\lozenge_{o_1} - \lozenge_{o_2} - \lozenge_{o_3})\lozenge_{o_1} \geq \lozenge_{o_2} \end{aligned}$$

by the axioms $\infty_{o_2} \geq 4\infty_{o_1}$ and $\infty_{o_3} \geq 4\infty_{o_1}$ and $\infty_{o_2} \geq 2\infty_{o_1}^2$, so we must have $f'_{\vec{n}}(c_{\vec{n}}) \leq_{o_1} 0$. By a similar argument, $f'_{\vec{n}}(c_{\vec{n}}) \geq_{o_1} 0$. □

The mean value theorem follows as a corollary:

Corollary 3.65. $\langle o_1 < o_2 \mid p < q \rangle$ *If* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *is a* $\lozenge_p$*-proper interval and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *has a* qpo_2*-derivative* $(f'_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$*, then*

$$(f'_{\vec{n}}(c_{\vec{n}})) =_{o_1} \left(\frac{f_{\vec{n}}(I_{R,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}})}{I_{R,\vec{n}} - I_{L,\vec{n}}} \right)$$

for some $(c_{\vec{n}}) \in (I_{\vec{n}})$.

Proof. Define $(h_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ by putting

$$h_{\vec{n}}(x) = \frac{f_{\vec{n}}(I_{R,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}})}{I_{R,\vec{n}} - I_{L,\vec{n}}}(x - I_{L,\vec{n}}) - f_{\vec{n}}(x)$$

for all $\vec{n}$ and $x \in I_{\vec{n}}$. Let $(h'_{\vec{n}})$ be a qpo_2-derivative of $(h_{\vec{n}})$. Since $(h_{\vec{n}}(I_{L,\vec{n}})) = (h_{\vec{n}}(I_{R,\vec{n}}))$, it follows from Rolle's theorem that

$$(h'_{\vec{n}}(c_{\vec{n}})) = \left(\frac{f_{\vec{n}}(I_{R,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}})}{I_{R,\vec{n}} - I_{L,\vec{n}}} - f'_{\vec{n}}(c_{\vec{n}}) \right) =_{o_1} 0$$

for some $(c_{\vec{n}}) \in (I_{\vec{n}})$. □

The mean value theorem has in turn the following corollary:

Corollary 3.66. $\langle h, o_1 < o_2 < o_3 \mid p < q \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has a* $\lozenge_p$*-appreciable and* ∞_h*-bounded length and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *has a* qpo_3*-derivative* $(f'_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$*. If* $(f'_{\vec{n}}) =_{o_2} 0$*, then* $(f_{\vec{n}}) =_{o_1} (f_{\vec{n}}(d_{\vec{n}}))$ *for all* $(d_{\vec{n}}) \in (I_{\vec{n}})$*.*

Proof. Take any $\vec{n}$ and $x \in I_{\vec{n}}$. If $d_{\vec{n}} =_p x$, then $f_{\vec{n}}(d_{\vec{n}}) =_{o_1} f_{\vec{n}}(x)$, since $(f_{\vec{n}})$ is po_1-continuous by Lemma 3.60. If $d_{\vec{n}} \neq_p x$, then by the mean value theorem there is $c_{\vec{n}} \in [\min(d_{\vec{n}}, x), \max(d_{\vec{n}}, x)]$ such that

$$\frac{f_{\vec{n}}(d_{\vec{n}}) - f_{\vec{n}}(x)}{d_{\vec{n}} - x} =_{o_2} f'_{\vec{n}}(c_{\vec{n}}) =_{o_2} 0,$$

so

$$|f_{\vec{n}}(d_{\vec{n}}) - f_{\vec{n}}(x)| \leq 2\Diamond_{o_2}|d_{\vec{n}} - x| \leq 2\Diamond_{o_2}\infty_h \leq \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \geq 2\infty_h\infty_{o_1}$. □

3.6 Riemann Integration

Each function has an integral in the following step function sense simply because its domain is a finite set:

Definition 3.67. $\langle t, u < v \rangle$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval and $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ be a function. Write $dx_t = 1/\infty_t!$. We call the real number $(\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\, dx_t) \in \mathbb{Q}_v$, defined by putting

$$\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\, dx_t = \sum_{I_{L,\vec{n}} \leq x < I_{R,\vec{n}}} f_{\vec{n}}(x) dx_t \tag{7}$$

for all $\vec{n}$, the integral of $(f_{\vec{n}})$ over $(I_{\vec{n}})$. We often write $(\int_{I_{\vec{n}}} f_{\vec{n}}\, dx_t)$ instead of $(\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\, dx_t)$. We define $(\int_{I_{R,\vec{n}}}^{I_{L,\vec{n}}} f_{\vec{n}}\, dx_t)$ by putting

$$\int_{I_{R,\vec{n}}}^{I_{L,\vec{n}}} f_{\vec{n}}\, dx_t = -\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\, dx_t$$

for all $\vec{n}$.

What remains to be done is to specify classes of functions for which the integral is a Cauchy real and which have convenient closure properties. We define below the class of Riemann integrable functions and in Chapter 4 the class of L^p-functions.

A Riemann integrable function is defined here as follows:

Definition 3.68. $\langle o < p \rangle$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be a p-Cauchy interval and $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ be a po-Cauchy function. We then say that $(f_{\vec{n}})$ is po-Riemann integrable and call $(\int_{I_{\vec{n}}} f_{\vec{n}}\, dx_t)$ its po-Riemann integral.

Note that the term defined by Equation (7) above contains ∞_t. However, the Riemann integral of a Riemann integrable function can be approximated by a term not containing it:

Lemma 3.69. $\langle h, o_1 < o_2 \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has an* ∞_h*-bounded length and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *is* po_2*-Riemann integrable. For each* $\vec{n}$*, if* N *is the smallest natural number such that* $|I_{\vec{n}}| \leq N \lozenge_p$ *and*

$$dx_i = \begin{cases} \lozenge_p & \text{if } i < N-1, \\ |I_{\vec{n}}| - (N-1)\lozenge_p & \text{if } i = N-1 \end{cases}$$

and $x_i = I_{L,\vec{n}} + \sum_{j=o}^{i-1} dx_j$*, then*

$$\int_{I_{\vec{n}}} f_{\vec{n}} \, dx_t =_{o_1} \sum_{i=0}^{N-1} f_{\vec{n}}(x_i) dx_i.$$

Proof. Since $(f_{\vec{n}})$ is po_2-Cauchy by the assumption, we have

$$\begin{aligned} \Bigg| \sum_{I_{L,\vec{n}} \leq x < I_{R,\vec{n}}} f_{\vec{n}}(x) dx_t - \sum_{i=0}^{N-1} f_{\vec{n}}(x_i) dx_i \Bigg| & \\ \leq dx_t \sum_{i=0}^{N-1} \sum_{x_i \leq x < x_i + dx_i} |f_{\vec{n}}(x) - f_{\vec{n}}(x_i)| & \\ \leq \lozenge_{o_2} |I_{\vec{n}}| \leq \lozenge_{o_2} \infty_h \leq \lozenge_{o_1} & \end{aligned}$$

for each $\vec{n}$ by the axiom $\infty_{o_2} \geq \infty_{o_1} \infty_h$. ☐

The Riemann integral of a Riemann integrable function is a Cauchy real number under some additional assumptions:

Lemma 3.70. $\langle h, o_1 < p \shortmid k, o_1 < o_2 \rangle$ *If* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *is a* p*-Cauchy interval having an* ∞_k*-bounded length and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *is an* ∞_h*-bounded* po_2*-Riemann integrable function, then* $(\int_{I_{\vec{n}}} f_{\vec{n}} \, dx_t)$ *is an* o_1*-Cauchy real number.*

Proof. Take any $\vec{n}_1, \vec{n}_2$. We divide the proof into two cases for the sake of clarity. If $\max(I_{L,\vec{n}_1}, I_{L,\vec{n}_2}) \geq \min(I_{R,\vec{n}_1}, I_{R,\vec{n}_2})$, then

$$\begin{aligned} \left| \int_{I_{\vec{n}_1}} f_{\vec{n}_1} \, dx_t - \int_{I_{\vec{n}_2}} f_{\vec{n}_2} \, dx_t \right| &\leq |I_{R,\vec{n}_1} - I_{L,\vec{n}_1}| \cdot \infty_h + |I_{R,\vec{n}_2} - I_{L,\vec{n}_2}| \cdot \infty_h \\ &\leq 2\lozenge_p \infty_h \leq \lozenge_{o_1} \end{aligned}$$

by the axiom $\infty_p \geq 2\infty_{o_1}\infty_h$. If $L = \max(I_{L,\vec{n}_1}, I_{L,\vec{n}_2}) < \min(I_{R,\vec{n}_1}, I_{R,\vec{n}_2}) = R$, then

$$\begin{aligned}\left|\int_{I_{\vec{n}_1}} f_{\vec{n}_1}\,dx_t - \int_{I_{\vec{n}_2}} f_{\vec{n}_2}\,dx_t\right| &\leq 2\Diamond_p\infty_h + \sum_{L\leq x<R} |f_{\vec{n}_1}(x) - f_{\vec{n}_2}(x)|dx_t \\ &\leq 2\Diamond_p\infty_h + \Diamond_{o_2}(R-L) \\ &\leq 2\Diamond_p\infty_h + \Diamond_{o_2}\infty_k \leq \Diamond_{o_1}\end{aligned}$$

by the axioms $\infty_p \geq 4\infty_{o_1}\infty_h$ and $\infty_{o_2} \geq 2\infty_{o_1}\infty_k$. □

A similar proof shows that equal functions have equal integrals:

Lemma 3.71. *$\langle h, o_1 < p \mid k, o_1 < o_2\rangle$ Suppose $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}, (J_{\vec{n}}) \subseteq \mathbb{Q}_t$ have ∞_k-bounded lengths. If $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ and $(g_{\vec{n}}\colon J_{\vec{n}} \to \mathbb{Q}_u)$ are ∞_h-bounded and ppo$_2$-equal, then $(\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t)$ and $(\int_{J_{\vec{n}}} g_{\vec{n}}\,dx_t)$ are o_1-equal.*

The proofs of the following two lemmas are immediate.

Lemma 3.72. *Let $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval, $(f_{\vec{n}}, g_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ be functions and $(a_{\vec{n}})$, $(b_{\vec{n}})$ be real numbers. Then*

$$\int_{I_{\vec{n}}} (a_{\vec{n}}f_{\vec{n}} + b_{\vec{n}}g_{\vec{n}})\,dx_t = a_{\vec{n}}\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t + b_{\vec{n}}\int_{I_{\vec{n}}} g_{\vec{n}}\,dx_t$$

for all $\vec{n}$. If $(f_{\vec{n}}) \leq (g_{\vec{n}})$, then

$$\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t \leq \int_{I_{\vec{n}}} g_{\vec{n}}\,dx_t$$

for all $\vec{n}$.

Lemma 3.73. *Let $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval, $(c_{\vec{n}}) \in (I_{\vec{n}})$ be a real number and $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ be a function. Then*

$$\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\,dx_t = \int_{I_{L,\vec{n}}}^{c_{\vec{n}}} f_{\vec{n}}\,dx_t + \int_{c_{\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\,dx_t$$

for all $\vec{n}$.

We define a primitive of a function as usual:

Definition 3.74. Let $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval and $(f_{\vec{n}}, F_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ be functions. We call $(F_{\vec{n}})$ a *qpo*-primitive of $(f_{\vec{n}})$ in case $(F_{\vec{n}})' =_{qpo} (f_{\vec{n}})$.

Taking derivatives is inverse to taking integrals:

Theorem 3.75. *Let* $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *be an interval,* $(c_{\vec{n}}) \in (I_{\vec{n}})$ *be a real number and* $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ *be a po-continuous function. If we define* $(F_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ *by putting*

$$F_{\vec{n}}(x) = \int_{c_{\vec{n}}}^{x} f_{\vec{n}}\, dx_t$$

for all $\vec{n}$ *and* $x \in I_{\vec{n}}$*, then* $(F_{\vec{n}})$ *is a qpo-primitive of* $(f_{\vec{n}})$*.*

Proof. Take any $\vec{n}$ and $x, y \in I_{\vec{n}}$ such that $x \neq_q y$ and $x =_p y$. We may assume that $x < y$. Then

$$\begin{aligned}\left|\frac{F_{\vec{n}}(y) - F_{\vec{n}}(x)}{y-x} - f_{\vec{n}}(x)\right| &= \left|\frac{\sum_{x\leq z<y} f_{\vec{n}}(z)dx_t}{y-x} - f_{\vec{n}}(x)\right| \\ &\leq \frac{\sum_{x\leq z<y}|f_{\vec{n}}(z) - f_{\vec{n}}(x)|dx_t}{y-x} \\ &\leq \max_{x\leq z<y}|f_{\vec{n}}(z) - f_{\vec{n}}(x)| \leq \Diamond_o\end{aligned}$$

by *po*-continuity. □

The fundamental theorem of calculus follows as a corollary:

Corollary 3.76. $\langle h, o_1 < o_2 < o_3 \mid p < q\rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has a* $\Diamond_p$*-appreciable and* ∞_h*-bounded length and* $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ *has a* po_3*-continuous* qpo_3*-derivative* $(f'_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$*. Then*

$$\int_{I_{\vec{n}}} f'_{\vec{n}}\, dx_t =_{o_1} f_{\vec{n}}(I_{R,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}}) \tag{8}$$

for all $\vec{n}$*.*

Proof. Define $(F_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ by putting $F_{\vec{n}}(x) = \int_{I_{L,\vec{n}}}^{x} f'_{\vec{n}}\, dx_t$ for all $\vec{n}$ and $x \in I_{\vec{n}}$. Then $(F_{\vec{n}} - f_{\vec{n}})' =_{qpo_2} 0$ by the preceding theorem and Lemma 3.62, so $F_{\vec{n}}(x) - f_{\vec{n}}(x) =_{o_1} F_{\vec{n}}(I_{L,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}})$ for all $\vec{n}$ and $x \in I_{\vec{n}}$ by Corollary 3.66. In particular, $F_{\vec{n}}(I_{R,\vec{n}}) - f_{\vec{n}}(I_{R,\vec{n}}) =_{o_1} F_{\vec{n}}(I_{L,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}})$. Since $F_{\vec{n}}(I_{L,\vec{n}}) = 0$ and $F_{\vec{n}}(I_{R,\vec{n}}) = \int_{I_{\vec{n}}} f'_{\vec{n}}\, dx_t$, Equation (8) holds. □

Another corollary is the formula for integration by parts:

Corollary 3.77. $\langle h, k, o_1 < o_2 < o_3 \mid p < q\rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has a* $\Diamond_p$*-appreciable and* ∞_h*-bounded length and* $(f_{\vec{n}}, f'_{\vec{n}}, g_{\vec{n}}, g'_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$ *are* ∞_k*-bounded and such that* $(f_{\vec{n}})' =_{qpo_3} (f'_{\vec{n}})$*,* $(g_{\vec{n}})' =_{qpo_3} (g'_{\vec{n}})$*. Then*

$$\int_{I_{\vec{n}}} f_{\vec{n}} g'_{\vec{n}}\, dx_t =_{o_1} f_{\vec{n}}(I_{R,\vec{n}})g_{\vec{n}}(I_{R,\vec{n}}) - f_{\vec{n}}(I_{L,\vec{n}})g_{\vec{n}}(I_{L,\vec{n}}) - \int_{I_{\vec{n}}} f'_{\vec{n}} g_{\vec{n}}\, dx_t$$

for all $\vec{n}$*.*

Proof. Since $(f_{\vec{n}}g_{\vec{n}})' =_{qpo_2} (f_{\vec{n}}g'_{\vec{n}} + f'_{\vec{n}}g_{\vec{n}})$ by Lemma 3.62 and $(f_{\vec{n}}g'_{\vec{n}} + f'_{\vec{n}}g_{\vec{n}})$ is po_2-continuous by Lemma 3.60, the formula follows from the fundamental theorem of calculus. □

Our last corollary is the change of variables theorem:

Corollary 3.78. $\langle h, k, o_1 < o_2 < o_3 < p_1 < p_2 < q_1 < q_2 < q_3 \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_u$ *has a* $\lozenge_{p_2}$*-appreciable and* ∞_h*-bounded length and* $(J_{\vec{n}}) \subseteq \mathbb{Q}_t$*. If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_v)$ *is* ∞_k*-bounded and* p_1o_3*-continuous and* $(g_{\vec{n}}, g'_{\vec{n}} \colon J_{\vec{n}} \to \mathbb{Q}_u)$ *are such that* $(g_{\vec{n}})$ *is* p_2p_1*-continuous,* $(g_{\vec{n}})' =_{q_1p_2o_3} (g'_{\vec{n}})$ *and* $(g'_{\vec{n}})$ *is* ∞_k*-bounded,* $(g_{\vec{n}}(J_{\vec{n}})) \subseteq (I_{\vec{n}})$*,* $(g_{\vec{n}}(J_{L,\vec{n}})) = (I_{L,\vec{n}})$ *and* $(g_{\vec{n}}(J_{R,\vec{n}})) = (I_{R,\vec{n}})$*, then*

$$\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\, dx_u =_{o_1} \int_{J_{L,\vec{n}}}^{J_{R,\vec{n}}} (f_{\vec{n}} \circ g_{\vec{n}}) \cdot g'_{\vec{n}}\, dx_t \tag{9}$$

for all $\vec{n}$*.*

Proof. Let $(F_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ be as in Theorem 3.75 with $(c_{\vec{n}}) = (I_{L,\vec{n}})$. It is easy to see that $(F_{\vec{n}})$ is q_3q_2-continuous. Moreover, $(F_{\vec{n}})' =_{q_3p_1o_3} (f_{\vec{n}})$ by Theorem 3.75. Thus for each $\vec{n}$,

$$\begin{aligned}\int_{I_{L,\vec{n}}}^{I_{R,\vec{n}}} f_{\vec{n}}\, dx_u &= F_{\vec{n}}(I_{R,\vec{n}}) - F_{\vec{n}}(I_{L,\vec{n}}) \\ &= F_{\vec{n}}(g_{\vec{n}}(J_{R,\vec{n}})) - F_{\vec{n}}(g_{\vec{n}}(J_{L,\vec{n}})) =_{o_1} \int_{J_{L,\vec{n}}}^{J_{R,\vec{n}}} (f_{\vec{n}} \circ g_{\vec{n}}) \cdot g'_{\vec{n}}\, dx_t\end{aligned}$$

by the fundamental theorem of calculus, since $(F_{\vec{n}} \circ g_{\vec{n}})$ is p_2o_2-continuous by Lemma 3.60 and $(F_{\vec{n}} \circ g_{\vec{n}})' =_{q_1p_2o_2} (F'_{\vec{n}} \circ g_{\vec{n}}) \cdot g'_{\vec{n}} = (f_{\vec{n}} \circ g_{\vec{n}}) \cdot g'_{\vec{n}}$ by the chain rule. □

3.7 Finite Sequences of Functions

We take a sequence of functions to be a "double" sequence of functions:

Definition 3.79. $\langle r < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. A sequence of functions is a finite sequence $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_r}$.

Equality of sequences of functions is defined as follows:

Definition 3.80. $\langle o, r < p_1 \le p_2 < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}, (T_{\vec{n}}) \subseteq \mathbb{Q}_t$. We say that two sequences of functions $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_r}$ and $(g_{m,\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_r}$ are p_2p_1o-equal in case $(f_{m,\vec{n}})$ is p_2p_1o-equal to $(g_{m,\vec{n}})$ for all $m \le \infty_r$. We write then $(f_{m,\vec{n}})_{m \le \infty_r} =_{p_2p_1o} (g_{m,\vec{n}})_{m \le \infty_r}$.

A sequence of Cauchy functions is defined as follows:

Definition 3.81. $\langle r, o < p < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. We say that $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_r}$ is a sequence of po-Cauchy functions if $(f_{m,\vec{n}})$ is po-Cauchy for all $m \le \infty_r$.

Equality is respected:

Lemma 3.82. $\langle o_1 < o_2 \mid p_1 < p_2 \rangle$ *Let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$, $(T_{\vec{n}}) \subseteq \mathbb{Q}_t$ *and let* $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_r}$ *be a sequence of* $p_1 o_2$*-Cauchy functions that is* $p_2 p_2 o_2$*-equal to* $(g_{m,\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_r}$. *Then the latter is a sequence of* $p_2 o_1$*-Cauchy functions.*

Proof. Take any $m \le \infty_r$ and $\vec{n}_1, \vec{n}_2$. Let $x \in T_{\vec{n}_1}$ and $y \in T_{\vec{n}_2}$ be such that $x =_{p_2} y$. Then $x =_{p_2} u$ and $v =_{p_2} y$ for some $u \in S_{\vec{n}_1}$ and $v \in S_{\vec{n}_2}$. Now $u =_{p_1} v$ by the axiom $\infty_{p_2} \ge 3\infty_{p_1}$, so

$$g_{m,\vec{n}_1}(x) =_{o_2} f_{m,\vec{n}_1}(u) =_{o_2} f_{m,\vec{n}_2}(v) =_{o_2} g_{m,\vec{n}_2}(y).$$

Hence $g_{m,\vec{n}_1}(x) =_{o_1} g_{m,\vec{n}_2}(y)$ by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$. □

We define next Cauchyness of a sequence of functions:

Definition 3.83. $\langle o < r_1 < r_2 < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. We call $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_{r_2}}$ an $r_1 o$-Cauchy sequence of functions if $f_{m_1,\vec{n}}(x) =_o f_{m_2,\vec{n}}(x)$ for all $\infty_{r_1} \le m_1, m_2 \le \infty_{r_2}$ and $\vec{n}$ and $x \in S_{\vec{n}}$.

The property of being a Cauchy sequence of functions respects equality:

Lemma 3.84. $\langle o_1 < o_2 \rangle$ *Let* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$, $(T_{\vec{n}}) \subseteq \mathbb{Q}_t$. *If* $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_{r_2}}$ *is an* $r_1 o_2$*-Cauchy sequence of functions* ppo_2*-equal to* $(g_{m,\vec{n}} \colon T_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_{r_2}}$, *then the latter is an* $r_1 o_1$*-Cauchy sequence of functions.*

Theorem 3.19 on the Cauchy completeness of the real numbers has the following analogue saying that a Cauchy sequence of Cauchy functions converges to a Cauchy function. Note that it is enough to assume that only "one" function of the given Cauchy sequence of functions be Cauchy:

Theorem 3.85. $\langle o_1 < o_2 \rangle$ *Suppose* $(S_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$. *If* $(f_{m,\vec{n}} \colon S_{\vec{n}} \to \mathbb{Q}_u)_{m \le \infty_{r_3}}$ *is an* $r_1 o_2$*-Cauchy sequence of functions such that* $(f_{\infty_{r_1},\vec{n}})$ *is* po_2*-Cauchy, then* $(f_{m,\vec{n}})_{\infty_{r_2} \le m \le \infty_{r_3}}$ *is a* po_1*-Cauchy function.*

Proof. Take any $\infty_{r_2} \le m_1, m_2 \le \infty_{r_3}$ and $\vec{n}_1, \vec{n}_2$. If $x \in S_{\vec{n}_1}$, $y \in S_{\vec{n}_2}$ are such that $x =_p y$, then

$$f_{m_1,\vec{n}_1}(x) =_{o_2} f_{\infty_{r_1},\vec{n}_1}(x) =_{o_2} f_{\infty_{r_1},\vec{n}_2}(y) =_{o_2} f_{m_2,\vec{n}_2}(y)$$

holds by the assumption, so $f_{m_1,\vec{n}_1}(x) =_{o_1} f_{m_2,\vec{n}_2}(y)$ follows by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$. □

3.8 Taylor's Theorem

We say first what we mean by a function that is differentiable up to an indefinitely large yet finite number of times:

Definition 3.86. $\langle k, o < p < q < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval and $N \leq \infty_k$ be a natural number. We say that $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ is N-times *qpo*-differentiable if there is a sequence $(g_{i,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{i \leq N}$ of functions such that $(g_{0,\vec{n}}) = (f_{\vec{n}})$ and $(g_{i,\vec{n}})' =_{qpo} (g_{i+1,\vec{n}})$ for all $i < N$. We write $(f_{\vec{n}}^{(i)})$ instead of $(g_{i,\vec{n}})$.

Taylor's theorem gets now the following form:

Theorem 3.87. $\langle o_1 < o_2 \shortmid h, k, l, o_2 < o_3 \shortmid p < q \rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *has a* $\lozenge_p$*-appreciable and* ∞_l*-bounded length,* $N < \infty_k$ *is a natural number and* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ *is* $(N+1)$*-times qpo*$_3$*-differentiable so that* $(f_{\vec{n}}^{(i)})$ *is* ∞_h*-bounded for all* $i \leq N$*. Let* $(c_{\vec{n}}) \in (I_{\vec{n}})$ *be a real number and define*

$$P_{N,\vec{n}}(x) = \sum_{i=0}^{N} \frac{f_{\vec{n}}^{(i)}(c_{\vec{n}})}{i!}(x - c_{\vec{n}})^i \colon I_{\vec{n}} \to \mathbb{Q}_u$$

and

$$E_{N,\vec{n}}(x) = f_{\vec{n}}(x) - P_{N,\vec{n}}(x) \colon I_{\vec{n}} \to \mathbb{Q}_u$$

for all $\vec{n}$ *and* $x \in I_{\vec{n}}$*. Then for each* $\vec{n}$ *and* $x \in I_{\vec{n}}$ *there is* $y \in I_{\vec{n}}$ *such that* $\min(x, c_{\vec{n}}) \leq y \leq \max(x, c_{\vec{n}})$ *and*

$$E_{N,\vec{n}}(x) =_{o_1} \frac{f_{\vec{n}}^{(N+1)}(y)}{N!}(x - y)^N (x - c_{\vec{n}}). \tag{10}$$

Proof. Take any $\vec{n}$ and $x \in I_{\vec{n}}$. Define $g_{x,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u$ by putting

$$g_{x,\vec{n}}(y) = \Big(f_{\vec{n}}(x) - \sum_{i=0}^{N} \frac{f_{\vec{n}}^{(i)}(y)}{i!}(x - y)^i\Big)(x - c_{\vec{n}}) - E_{N,\vec{n}}(x)(x - y).$$

By Lemma 3.62, $g_{x,\vec{n}}$ has a *qpo*$_2$-derivative $g'_{x,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u$ given by

$$g'_{x,\vec{n}}(y) = -\frac{f_{\vec{n}}^{(N+1)}(y)}{N!}(x - y)^N (x - c_{\vec{n}}) + E_{N,\vec{n}}(x).$$

Now, if $x =_p c_{\vec{n}}$, we can choose $y = x$. Then $g'_{x,\vec{n}}(y) = E_{N,\vec{n}}(x) =_{o_1} 0$ because of *po*$_1$-continuity by Lemma 3.46. If $x \neq_p c_{\vec{n}}$, then by Rolle's theorem (since $g_{x,\vec{n}}(x) = g_{x,\vec{n}}(c_{\vec{n}}) = 0$), there is some $\min(x, c_{\vec{n}}) \leq y \leq \max(x, c_{\vec{n}})$ such that $g'_{x,\vec{n}}(y) =_{o_1} 0$. So Equation (10) holds in both cases. □

As immediate corollaries we have

Corollary 3.88. *If* $(x_{\vec{n}}) \in (I_{\vec{n}})$, *then*

$$(E_{N,\vec{n}}(x_{\vec{n}})) =_{o_1} \left(\frac{f_{\vec{n}}^{(N+1)}(y_{\vec{n}})}{N!} (x_{\vec{n}} - y_{\vec{n}})^N (x_{\vec{n}} - c_{\vec{n}}) \right)$$

for some $(\min(x_{\vec{n}}, c_{\vec{n}})) \leq (y_{\vec{n}}) \leq (\max(x_{\vec{n}}, c_{\vec{n}}))$.

Corollary 3.89. *If* $(f_{\vec{n}}^{(N+1)})$ *is* $(M_{\vec{n}})$*-bounded, then*

$$|E_{N,\vec{n}}(x)| \leq_{o_1} \frac{M_{\vec{n}}}{N!} |x - c_{\vec{n}}|^{N+1}$$

for all $\vec{n}$ *and* $x \in I_{\vec{n}}$.

A function has a Taylor series expansion, for instance, under the assumption (11):

Corollary 3.90. $\langle o_1 < o_2 \shortmid h, l, o_2, r_2 < o_3 \shortmid p < q \rangle$ *Suppose the length of* $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ *is* $\Diamond_p$*-appreciable and* ∞_l*-bounded and* $(c_{\vec{n}}) \in (I_{\vec{n}})$ *is a real number. If* $(f_{i,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{i \leq \infty_{r_2}}$ *is a sequence of functions such that* $(f_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $i \leq \infty_{r_2}$ *and* $(f_{i,\vec{n}})' =_{qpo_3} (f_{i+1,\vec{n}})$ *for all* $i < \infty_{r_2}$ *and*

$$\frac{|I_{\vec{n}}|^{i+1} \cdot f_{i+1,\vec{n}}(x)}{i!} =_{o_2} 0 \tag{11}$$

for all $\infty_{r_1} \leq i < \infty_{r_2}$ *and* $\vec{n}$ *and* $x \in I_{\vec{n}}$, *then*

$$f_{0,\vec{n}}(x) =_{o_1} \sum_{i=0}^{m} \frac{f_{i,\vec{n}}(c_{\vec{n}})}{i!} (x - c_{\vec{n}})^i \tag{12}$$

for all $\infty_{r_1} \leq m < \infty_{r_2}$ *and* $\vec{n}$ *and* $x \in I_{\vec{n}}$.

Proof. Take any $\infty_{r_1} \leq m < \infty_{r_2}$ and $\vec{n}$ and $x \in I_{\vec{n}}$. Since

$$E_{m,\vec{n}}(x) =_{o_2} \frac{f_{m+1,\vec{n}}(y)}{m!} (x - y)^m (x - c_{\vec{n}})$$

by Taylor's theorem and

$$\left| \frac{f_{m+1,\vec{n}}(y)}{m!} (x - y)^m (x - c_{\vec{n}}) \right| \leq \frac{|I_{\vec{n}}|^{m+1} \cdot |f_{m+1,\vec{n}}(x)|}{m!} =_{o_2} 0$$

by the assumption, we get Equation (12) by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. □

3.9 Some Elementary Functions

The exponential function:

Definition 3.91. The exponential function $(\exp_n\colon \mathbb{Q}_t \to \mathbb{Q}_u)_{\infty_{s_1} \leq n \leq \infty_{s_2}}$ is defined by putting

$$\exp_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!} \tag{13}$$

for all n and $x \in \mathbb{Q}_t$. We leave it to the reader to check that this function really maps elements of $\mathbb{Q}_t$ to elements of $\mathbb{Q}_u$.

We show next by straightforward calculations some properties of the exponential function. Some of the calculations are adapted from the corresponding ones by Stroyan and Luxemburg (1976).

Theorem 3.92. $\langle h, o_1 < o_2 < p < q < s_1 < s_2 \rangle$

(*a*) $(\exp_n)_{\infty_{s_1} \leq n \leq \infty_{s_2}}$ *is* po_1*-Cauchy on* $([-\infty_h, \infty_h])$,

(*b*) $(\exp_n)' =_{qpo_1} (\exp_n)$ *on* $([-\infty_h, \infty_h])$,

(*c*) $\exp_n(x+y) =_{o_1} \exp_n(x) \cdot \exp_n(y)$ *for all* n *and* $x, y \in [-\infty_h, \infty_h]$,

where $([-\infty_h, \infty_h]) \subseteq \mathbb{Q}_t$.

Proof. Note first that $(\sum_{i=0}^{n} \infty_h^i / i!)$ $s_1 o_2$-converges by the ratio test. Moreover, it is not difficult to see that $\exp_n[-\infty_h, \infty_h] \subseteq [3^{-\infty_h}, 3^{\infty_h}]$.

(*a*) Let $\infty_{s_1} \leq n_1 < n_2 \leq \infty_{s_2}$ and let $x, y \in [-\infty_h, \infty_h]$ be such that $x =_p y$. Then

$$\begin{aligned}
\left| \sum_{i=0}^{n_2} \frac{x^i}{i!} - \sum_{i=0}^{n_1} \frac{y^i}{i!} \right| &\leq \sum_{i=n_1+1}^{n_2} \frac{|x|^i}{i!} + \sum_{i=1}^{n_1} \frac{|x^i - y^i|}{i!} \\
&\leq \sum_{i=n_1+1}^{n_2} \frac{\infty_h^i}{i!} + |x-y| \sum_{i=0}^{n_1-1} \frac{1}{(i+1)!} \sum_{j=0}^{i} |x|^j |y|^{i-j} \\
&\leq \Diamond_{o_2} + |x-y| \sum_{i=0}^{n_1-1} \frac{1}{(i+1)!} \sum_{j=0}^{i} \infty_h^i \\
&\leq \Diamond_{o_2} + |x-y| \sum_{i=0}^{\infty_{s_2}} \frac{\infty_h^i}{i!} \\
&\leq \Diamond_{o_2} + \Diamond_p 3^{\infty_h} \leq \Diamond_{o_1}
\end{aligned}$$

by the axioms $\infty_{o_2} \geq 2\infty_{o_1}$ and $\infty_p \geq 2\infty_{o_1} 3^{\infty_h}$.

(b) Let $\infty_{s_1} \leq n \leq \infty_{s_2}$ and let $x, y \in [-\infty_h, \infty_h]$ be such that $x \neq_q y$ and $x =_p y$. Then

$$
\begin{aligned}
\frac{1}{x-y}&\left(\sum_{i=0}^{n} \frac{x^i}{i!} - \sum_{i=0}^{n} \frac{y^i}{i!}\right) - \sum_{i=0}^{n} \frac{x^i}{i!} \\
&= \sum_{i=1}^{n} \frac{x^i - y^i}{i!(x-y)} - \sum_{i=1}^{n+1} \frac{ix^{i-1}}{i(i-1)!} \\
&= \sum_{i=1}^{n} \frac{1}{i!}\left(\frac{x^i - y^i}{x-y} - ix^{i-1}\right) + \frac{x^n}{n!} \\
&= \sum_{i=2}^{n} \frac{1}{i!}\left(\sum_{j=0}^{i-1} x^j y^{i-(j+1)} - ix^{i-1}\right) + \frac{x^n}{n!} \\
&= \sum_{i=2}^{n} \frac{1}{i!}\left(\sum_{j=1}^{i-1} jx^{j-1}y^{i-j} - \sum_{j=1}^{i-1} jx^j y^{i-(j+1)}\right) + \frac{x^n}{n!} \\
&= \sum_{i=2}^{n} \frac{1}{i!}\left(\sum_{j=1}^{i-1} (y-x)jx^{j-1}y^{i-(j+1)}\right) + \frac{x^n}{n!} \\
&= (y-x)\sum_{i=2}^{n} \frac{1}{i!}\left(\sum_{j=1}^{i-1} jx^{j-1}y^{i-(j+1)}\right) + \frac{x^n}{n!}.
\end{aligned}
$$

Since $|x|, |y| \leq \infty_h$, the term on the last line has an upper bound

$$
\begin{aligned}
&\left|(y-x)\sum_{i=2}^{n} \frac{1}{i!}\left(\sum_{j=1}^{i-1} jx^{j-1}y^{i-(j+1)}\right) + \frac{x^n}{n!}\right| \\
&\qquad \leq |x-y| \sum_{i=2}^{n} \frac{1}{i!}\left|\sum_{j=1}^{i-1} jx^{j-1}y^{i-(j+1)}\right| + \frac{|x|^n}{n!} \\
&\qquad \leq |x-y| \sum_{i=2}^{n} \frac{i(i-1)\infty_h^{i-2}}{2i!} + \frac{\infty_h^n}{n!} \\
&\qquad \leq \frac{1}{2}|x-y| \sum_{i=0}^{\infty_{s_2}} \frac{\infty_h^i}{i!} + \frac{\infty_h^n}{n!} \\
&\qquad \leq \frac{1}{2}\lozenge_p 3^{\infty_h} + \lozenge_{o_2} \leq \lozenge_{o_1}
\end{aligned}
$$

by the axioms $\infty_{o_2} \geq 2\infty_{o_1}$ and $\infty_p \geq \infty_{o_1} 3^{\infty_h}$.

(c) Let $\infty_{s_1} \leq n \leq \infty_{s_2}$ and let $x, y \in [-\infty_h, \infty_h]$. Since all the three series $\left(\sum_{i=0}^{n} x^i/i!\right)_{\infty_{s_1} \leq n \leq \infty_{s_2}}$, $\left(\sum_{i=0}^{n} |x|^i/i!\right)$ and $\left(\sum_{i=0}^{n} y^i/i!\right)$ s_1o_2-converge by the comparison test, Newton's binomial formula and Theorem 3.32 yield

$$\begin{aligned}\sum_{i=0}^{n}\frac{(x+y)^i}{i!} &= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{i}{j}\frac{x^j y^{i-j}}{i!}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}\frac{x^j}{j!}\frac{y^{i-j}}{(i-j)!} =_{o_1} \sum_{i=0}^{n}\frac{x^i}{i!}\sum_{i=0}^{n}\frac{y^i}{i!}.\end{aligned}$$

Thus $\exp_n(x+y) =_{o_1} \exp_n(x)\cdot\exp_n(y)$. □

The exponential function $(\mathrm{Exp}_n\colon \mathbb{Q}_t \to \mathbb{Q}_u)_{\infty_{s_1}\le n\le\infty_{s_2}}$ can equally well be defined by putting $\mathrm{Exp}_n(x) = (1+x/n)^n$ for all n and $x\in\mathbb{Q}_t$. We leave it to the reader to verify that $\mathrm{Exp}_n(x) =_o \exp_n(x)$ for all n and $x\in[-\infty_h,\infty_h]$.

The logarithmic function:

Definition 3.93. $\langle t < u\rangle$ The logarithmic function $\ln\colon \mathbb{Q}_t^+ \to \mathbb{Q}_u$ is defined by putting

$$\ln(x) = \int_1^x \frac{dz_t}{z} \tag{14}$$

for all $x\in\mathbb{Q}_t^+$. Here $\mathbb{Q}_t^+ = \{x\in\mathbb{Q}_t : x>0\}$.

Note that ∞_t occurs on the right hand side of (14), so the value $\ln(x)$ depends on the partition of $\mathbb{Q}_t$. We can eliminate ∞_t according to Lemma 3.69, but we prefer using (14) because of its simplicity.

We prove now some properties of the logarithmic function.

Theorem 3.94. $\langle h, o < p \mid h < t\rangle$

(*a*) $\ln$ *is po-Cauchy on* $[\Diamond_h,\infty_h]$,

(*b*) $\ln$ *has* $(x\mapsto 1/x)$ *as a qpo-derivative on* $[\Diamond_h,\infty_h]$,

where $[\Diamond_h,\infty_h]\subseteq\mathbb{Q}_t^+$. *Moreover,* $\ln[\Diamond_h,\infty_h]\subseteq[-\infty_h,\infty_h]$.

Proof. (*a*) Let $x,y\in[\Diamond_h,\infty_h]$ be such that $x =_p y$. We may assume that $x\le y$. Then

$$|\ln(x)-\ln(y)| = \int_x^y \frac{dz_t}{z} = \sum_{x\le z<y}\frac{1}{z\cdot\infty_t!} \le \Diamond_p\infty_h \le \Diamond_o$$

by the axiom $\infty_p \ge \infty_o\infty_h$.

(b) Let $x, y \in [\lozenge_h, \infty_h]$ be such that $x \neq_q y$ and $x =_p y$. We may assume that $x < y$. Now, if we write $N = \infty_t!(y - x)$, then

$$\begin{aligned}
\left|\frac{\ln(y) - \ln(x)}{y - x} - \frac{1}{x}\right| &= \left|\frac{1}{y-x} \sum_{x \le z < y} \frac{dz_t}{z} - \frac{1}{x}\right| = \frac{1}{Nx} \sum_{i=0}^{N-1} \frac{i}{x \cdot \infty_t! + i} \\
&\le \frac{1}{Nx} \cdot N \cdot \frac{N-1}{x \cdot \infty_t! + N - 1} \le \lozenge_p \infty_h^2 \frac{\infty_t!}{\infty_t! - \infty_h} \\
&\le 2\lozenge_p \infty_h^2 \le \lozenge_o
\end{aligned}$$

by the axioms $\infty_t \ge 2\infty_h$ and $\infty_p \ge 2\infty_o\infty_h^2$. □

The following two facts will be needed in the proof of Lemma 5.15:

Lemma 3.95. $\langle k, o_1 < o_2 \rangle$ *Let* $N \le \infty_k$ *be a natural number and let* $x_i \in [\lozenge_h, \infty_h] \subseteq \mathbb{Q}_t$ *for all* $i \le N$*, then*

$$\ln\left(\prod_{i=1}^{N} x_i\right) =_{o_1} \sum_{i=1}^{N} \ln(x_i).$$

Here $\ln\colon \mathbb{Q}_u^+ \to \mathbb{Q}_v$ *with* $t < u < v$.

Proof. It is enough to prove $\ln(x_1 x_2) =_{o_2} \ln(x_1) + \ln(x_2)$, since the claim follows from this by the axiom $\infty_{o_2} \ge \infty_{o_1}(\infty_k - 1)$. But we get this by differentiating $\ln(x_1x_2) - \ln(x_1) - \ln(x_2)$ w.r.t. x_1 and using Corollary 3.66. □

Lemma 3.96. $\langle o < t \rangle$ *We have* $0 \le x - \ln(1+x) \le_o x^2$ *for all* $x \in [-\frac{1}{2}, \infty_h] \subseteq \mathbb{Q}_t$.

Proof. Take any $x \in [-\frac{1}{2}, \infty_h]$. In case $-\frac{1}{2} \le x < 0$, we have

$$\begin{aligned}
0 \le x - \int_1^{1+x} \frac{dz_t}{z} &= \sum_{i=0}^{-x\cdot\infty_t!-1} \frac{-x \cdot \infty_t! - i}{\infty_t!((1+x)\cdot\infty_t! + i)} \\
&\le \sum_{i=0}^{-x\cdot\infty_t!-1} \frac{-x \cdot \infty_t! - i}{(1+x)\cdot(\infty_t!)^2} = \frac{x^2 \cdot (\infty_t!)^2 - x \cdot \infty_t!}{2(1+x)\cdot(\infty_t!)^2} \\
&= \frac{x^2}{2(1+x)} + \frac{-x}{2(1+x)\cdot\infty_t!} \le x^2 + \frac{1}{2\infty_t!} \le_o x^2
\end{aligned}$$

by the axiom $\infty_t \ge \infty_o$. The case $0 \le x \le \infty_h$ has a similar proof. □

The exponential and logarithmic functions are inverses of each other in the following sense:

Theorem 3.97. $\langle h, k, o_1 < o_2 \mid p < q\rangle$

(*a*) $(\ln \circ \exp_n)(x) =_{o_1} x$ *for all* $x \in [-\infty_h, \infty_h] \subseteq \mathbb{Q}_t$,

(*b*) $(\exp_n \circ \ln)(x) =_{o_1} x$ *for all* $x \in [\Diamond_k, \infty_k] \subseteq \mathbb{Q}_t$.

Proof. (*a*) Since $\ln(\exp_n(0)) = 0$ and $(\ln \circ \exp_n)' =_{qpo_2} (\exp_n^{-1} \cdot \exp_n) = (x \mapsto 1)$ by the chain rule, $\ln(\exp_n(x)) =_{o_1} x$ for all n and $x \in [-\infty_h, \infty_h]$ by Corollary 3.66.

(*b*) Take any n and $x \in [\Diamond_k, \infty_k]$. Suppose $\exp_n(\ln(x)) >_{o_1} x$. Writing $N = \infty_u!(\exp_n(\ln(x)) - x)$, where u is such that $\exp_n(\ln(x)) - x \in \mathbb{Q}_u$, we get a contradiction

$$\begin{aligned}\ln(\exp_n(\ln(x))) - \ln(x) &= \sum_{i=0}^{N-1} \frac{1}{x \cdot \infty_u! + i} > \frac{N}{\infty_k \infty_u! + N} \\ &= \frac{\exp_n(\ln(x)) - x}{\infty_k + \exp_n(\ln(x)) - x} \\ &\geq \frac{1}{\infty_{o_1}(\infty_k + 3^{\infty_k})} \geq \Diamond_{o_2}\end{aligned}$$

by the axiom $\infty_{o_2} \geq \infty_{o_1}(\infty_k + 3^{\infty_k})$, since $\ln(\exp_n(\ln(x))) - \ln(x) =_{o_2} 0$ by (*a*). Assuming $x >_{o_1} \exp_n(\ln(x))$ results similarly in a contradiction, so we must have $\exp_n(\ln(x)) =_{o_1} x$. □

We will need the following function when defining L^p-spaces in Section 4.

Definition 3.98. Let $(a_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ be a positive real number. We define $(x \mapsto x^{a_{\vec{n}}} : \mathbb{Q}_t^{0+} \to \mathbb{Q}_u)$ by putting

$$x^{a_{\vec{n}}} = \begin{cases} \exp_{\vec{n}}(a_{\vec{n}} \ln(x)) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

for all $\vec{n}$ and $x \in \mathbb{Q}_t^{0+}$. Here $\mathbb{Q}_t^{0+} = \{x \in \mathbb{Q}_t : x \geq 0\}$.

If $(a_n) = \frac{1}{2}$, we simply write $\sqrt{\cdot}$ without any indices. We leave the proofs of the next two lemmas to the reader.

Lemma 3.99. $\langle h, k, l, o < p\rangle$ *Let* $(a_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ *be a* $\Diamond_k$*-appreciable and* ∞_k*-bounded positive* l*-Cauchy real number. Then* $(x \mapsto x^{a_{\vec{n}}})$ *is* po*-Cauchy on* $([0, \infty_h]) \subseteq \mathbb{Q}_t$.

Lemma 3.100. *If* $(a_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ *is a* $\Diamond_k$*-appreciable and* ∞_k*-bounded positive real number, then*

(a) $x^{a_{\vec{n}}} \cdot y^{a_{\vec{n}}} =_o (xy)^{a_{\vec{n}}}$,

(b) $(x^{a_{\vec{n}}})^{\frac{1}{a_{\vec{n}}}} =_o x$

for all $\vec{n}$ *and* $x, y \in [0, \infty_h] \subseteq \mathbb{Q}_t$.

We also leave it to the reader to define trigonometric functions and π.

3.10 Ordinary First Order Differential Equations

Recall that a function, in general, takes elements of its domain $(S_{\vec{n}}) \subseteq \mathbb{Q}_t$ to elements of $\mathbb{Q}_u$, where $t < u$. This causes some extra complications in the following statement and proof of Peano's existence theorem due to the recursion equations (16) below.

Theorem 3.101. $\langle o_1 < o_2 \shortmid h, p_1 < p_2 \rangle$ *Let* $(a_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (b_{\vec{n}}) \in \mathbb{Q}_t$ *be positive real numbers and let* $([0, a_{\vec{n}}]) \subseteq \mathbb{Q}_u$ *and* $([-b_{\vec{n}}, b_{\vec{n}}]) \subseteq \mathbb{Q}_v$ *be intervals. If* $(f_{\vec{n}} \colon [0, a_{\vec{n}}] \times [-b_{\vec{n}}, b_{\vec{n}}] \to \mathbb{Q}_w)$ *is* $p_1 o_2$*-continuous and* ∞_h*-bounded, then there is* $(z_{\vec{n}} \colon [0, \min(a_{\vec{n}}, b_{\vec{n}} \Diamond_h)] \to \mathbb{Q}_v)$ *such that*

$$z_{\vec{n}}(0) =_{o_1} 0 \quad \text{and} \quad z'_{\vec{n}}(x) =_{qp_2o_1} f_{\vec{n}}(x, z_{\vec{n}}(x)) \tag{15}$$

for all $\vec{n}$ *and* $x \in [0, \min(a_{\vec{n}}, b_{\vec{n}} \Diamond_h)] \subseteq \mathbb{Q}_u$.

Proof. Take any $\vec{n}$ and let N be the smallest natural number such that $\min(a_{\vec{n}}, b_{\vec{n}} \Diamond_h)/N \leq \Diamond_q$. Put $dx = \min(a_{\vec{n}}, b_{\vec{n}} \Diamond_h)/N \in \mathbb{Q}_u$ and $x_i = i \cdot dx \in \mathbb{Q}_u$. Let g be such that $f_{\vec{n}}(x, y) =_{o_2} g(x, y) \in \mathbb{Q}_u$ for all $(x, y) \in [0, a_{\vec{n}}] \times [-b_{\vec{n}}, b_{\vec{n}}]$. There is such a g since $f_{\vec{n}}$ is ∞_h-bounded by the assumption. Define now $z_{\vec{n}} \colon [0, \min(a_{\vec{n}}, b_{\vec{n}} \Diamond_h)] \to \mathbb{Q}_v$ by putting

$$\begin{cases} z_{\vec{n}}(x_0) = 0, \\ z_{\vec{n}}(x_{i+1}) = z_{\vec{n}}(x_i) + dx \cdot g(x_i, z_{\vec{n}}(x_i)), \\ z_{\vec{n}} \text{ is piecewise linear in points between } x_i \text{ and } x_{i+1}. \end{cases} \tag{16}$$

Take then any $x, y \in [0, \min(a_{\vec{n}}, b_{\vec{n}} \Diamond_h)]$ such that $x \neq_q y$ and $x =_{p_2} y$. We can assume that $x < y$. Let i and j be the smallest natural numbers such that $x_i \leq x \leq x_{i+1}$ and $x_{i+j} \leq y \leq x_{i+j+1}$. Note that $j > 0$, since $dx \leq \Diamond_q$. Write

$$dx_k = \begin{cases} x_{i+1} - x & \text{if } k = 0, \\ dx & \text{if } 0 < k < j, \\ y - x_{i+j} & \text{if } k = j. \end{cases}$$

Now, since for any $k \leq j$,

$$|z_{\vec{n}}(x_{i+k}) - z_{\vec{n}}(x)| \leq \sum_{l=0}^{k-1} dx_l \cdot |g(x_{i+l}, z_{\vec{n}}(x_{i+l}))| \leq \Diamond_{p_2} \infty_h \leq \Diamond_{p_1}$$

by the axiom $\infty_{p_2} \geq \infty_h \infty_{p_1}$, it follows from $p_1 o_2$-continuity of $f_{\vec{n}}$ that

$$\begin{aligned}
&\left| \frac{z_{\vec{n}}(y) - z_{\vec{n}}(x)}{y - x} - f_{\vec{n}}(x, z_{\vec{n}}(x)) \right| \\
&\quad \leq \left| \frac{\sum_{k=0}^{j} dx_k \cdot g(x_{i+k}, z_{\vec{n}}(x_{i+k}))}{y - x} - g(x, z_{\vec{n}}(x)) \right| + \Diamond_{o_2} \\
&\quad \leq \frac{\sum_{k=0}^{j} dx_k \cdot |g(x_{i+k}, z_{\vec{n}}(x_{i+k})) - g(x, z_{\vec{n}}(x))|}{y - x} + \Diamond_{o_2} \\
&\quad \leq \max_{k \leq j} |g(x_{i+k}, z_{\vec{n}}(x_{i+k})) - g(x, z_{\vec{n}}(x))| + \Diamond_{o_2} \leq 4\Diamond_{o_2} \leq \Diamond_{o_1}
\end{aligned}$$

by the axiom $\infty_{o_2} \geq 4\infty_{o_1}$. Thus $(z_{\vec{n}})$ satisfies (15). □

The solution is unique and thus Cauchy if $(f_{\vec{n}})$ is Cauchy and satisfies the Lipschitz condition:

Definition 3.102. $(o < \vec{s}_1 < \vec{s}_2 < t \leq u \leq v)$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ and $(J_{\vec{n}}) \subseteq \mathbb{Q}_u$ be intervals, $(L_{\vec{n}})$ be a real number and $(f_{\vec{n}} \colon I_{\vec{n}} \times J_{\vec{n}} \to \mathbb{Q}_v)$ be a function. We say that $(f_{\vec{n}})$ is o-$(L_{\vec{n}})$-Lipschitz if

$$|f_{\vec{n}}(x, y) - f_{\vec{n}}(x, z)| \leq_o L_{\vec{n}} |y - z|$$

for all $\vec{n}$ and $x \in I_{\vec{n}}$ and $y, z \in J_{\vec{n}}$.

Equality is respected under some additional assumptions:

Lemma 3.103. $\langle o_1 < o_2 \mid h, o_2 < o_3 \mid k, o_2 < p \rangle$ *Suppose* $(f_{\vec{n}} \colon I_{1,\vec{n}} \times J_{1,\vec{n}} \to \mathbb{Q}_v)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is* ppo_2*-equal to* $(g_{\vec{n}} \colon I_{2,\vec{n}} \times J_{2,\vec{n}} \to \mathbb{Q}_v)$ *and* $(L_{\vec{n}})$ *is* o_3*-equal to* $(M_{\vec{n}})$*. If* $(f_{\vec{n}})$ *is* o_2*-*$(L_{\vec{n}})$*-Lipschitz,* $(L_{\vec{n}})$ *is* ∞_k*-bounded and* $(J_{2,\vec{n}})$ *has an* ∞_h*-bounded length, then* $(g_{\vec{n}})$ *is* o_1*-*$(M_{\vec{n}})$*-Lipschitz.*

Proof. Take any $\vec{n}$ and $x \in I_{2,\vec{n}}$ and $y, z \in J_{2,\vec{n}}$. Then there are $u \in I_{1,\vec{n}}$ and $v, w \in J_{1,\vec{n}}$ such that $u =_p x$, $v =_p y$ and $w =_p z$. Now, since

$$\begin{aligned}
\big| L_{\vec{n}} |v - w| - M_{\vec{n}} |y - z| \big| &\leq |L_{\vec{n}} - M_{\vec{n}}| |y - z| + |L_{\vec{n}}| 2\Diamond_p \\
&\leq \Diamond_{o_3} \infty_h + 2\infty_k \Diamond_p \leq \Diamond_{o_2}
\end{aligned}$$

by the axioms $\infty_{o_3} \geq 2\infty_{o_2}\infty_h$ and $\infty_p \geq 4\infty_{o_2}\infty_k$, we get

$$|g_{\vec{n}}(x, y) - g_{\vec{n}}(x, z)| =_{o_2} |f_{\vec{n}}(u, v) - f_{\vec{n}}(u, w)| \leq_{o_2} L_{\vec{n}} |v - w| =_{o_2} M_{\vec{n}} |y - z|,$$

so $|g_{\vec{n}}(x, y) - g_{\vec{n}}(x, z)| \leq_{o_1} M_{\vec{n}} |y - z|$ holds by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. □

We can generalize Definition 3.102 a little in case $(f_{\vec{n}})$ really is Cauchy:

Lemma 3.104. *$\langle o_1 < o_2 \rangle$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq \mathbb{Q}_t$ be an interval, $(L_{\vec{n}})$ be a real number and $(f_{\vec{n}} \colon I_{\vec{n}} \times \mathbb{Q}_u \to \mathbb{Q}_v)$ be po_2-Cauchy and o_2-$(L_{\vec{n}})$-Lipschitz. Then*

$$|f_{\vec{n}_1}(x_1, y_1) - f_{\vec{n}_2}(x_2, y_2)| \leq_{o_1} L_{\vec{n}_1}|y_1 - y_2|$$

for all $\vec{n}_1, \vec{n}_2$ and $x_1 \in I_{\vec{n}_1}$, $x_2 \in I_{\vec{n}_2}$ and $y_1, y_2 \in \mathbb{Q}_u$ such that $x_1 =_p x_2$ and $y_1 =_p y_2$.

Proof. For all $\vec{n}_1, \vec{n}_2$ and $x_1 \in I_{\vec{n}_1}$, $x_2 \in I_{\vec{n}_2}$ and $y_1, y_2 \in \mathbb{Q}_u$ such that $x_1 =_p x_2$ and $y_1 =_p y_2$,

$$\begin{aligned}
&|f_{\vec{n}_1}(x_1, y_1) - f_{\vec{n}_2}(x_2, y_2)| \\
&\qquad \leq |f_{\vec{n}_1}(x_1, y_1) - f_{\vec{n}_1}(x_1, y_2)| + |f_{\vec{n}_1}(x_1, y_2) - f_{\vec{n}_2}(x_2, y_2)| \\
&\qquad \leq L_{\vec{n}_1}|y_1 - y_2| + \lozenge_{o_2} + \lozenge_{o_2} \leq L_{\vec{n}_1}|y_1 - y_2| + \lozenge_{o_1}
\end{aligned}$$

holds by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. □

We prove now Picard's existence theorem:

Theorem 3.105. *$\langle o_1 < o_2 < o_3 \mid k, l, o_3 < o_4 \mid h, o_3, p_1 < p_2 \rangle$ Suppose $(L_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ is a $\lozenge_k$-appreciable and ∞_k-bounded positive real number and $(a_{\vec{n}}), (b_{\vec{n}}) \in \mathbb{Q}_t$ are positive real numbers such that $(\min(a_{\vec{n}}, b_{\vec{n}}\lozenge_h))$ is ∞_l-bounded. If $([0, a_{\vec{n}}]) \subseteq \mathbb{Q}_u$ and $([-b_{\vec{n}}, b_{\vec{n}}]) \subseteq \mathbb{Q}_v$ and $(f_{\vec{n}} \colon [0, a_{\vec{n}}] \times [-b_{\vec{n}}, b_{\vec{n}}] \to \mathbb{Q}_w)$ is $p_1 o_4$-Cauchy, ∞_h-bounded and o_4-$(L_{\vec{n}})$-Lipschitz, then there is a $p_2 o_1$-unique $p_2 o_1$-Cauchy $(z_{\vec{n}} \colon [0, \min(a_{\vec{n}}, b_{\vec{n}}\lozenge_h)] \to \mathbb{Q}_v)$ such that*

$$z_{\vec{n}}(0) =_{o_4} 0 \quad \text{and} \quad z'_{\vec{n}}(x) =_{qp_2 o_4} f_{\vec{n}}(x, z_{\vec{n}}(x)) \tag{17}$$

for all $\vec{n}$ and $x \in [0, \min(a_{\vec{n}}, b_{\vec{n}}\lozenge_h)] \subseteq \mathbb{Q}_u$.

Proof. Uniqueness: Let $(z_{\vec{n}})$ be the $qp_2 o_4$-differentiable solution defined by Equations (16) and assume $(w_{\vec{n}})$ is another $qp_2 o_4$-differentiable solution. Take any $\vec{n}$ and let N be the biggest natural number such that $N \cdot \lozenge_{p_2} \leq \min(a_{\vec{n}}, b_{\vec{n}}\lozenge_h)$ and write $x_i = i \cdot \lozenge_{p_2} \in \mathbb{Q}_u$ for all $i \leq N$. Since for each $i < N$,

$$\begin{aligned}
|z_{\vec{n}}(x_{i+1}) - w_{\vec{n}}(x_{i+1})| &\leq |z_{\vec{n}}(x_i) - w_{\vec{n}}(x_i)| \\
&\qquad + \lozenge_{p_2}|f_{\vec{n}}(x_i, z_{\vec{n}}(x_i)) - f_{\vec{n}}(x_i, w_{\vec{n}}(x_i))| + 2\lozenge_{p_2}\lozenge_{o_4} \\
&\leq (1 + L_{\vec{n}}\lozenge_{p_2})|z_{\vec{n}}(x_i) - w_{\vec{n}}(x_i)| + 3\lozenge_{p_2}\lozenge_{o_4},
\end{aligned}$$

iteration yields

$$\begin{aligned}
&|z_{\vec{n}}(x_{i+1}) - w_{\vec{n}}(x_{i+1})| \\
&\quad \leq (1 + L_{\vec{n}}\Diamond_{p_2})^{i+1}|z_{\vec{n}}(x_0) - w_{\vec{n}}(x_0)| + 3\Diamond_{p_2}\Diamond_{o_4}\sum_{j=0}^{i}(1 + L_{\vec{n}}\Diamond_{p_2})^j \\
&\quad \leq \left(1 + \frac{L_{\vec{n}}\min(a_{\vec{n}}, b_{\vec{n}}\Diamond_h)}{N}\right)^N \left(|z_{\vec{n}}(x_0) - w_{\vec{n}}(x_0)| + \frac{3\Diamond_{o_4}}{L_{\vec{n}}}\right) \\
&\quad \leq 3^{\infty_k\infty_l}(\Diamond_{o_4} + 3\Diamond_{o_4}\infty_k) \leq \Diamond_{o_3}
\end{aligned}$$

by the axiom $\infty_{o_4} \geq \infty_{o_3}3^{\infty_k\infty_l}(1 + 3\infty_k)$. Let now $x, y \in [0, \min(a_{\vec{n}}, b_{\vec{n}}\Diamond_h)]$ be such that $x =_{p_2} y$ and let $i \leq N$ be such that $x =_{p_2} x_i =_{p_2} y$. Since both $(z_{\vec{n}})$ and $(w_{\vec{n}})$ are p_2o_3-continuous by Lemma 3.60, we get

$$\begin{aligned}
&|z_{\vec{n}}(x) - w_{\vec{n}}(y)| \\
&\quad \leq |z_{\vec{n}}(x) - z_{\vec{n}}(x_i)| + |z_{\vec{n}}(x_i) - w_{\vec{n}}(x_i)| + |w_{\vec{n}}(x_i) - w_{\vec{n}}(y)| \leq 3\Diamond_{o_3} \leq \Diamond_{o_2}
\end{aligned}$$

by the axiom $\infty_{o_3} \geq 3\infty_{o_2}$. Hence $(z_{\vec{n}})$ is p_2o_2-equal to $(w_{\vec{n}})$.

Cauchyness: Take any $\vec{n}_1, \vec{n}_2$. If $\min(a_{\vec{n}_1}, b_{\vec{n}_1}\Diamond_h) \leq \min(a_{\vec{n}_2}, b_{\vec{n}_2}\Diamond_h)$, let w be the restriction of $z_{\vec{n}_2}$ to $[0, \min(a_{\vec{n}_1}, b_{\vec{n}_1}\Diamond_h)]$. Then w is obviously qp_2o_4-differentiable and satisfies (17), so $z_{\vec{n}_1}$ is p_2o_2-equal to w by the first part of this proof and w is p_2o_2-equal to $z_{\vec{n}_2}$ by the definition. Hence $z_{\vec{n}_1}$ is p_2o_1-equal to $z_{\vec{n}_2}$ by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. The proof goes analogously in case $\min(a_{\vec{n}_1}, b_{\vec{n}_1}\Diamond_h) > \min(a_{\vec{n}_2}, b_{\vec{n}_2}\Diamond_h)$. It follows that $(z_{\vec{n}})$ is p_2o_1-Cauchy. □

4 Finite L^p-Spaces in $\mathbf{HA}^*$

Nelson (1987) gives a simple definition of an L^1-function on a finite space following the theory of S-integration. This definition is too general since both integrability of a function and integral of an integrable function may depend on the chosen discretization of its domain. To overcome these problems, Cartier and Perrin (1995) add to the definition a condition of almost continuity. Since the resulting definition seems to be difficult to handle in our theory, we have chosen here a different definition but with a similar flavour.

We assume throughout this section that $(I_{\vec{n}}) \subseteq \mathbb{Q}_t$ is an interval and start by saying what we mean by an L^1-norm:

Definition 4.1. Let $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ be a function. We define a real number $(\|f_{\vec{n}}\|_1)$ by putting

$$\|f_{\vec{n}}\|_1 = \int_{I_{\vec{n}}} |f_{\vec{n}}| \, dx_t$$

for all $\vec{n}$.

It is immediate that $\|\cdot\|_1$ is a seminorm in the following sense:

Lemma 4.2. *If* $(f_{\vec{n}}, g_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *are functions and* $(c_{\vec{n}})$ *is a real number, then*

(*a*) $\|c_{\vec{n}} f_{\vec{n}}\|_1 = |c_{\vec{n}}| \cdot \|f_{\vec{n}}\|_1$,

(*b*) $\|f_{\vec{n}} + g_{\vec{n}}\|_1 \leq \|f_{\vec{n}}\|_1 + \|g_{\vec{n}}\|_1$

for all $\vec{n}$.

We define now an L^1-function as follows:

Definition 4.3. $\langle o_1 < r_1 \shortmid o_2, r_2 < p \rangle$ We call $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ a $po_2r_1o_1$-L^1-function if there is a sequence $(\phi_{m,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{m \leq \infty_{r_2}}$ of po_2-continuous functions such that

$$\|f_{\vec{n}} - \phi_{m,\vec{n}}\|_1 =_{o_1} 0$$

for all $\infty_{r_1} \le m \le \infty_{r_2}$ and $\vec{n}$. We call the sequence $(\phi_{m,\vec{n}})_{m\le\infty_{r_2}}$ a $po_2r_1o_1$-approximation of $(f_{\vec{n}})$ and say that it is po_2-Cauchy if $(\phi_{m,\vec{n}})$ is po_2-Cauchy for all $m \le \infty_{r_2}$.

Note that a continuous function is L^1 and has itself as an approximation. The integral of an L^1-function is a Cauchy real number if the function has a Cauchy approximation:

Lemma 4.4. $\langle o_1 < o_2 \shortmid h, o_2 < p \shortmid k, o_2 < o_3\rangle$ *If* $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ *has a* $po_3r_1o_2$*-approximation* $(\phi_{m,\vec{n}})_{m\le\infty_{r_2}}$*, then*

$$\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t =_{o_2} \int_{I_{\vec{n}}} \phi_{m,\vec{n}}\,dx_t$$

for all $\infty_{r_1} \le m \le \infty_{r_2}$ *and* $\vec{n}$*. If in addition* $(I_{\vec{n}})$ *is p-Cauchy having an* ∞_h*-bounded length and* $(\phi_{m,\vec{n}})_{m\le\infty_{r_2}}$ *is* po_3*-Cauchy and* ∞_k*-bounded, then* $(\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t)$ *is an* o_1*-Cauchy real number.*

Proof. For each $\infty_{r_1} \le m \le \infty_{r_2}$ and $\vec{n}$,

$$\left|\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t - \int_{I_{\vec{n}}} \phi_{m,\vec{n}}\,dx_t\right| \le \int_{I_{\vec{n}}} |f_{\vec{n}} - \phi_{m,\vec{n}}| dx_t = \|f_{\vec{n}} - \phi_{m,\vec{n}}\|_1 \le \Diamond_{o_2}.$$

Under the additional assumptions, $(\int_{I_{\vec{n}}} \phi_{\infty_{r_1},\vec{n}}\,dx_t)$ is an o_2-Cauchy real number by Lemma 3.70, so

$$\int_{I_{\vec{n}_1}} f_{\vec{n}_1}\,dx_t =_{o_2} \int_{I_{\vec{n}_1}} \phi_{\infty_{r_1},\vec{n}_1}\,dx_t =_{o_2} \int_{I_{\vec{n}_2}} \phi_{\infty_{r_1},\vec{n}_2}\,dx_t =_{o_2} \int_{I_{\vec{n}_2}} f_{\vec{n}_2}\,dx_t$$

for all $\vec{n}_1, \vec{n}_2$. It follows now by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$ that $(\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t)$ is an o_1-Cauchy real number. □

The set of L^1-functions is closed under the operations of addition, multiplication, maximum, minimum and absolute value:

Lemma 4.5. $\langle h, k, o_1 < o_2 \shortmid h, k, o_3 < o_4\rangle$ *Let* $N \le \infty_k$ *be a natural number. If* $(f_{i,\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ *are* $po_4r_1o_2$*-*L^1*-functions for all* $i \le N$*, then*

(*a*) $(\sum_{i=1}^N f_{i,\vec{n}})$,

(*b*) $(\prod_{i=1}^N f_{i,\vec{n}})$,

(*c*) $(\max\{f_{i,\vec{n}} : i \le N\})$,

(*d*) $(\min\{f_{i,\vec{n}} : i \le N\})$,

(e) $(|f_{1,\vec{n}}|)$

are $po_3r_1o_1$*-*L^1*-functions. In* (b) *we assume that* $(f_{i,\vec{n}})$ *is* ∞_h*-bounded for all* $1 < i \leq N$ *and that* $(f_{1,\vec{n}})$ *has an* ∞_h*-bounded* $po_4r_1o_2$*-approximation.*

Proof. Let $(\phi^i_{m,\vec{n}})_{m\leq\infty_{r_2}}$ be a $po_4r_1o_2$-approximation of $(f_{i,\vec{n}})$ for each $i \leq N$ and take any $\infty_{r_1} \leq m \leq \infty_{r_2}$ and $\vec{n}$.

(a) We have

$$\int_{I_{\vec{n}}} \left|\sum_{i=1}^{N} f_{i,\vec{n}} - \sum_{i=1}^{N} \phi^i_{m,\vec{n}}\right| dx_t \leq \sum_{i=1}^{N} \int_{I_{\vec{n}}} |f_{\vec{n}} - \phi^i_{m,\vec{n}}| dx_t \leq \infty_k \Diamond_{o_2} \leq \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \geq \infty_k \infty_{o_1}$. In addition, $(\sum_{i=1}^N \phi^i_{m,\vec{n}})_{m\leq\infty_{r_2}}$ is a sequence of po_3-continuous functions by Lemma 3.46.

(b) We may assume that $(\phi^i_{m,\vec{n}})$ is ∞_h-bounded for all $i \leq N$ and $m \leq \infty_{r_2}$. Then

$$\begin{aligned}
&\int_{I_{\vec{n}}} \left|\prod_{i=1}^{N} f_{i,\vec{n}} - \prod_{i=1}^{N} \phi^i_{m,\vec{n}}\right| dx_t \\
&\qquad \leq \int_{I_{\vec{n}}} \sum_{i=1}^{N} \left(|f_{i,\vec{n}} - \phi^i_{m,\vec{n}}| \prod_{j=i+1}^{N} |f_{j,\vec{n}}| \prod_{k=1}^{i-1} |\phi^k_{m,\vec{n}}| \right) dx_t \\
&\qquad \leq \infty_h^{\infty_k - 1} \sum_{i=1}^{N} \int_{I_{\vec{n}}} |f_{i,\vec{n}} - \phi^i_{m,\vec{n}}| dx_t \leq \infty_h^{\infty_k - 1} \infty_k \Diamond_{o_2} \leq \Diamond_{o_1}
\end{aligned}$$

by the axiom $\infty_{o_2} \geq \infty_h^{\infty_k - 1} \infty_k \infty_{o_1}$. Moreover, $(\prod_{i=1}^N \phi^i_{m,\vec{n}})_{m\leq\infty_{r_2}}$ is a sequence of po_3-continuous functions by Lemma 3.46.

(c) We have

$$\begin{aligned}
&\int_{I_{\vec{n}}} |\max\{f_{i,\vec{n}} : i \leq N\} - \max\{\phi^i_{m,\vec{n}} : i \leq N\}| dx_t \\
&\qquad \leq \int_{I_{\vec{n}}} \max\{|f_{i,\vec{n}} - \phi^i_{m,\vec{n}}| : i \leq N\} dx_t \\
&\qquad \leq \sum_{i=1}^{N} \int_{I_{\vec{n}}} |f_{i,\vec{n}} - \phi^i_{m,\vec{n}}| dx_t \leq \infty_k \Diamond_{o_2} \leq \Diamond_{o_1}
\end{aligned}$$

by the axiom $\infty_{o_2} \geq \infty_k \infty_{o_1}$. Moreover, $(\max\{\phi^i_{m,\vec{n}} : i \leq N\})_{m\leq\infty_{r_2}}$ is a sequence of po_3-continuous functions by Lemma 3.46. The proof of (d) is similar.

(e) We have

$$\int_{I_{\vec{n}}} \big||f_{1,\vec{n}}| - |\phi^1_{m,\vec{n}}|\big| dx_t \leq \int_{I_{\vec{n}}} |f_{1,\vec{n}} - \phi^1_{m,\vec{n}}| dx_t \leq \Diamond_{o_1},$$

where $(|\phi^1_{m,\vec{n}}|)$ is po_3-continuous for each $m \leq \infty_{r_2}$ by Lemma 3.46. □

A similar lemma with a similar proof holds for L^1-functions with Cauchy-approximations as well.

It does not seem possible to weaken the assumptions made in (b) above: Let $k, o_1 < s_1 \shortmid k, s_2 < u$ and let c be any ∞_k-bounded natural number. If $(f_n\colon [0,1] \to \mathbb{Q}_u)$ is the "unbounded" function defined by putting

$$f_n(x) = \begin{cases} cn & \text{if } x \leq 1/n^2, \\ 0 & \text{otherwise} \end{cases}$$

for all $\infty_{s_1} \leq n \leq \infty_{s_2}$ and $x \in [0,1] \subseteq \mathbb{Q}_t$, then $\|f_n\|_1 = c/n \leq \infty_k \Diamond_{s_1} \leq \Diamond_{o_1}$ by the axiom $\infty_{s_1} \geq \infty_{o_1}\infty_k$, so (f_n) has the constant sequence zero as a $po_2r_1o_1$-approximation. But $\|f_nf_n\|_1 = c^2$.

We say that two L^1-functions are equal in case they have equal approximations:

Definition 4.6. $\langle o_1 < r_1 \shortmid o_2, o_3, r_2 < p_1 \leq p_2 \rangle$ Let $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (J_{\vec{n}}) \subseteq \mathbb{Q}_t$ and let $(f_{\vec{n}}\colon I_{\vec{n}} \to \mathbb{Q}_u)$, $(g_{\vec{n}}\colon J_{\vec{n}} \to \mathbb{Q}_u)$ be $p_1o_3r_1o_1$-L^1-functions. We say that $(f_{\vec{n}})$ and $(g_{\vec{n}})$ are $p_2p_1o_3o_2$-equal if they have $p_1o_3r_1o_1$-approximations $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ and $(\psi_{m,\vec{n}})_{m \leq \infty_{r_2}}$, respectively, such that $(\phi_{m,\vec{n}}) =_{p_2p_1o_2} (\psi_{m,\vec{n}})$ for all $m \leq \infty_{r_2}$. We write then $(f_{\vec{n}}) =_{p_2p_1o_3o_2} (g_{\vec{n}})$.

Equality of L^1-functions inherits properties of equality of continuous functions. For instance, it follows from Lemmas 3.43 and 3.46 that addition, multiplication, maximum, minimum and absolute value preserve equality:

Lemma 4.7. $\langle h, k, o_1 < o_2 \shortmid h, ko_3 < o_4 \rangle$ *Let $N \leq \infty_k$ be a natural number and let $(f_{i,\vec{n}}\colon S_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ and $(g_{i,\vec{n}}\colon T_{\vec{n}} \to \mathbb{Q}_u)$ be such that $(f_{i,\vec{n}}) =_{p_2p_1o_4o_2} (g_{i,\vec{n}})$ for all $i \leq N$. Then*

(a) $(\sum_{i=1}^N f_{i,\vec{n}}) =_{p_2p_1o_3o_1} (\sum_{i=1}^N g_{i,\vec{n}})$,

(b) $(\prod_{i=1}^N f_{i,\vec{n}}) =_{p_2p_1o_3o_1} (\prod_{i=1}^N g_{i,\vec{n}})$,

(c) $(\max\{f_{i,\vec{n}} : i \leq N\}) =_{p_2p_1o_3o_1} (\max\{g_{i,\vec{n}} : i \leq N\})$,

(d) $(\min\{f_{i,\vec{n}} : i \leq N\}) =_{p_2p_1o_3o_1} (\min\{g_{i,\vec{n}} : i \leq N\})$,

(e) $(|f_{1,\vec{n}}|) =_{p_2p_1o_3o_1} (|g_{1,\vec{n}}|)$.

In (b) we assume that $(f_{i,\vec{n}})$ and $(g_{i,\vec{n}})$ are ∞_h-bounded for all $i \leq N$.

Equal L^1-functions have equal integrals:

Lemma 4.8. $\langle o_1 < o_2 \shortmid h, o_2 < p \shortmid k, o_2 < o_3 \rangle$ *Suppose $(I_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$, $(J_{\vec{n}}) \subseteq \mathbb{Q}_t$ have ∞_k-bounded lengths and $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$, $(g_{\vec{n}} \colon J_{\vec{n}} \to \mathbb{Q}_u)$ have ∞_h-bounded $po_4r_1o_3$-approximations witnessing $(f_{\vec{n}}) =_{ppo_4o_3} (g_{\vec{n}})$. Then $(\int_{I_{\vec{n}}} f_{\vec{n}}\, dx_t) =_{o_1} (\int_{J_{\vec{n}}} g_{\vec{n}}\, dx_t)$.*

Proof. Let $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$, $(\psi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ be ∞_h-bounded $po_4r_1o_3$-approximations of $(f_{\vec{n}})$, $(g_{\vec{n}})$, respectively, such that $(\phi_{m,\vec{n}}) =_{ppo_3} (\psi_{m,\vec{n}})$ for all $m \leq \infty_{r_2}$. Take now any $\infty_{r_1} \leq m \leq \infty_{r_2}$ and $\vec{n}$. Then

$$\int_{I_{\vec{n}}} f_{\vec{n}}\, dx_t =_{o_2} \int_{I_{\vec{n}}} \phi_{m,\vec{n}}\, dx_t =_{o_2} \int_{I_{\vec{n}}} \psi_{m,\vec{n}}\, dx_t =_{o_2} \int_{I_{\vec{n}}} g_{\vec{n}}\, dx_t$$

by Lemmas 3.71 and 4.4, so $\int_{I_{\vec{n}}} f_{\vec{n}}\, dx_t =_{o_1} \int_{J_{\vec{n}}} g_{\vec{n}}\, dx_t$ follows by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. □

By Lemma 4.2, $\|\cdot\|_1$ is a seminorm. It is a norm in the following sense:

Lemma 4.9. $\langle o_1 < o_2 \rangle$ *Let $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ be $po_3r_1o_2$-L^1. If $(\|f_{\vec{n}}\|_1) =_{o_2} 0$, then $(f_{\vec{n}}) =_{ppo_3o_3} 0$.*

Proof. Suppose $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ is a $po_3r_1o_2$-approximation of $(f_{\vec{n}})$. Since for every $\infty_{r_1} \leq m \leq \infty_{r_2}$ and $\vec{n}$,

$$\|0 - \phi_{m,\vec{n}}\|_1 = \|\phi_{m,\vec{n}}\|_1 \leq \|\phi_{m,\vec{n}} - f_{\vec{n}}\|_1 + \|f_{\vec{n}}\|_1 \leq 2\lozenge_{o_2} \leq \lozenge_{o_1}$$

by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$, the constant function zero is $po_3r_1o_1$-L^1 having $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ as an approximation. The claim follows now from the fact that each $(\phi_{m,\vec{n}})$ is ppo_3-equal to itself. □

Lemma 4.10. $\langle o_1 < o_2 \rangle$ *If $(f_{\vec{n}}, g_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ are functions such that $(f_{\vec{n}})$ is $po_3r_1o_2$-L^1 and $(\|f_{\vec{n}} - g_{\vec{n}}\|_1) =_{o_2} 0$, then $(g_{\vec{n}})$ is $po_3r_1o_1$-L^1.*

Proof. Let $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ be a $po_3r_1o_2$-approximation of $(f_{\vec{n}})$. Now for each $\infty_{r_1} \leq m \leq \infty_{r_2}$ and $\vec{n}$,

$$\|g_{\vec{n}} - \phi_{m,\vec{n}}\|_1 \leq \|g_{\vec{n}} - f_{\vec{n}}\|_1 + \|f_{\vec{n}} - \phi_{m,\vec{n}}\|_1 \leq 2\lozenge_{o_2} \leq \lozenge_{o_1}$$

by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$, so $(g_{\vec{n}})$ is $po_3r_1o_1$-L^1 having $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ as an approximation. □

We prove next our version of a theorem saying that the set of L^1-functions on a given interval $(I_{\vec{n}})$ is Cauchy complete w.r.t. $\|\cdot\|_1$, i.e. that the pair $(L^1, \|\cdot\|_1)$ makes up a Banach space. For this just "one" function of the given Cauchy sequence of functions needs to be L^1:

Theorem 4.11. $\langle o_1 < o_2 \rangle$ *Suppose* $(f_{m,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{m \leq \infty_{r_3}, \infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is an* $r_1 o_2$*-Cauchy sequence w.r.t.* $\|\cdot\|_1$ *and* $(f_{\infty_{r_1},\vec{n}})$ *is a* $po_3 r_1 o_2$*-*L^1*-function. Then* $(f_{m,\vec{n}})_{\infty_{r_1} \leq m \leq \infty_{r_3}}$ *is a* $po_3 r_1 o_1$*-*L^1*-function.*

Proof. Let $(\phi_{l,\vec{n}})_{l \leq \infty_{r_2}}$ be a $po_3 r_1 o_2$-approximation of $(f_{\infty_{r_1},\vec{n}})$. Then for all $\infty_{r_1} \leq l \leq \infty_{r_2}$ and $\infty_{r_1} \leq m \leq \infty_{r_3}$ and $\vec{n}$,

$$\|f_{m,\vec{n}} - \phi_{l,\vec{n}}\|_1 \leq \|f_{m,\vec{n}} - f_{\infty_{r_1},\vec{n}}\|_1 + \|f_{\infty_{r_1},\vec{n}} - \phi_{l,\vec{n}}\|_1 \leq 2\Diamond_{o_2} \leq \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. Hence $(f_{m,\vec{n}})_{\infty_{r_1} \leq m \leq \infty_{r_3}}$ is a $po_3 r_1 o_1$-L^1-function having $(\phi_{l,\vec{n}})_{l \leq \infty_{r_2}}$ as an approximation. □

We say next what we mean by an L^1-subset:

Definition 4.12. $\langle o_1 < r_1 \mid r_2, o_2 < p \rangle$ Let $(J_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq (I_{\vec{n}})$. The characteristic function $(\chi_{J_{\vec{n}}} \colon I_{\vec{n}} \to \mathbb{Q}_u)$ of the subset $(J_{\vec{n}})$ is defined by putting

$$\chi_{J_{\vec{n}}}(x) = \begin{cases} 1 & \text{if } x \in J_{\vec{n}}, \\ 0 & \text{if } x \notin J_{\vec{n}} \end{cases}$$

for all $\vec{n}$ and $x \in I_{\vec{n}}$. We say that $(J_{\vec{n}})$ is $po_2 r_1 o_1$-L^1 if $(\chi_{J_{\vec{n}}})$ is $po_2 r_1 o_1$-L^1.

Equality of two L^1-subsets is defined now as equality of their characteristic functions.

The set of L^1-subsets is closed under Boolean operations:

Theorem 4.13. $\langle o_1 < o_2 \mid o_3 < o_4 \rangle$ *If* $(J_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}, (K_{\vec{n}}) \subseteq (I_{\vec{n}})$ *are* $po_4 r_1 o_2$*-*L^1*, then* $(I_{\vec{n}} \setminus J_{\vec{n}})$*,* $(J_{\vec{n}} \cap K_{\vec{n}})$ *and* $(J_{\vec{n}} \cup K_{\vec{n}})$ *are* $po_3 r_1 o_1$*-*L^1*.*

Proof. For every $\vec{n}$ and $x \in I_{\vec{n}}$,

$$\begin{aligned} \chi_{I_{\vec{n}} \setminus J_{\vec{n}}}(x) &= 1 - \chi_{J_{\vec{n}}}(x), \\ \chi_{J_{\vec{n}} \cap K_{\vec{n}}}(x) &= \chi_{J_{\vec{n}}}(x) \cdot \chi_{K_{\vec{n}}}(x), \\ \chi_{J_{\vec{n}} \cup K_{\vec{n}}}(x) &= \chi_{J_{\vec{n}}}(x) + \chi_{K_{\vec{n}}}(x) - \chi_{J_{\vec{n}}}(x) \cdot \chi_{K_{\vec{n}}}(x). \end{aligned}$$

Since $(\chi_{J_{\vec{n}}})$, $(\chi_{K_{\vec{n}}})$ and $(x \mapsto 1)$ are $po_4 r_1 o_2$-L^1 by the assumption, the above characteristic functions are $po_2 r_1 o_1$-L^1 by Lemma 4.5. □

The set of L^1-subsets is also closed under indefinitely long yet finite intersections and unions. These are our substitutes for countable intersections and unions:

Theorem 4.14. $\langle k, o_1 < o_2 \rangle$ *Let* $N \leq \infty_k$ *be a natural number and let* $(J_{i,\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq (I_{\vec{n}})$ *be* $po_4r_1o_2$-L^1 *for all* $i \leq N$. *Then* $(\bigcap_{i=1}^N J_{i,\vec{n}})$ *and* $(\bigcup_{i=1}^N J_{i,\vec{n}})$ *are* $po_3r_1o_1$-L^1.

Proof. The proof that $(\bigcap_{i=1}^N J_{i,\vec{n}})$ is $po_3r_1o_1$-L^1 is similar to the proof of (b) in Lemma 4.5. It follows now from this fact together with Theorem 4.13 and the identity

$$\bigcup_{i=1}^N J_{i,\vec{n}} = I_{\vec{n}} \setminus \bigcap_{i=1}^N (I_{\vec{n}} \setminus J_{i,\vec{n}})$$

that $(\bigcup_{i=1}^N J_{i,\vec{n}})$ is $po_3r_1o_1$-L^1. □

We define next measure of a subset and say when a subset has measure zero:

Definition 4.15. Let $(J_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \subseteq (I_{\vec{n}})$. The measure of $(J_{\vec{n}})$ is the real number $(\mathrm{m}(J_{\vec{n}}))$ defined by putting

$$\mathrm{m}(J_{\vec{n}}) = \int_{I_{\vec{n}}} \chi_{J_{\vec{n}}}\, dx_t$$

for all $\vec{n}$. We say that $(J_{\vec{n}})$ has measure $\Diamond_o$ in case $(\mathrm{m}(J_{\vec{n}})) =_o 0$.

Note that if a subset has measure zero, then it has the constant sequence zero as an approximation.

Inequality (18) of the following lemma is called Chebyshev's inequality:

Lemma 4.16. *Let* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *be a function and* $(c_{\vec{n}})$ *be a positive real number. Define* $(J_{\vec{n}}) \subseteq (I_{\vec{n}})$ *by putting* $J_{\vec{n}} = \{x \in I_{\vec{n}} : |f_{\vec{n}}(x)| > c_{\vec{n}}\}$ *for all* $\vec{n}$. *Then*

$$\mathrm{m}(J_{\vec{n}}) \leq \frac{1}{c_{\vec{n}}} \int_{I_{\vec{n}}} |f_{\vec{n}}|\, dx_t = \frac{\|f_{\vec{n}}\|_1}{c_{\vec{n}}} \tag{18}$$

for all $\vec{n}$.

Proof. Since

$$c_{\vec{n}} \cdot \sum_{I_{L,\vec{n}} \leq x < I_{R,\vec{n}}} \chi_{J_{\vec{n}}}(x) dx_t \leq \sum_{I_{L,\vec{n}} \leq x < I_{R,\vec{n}}} |f_{\vec{n}}(x)| dx_t$$

for every $\vec{n}$, Chebyshev's inequality holds. □

As the proof shows, Inequality (18) is valid also when we have $\geq$ instead of $>$ in the definition of $(J_{\vec{n}})$.

We define next a truncated function:

Definition 4.17. Let $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ be a function and $(c_{\vec{n}})$ be a nonnegative real number. We define a truncated function $(f_{\vec{n}}[c_{\vec{n}}] \colon I_{\vec{n}} \to \mathbb{Q}_u)$ by putting

$$f_{\vec{n}}[c_{\vec{n}}](x) = \begin{cases} f_{\vec{n}}(x) & \text{if } |f_{\vec{n}}(x)| \le c_{\vec{n}}, \\ 0 & \text{otherwise} \end{cases}$$

for all $\vec{n}$ and $x \in I_{\vec{n}}$.

Note that $(f_{\vec{n}}[c_{\vec{n}}])$ may not be L^1 even if $(f_{\vec{n}})$ were L^1.

The following three lemmas show that there is a certain similarity between the present approach and the theory of S-integration.

Lemma 4.18. $\langle o_1 < o_2 \rangle$ *If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *is* $po_3r_1o_2$-L^1 *and* $(\|f_{\vec{n}} - f_{\vec{n}}[\infty_h]\|_1) =_{o_2} 0$*, then* $(f_{\vec{n}})$ *has an* ∞_h*-bounded* $po_3r_1o_1$*-approximation.*

Proof. Let $(\phi_{m,\vec{n}})_{m \le \infty_{r_2}}$ be a $po_3r_1o_2$-approximation of $(f_{\vec{n}})$. Define a subset $(J_{\vec{n}})$ and a sequence $(K_{m,\vec{n}})_{m \le \infty_{r_2}}$ of subsets of $(I_{\vec{n}})$ and a sequence of functions $(\psi_{m,\vec{n}})_{m \le \infty_{r_2}}$ by putting

$$\begin{aligned} J_{\vec{n}} &= \{x \in I_{\vec{n}} : |f_{\vec{n}}(x)| > \infty_h\}, \\ K_{m,\vec{n}} &= \{x \in I_{\vec{n}} : |\phi_{m,\vec{n}}(x)| > \infty_h\}, \\ \psi_{m,\vec{n}} &= \operatorname{sgn}(\phi_{m,\vec{n}})\infty_h \chi_{K_{m,\vec{n}}} + \phi_{m,\vec{n}}\chi_{(I_{\vec{n}} \smallsetminus K_{m,\vec{n}})} \end{aligned}$$

for every $m \le \infty_{r_2}$ and $\vec{n}$. Here $\operatorname{sgn}(x) = -1$ in case $x < 0$ and $\operatorname{sgn}(x) = +1$ otherwise. Obviously $(\psi_{m,\vec{n}})$ is po_3-Cauchy and ∞_h-bounded for every $m \le \infty_{r_2}$. Moreover, if we take any $\infty_{r_1} \le m \le \infty_{r_2}$ and $\vec{n}$, then

$$\begin{aligned} |f_{\vec{n}} - \psi_{m,\vec{n}}| &= |f_{\vec{n}} - \psi_{m,\vec{n}}|\chi_{J_{\vec{n}}} + |f_{\vec{n}} - \psi_{m,\vec{n}}|\chi_{(I_{\vec{n}} \smallsetminus J_{\vec{n}})} \\ &\le 2|f_{\vec{n}}|\chi_{J_{\vec{n}}} + |f_{\vec{n}} - \phi_{m,\vec{n}}|\chi_{(I_{\vec{n}} \smallsetminus J_{\vec{n}})} \\ &\le 2|f_{\vec{n}} - f_{\vec{n}}[\infty_h]| + |f_{\vec{n}} - \phi_{m,\vec{n}}|, \end{aligned}$$

so we have

$$\begin{aligned} \int_{I_{\vec{n}}} |f_{\vec{n}} - \psi_{m,\vec{n}}| dx_t &\le 2\int_{I_{\vec{n}}} |f_{\vec{n}} - f_{\vec{n}}[\infty_h]| dx_t + \int_{I_{\vec{n}}} |f_{\vec{n}} - \phi_{m,\vec{n}}| dx_t \\ &= 2\|f_{\vec{n}} - f_{\vec{n}}[\infty_h]\|_1 + \|f_{\vec{n}} - \phi_{m,\vec{n}}\|_1 \\ &\le 2\lozenge_{o_2} + \lozenge_{o_2} \le \lozenge_{o_1} \end{aligned}$$

by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$. Hence $(\psi_{m,\vec{n}})_{m \le \infty_{r_2}}$ is a $po_3r_1o_1$-approximation of $(f_{\vec{n}})$. □

We can invert the above lemma in case $(f_{\vec{n}})$ has a bounded norm:

Lemma 4.19. $\langle o_1, h, k < l \mid o_1 < o_2 \rangle$ *If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *has an* ∞_h-*bounded* $po_3r_1o_2$-L^1-*approximation and* $(\|f_{\vec{n}}\|_1)$ *is* ∞_k-*bounded, then* $(\|f_{\vec{n}} - f_{\vec{n}}[c_{\vec{n}}]\|_1) =_{o_1} 0$ *for all real numbers* $(c_{\vec{n}}) \ge \infty_l$.

Proof. Let $(\phi_{m,\vec{n}})_{m \le \infty_{r_2}}$ be an ∞_h-bounded $po_3r_1o_2$-approximation of $(f_{\vec{n}})$ and let $(c_{\vec{n}}) \ge \infty_l$. Define $(J_{\vec{n}}), (K_{\vec{n}}) \subseteq (I_{\vec{n}})$ by putting

$$J_{\vec{n}} = \{x \in I_{\vec{n}} : |f_{\vec{n}}(x)| > c_{\vec{n}}\},$$
$$K_{\vec{n}} = \{x \in I_{\vec{n}} : |f_{\vec{n}}(x)| > \infty_l\}$$

for all $\vec{n}$. Take now any $\vec{n}$. Since $J_{\vec{n}} \subseteq K_{\vec{n}}$ and $\mathrm{m}(K_{\vec{n}}) \le \lozenge_l \|f_{\vec{n}}\|_1 \le \lozenge_l \infty_k$ by Chebyshev's inequality, we have

$$|f_{\vec{n}}|\chi_{J_{\vec{n}}} \le |f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}| + |\phi_{\infty_{r_1},\vec{n}}|\chi_{K_{\vec{n}}} \le |f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}| + \infty_h \chi_{K_{\vec{n}}},$$

so

$$\begin{aligned}\|f_{\vec{n}} - f_{\vec{n}}[c_{\vec{n}}]\|_1 &= \int_{I_{\vec{n}}} |f_{\vec{n}}|\chi_{J_{\vec{n}}} dx_t \\ &\le \|f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}\|_1 + \infty_h \,\mathrm{m}(K_{\vec{n}}) \le \lozenge_{o_2} + \infty_h \lozenge_l \infty_k \le \lozenge_{o_1}\end{aligned}$$

by the axioms $\infty_{o_2} \ge 2\infty_{o_1}$ and $\infty_l \ge 2\infty_{o_1}\infty_h\infty_k$. □

If an L^1-function has a bounded approximation, then its integral over a set of measure zero is zero:

Lemma 4.20. $\langle h, o_1 < o_3 \mid o_1 < o_2 \rangle$ *If* $(f_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *has an* ∞_h-*bounded* $po_4r_1o_2$-*approximation and* $(J_{\vec{n}}) \subseteq (I_{\vec{n}})$ *has measure* $\lozenge_{o_3}$, *then* $(\int_{I_{\vec{n}}} |f_{\vec{n}}|\chi_{J_{\vec{n}}}\, dx_t) =_{o_1} 0$.

Proof. Let $(\phi_{m,\vec{n}})_{m \le \infty_{r_2}}$ be an ∞_h-bounded $po_4r_1o_2$-approximation of $(f_{\vec{n}})$. For each $\vec{n}$, since

$$|f_{\vec{n}}|\chi_{J_{\vec{n}}} \le |f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}|\chi_{J_{\vec{n}}} + |\phi_{\infty_{r_1},\vec{n}}|\chi_{J_{\vec{n}}} \le |f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}| + \infty_h \chi_{J_{\vec{n}}},$$

we get

$$\begin{aligned}\int_{I_{\vec{n}}} |f_{\vec{n}}|\chi_{J_{\vec{n}}}\, dx_t &\le \int_{I_{\vec{n}}} |f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}| dx_t + \int_{I_{\vec{n}}} \infty_h \chi_{J_{\vec{n}}} dx_t \\ &= \|f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}\|_1 + \infty_h \,\mathrm{m}(J_{\vec{n}}) \le \lozenge_{o_2} + \infty_h \lozenge_{o_3} \le \lozenge_{o_1}\end{aligned}$$

by the axioms $\infty_{o_2} \ge 2\infty_{o_1}$ and $\infty_{o_3} \ge 2\infty_{o_1}\infty_h$. □

We say next what it means for a property to hold pointwise almost everywhere:

Definition 4.21. Let $(P_{\vec{n}})_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ be a property possibly depending on $(I_{\vec{n}})$. We say that $(P_{\vec{n}})$ holds o-almost everywhere (o-a.e. for short) if there is a subset $(J_{\vec{n}})\subseteq(I_{\vec{n}})$ having measure $\Diamond_o$ such that $P_{\vec{n}}(x)$ holds for all $\vec{n}$ and $x\in I_{\vec{n}}\smallsetminus J_{\vec{n}}$.

Lemma 4.22. $\langle k, o_1 < o_2 \shortmid h, o_2 < o_3\rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ *has an* ∞_k-*bounded length and* $(f_{\vec{n}}\colon I_{\vec{n}}\to\mathbb{Q}_u)$ *has an* ∞_h-*bounded* $po_4r_1o_3$-*approximation.*

(a) *If* $(f_{\vec{n}}) =_{o_2} 0$ o_3-*a.e., then* $(\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t) =_{o_1} 0$.

(b) *If* $(f_{\vec{n}}) \ge_{o_2} 0$ *and* $(\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t) =_{o_2} 0$, *then* $(f_{\vec{n}}) =_{o_1} 0$ o_1-*a.e.*

Proof. (a) Let $(J_{\vec{n}})\subseteq(I_{\vec{n}})$ be such that $\mathrm{m}(J_{\vec{n}}) =_{o_3} 0$ and $f_{\vec{n}}(x) =_{o_2} 0$ for all $\vec{n}$ and $x\in I_{\vec{n}}\smallsetminus J_{\vec{n}}$. Lemma 4.20 and the axiom $\infty_{o_2}\ge\infty_{o_1}(\infty_k+1)$ yield now that

$$\left|\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t\right| \le \int_{I_{\vec{n}}} |f_{\vec{n}}|\chi_{(I_{\vec{n}}\smallsetminus J_{\vec{n}})}dx_t + \int_{I_{\vec{n}}} |f_{\vec{n}}|\chi_{J_{\vec{n}}}dx_t$$
$$\le \Diamond_{o_2}|I_{\vec{n}}| + \Diamond_{o_2} \le \Diamond_{o_2}(\infty_k+1) \le \Diamond_{o_1}$$

holds for each $\vec{n}$.

(b) Define $(J_{\vec{n}})\subseteq(I_{\vec{n}})$ by putting $J_{\vec{n}} = \{x\in I_{\vec{n}} : f_{\vec{n}}(x) > \Diamond_{o_1}\}$ for each $\vec{n}$. Take then any $\vec{n}$. Now $f_{\vec{n}}(x) =_{o_1} 0$ for all $x\in I_{\vec{n}}\smallsetminus J_{\vec{n}}$. Moreover, since $\chi_{J_{\vec{n}}} \le \infty_{o_1} f_{\vec{n}} + \infty_{o_1}\Diamond_{o_2}$, we get

$$\mathrm{m}(J_{\vec{n}}) = \int_{I_{\vec{n}}} \chi_{J_{\vec{n}}}dx_t$$
$$\le \infty_{o_1}\int_{I_{\vec{n}}} f_{\vec{n}}\,dx_t + \infty_{o_1}\Diamond_{o_2}|I_{\vec{n}}| \le \infty_{o_1}\Diamond_{o_2}(1+\infty_k) \le \Diamond_{o_1}$$

by the axiom $\infty_{o_2}\ge\infty_{o_1}^2(\infty_k+1)$, so the claim holds. □

The following two lemmas show that pointwise equality a.e. and equality as L^1-functions are the same thing for functions defined on the same interval. However, since pointwise equality a.e. is quite a weak notion, we need some additional assumptions:

Lemma 4.23. $\langle k, o_1 < o_2 \shortmid h, o_2 < o_3 < r_1 < r_2 \shortmid o_4, r_2 < p\rangle$ *Suppose* $(I_{\vec{n}})_{\infty_{\vec{s}_1}\le\vec{n}\le\infty_{\vec{s}_2}}$ *has an* ∞_k-*bounded length and* $(f_{\vec{n}}, g_{\vec{n}}\colon I_{\vec{n}}\to\mathbb{Q}_u)$ *have* ∞_h-*bounded* $po_4r_1o_3$-*approximations. If* $(f_{\vec{n}}) =_{o_2} (g_{\vec{n}})$ *(pointwise)* o_3-*a.e., then* $(f_{\vec{n}}) =_{ppo_4o_4} (g_{\vec{n}})$ *(as* L^1-*functions).*

Proof. Let $(J_{\vec{n}}) \subseteq (I_{\vec{n}})$ be such that $\mathrm{m}(J_{\vec{n}}) =_{o_3} 0$ and $f_{\vec{n}}(x) =_{o_2} g(x)$ for all $\vec{n}$ and $x \in I_{\vec{n}} \smallsetminus J_{\vec{n}}$. Let $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ be an ∞_h-bounded $p_1 o_4 r_1 o_2$-approximation of $(f_{\vec{n}})$. Then for each $\infty_{r_1} \leq m \leq \infty_{r_2}$ and $\vec{n}$, since

$$\begin{aligned}|g_{\vec{n}} - \phi_{m,\vec{n}}| &\leq |g_{\vec{n}} - f_{\vec{n}}|\chi_{J_{\vec{n}}} + |g_{\vec{n}} - f_{\vec{n}}|\chi_{(I_{\vec{n}} \smallsetminus J_{\vec{n}})} + |f_{\vec{n}} - \phi_{m,\vec{n}}| \\ &\leq |f_{\vec{n}}|\chi_{J_{\vec{n}}} + |g_{\vec{n}}|\chi_{J_{\vec{n}}} + |g_{\vec{n}} - f_{\vec{n}}|\chi_{(I_{\vec{n}} \smallsetminus J_{\vec{n}})} + |f_{\vec{n}} - \phi_{m,\vec{n}}|,\end{aligned}$$

it follows from Lemma 4.20 that

$$\begin{aligned}\int_{I_{\vec{n}}} |g_{\vec{n}} - \phi_{m,\vec{n}}| dx_t &\leq \int_{I_{\vec{n}}} |g_{\vec{n}}|\chi_{J_{\vec{n}}} dx_t + \int_{I_{\vec{n}}} |f_{\vec{n}}|\chi_{J_{\vec{n}}} dx_t \\ &\quad + \int_{I_{\vec{n}}} |f_{\vec{n}} - g_{\vec{n}}|\chi_{(I_{\vec{n}} \smallsetminus J_{\vec{n}})} dx_t + \int_{I_{\vec{n}}} |f_{\vec{n}} - \phi_{m,\vec{n}}| dx_t \\ &\leq 2\lozenge_{o_2} + \lozenge_{o_2}|I_{\vec{n}}| + \lozenge_{o_2} \leq (\infty_k + 3)\lozenge_{o_2} \leq \lozenge_{o_1}\end{aligned}$$

by the axiom $\infty_{o_2} \geq \infty_{o_1}(\infty_k + 3)$. We have thus shown that $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ is a $po_4 r_1 o_1$-approximation of $(g_{\vec{n}})$. The claim follows now because each $(\phi_{m,\vec{n}})$ is ppo_4-equal to itself. □

The opposite direction can be shown without assuming the underlying set and the approximations to be bounded:

Lemma 4.24. $\langle o_1 < o_2 < o_3 \rangle$ *Let* $(f_{\vec{n}}, g_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *be* $po_4 r_1 o_3$-L^1. *If* $(f_{\vec{n}}) =_{ppo_4 o_2} (g_{\vec{n}})$ *(as* L^1*-functions), then* $(f_{\vec{n}}) =_{o_1} (g_{\vec{n}})$ *(pointwise)* o_1*-a.e.*

Proof. Let $(\phi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ and $(\psi_{m,\vec{n}})_{m \leq \infty_{r_2}}$ be the $po_4 r_1 o_3$-approximations of $(f_{\vec{n}})$ and $(g_{\vec{n}})$, respectively, witnessing $(f_{\vec{n}}) =_{ppo_4 o_2} (g_{\vec{n}})$. Define $(J_{\vec{n}}), (K_{\vec{n}}) \subseteq (I_{\vec{n}})$ by putting

$$\begin{aligned}J_{\vec{n}} &= \{x \in I_{\vec{n}} : |f_{\vec{n}}(x) - \phi_{\infty_{r_1},\vec{n}}(x)| > \lozenge_{o_2}\}, \\ K_{\vec{n}} &= \{x \in I_{\vec{n}} : |g_{\vec{n}}(x) - \psi_{\infty_{r_1},\vec{n}}(x)| > \lozenge_{o_2}\}\end{aligned}$$

for all $\vec{n}$. Take now any $\vec{n}$. By Chebyshev's inequality,

$$\mathrm{m}(J_{\vec{n}}) \leq \infty_{o_2} \|f_{\vec{n}} - \phi_{\infty_{r_1},\vec{n}}\|_1 \leq \infty_{o_2} \lozenge_{o_3}.$$

Similarly $\mathrm{m}(K_{\vec{n}}) \leq \infty_{o_2}\lozenge_{o_3}$. Thus $\mathrm{m}(J_{\vec{n}} \cup K_{\vec{n}}) \leq 2\infty_{o_2}\lozenge_{o_3} \leq \lozenge_{o_1}$ by the axiom $\infty_{o_3} \geq 2\infty_{o_1}\infty_{o_2}$. Moreover, for all $x \in I_{\vec{n}} \smallsetminus (J_{\vec{n}} \cup K_{\vec{n}})$,

$$\begin{aligned}&|f_{\vec{n}}(x) - g_{\vec{n}}(x)| \\ &\quad \leq |f_{\vec{n}}(x) - \phi_{\infty_{r_1},\vec{n}}(x)| + |\phi_{\infty_{r_2},\vec{n}}(x) - \psi_{\infty_{r_1},\vec{n}}(x)| + |\psi_{\infty_{r_2},\vec{n}}(x) - g_{\vec{n}}(x)| \\ &\quad \leq \lozenge_{o_2} + \lozenge_{o_2} + \lozenge_{o_2} = 3\lozenge_{o_2} \leq \lozenge_{o_1}\end{aligned}$$

by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. Thus $(f_{\vec{n}}) =_{o_1} (g_{\vec{n}})$ (pointwise) o_1-a.e. □

We define next pointwise convergence a.e. This notion defines uniform convergence a.e. but is not so restrictive in view of Egorov's theorem saying that a sequence of functions converging pointwise at every point of a set having a finite measure converges uniformly on a subset of that set having a measure arbitrarily close to it.

Definition 4.25. $\langle o_1, o_2 < r_1 < r_2 \rangle$ Let $(g_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ be a function and $(f_{m,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{m \leq \infty_{r_2}}$ be a sequence of functions. We say that $(f_{m,\vec{n}})_{m \leq \infty_{r_2}}$ r_1o_1-converges to $(g_{\vec{n}})$ pointwise o_2-a.e. in case $(f_{m,\vec{n}}) =_{o_1} (g_{\vec{n}})$ for all $\infty_{r_1} \leq m \leq \infty_{r_2}$ o_2-a.e.

We did not assume anything about the functions in the above definition. The same holds for the following lemma expressing the familiar connection between pointwise convergence w.r.t. $\|\cdot\|_1$ and pointwise convergence a.e.

Lemma 4.26. $\langle r_1, r_2, o_1 < o_2 \shortmid r_1, r_2, r_3 < r_4 \rangle$ *If* $(g_{\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is a function and* $(f_{m,\vec{n}} \colon I_{\vec{n}} \to \mathbb{Q}_u)_{m \leq \infty_{r_4}}$ *is a sequence of functions* r_3o_2*-converging to* $(g_{\vec{n}})$ *w.r.t.* $\|\cdot\|_1$*, then it has a subsequence* $(f_{\alpha(m),\vec{n}})_{m \leq \infty_{r_2}}$ r_1o_1*-converging to* $(g_{\vec{n}})$ *pointwise* o_1*-a.e.*

Proof. Take any $\vec{n}$ and define a sequence $(J_{m,\vec{n}})_{m \leq \infty_{r_4}}$ of subsets of $I_{\vec{n}}$ by putting

$$J_{m,\vec{n}} = \{x \in I_{\vec{n}} : |f_{m,\vec{n}}(x) - g_{\vec{n}}(x)| > \Diamond_{o_1}\}$$

for all $m \leq \infty_{r_4}$. Now for each $\infty_{r_3} \leq m \leq \infty_{r_4}$, since $\|f_{m,\vec{n}} - g_{\vec{n}}\|_1 =_{o_2} 0$ by the assumption, $\mathrm{m}(J_{m,\vec{n}}) \leq \infty_{o_1} \Diamond_{o_2}$ by Chebyshev's inequality. Define then $K_{\vec{n}} \subseteq I_{\vec{n}}$ by putting $K_{\vec{n}} = \bigcup_{i=\infty_{r_1}}^{\infty_{r_2}} J_{\alpha(i),\vec{n}}$, where $\alpha(i) = \infty_{r_4} - \infty_{r_2} + i \geq \infty_{r_3}$ by the axiom $\infty_{r_4} \geq \infty_{r_3} + \infty_{r_2} - \infty_{r_1}$. We get

$$\mathrm{m}(K_{\vec{n}}) \leq \sum_{i=\infty_{r_1}}^{\infty_{r_2}} \mathrm{m}(J_{\alpha(i),\vec{n}}) \leq (\infty_{r_2} - \infty_{r_1} + 1)\infty_{o_1} \Diamond_{o_2} \leq \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \geq \infty_{o_1}^2(\infty_{r_2} - \infty_{r_1} + 1)$. Moreover, $f_{\alpha(m),\vec{n}}(x) =_{o_1} g_{\vec{n}}(x)$ for all $\infty_{r_1} \leq m \leq \infty_{r_2}$ and $x \in I_{\vec{n}} \setminus K_{\vec{n}}$. Thus $(f_{\alpha(m),\vec{n}})_{m \leq \infty_{r_2}}$ r_1o_1-converges to $(g_{\vec{n}})$ pointwise o_1-a.e. $\square$

We finally define an L^p-function:

Definition 4.27. $\langle o_1 < r_1 < r_2 \shortmid r_2, o_2 < p \rangle$ Let $(p_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ be a positive real number and $(f_{\vec{n}} \colon I_{\vec{n}} \to Q_u)$ be any function. We say that $(f_{\vec{n}})$ is $po_2r_1o_1$-$L^{(p_{\vec{n}})}$ if $(|f_{\vec{n}}|^{p_{\vec{n}}})$ is $po_2r_1o_1$-L^1.

5 Finite Probability Spaces in HA*

Nelson (1977) introduced an axiomatic system for nonstandard set theory called Internal Set Theory (**IST** for short) and gave then a short exposition of finite probability spaces in that theory. In this chapter we more or less follow his exposition just to see if and how his arguments can be reworked in our theory. We also study briefly the Black-Scholes option pricing model. The first nonstandard treatment of the model using the full machinery of nonstandard analysis was given by Cutland, Kopp and Willinger (1991). Following Nelson's (1987) approach to finite stochastic processes the model has been worked out in **IST** by van den Berg and Koudjeti (1997) and van den Berg (2000). Our treatment of the model is based on their work.

There is a noteworthy difference between our approach and Nelson's approach. In our approach each mathematical object is finite whereas in Nelson's approach intervals and sequences of real numbers, probability spaces, random variables, stochastic processes etc. are treated as finite mathematical objects but the real numbers themselves are not. Our $=$ and $=_o$ are the strict and approximative equalities, respectively, on rational numbers while Nelson's $=$ and $\approx$ are the strict and approximative equalities, respectively, on real numbers.

5.1 Basic Notions

So far we have been working with the finite set $\mathbb{Q}_t$ equipped with the metric induced by the absolute value. We now need a notion of a metric space:

Definition 5.1. Let S be a finite set and $d\colon S \times S \to \mathbb{Q}_u$ be a metric on S, i.e.

(a) $d(x, y) \geq 0$,

(b) $d(x, x) = 0$,

(c) $d(x, y) = d(y, x)$,

(d) $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in S$. We call the pair (S, d) a metric space. Strict and approximative equality relations can now be defined on (S, d) by putting

(e) $x = y$ if $d(x, y) = 0$,

(f) $x =_o y$ if $d(x, y) =_o 0$

for all $x, y \in S$.

We give next definitions of some basic notions of probability theory:

Definition 5.2. $\langle o < \vec{s}_1 < \vec{s}_2 \rangle$ A finite o-probability space on a metric space (S, d) is a finite sequence of pairs $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$, where $\Omega_{\vec{n}} \subseteq S$ and $\mathrm{pr}_{\vec{n}} \colon \Omega_{\vec{n}} \to [0, 1] \subseteq \mathbb{Q}_u$ is such that

$$\sum_{\omega \in \Omega_{\vec{n}}} \mathrm{pr}_{\vec{n}}(\omega) =_o 1.$$

A subset $(A_{\vec{n}}) \subseteq (\Omega_{\vec{n}})$ is called an event. Its probability is the real number $\mathrm{Pr}(A_{\vec{n}})$ defined by putting

$$\mathrm{Pr}\, A_{\vec{n}} = \sum_{\omega \in \Omega_{\vec{n}}} pr_{\vec{n}}(\omega)$$

for all $\vec{n}$. A function $(X_{\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)$ is called a random variable. Its mean and variance are the real numbers $\mathrm{E}(X_{\vec{n}})$ and $\mathrm{Var}(X_{\vec{n}})$ defined by putting

$$\mathrm{E}\, X_{\vec{n}} = \sum_{\omega \in \Omega_{\vec{n}}} X_{\vec{n}}(\omega)\, \mathrm{pr}_{\vec{n}}(\omega),$$
$$\mathrm{Var}\, X_{\vec{n}} = \mathrm{E}(X_{\vec{n}} - \mathrm{E}\, X_{\vec{n}})^2$$

for all $\vec{n}$. Its probability distribution is the function $(f_{\vec{n}} \colon \mathbb{Q}_u \to \mathbb{Q}_u)$ defined by putting

$$f_{\vec{n}}(x) = \mathrm{Pr}\{X_{\vec{n}} = x\}$$

for all $\vec{n}$ and $x \in \mathbb{Q}_u$.

The following is a standard example of a (hyper)finite probability space. Let $o_1, o_2 < s_1 < s_2 < t$ and define a finite set S_t and a metric d on S_t by putting

$$S_t = \{(x_1, x_2, \ldots, x_n) : n \leq \infty_t \text{ and } x_i \in \{-1, 1\} \text{ for all } i \leq n\}$$

and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y \text{ and } n = \max\{j : x_i = y_i \text{ for all } i \leq j\}. \end{cases}$$

We define now $(\Omega_n) \subseteq S_t$ and $(\mathrm{pr}_n \colon \Omega_n \to \mathbb{Q}_t)$ by putting

$$\Omega_n = \{(x_1, x_2, \ldots, x_n) : x_i \in \{-1, 1\} \text{ for all } i \leq n\}$$

and

$$\mathrm{pr}_n(x_1, x_2, \ldots, x_n) = 2^{-n}$$

for all $\infty_{s_1} \leq n \leq \infty_{s_2}$. Then $(\Omega_n, \mathrm{pr}_n)$ is a finite o_1-probability space modelling indefinitely long yet finite coin tossing. The following example is our version of a corresponding example by Loeb (1975). We consider coin tossing of length $\infty_{s_1} \leq n \leq \infty_{s_2}$. If $(A_n^m) \subseteq (\Omega_n)$ is the event "the first $m-1$ tosses are tails but the mth toss is a head", then $(A_n) \subseteq (\Omega_n)$ defined by putting

$$A_n = \begin{cases} \bigcup_{m=1}^{n/2} A_n^{2m} & \text{if } n \text{ is even,} \\ \bigcup_{m=1}^{(n-1)/2} A_n^{2m} & \text{if } n \text{ is odd} \end{cases}$$

is the event "the first head occurs at an even-numbered toss". Now for even n,

$$\Pr A_n = \sum_{m=1}^{n/2} \Pr A_n^{2m} = \sum_{m=1}^{n/2} \frac{1}{2^{2m}} = \frac{1}{3} - \frac{1}{3 \cdot 2^n} =_{o_2} \frac{1}{3}$$

by the axiom $\infty_{s_1} \geq \min\{i \leq \infty_{o_2} : 3 \cdot 2^i \geq \infty_{o_2}\}$, and similarly for odd n. Thus $\Pr(A_n) =_{o_2} 1/3$.

The next lemma is our version of the Mass Concentration Lemma of van den Berg (2000):

Lemma 5.3. $\langle h_1, h_2, o_1 < o_2 < s_1 \mid o_1 < k \rangle$ *If* $(X_{\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is a random variable on a finite* o_2*-probability space* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})$ *so that* $\mathrm{Var}(X_{\vec{n}})$ *is* $\Diamond_{h_1}$*-appreciable and* ∞_{h_2}*-bounded, then*

$$\Pr\left\{\left|\frac{X_{\vec{n}} - \mathrm{E}\, X_{\vec{n}}}{\sqrt{\mathrm{Var}\, X_{\vec{n}}}}\right| \leq \infty_k\right\} =_{o_1} 1$$

for all $\vec{n}$.

Proof. Take any $\vec{n}$. Since $\mathrm{Var}\, X_{\vec{n}} =_{o_2} (\sqrt{\mathrm{Var}\, X_{\vec{n}}}\,)^2$ by Lemma 3.100, we get

$$(\sqrt{\mathrm{Var}\, X_{\vec{n}}}\,)^2 = \mathrm{Var}\, X_{\vec{n}} + ((\sqrt{\mathrm{Var}\, X_{\vec{n}}}\,)^2 - \mathrm{Var}\, X_{\vec{n}}) \geq \Diamond_{h_1} - \Diamond_{o_2} > 0$$

by the axiom $\infty_{o_2} \geq \infty_{h_1} + 1$, so

$$\frac{\mathrm{Var}\, X_{\vec{n}}}{(\sqrt{\mathrm{Var}\, X_{\vec{n}}}\,)^2} \leq 1 + \frac{\Diamond_{o_2}}{(\sqrt{\mathrm{Var}\, X_{\vec{n}}}\,)^2} \leq \frac{\infty_{o_2}}{\infty_{o_2} - \infty_{h_1}}.$$

It follows now from this and Chebyshev's inequality that the probability of the complementary event has the upper bound

$$\Pr\{(X_{\vec{n}} - \operatorname{E} X_{\vec{n}})^2 > \infty_k^2(\sqrt{\operatorname{Var} X_{\vec{n}}}\,)^2\} \\ \le \frac{\operatorname{Var} X_{\vec{n}}}{\infty_k^2(\sqrt{\operatorname{Var} X_{\vec{n}}}\,)^2} \le \frac{\infty_{o_2}}{\infty_k^2(\infty_{o_2} - \infty_{h_1})}.$$

Moreover, $\Pr \Omega_{\vec{n}} =_{o_2} 1$ by the assumption. Hence we get the claim by the axioms $\infty_{o_2} \ge 2\infty_{o_1}$ and

$$\infty_k \ge \min\{n : n^2 \ge \lceil 2\infty_{o_1}\infty_{o_2}/(\infty_{o_2} - \infty_{h_1})\rceil\}. \qquad \square$$

Independence of random variables is defined as usual:

Definition 5.4. $\langle k, o < \vec{s}_1 < \vec{s}_2\rangle$ Let $N \le \infty_k$ be a natural number and let $(X_{i,\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ be random variables for all $i \le N$. We call them o-independent in case we have

$$\Pr\{X_{i,\vec{n}} \in S_i \text{ for all } i \le N\} =_o \prod_{i=0}^{N} \Pr\{X_{i,\vec{n}} \in S_i\}$$

for all $\vec{n}$ and $S_i \subseteq \mathbb{Q}_u$.

Note that if $(X_{i,\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)$, where $i \le N$, are o-independent random variables and $(f_{i,\vec{n}} \colon \mathbb{Q}_u \to \mathbb{Q}_v)$ are any functions, then $(f_{i,\vec{n}}(X_{i,\vec{n}}) \colon \Omega_{\vec{n}} \to \mathbb{Q}_v)$ are also o-independent random variables.

The following lemma shows that expectation commutes with multiplication:

Lemma 5.5. $\langle h, o_1 < o_2 < o_3 < o_4\rangle$ *Let* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *be a finite* o_2*-probability space and let* $(X_{\vec{n}}, Y_{\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)$ *be* o_4*-independent random variables. If* $(X_{\vec{n}})$ *has an* ∞_h*-bounded* $po_5r_1o_2$*-*L^1*-approximation (or is* ∞_h*-bounded) and* $(Y_{\vec{n}})$ *is* ∞_h*-bounded, then*

$$\operatorname{E}(X_{\vec{n}}Y_{\vec{n}}) =_{o_1} \operatorname{E}(X_{\vec{n}})\operatorname{E}(Y_{\vec{n}})$$

for all $\vec{n}$*.*

Proof. Write $z_i = i\Diamond_{o_3}\infty_h$ for all $i = 0, 1, \dots, \infty_{o_3}$ and define an auxiliary function $f \colon \mathbb{Q}_u \to \mathbb{Q}_u$ by putting

$$f(x) = \begin{cases} \operatorname{sgn}(x) z_i & \text{if } z_i \le |x| < z_{i+1}, \\ \operatorname{sgn}(x)\infty_h & \text{if } |x| \ge \infty_h. \end{cases}$$

Then $(f(X_{\vec n}))$ and $(f(Y_{\vec n}))$ are o_4-independent random variables by the above remark. Since

$$\sum_{x,y\in\{\pm z_i\}} |x||y| = \left(2\lozenge_{o_3}\infty_h \sum_{i=1}^{\infty_{o_3}} i\right)^2 = \infty_h^2(\infty_{o_3}+1)^2,$$

we get for each $\vec n$,

$$\begin{aligned}
\mathrm{E}(f(X_{\vec n})f(Y_{\vec n})) &= \sum_{x,y\in\{\pm z_i\}} xy\Pr\{f(X_{\vec n})=x \text{ and } f(Y_{\vec n})=y\} \\
&=_{o_2} \sum_{x,y\in\{\pm z_i\}} xy\Pr\{f(X_{\vec n})=x\}\Pr\{f(Y_{\vec n})=y\} \\
&= \sum_{x\in\{\pm z_i\}} x\Pr\{f(X_{\vec n})=x\} \sum_{y\in\{\pm z_i\}} y\Pr\{f(Y_{\vec n})=y\} \\
&= \mathrm{E}(f(X_{\vec n}))\,\mathrm{E}(f(Y_{\vec n}))
\end{aligned}$$

by the axiom $\infty_{o_4} \ge \infty_{o_2}\infty_h^2(\infty_{o_3}+1)^2$. Assume now that $(\phi_{m,\vec n})_{m\le\infty_{r_2}}$ is an ∞_h-bounded $po_5r_1o_2$-L^1-approximation of $(X_{\vec n})$. Take any $\vec n$. If we write $\bar X_{\vec n}(\omega) = \operatorname{sgn}(X_{\vec n}(\omega))\cdot\min\{|X_{\vec n}(\omega)|,\infty_h\}$, then we have

$$\begin{aligned}
|X_{\vec n}(\omega) - f(X_{\vec n})(\omega)| &\le |X_{\vec n}(\omega) - \bar X_{\vec n}(\omega)| + |\bar X_{\vec n}(\omega) - f(X_{\vec n})(\omega)| \\
&\le |X_{\vec n}(\omega) - \phi_{\infty_{r_1},\vec n}(\omega)| + \lozenge_{o_3}\infty_h
\end{aligned}$$

for all $\omega\in\Omega_{\vec n}$, so

$$\begin{aligned}
|\mathrm{E}\,X_{\vec n} - \mathrm{E}\,f(X_{\vec n})| &\le \mathrm{E}|X_{\vec n} - f(X_{\vec n})| \\
&\le \mathrm{E}|X_{\vec n} - \phi_{\infty_{r_1},\vec n}| + \lozenge_{o_3}\infty_h\Pr(\Omega_{\vec n}) \\
&\le \lozenge_{o_2} + \lozenge_{o_3}\infty_h(1+\lozenge_{o_2}) \le 2\lozenge_{o_2}
\end{aligned}$$

by the axiom $\infty_{o_3} \ge \infty_h(\infty_{o_2}+1)$. Also

$$|\mathrm{E}\,Y_{\vec n} - \mathrm{E}\,f(Y_{\vec n})| \le \lozenge_{o_3}\infty_h\Pr(\Omega_{\vec n}) \le \lozenge_{o_3}\infty_h(1+\lozenge_{o_2}) \le \lozenge_{o_2}$$

by the axiom $\infty_{o_3} \ge \infty_h(\infty_{o_2}+1)$, since $(Y_{\vec n})$ is ∞_h-bounded by the assumption. We thus have

$$\begin{aligned}
&|\mathrm{E}(X_{\vec n}Y_{\vec n}) - \mathrm{E}(f(X_{\vec n})f(Y_{\vec n}))| \\
&\qquad \le \infty_h\big(\mathrm{E}|X_{\vec n} - f(X_{\vec n})| + \mathrm{E}|Y_{\vec n} - f(Y_{\vec n})|\big) \le 3\infty_h\lozenge_{o_2}
\end{aligned}$$

and

$$\begin{aligned}
&|\mathrm{E}(f(X_{\vec n}))\,\mathrm{E}(f(Y_{\vec n})) - \mathrm{E}(X_{\vec n})\,\mathrm{E}(Y_{\vec n})| \\
&\qquad \le \infty_h\Pr(\Omega_{\vec n})\big(|\mathrm{E}\,X_{\vec n} - \mathrm{E}\,f(X_{\vec n})| + |\mathrm{E}\,Y_{\vec n} - \mathrm{E}\,f(Y_{\vec n})|\big) \\
&\qquad \le 3\infty_h(1+\lozenge_{o_2})\lozenge_{o_2} \le 3(\infty_h+1)\lozenge_{o_2}.
\end{aligned}$$

Putting the above estimates together finally yields

$$|\mathrm{E}(X_{\vec{n}}Y_{\vec{n}}) - \mathrm{E}(X_{\vec{n}})\,\mathrm{E}(Y_{\vec{n}})| \leq 3\infty_h \Diamond_{o_2} + \Diamond_{o_2} + 3(\infty_h + 1)\Diamond_{o_2} \leq \Diamond_{o_1}$$

by the axiom $\infty_{o_2} \geq \infty_{o_1}(6\infty_h + 4)$. □

Finally, we say what it means for a property to hold pointwise almost surely:

Definition 5.6. $\langle o_1, o_2 < \vec{s}_1 < \vec{s}_2 \rangle$ Let $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ be a finite o_2-probability space and $(P_{\vec{n}})$ be a property possibly depending on $(\Omega_{\vec{n}})$. We say that $(P_{\vec{n}})$ holds o_1-almost surely (o_1-a.s. for short) if there is an event $(A_{\vec{n}})$ having measure $\Diamond_{o_1}$ such that $P_{\vec{n}}(\omega)$ holds for all $\vec{n}$ and $\omega \in \Omega_{\vec{n}} \setminus A_{\vec{n}}$.

5.2 On Laws of Large Numbers

We will need Chebyshev's inequality in the following form:

Lemma 5.7. *Suppose* $(X_{\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *is a random variable on a finite o-probability space* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})$ *and* $(c_{\vec{n}})$ *is a positive real number. Then*

$$\Pr\{|X_{\vec{n}}| > c_{\vec{n}}\} \leq \frac{\mathrm{E}\, X_{\vec{n}}^2}{c_{\vec{n}}^2}$$

for all $\vec{n}$.

Proof. We have

$$\begin{aligned} \mathrm{E}\, X_{\vec{n}}^2 &= \sum_{\omega \in \Omega_{\vec{n}}} X_{\vec{n}}^2(\omega)\, \mathrm{pr}_{\vec{n}}(\omega) \\ &\geq \sum_{\omega \in \{|X_{\vec{n}}| > c_{\vec{n}}\}} X_{\vec{n}}^2(\omega)\, \mathrm{pr}_{\vec{n}}(\omega) \geq c_{\vec{n}}^2 \Pr\{|X_{\vec{n}}| > c_{\vec{n}}\} \end{aligned}$$

for all $\vec{n}$. □

The next theorem is our version of the weak law of large numbers:

Theorem 5.8. $\langle o_1 < o_2 \wr h, o_2 < o_3 < o_4 \wr k, o_1 < o_3, r_1 \rangle$ *Let* $(X_{m,\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{m \leq \infty_{r_2}, \infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *be a sequence of* o_4*-independent* ∞_h*-bounded random variables on a finite* o_3*-probability space* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})$ *such that* $\mathrm{E}(X_{i,\vec{n}})$ *and* $\mathrm{Var}(X_{i,\vec{n}})$ *are* ∞_k*-bounded. Then*

$$\frac{X_{1,\vec{n}} + \cdots + X_{m,\vec{n}}}{m} =_{o_1} \frac{\mathrm{E}\, X_{1,\vec{n}} + \cdots + \mathrm{E}\, X_{m,\vec{n}}}{m}$$

holds o_1*-a.s. for all* $\infty_{r_1} \leq m \leq \infty_{r_2}$ *and* $\vec{n}$*. Note that here the exceptional set depends on* m*.*

Proof. Take any $\infty_{r_1} \le m \le \infty_{r_2}$ and $\vec{n}$. Write $\mu_i = \mathrm{E}\, X_{i,\vec{n}}$,

$$\mu = \frac{\mu_1 + \cdots + \mu_m}{m}$$

and $S_m = X_{1,\vec{n}} + \cdots + X_{m,\vec{n}}$. Since $(X_{i,\vec{n}} - \mu_i)$ are o_4-independent by the assumption and

$$|\mathrm{E}(X_{i,\vec{n}} - \mu_i)| = |\mu_i(1 - \Pr \Omega_{\vec{n}})| \le |\mu_i| \Diamond_{o_3} \le \infty_k \Diamond_{o_3},$$

we have

$$\begin{aligned}
&\mathrm{E}\left(\frac{S_m}{m} - \mu\right)^2 \\
&\quad = \frac{1}{m^2}\left(\sum_{i=1}^{m} \mathrm{E}(X_{i,\vec{n}} - \mu_i)^2 + 2 \sum_{\substack{i,j=1 \\ i \ne j}}^{m} \mathrm{E}((X_{i,\vec{n}} - \mu_i)(X_{j,\vec{n}} - \mu_j))\right) \\
&\quad \le \frac{\infty_k m + 2(\infty_k^2 \Diamond_{o_3}^2 + \Diamond_{o_2}) m(m-1)}{m^2} \le \infty_k \Diamond_{r_1} + 2(\infty_k^2 \Diamond_{o_3}^2 + \Diamond_{o_2})
\end{aligned}$$

by Lemma 5.5. Chebyshev's inequality yields now

$$\begin{aligned}
\Pr\left\{\left|\frac{S_m}{m} - \mu\right| > \Diamond_{o_1}\right\} &\le \infty_{o_1}^2\, \mathrm{E}\left(\frac{S_m}{m} - \mu\right)^2 \\
&\le \infty_{o_1}^2 (\infty_k \Diamond_{r_1} + 2(\infty_k^2 \Diamond_{o_3}^2 + \Diamond_{o_2})) \le \Diamond_{o_1}
\end{aligned}$$

by the axioms $\infty_{r_1} \ge 2\infty_{o_1}^3 \infty_k$ and $\infty_{o_3} \ge 3\infty_{o_1}^2 \infty_k$ and $\infty_{o_2} \ge 8\infty_{o_1}^3$. □

We prove two lemmas before we prove our version of the strong law of large numbers. Inequality (19) of the first lemma is known as Kolmogorov's inequality:

Lemma 5.9. $\langle h, p, r, o_1 < o_2 < o_3 < o_4 \rangle$ *Let* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *be a finite* o_3*-probability space. If* $N \le \infty_r$ *is a natural number and* $(X_{i,\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)$ *are* o_4*-independent* ∞_h*-bounded random variables such that* $\mathrm{E}(X_{i,\vec{n}}) =_{o_2} 0$ *for all* $i \le N$*, then*

$$\Pr\left\{\max_{i \le N} |X_{1,\vec{n}} + \cdots + X_{i,\vec{n}}| \ge c_{\vec{n}}\right\} \le_{o_1} \frac{1}{c_{\vec{n}}^2} \sum_{i=1}^{N} \mathrm{Var}\, X_{i,\vec{n}} \tag{19}$$

holds for all $\vec{n}$*, where* $(c_{\vec{n}})$ *is any real number* $\ge \Diamond_p$*.*

Proof. Take any $\vec{n}$ and define events A and A_j by putting

$$A = \{\mathrm{max}_{i \le N} |S_i| \ge c_{\vec{n}}\},$$
$$A_j = \{|S_i| < c_{\vec{n}} \text{ for all } i < j \text{ and } |S_j| \ge c_{\vec{n}}\},$$

where $j \le N$ and $S_i = X_{1,\vec{n}} + \cdots + X_{i,\vec{n}}$. Note that A is a disjoint union of A_j's. Now, since $\mathrm{E}\, X_{i,\vec{n}} =_{o_2} 0$ and $\Pr \Omega_{\vec{n}} =_{o_3} 1$ by the assumptions, we get

$$\begin{aligned}\mathrm{Var}\, X_{i,\vec{n}} &= \mathrm{E}(X_{i,\vec{n}} - \mathrm{E}\, X_{i,\vec{n}})^2 \\ &= \mathrm{E}\, X_{i,\vec{n}}^2 - (\mathrm{E}\, X_{i,\vec{n}})^2(2 - \Pr \Omega_{\vec{n}}) \ge \mathrm{E}\, X_{i,\vec{n}}^2 - \Diamond_{o_2}\end{aligned}$$

by the axiom $\infty_{o_2} \ge 2$. Moreover,

$$\begin{aligned}\sum_{i=1}^{N} \mathrm{E}\, X_{i,\vec{n}}^2 &= \mathrm{E}\, S_N^2 - \sum_{\substack{i,j=1 \\ i \ne j}}^{N} \mathrm{E}(X_{i,\vec{n}} X_{j,\vec{n}}) \\ &\ge \mathrm{E}\, S_N^2 - \sum_{\substack{i,j=1 \\ i \ne j}}^{N} \mathrm{E}(X_{i,\vec{n}})\, \mathrm{E}(X_{j,\vec{n}}) - \Diamond_{o_2} \infty_r (\infty_r - 1) \\ &\ge \mathrm{E}\, S_N^2 - 2\Diamond_{o_2} \infty_r (\infty_r - 1)\end{aligned}$$

follows by Lemma 5.5. Thus, letting $\eta = \Diamond_{o_2} \infty_r (2\infty_r - 1)$,

$$\begin{aligned}\sum_{i=1}^{N} \mathrm{Var}\, X_{i,\vec{n}} + \eta \ge \mathrm{E}\, S_N^2 \ge \mathrm{E}(S_N^2 \chi_A) &= \sum_{j=1}^{N} \mathrm{E}(S_N^2 \chi_{A_j}) \\ &= \sum_{j=1}^{N} \mathrm{E}(S_j + (S_N - S_j))^2 \chi_{A_j}) \\ &\ge \sum_{j=1}^{N} (\mathrm{E}(S_j^2 \chi_{A_j}) + 2\, \mathrm{E}(S_j (S_N - S_j) \chi_{A_j})).\end{aligned}$$

Furthermore, since $|\mathrm{E}(X_{l,\vec{n}} \chi_{A_j})| \le \mathrm{E}|X_{l,\vec{n}} \chi_{A_j}| \le \infty_h \Pr \Omega_{\vec{n}} \le 2\infty_h$ and

$$2 \sum_{j=1}^{N} \sum_{k=1}^{j} \sum_{l=j+1}^{N} 1 = 2 \sum_{j=1}^{N} j(N - j) = (N-1)N(N+1)/3,$$

we get

$$
\begin{aligned}
&2\sum_{j=1}^{N} \mathrm{E}(S_j(S_N - S_j)\chi_{A_j})) \\
&\quad = 2\sum_{j=1}^{N}\sum_{k=1}^{j}\sum_{l=j+1}^{N} \mathrm{E}(X_{k,\vec{n}}X_{l,\vec{n}}\chi_{A_j}) \\
&\quad \geq 2\sum_{j=1}^{N}\sum_{k=1}^{j}\sum_{l=j+1}^{N} \mathrm{E}(X_{k,\vec{n}})\,\mathrm{E}(X_{l,\vec{n}}\chi_{A_j}) - \Diamond_{o_2}(\infty_r - 1)\infty_r(\infty_r + 1)/3 \\
&\quad \geq -\Diamond_{o_2}(1 + 2\infty_h)(\infty_r - 1)\infty_r(\infty_r + 1)/3
\end{aligned}
$$

again by Lemma 5.5. Putting the above estimates together finally yields

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathrm{Var}\, X_{i,\vec{n}} \\
&\quad \geq \sum_{j=1}^{N} \mathrm{E}(S_j^2\chi_{A_j}) - \Diamond_{o_2}\infty_r(2\infty_r - 1 + (1 + 2\infty_h)(\infty_r - 1)(\infty_r + 1)/3) \\
&\quad \geq c_{\vec{n}}^2 \Pr A - \Diamond_{o_2}\infty_r(2\infty_r - 1 + (1 + 2\infty_h)(\infty_r - 1)(\infty_r + 1)/3),
\end{aligned}
$$

so Equation (19) holds by the axiom

$$
\infty_{o_2} \geq \lceil \infty_{o_1}\infty_p^2\infty_r(2\infty_r - 1 + (1 + 2\infty_h)(\infty_r - 1)(\infty_r + 1)/3)\rceil. \qquad \square
$$

The second lemma which we prove next is a straightforward corollary of Kolmogorov's inequality:

Lemma 5.10. $\langle o_1 < o_2 \mid h, r_2, o_2 < o_3 < o_4 < o_5 \rangle$ *Let* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}}$ *be a finite* o_4*-probability space and let* $(X_{i,\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{i \leq \infty_{r_2}}$ *be* o_5*-independent* ∞_h*-bounded random variables such that* $\mathrm{E}(X_{i,\vec{n}}) =_{o_3} 0$ *for all* $i \leq \infty_{r_2}$ *and* $\vec{n}$*. If* $(\sum_{i=1}^{m} \mathrm{Var}\, X_{i,\vec{n}})_{m \leq \infty_{r_2}}$ r_1o_3*-converges, then* $(\sum_{i=1}^{m} X_{i,\vec{n}})_{m \leq \infty_{r_2}}$ r_1o_1*-converges pointwise* o_1*-a.s.*

Proof. Take any $\vec{n}$. Let A be the event defined by putting

$$
A = \left\{ \max_{\infty_{r_1} \leq i \leq \infty_{r_2}} |X_{i,\vec{n}} + \cdots + X_{\infty_{r_2},\vec{n}}| \geq \Diamond_{o_2} \right\}.
$$

Then

$$
\Pr A \leq \infty_{o_2}^2 \sum_{i=\infty_{r_1}}^{\infty_{r_2}} \mathrm{Var}\, X_{i,\vec{n}} + \Diamond_{o_2} \leq \infty_{o_2}^2 \Diamond_{o_3} + \Diamond_{o_2} \leq \Diamond_{o_1}
$$

holds by Kolmogorov's inequality and the axioms $\infty_{o_2} \geq 2\infty_{o_1}$ and $\infty_{o_3} \geq 2\infty_{o_1}\infty_{o_2}^2$. Moreover, for all $\omega \in \Omega_{\vec{n}} \smallsetminus A$ and $\infty_{r_1} \leq m_1 < m_2 \leq \infty_{r_2}$,

$$\left|\sum_{i=m_1+1}^{m_2} X_{i,\vec{n}}(\omega)\right| = \left|\sum_{i=m_1+1}^{\infty_{r_2}} X_{i,\vec{n}}(\omega) - \sum_{i=m_2+1}^{\infty_{r_2}} X_{i,\vec{n}}(\omega)\right| \leq 2\Diamond_{o_2} \leq \Diamond_{o_1}$$

holds by the axiom $\infty_{o_2} \geq 2\infty_{o_1}$. Hence $(\sum_{i=1}^{m} X_{i,\vec{n}})_{m\leq\infty_{r_2}}$ r_1o_1-converges pointwise o_1-a.s. □

We finally prove our version of the strong law of large numbers:

Theorem 5.11. $\langle o_1 < o_2 < o_3 \mid h, k, o_2 < r_2 \mid r_3, o_3 < o_4 < o_5 < o_6\rangle$ *Suppose* $(\Omega_{\vec{n}}, \mathrm{pr}_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ *is a finite* o_5*-probability space and* $(X_{m,\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{m\leq\infty_{r_3}}$ *is a sequence of* o_6*-independent* ∞_h*-bounded random variables. If* $\mathrm{E}(X_{i,\vec{n}})$ *and* $\mathrm{Var}(X_{i,\vec{n}})$ *are* ∞_k*-bounded for all* $i \leq \infty_{r_3}$*, then*

$$\frac{X_{1,\vec{n}} + \cdots + X_{m,\vec{n}}}{m} =_{o_1} \frac{\mathrm{E}\,X_{1,\vec{n}} + \cdots + \mathrm{E}\,X_{m,\vec{n}}}{m}$$

holds for all $\infty_{r_2} \leq m \leq \infty_{r_3}$ *and* $\vec{n}$ o_1*-a.s. Note that here the exceptional set does not depend on* m*.*

Proof. Take any $\vec{n}$ and write $Y_i = X_{i,\vec{n}} - \mathrm{E}\,X_{i,\vec{n}}$. Then Y_i is an ∞_l-bounded random variable by the axiom $\infty_l \geq \infty_h + \infty_k$. Moreover,

$$|\mathrm{E}\,Y_i| = |\mathrm{E}(X_{i\vec{n}})(1 - \Pr\Omega_{\vec{n}})| \leq \Diamond_{o_5}\infty_k \leq \Diamond_{o_4}$$

by the axiom $\infty_{o_5} \geq \infty_{o_4}\infty_k$. Also, $\mathrm{Var}\,Y_i$ is ∞_k-bounded. Let $(Z_m)_{m\leq\infty_{r_3}}$ be a sequence of o_6-independent ∞_l-bounded random variables defined by putting

$$Z_m = \frac{Y_m}{m}$$

for all $m \leq \infty_{r_3}$. Since $(\sum_{i=1}^{m} 1/i^2)_{m\leq\infty_{r_3}}$ r_1o_5-converges as was shown in Section 3.3, $(\sum_{i=1}^{m} \mathrm{Var}\,Y_i/i^2)_{m\leq\infty_{r_3}}$ r_1o_4-converges by the axiom $\infty_{o_5} \geq \infty_{o_4}\infty_k$. Writing $S_m = Z_1 + \cdots + Z_m$, Kolmogorov's inequality yields now

$$\Pr\left\{\max_{j\leq\infty_{r_3}} |S_j| \geq \infty_l\right\} \leq \Diamond_l^2 \sum_{i=1}^{\infty_{r_3}} \frac{\mathrm{Var}\,Y_i}{i^2} + \Diamond_{o_3} \leq 2\Diamond_l^2\infty_k + \Diamond_{o_3} \leq \Diamond_{o_2}$$

by the axioms $\infty_l \geq \min\{i : i^2 \geq 4\infty_{o_2}\infty_k\}$ and $\infty_{o_3} \geq 2\infty_{o_2}$. In addition, $(S_m)_{m\leq\infty_{r_3}}$ r_1o_3-converges pointwise o_3-a.s. by Lemma 5.10. It follows now from these two facts together with Lemma 3.20 that

$$S_m - \frac{1}{m}\sum_{j=1}^{m} S_j =_{o_2} 0 \text{ and } \frac{1}{m}S_m =_{o_2} 0$$

hold for all $\infty_{r_2} \le m \le \infty_{r_3}$ and $\vec{n}$ o_1-a.s. Here the axioms $\infty_{r_2} \ge \infty_{o_2}\infty_l$ and $\infty_{o_2} \ge 2\infty_{o_1}$ and $\infty_{o_3} \ge 2\infty_{o_1}$ were needed. Finally, since

$$\frac{Y_1 + \cdots + Y_m}{m} = S_m - \frac{1}{m}\sum_{j=1}^{m} S_j + \frac{1}{m}S_m,$$

the claim follows by the axiom $\infty_{o_2} \ge 2\infty_{o_1}$. □

5.3 A Derivation of the Black-Scholes Formula

We sketch in this section a derivation of the Black-Scholes option pricing formula for a European call option by adapting the derivation of the formula by van den Berg and Koudjeti (1997) and van den Berg (2000) to our approach.

First we define the notion of a normally distributed random variable:

Definition 5.12. Let $(X_{\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ be an ∞_h-bounded random variable. We say that it is o-normally distributed if

$$\Pr\{X_{\vec{n}} = x\} \cdot \infty_u! =_o \frac{1}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-x^2/2)$$

for all $\vec{n}$ and $x \in [-\infty_h, \infty_h] \subseteq \mathbb{Q}_u$.

Lemma 5.13. $\langle o_1 < o_2 \shortmid h, o_2 < o_3 \rangle$ *If* $(X_{\vec{n}} \colon \Omega_{\vec{n}} \to \mathbb{Q}_u)_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *is* ∞_h-*bounded and* o_3-*normally distributed, then* $\mathrm{E}(X_{\vec{n}}) =_{o_1} 0$ *and* $\mathrm{Var}(X_{\vec{n}}) =_{o_1} 1$.

Proof. Take any $\vec{n}$. Since

$$\sum_{0 < x \le \infty_h} x = \frac{1}{\infty_u!} \sum_{i=1}^{\infty_h \infty_u!} i = \frac{(1 + \infty_h \infty_u!)\infty_h}{2},$$

we have

$$\mathrm{E}\, X_{\vec{n}} = \sum_{|x| \le \infty_h} x \Pr\{X_{\vec{n}} = x\} =_{o_2} \sum_{|x| \le \infty_h} \frac{x}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-x^2/2) dx_u$$

by the axiom $\infty_{o_3} \ge \infty_{o_2}\infty_h^2 + 1$. Writing $\infty_h^+ = \infty_h + 1/\infty_u!$, it follows from the fundamental theorem of calculus and the chain rule that

$$\begin{aligned}\sum_{|x| \le \infty_h} \frac{x}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-x^2/2) dx_u &= -\frac{1}{\sqrt{2\pi_{\vec{n}}}} \int_{-\infty_h}^{\infty_h^+} -x \cdot \exp_{\vec{n}}(-x^2/2) dx_u \\ &=_{o_2} \exp_{\vec{n}}(-(\infty_h^+)^2/2) - \exp_{\vec{n}}(-\infty_h^2/2) =_{o_2} 0.\end{aligned}$$

Thus $\mathrm{E}\, X_{\vec{n}} =_{o_1} 0$ by the axiom $\infty_{o_2} \ge 3\infty_{o_1}$. The other part of the proof is left to the reader. □

We make first some abbreviations:

Definition 5.14. Let $(p_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ be a real number and let $N \leq \infty_l$ be a natural number. We write

$$\begin{aligned} B_{N,\vec{n}}(j) &= \binom{N}{j} p_{\vec{n}}^j (1-p_{\vec{n}})^{N-j}, \\ \mu_{\vec{n}} &= N p_{\vec{n}}, \\ \sigma_{\vec{n}} &= \sqrt{N p_{\vec{n}}(1-p_{\vec{n}})}, \\ x_{\vec{n},j} &= (j - \mu_{\vec{n}})/\sigma_{\vec{n}}, \\ \Omega_{N,\vec{n}} &= \{x_{\vec{n},j} : j = 0, 1, \ldots, N\}. \end{aligned}$$

Then $(\Omega_{N,\vec{n}}, B_{N,\vec{n}})$ is a finite o-probability space.

Binomial distibution can be approximated by normal distribution:

Lemma 5.15. $\langle o_1 < o_2 \mid o_2, k < l_1 \mid l_2 < o_3 \rangle$ *Let* $(p_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ *be a real number such that* $(p_{\vec{n}}) =_{o_3} 1/2$ *and let* $\infty_{l_1} \leq N \leq \infty_{l_2}$ *be a natural number. If* $(x_{\vec{n},j})$ *is* ∞_k*-bounded, then*

$$B_{N,\vec{n}}(j)\sigma_{\vec{n}} =_{o_1} \frac{1}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-x_{\vec{n},j}^2/2)$$

for all $\vec{n}$. *If* $(x_{\vec{n},j})$ *is not* ∞_k*-bounded, then*

$$B_{N,\vec{n}}(j)\sigma_{\vec{n}} \leq \frac{4}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-(j-\mu_{\vec{n}})^2/2N(1-p_{\vec{n}}))$$

for all $\vec{n}$.

Proof. Take any $\vec{n}$. Let $j \geq \mu_{\vec{n}}$ be such that $|x_{\vec{n},j}| \leq \infty_k$. Note first that for all $\mu_{\vec{n}} \leq i < j$,

$$\frac{B_{N,\vec{n}}(i+1)}{B_{N,\vec{n}}(i)} = 1 - \frac{i - \lceil \mu_{\vec{n}} \rceil}{N p_{\vec{n}}(1-p_{\vec{n}})} + \frac{M_i}{N p_{\vec{n}}(1-p_{\vec{n}})},$$

where

$$M_i = \frac{x_{\vec{n},i}^2 + x_{\vec{n},i}/\sigma_{\vec{n}} - N p_{\vec{n}}(1-p_{\vec{n}})/\sigma_{\vec{n}}^2}{(i+1)/\sigma_{\vec{n}}^2} - \lceil \mu_{\vec{n}} \rceil + \mu_{\vec{n}}$$

is bounded by ∞_k^2. Then

$$\begin{aligned}
\frac{B_{N,\vec{n}}(j)}{B_{N,\vec{n}}(\lceil\mu_{\vec{n}}\rceil)} &=_{o_2} \exp_{\vec{n}}\left(\sum_{i=\lceil\mu_{\vec{n}}\rceil}^{j-1} \ln\left(1 - \frac{i-\lceil\mu_{\vec{n}}\rceil}{Np_{\vec{n}}(1-p_{\vec{n}})} + \frac{M_i}{Np_{\vec{n}}(1-p_{\vec{n}})}\right)\right) \\
&=_{o_2} \exp_{\vec{n}}\left(\sum_{i=\lceil\mu_{\vec{n}}\rceil}^{j-1} \left(-\frac{i-\lceil\mu_{\vec{n}}\rceil}{Np_{\vec{n}}(1-p_{\vec{n}})} + \frac{M_i}{Np_{\vec{n}}(1-p_{\vec{n}})}\right)\right) \\
&= \exp_{\vec{n}}\left(-\frac{(j-\lceil\mu_{\vec{n}}\rceil)^2-(j-\lceil\mu_{\vec{n}}\rceil)}{2Np_{\vec{n}}(1-p_{\vec{n}})} + \sum_{i=\lceil\mu_{\vec{n}}\rceil}^{j-1} \frac{M_i}{Np_{\vec{n}}(1-p_{\vec{n}})}\right) \\
&=_{o_2} \exp_{\vec{n}}(-x_{\vec{n},j}^2/2),
\end{aligned}$$

so

$$|B_{N,\vec{n}}(j)\sigma_{\vec{n}} - B_{N,\vec{n}}(\lceil\mu_{\vec{n}}\rceil)\sigma_{\vec{n}}\exp_{\vec{n}}(-x_{\vec{n},j}^2/2)| \le 3\Diamond_{o_2}.$$

It remains to show that

$$B_{N,\vec{n}}(\lceil\mu_{\vec{n}}\rceil)\sigma_{\vec{n}} =_{o_2} \frac{1}{\sqrt{2\pi_{\vec{n}}}}.$$

This follows from Stirling's formula the proof of which in the present context we leave to the reader. The claim follows now by the axiom $\infty_{o_2} \ge 4\infty_{o_1}$. The proof goes similarly in case $j < \mu_{\vec{n}}$ is such that $|x_{\vec{n},j}| \le \infty_k$.

Let then $j \ge \mu_{\vec{n}}$ be such that $|x_{\vec{n},j}| > \infty_k$. Since for all $\mu_{\vec{n}} \le i < j$,

$$\frac{B_{N,\vec{n}}(i+1)}{B_{N,\vec{n}}(i)} = \frac{N-i}{i+1}\cdot\frac{p_{\vec{n}}}{1-p_{\vec{n}}} \le 1 - \frac{i-\mu_{\vec{n}}}{N(1-p_{\vec{n}})},$$

we get

$$\begin{aligned}
\frac{B_{N,\vec{n}}(j)}{B_{N,\vec{n}}(\lceil\mu_{\vec{n}}\rceil)} &\le \prod_{i=\lceil\mu_{\vec{n}}\rceil}^{j-1}\left(1-\frac{i-\mu_{\vec{n}}}{N(1-p_{\vec{n}})}\right) \\
&=_{o_3} \exp_{\vec{n}}\left(\sum_{i=\lceil\mu_{\vec{n}}\rceil}^{j-1} \ln\left(1-\frac{i-\mu_{\vec{n}}}{N(1-p_{\vec{n}})}\right)\right) \\
&\le_{o_3} \exp_{\vec{n}}\left(\sum_{i=\lceil\mu_{\vec{n}}\rceil}^{j-1} -\frac{i-\mu_{\vec{n}}}{N(1-p_{\vec{n}})}\right) \\
&= \exp_{\vec{n}}(H_{\vec{n}} - (j-\mu_{\vec{n}})^2/2N(1-p_{\vec{n}})) \\
&=_{o_3} \exp_{\vec{n}}(H_{\vec{n}})\cdot\exp_{\vec{n}}(-(j-\mu_{\vec{n}})^2/2N(1-p_{\vec{n}})),
\end{aligned}$$

where $|H_{\vec{n}}| \leq 1/2$. Note that here ∞_{l_2} does not depend on ∞_{o_3}, so we can choose ∞_{o_3} so big that

$$3\Diamond_{o_3} \leq \frac{2}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-(j-\mu_{\vec{n}})^2/2N(1-p_{\vec{n}})).$$

Thus

$$B_{N,\vec{n}}(j)\sigma_{\vec{n}} \leq \frac{4}{\sqrt{2\pi_{\vec{n}}}} \exp_{\vec{n}}(-(j-\mu_{\vec{n}})^2/2N(1-p_{\vec{n}})).$$

The proof is similar in case $j < \mu_{\vec{n}}$ is such that $|x_{\vec{n},j}| > \infty_k$. □

Development of stock price is modelled as a discrete geometric Brownian motion:

Definition 5.16. Let $(s_{\vec{n}})_{\infty_{\vec{s}_1} \leq \vec{n} \leq \infty_{\vec{s}_2}} \in \mathbb{Q}_t$ be a positive real number and T be a positive natural number. Write $dt = \Diamond_p$ and $dx = \sqrt{dt}$ and $N = T\infty_p$

$$C_{\vec{n}} = \{(idt, (2j-i)dx) \in \mathbb{Q}_t \times \mathbb{Q}_t : 0 \leq i \leq N \text{ and } 0 \leq j \leq i\}.$$

Let $(\mu_{\vec{n}}) \in \mathbb{Q}_t$ be a real number and $(\rho_{\vec{n}}) \in \mathbb{Q}_t$ be a positive real number. We define recursively a finite stochastic process $(S_{\vec{n}} \colon C_{\vec{n}} \to \mathbb{Q}_u)$ by putting $S_{\vec{n}}(0,0) = s_{\vec{n}}$ and

$$\begin{cases} S_{\vec{n}}(t+dt, x+dx) = S_{\vec{n}}(t,x) \cdot (1 + \mu_{\vec{n}}dt + \rho_{\vec{n}}dx) \text{ with prob. } p_{\vec{n}}, \\ S_{\vec{n}}(t+dt, x-dx) = S_{\vec{n}}(t,x) \cdot (1 + \mu_{\vec{n}}dt - \rho_{\vec{n}}dx) \text{ with prob. } 1-p_{\vec{n}} \end{cases}$$

for all $\vec{n}$ and $t = idt$ and $x = (2j-i)dx$ with $0 \leq i < N$ and $0 \leq j \leq i$. This process is called a discrete geometric Brownian motion. Note that the random variables of the process have binomial distributions. With $(p_{\vec{n}}) = 1/2$ this process models the price of a stock. Then $(\mu_{\vec{n}})$ is the rate of return of the price process and $(\rho_{\vec{n}})$ its volatility.

Let now $(K_{\vec{n}}) \in \mathbb{Q}_v$ be a positive real number and write

$$C_{T,\vec{n}} = \{(2j-N)dx \in \mathbb{Q}_t : 0 \leq j \leq N\}.$$

The random variable $(O_{\vec{n}} \colon C_{T,\vec{n}} \to \mathbb{Q}_v)$ defined by putting

$$O_{\vec{n}}(x) = \max(S_{\vec{n}}(T,x) - K_{\vec{n}}, 0)$$

for all $\vec{n}$ and $x \in C_{T,\vec{n}}$ is called a European call option having T as the exercise time and $(K_{\vec{n}})$ as the exercise price. This means that the owner of the option has the right to buy the stock at T years in the future at price $(K_{\vec{n}})$.

The stock price at time T can be approximated as follows:

Theorem 5.17. *$\langle o_1 < o_2 \shortmid o_2, h < o_3 < p\rangle$ Suppose $(s_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ and $(\mu_{\vec{n}})$, $(\rho_{\vec{n}})$, T are ∞_h-bounded. Then*

$$S_{\vec{n}}(T, x) =_{o_1} s_{\vec{n}} \cdot \exp_{\vec{n}}\left(\left(\mu_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) T + \rho_{\vec{n}} x\right)$$

for all $\vec{n}$ and $x \in C_{T,\vec{n}}$.

Proof. We only treat the case $x = (2j - N)dx$ with even N and $j \ge N/2$. The cases with $j < N/2$ as well as with odd N are similar. Now for all $\vec{n}$,

$$\begin{aligned}
S_{\vec{n}}(T, 0) &=_{o_3} s_{\vec{n}} \cdot \left(1 + \left(\mu_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) 2dt + \mu_{\vec{n}}^2 dt^2\right)^{T/2dt} \\
&=_{o_3} s_{\vec{n}} \cdot \exp_{\vec{n}}\left(\frac{T}{2dt} \ln\left(1 + \left(\mu_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) 2dt + \mu_{\vec{n}}^2 dt^2\right)\right) \\
&=_{o_3} s_{\vec{n}} \cdot \exp_{\vec{n}}\left(\frac{T}{2dt}\left(\left(\mu_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) 2dt + \mu_{\vec{n}}^2 dt^2\right)\right) \\
&=_{o_3} s_{\vec{n}} \cdot \exp_{\vec{n}}\left(\left(\mu_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) T\right)
\end{aligned}$$

and

$$\begin{aligned}
S_{\vec{n}}(T, x) &=_{o_2} S_{\vec{n}}(T, 0) \cdot \left(\frac{1 + \mu_{\vec{n}} dt + \rho_{\vec{n}} dx}{1 + \mu_{\vec{n}} dt - \rho_{\vec{n}} dx}\right)^{x/2dx} \\
&=_{o_2} S_{\vec{n}}(T, 0) \cdot \exp_{\vec{n}}\left(\frac{x}{2dx} \ln\left(\frac{1 + \mu_{\vec{n}} dt + \rho_{\vec{n}} dx}{1 + \mu_{\vec{n}} dt - \rho_{\vec{n}} dx}\right)\right) \\
&=_{o_2} S_{\vec{n}}(T, 0) \cdot \exp_{\vec{n}}\left(\frac{x}{2dx} \ln(1 + 2\rho_{\vec{n}} dx)\right) \\
&=_{o_2} S_{\vec{n}}(T, 0) \cdot \exp_{\vec{n}}(\rho_{\vec{n}} x).
\end{aligned}$$

Thus

$$S_{\vec{n}}(T, x) =_{o_1} s_{\vec{n}} \cdot \exp_{\vec{n}}\left(\left(\mu_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) T + \rho_{\vec{n}} x\right).$$ □

Let $(r_{\vec{n}}) \in \mathbb{Q}_t$ be a positive real number, the risk-free rate of interest. By an argument of Cox, Ross and Rubinstein (1979), the right price of the option $(O_{\vec{n}})$ is the expectation of its discounted value in a risk-neutral world, i.e. in a world where the rate of return $(\mu_{\vec{n}})$ of the price process is $(r_{\vec{n}})$. This can also be achieved by changing the probability $p_{\vec{n}}$ of the price process from 1/2 to

$$p_{\vec{n}} = \frac{1}{2} + \frac{r_{\vec{n}} - \mu_{\vec{n}}}{2\rho_{\vec{n}}} \cdot dx.$$

So let $(p_{\vec{n}})$ be defined in this way. The price of the option is now the real number defined by putting

$$\frac{\mathrm{E}\, O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))}{(1+r_{\vec{n}}dt)^{T/dt}} = \frac{\mathrm{E}\, O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))}{(1+r_{\vec{n}}T/N)^{N}}$$

for all $\vec{n}$. We have the following corollary:

Corollary 5.18. $\langle o_1 < o_2 \mid o_1, h < p \rangle$ *Suppose* $(s_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *and* $(\mu_{\vec{n}})$, $(\rho_{\vec{n}})$, T *are* ∞_h*-bounded and* $(\rho_{\vec{n}})$ *is* $\Diamond_h$*-appreciable. If* $(f_{\vec{n}} \colon \Omega_{N,\vec{n}} \to C_{T,\vec{n}})$ *is the affine mapping defined by putting*

$$f_{\vec{n}}(x_{j,\vec{n}}) = N dx(2p_{\vec{n}} - 1) + 2dx\rho_{\vec{n}} x_{j,\vec{n}}$$

for all $\vec{n}$ *and* $0 \le j \le N$*, then*

$$f_{\vec{n}}(x_{j,\vec{n}}) =_{o_2} \frac{r_{\vec{n}} - \mu_{\vec{n}}}{\rho_{\vec{n}}} \cdot T + \sqrt{T} \cdot x_{j,\vec{n}}$$

and

$$S_{\vec{n}}(T, f_{\vec{n}}(x_{j,\vec{n}})) =_{o_1} s_{\vec{n}} \cdot \exp_{\vec{n}}\left(\left(r_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) T + \rho_{\vec{n}}\sqrt{T} x_{j,\vec{n}}\right)$$

for all $\vec{n}$*.*

The price of the option can now be expressed as follows:

Theorem 5.19. $\langle o_1 < o_2 \mid o_2, h, k < p \mid h < k \rangle$ *Let* $(s_{\vec{n}})_{\infty_{\vec{s}_1} \le \vec{n} \le \infty_{\vec{s}_2}}$ *and* $(\mu_{\vec{n}})$, $(\rho_{\vec{n}})$, $(r_{\vec{n}})$, T *be* ∞_h*-bounded and let* $(\rho_{\vec{n}})$ *be* $\Diamond_h$*-appreciable. Then*

$$\frac{\mathrm{E}\, O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))}{(1+r_{\vec{n}}T/N)^{N}} =_{o_1} \frac{\exp_{\vec{n}}(-r_{\vec{n}}T)}{\sqrt{2\pi_{\vec{n}}}} \int_{-\infty_k}^{\infty_k} A_{\vec{n}}(x) \exp_{\vec{n}}(-x^2/2) dx_t$$

for all $\vec{n}$*, where*

$$A_{\vec{n}}(x) = \max\left(s_{\vec{n}} \exp_{\vec{n}}\left(\left(r_{\vec{n}} - \frac{\rho_{\vec{n}}^2}{2}\right) T + \rho_{\vec{n}}\sqrt{T} x\right) - K_{\vec{n}}, 0\right)$$

is a function from $\mathbb{Q}_t$ *to* $\mathbb{Q}_v$*.*

Proof. Note first that $\mathrm{E}\, O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))$ can be divided into the sum

$$\sum_{j : |x_{j,\vec{n}}| \le \infty_k} O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}})) B_{N,\vec{n}}(j) + \sum_{j : |x_{j,\vec{n}}| > \infty_k} O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}})) B_{N,\vec{n}}(j).$$

The first term of this sum has at most $2\infty_k\lceil\sigma_{\vec{n}}\rceil + 1$ members, so it has the estimate

$$\sum_{j:|x_{j,\vec{n}}|\leq\infty_k} O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))B_{N,\vec{n}}(j)$$

$$=_{o_2} \frac{1}{\sqrt{2\pi_{\vec{n}}}} \sum_{j:|x_{j,\vec{n}}|\leq\infty_k} O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}})) \exp_{\vec{n}}(-x_{j,\vec{n}}^2/2)\frac{1}{\sigma_{\vec{n}}}$$

$$=_{o_2} \frac{1}{\sqrt{2\pi_{\vec{n}}}} \int_{-\infty_k}^{\infty_k} A_{\vec{n}}(x) \exp_{\vec{n}}(-x^2/2)\, dx_t$$

by Lemma 5.15. For the second term of the sum, we have

$$\sum_{j:|x_{j,\vec{n}}|>\infty_k} O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))B_{N,\vec{n}}(j) =_{o_2} 0$$

by Lemma 5.15 and Corollary 5.18. The claim follows now by the axiom $\infty_{o_2} \geq 3\infty_{o_1}$. □

A simple transformation of the integral gives finally the Black-Scholes formula:

Corollary 5.20. $\langle o_1, h, k < p \mid h < k \rangle$ *Let* $(s_{\vec{n}})_{\infty_{\vec{s}_1}\leq\vec{n}\leq\infty_{\vec{s}_2}}$ *and* $(\mu_{\vec{n}})$, $(\rho_{\vec{n}})$, $(r_{\vec{n}})$, T *be* ∞_h*-bounded and let* $(\rho_{\vec{n}})$, $(K_{\vec{n}})$ *be* $\Diamond_h$*-appreciable. If we put*

$$c_{0,\vec{n}} = \frac{\ln(s_{\vec{n}}/K_{\vec{n}}) + (r_{\vec{n}} - \rho_{\vec{n}}^2/2)T}{\rho_{\vec{n}}\sqrt{T}}$$

then

$$\frac{\mathrm{E}\, O_{\vec{n}}(f_{\vec{n}}(x_{j,\vec{n}}))}{(1 + r_{\vec{n}}T/N)^N} =_{o_1} s_{\vec{n}}N_k(c_{0,\vec{n}} + \rho_{\vec{n}}\sqrt{T}) - K_{\vec{n}} \exp_{\vec{n}}(-r_{\vec{n}}T)N_k(c_{0,\vec{n}})$$

for all $\vec{n}$*, where*

$$N_k(x) = \frac{1}{\sqrt{2\pi_{\vec{n}}}} \int_{-\infty_k}^{x} \exp_{\vec{n}}(-x^2/2)\, dx_t.$$

6 Examples of the Conversion Algorithm

In this final chapter we give four simple examples just to illustrate the conversion algorithm given in Section 2.2. Here are the examples.

Example 1. Define a sequence $(x_{m,n})_{m\le\infty_3,\infty_4\le n\le\infty_5}$ of real numbers by putting

$$x_{m,n} = \begin{cases} 1, & \text{if } n < m, \\ (m+1)/mn, & \text{if } n \ge m. \end{cases}$$

Then, if $\infty_2 \le m_1, m_2 \le \infty_3$ and $\infty_4 \le n \le \infty_5$, we have

$$|x_{m_1,n} - x_{m_2,n}| = \frac{1}{n}\left|\frac{1}{m_1} - \frac{1}{m_2}\right| \le \frac{1}{n}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \le 2\Diamond_4\Diamond_2 \le \Diamond_1$$

by the axioms $\infty_4 \ge \infty_3$ and $\infty_4 \ge t$ and $\infty_2 \ge t$, where $t = \min\{n \le 2\infty_1 : n^2 \ge 2\infty_1\}$, so $(x_{m,n})_{m\le\infty_3}$ is a 2.1-Cauchy sequence of real numbers. Moreover, if $m \le \infty_3$ and $\infty_4 \le n_1, n_2 \le \infty_5$, we have

$$\begin{aligned} |x_{m,n_1} - x_{m,n_2}| &= \left(1 + \frac{1}{m}\right)\left|\frac{1}{n_1} - \frac{1}{n_2}\right| \le \left(1 + \frac{1}{m}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \\ &\le 2(1 + \Diamond_2)\Diamond_4 \le \Diamond_1 \end{aligned}$$

by the axioms $\infty_4 \ge \infty_3$ and $\infty_4 \ge 2\infty_1\lceil 1+\Diamond_2\rceil$, so $(x_{m,n})$ is a 1-Cauchy real number for all $m \le \infty_3$. Now, by Theorem 3.19, $(x_{m,n})_{\infty_2\le m\le\infty_3,\infty_4\le n\le\infty_5}$ is a 0-Cauchy real number. Recall that the proof of Theorem 3.19 used the axiom $\infty_1 \ge 2\infty_0$. Note also the dependence of ∞_4 on ∞_3.

Now $\infty_0, \infty_1, \infty_2, \infty_3, \infty_4$ and ∞_5 get converted into the following terms of **HA**:

$$\begin{aligned}
&\infty_0 \rightsquigarrow z_0, \\
&\infty_1 \rightsquigarrow 2z_0 + z_1, \\
&\infty_2 \rightsquigarrow \min\{n \le 2(2z_0 + z_1) : n^2 \ge 2(2z_0 + z_1)\} + z_2 = s + z_2, \\
&\infty_3 \rightsquigarrow z_3, \\
&\infty_4 \rightsquigarrow \max\{z_3, s, 2(xz_0 + z_1)\lceil 1 + 1/(s + z_2)\rceil\} + z_4, \\
&\infty_5 \rightsquigarrow z_5,
\end{aligned}$$

where z_0, z_1, z_2, z_3, z_4 and z_5 are fresh variables. If we choose $z_1 = z_2 = z_4 = 0$, then the above conversions of ∞_2 and ∞_4 tell us how far in the sequence we have to go to get an approximation up to $1/z_0$.

Example 2. We show that the harmonic series $(\sum_{i=1}^{m} 1/i)_{\infty_1 \leq m \leq \infty_2}$ is 1.0-divergent. To this end, take any $\infty_1 \leq m_1 < m_2 \leq \infty_2$ such that m_2 is even and $m_1 = m_2/2$. This is possible by the axiom $\infty_2 \geq 2\infty_1$. The claim follows now immediately, since

$$\sum_{i=m_1+1}^{m_2} \frac{1}{i} \geq \sum_{i=m_1+1}^{m_2} \frac{1}{m_2} = \frac{m_2 - m_1}{m_2} = \frac{1}{2} \geq \Diamond_0$$

holds by the axiom $\infty_0 \geq 2$. Here ∞_0, ∞_1 and ∞_2 get converted into terms of **HA** as follows:

$$\begin{aligned} \infty_0 &\rightsquigarrow 2 + z_0, \\ \infty_1 &\rightsquigarrow z_1, \\ \infty_2 &\rightsquigarrow 2z_1 + z_2, \end{aligned}$$

where z_0, z_1, z_2 are fresh variables. If we choose $z_0 = 0$, then we have for all z_1 and z_2 that

$$\sum_{i=m_1+1}^{m_2} \frac{1}{i} \geq \frac{1}{2}$$

holds whenever $z_1 \leq m_1 < m_2 \leq 2z_1 + z_2$ are such that m_2 is even and $m_1 = m_2/2$. If we now take $m_2 = 2z_1$ and $m_1 = z_1$, then we get

$$\sum_{i=1}^{2^{2m}} \frac{1}{i} \geq 2m \cdot \frac{1}{2} = m \tag{20}$$

for all m by induction on z_1.

Example 3. It is easy to see that the reciprocal function $f\colon \mathbb{Q}_2 \smallsetminus \{0\} \to \mathbb{Q}_3$, where $f(x) = 1/x$, is well-defined by using the axiom $\infty_3 \geq \infty_2!$. Moreover, $\infty_1 \in \mathbb{Q}_2 \smallsetminus \{0\}$ by the axiom $\infty_2 \geq \min\{n \leq \infty_1 : n! \geq \infty_1\}$. Now the following "improper" integral of f (which is $\ln(\infty_1)$) has an indefinitely large yet finite lower bound

$$\int_1^{\infty_1} \frac{dx_2}{x} \geq \sum_{i=2}^{\infty_1} \frac{1}{i} \geq \sum_{i=2}^{2^{2(\infty_0+1)}} \geq \infty_0$$

by (20) and the axiom $\infty_1 \geq 2^{2(\infty_0+1)}$. The conversion of $\infty_0, \infty_1, \infty_2$ and ∞_3 gives us the following terms of **HA**:

$$\begin{aligned}
\infty_0 &\rightsquigarrow z_0,\\
\infty_1 &\rightsquigarrow 2^{2(z_0+1)} + z_1,\\
\infty_2 &\rightsquigarrow \min\{n \leq 2^{2(z_0+1)} + z_1 : n! \geq 2^{2(z_0+1)} + z_1\} + z_2,\\
\infty_3 &\rightsquigarrow (\min\{n \leq 2^{2(z_0+1)} + z_1 : n! \geq 2^{2(z_0+1)} + z_1\} + z_2)! + z_3,
\end{aligned}$$

where z_0, z_1, z_2 and z_3 are fresh variables. So we reach the lower bound z_0, if we integrate the reciprocal function from 1 to $2^{2(z_0+1)} + z_1$.

Example 4. Let $f\colon \mathbb{Q}_3 \to \mathbb{Q}_4$ be the function $f(x) = x^2$. Note that $x^2 \in \mathbb{Q}_4$ for all $x \in \mathbb{Q}_3$ by the axiom $\infty_4 \geq (\infty_3!)^2$. Now f has a 2.1.0-derivative $f'\colon \mathbb{Q}_3 \to \mathbb{Q}_4$, where $f'(x) = 2x$, since for all $x, x+h \in \mathbb{Q}_3$ with $h \neq_2 0$ and $h =_1 0$,

$$\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h =_0 2x$$

by the axiom $\infty_1 \geq \infty_0$. Hence $\infty_0, \infty_1, \infty_2, \infty_3$ and ∞_4 get converted into the following terms of **HA**:

$$\begin{aligned}
\infty_0 &\rightsquigarrow z_0,\\
\infty_1 &\rightsquigarrow z_0 + z_1,\\
\infty_2 &\rightsquigarrow z_2,\\
\infty_3 &\rightsquigarrow z_3,\\
\infty_4 &\rightsquigarrow (z_3!)^2 + z_4,
\end{aligned}$$

where z_0, z_1, z_2, z_3 and z_4 are fresh variables. If we choose $z_1 = 0$, then the difference quotient approximates the derivative up to $1/z_0$ for all rational numbers h with $0 < |h| \leq 1/z_0$.

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