

Composition operators on vector-valued BMOA and related function spaces

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Academic dissertation

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Jussi Laitila

List of included articles

This thesis consists of an introductory part and the following four articles, which will be referred to in the sequel by [A], [B], [C], and [D]:

- [A] J. Laitila, *Weakly compact composition operators on vector-valued BMOA*, Journal of Mathematical Analysis and Applications 308 (2005) 730–745.
- [B] J. Laitila and H.-O. Tylli, *Composition operators on vector-valued harmonic functions and Cauchy transforms*, Indiana University Mathematics Journal 55 (2006) 719–746.
- [C] J. Laitila, *Composition operators and vector-valued BMOA*, Reports in Mathematics, Preprint 425, Department of Mathematics and Statistics, University of Helsinki, Helsinki 2005.
- [D] J. Laitila, *Weighted composition operators on BMOA*, Reports in Mathematics, Preprint 437, Department of Mathematics and Statistics, University of Helsinki, Helsinki 2006.

1. Introduction

The main objective of this thesis is to study properties of composition operators and weighted composition operators on certain Banach spaces of analytic or harmonic functions on the unit disk of the complex plane.

Articles [A], [B], and [C] mainly concern compactness properties of composition operators on certain spaces of vector-valued functions, i.e., functions which take values in a complex Banach space. The function spaces of interest include vector-valued BMOA spaces, harmonic Hardy spaces, and spaces of Cauchy transforms. It turns out that composition operators are usually not compact on the relevant vector-valued function spaces. On the other hand, many previously known necessary and sufficient conditions for the weak compactness of a composition operator generalize from the scalar-valued setting to a more general vector-valued one. A novel feature in the vector-valued theory is that there often are several natural ways to introduce a vector-valued counterpart of a given scalar-valued function space. Various examples illustrate the differences between versions of such vector-valued function spaces.

In Article [D] characterizations are given of the boundedness and compactness of weighted composition operators on the classical scalar-valued BMOA space and its subspace VMOA. In addition, the essential norm of a weighted composition operator on VMOA is estimated. These results generalize various previously known results about both pointwise multipliers and composition operators on BMOA and VMOA.

The main results of Articles [A], [B], [C], and [D] will be discussed in Sections 2 to 4. In the remaining part of this section a short introduction is provided to composition operators and some relevant function spaces.

Composition operators and classical Hardy spaces

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane \mathbb{C} and let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic map. Then the *composition operator* C_φ associated with φ is the linear map

$$C_\varphi: f \mapsto f \circ \varphi,$$

defined on the linear space $\mathcal{H}(\mathbb{D})$ of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$. A fundamental problem concerning composition operators is to relate operator-theoretic properties of C_φ , such as the norm, compactness, and spectra, to function-theoretic properties of the map φ when C_φ is restricted to a suitable Banach space of analytic or, more generally, harmonic functions on \mathbb{D} .

As a linear operator C_φ was first studied in the 1960s, by J. Ryff, E. Nordgren, and H. Schwartz, in the context of analytic Hardy spaces H^p . Since these spaces are closely related to other function spaces of our interest, it will be useful to recall some fundamental results related to composition operators on H^p . We will restrict ourselves to results which are relevant

in the sequel. The reader is referred to [23, 26, 30] for the H^p theory, and to [19, 43] for the basic results about composition operators on classical function spaces.

For $1 \leq p < \infty$, the *analytic Hardy space* H^p consists of the analytic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left(\int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) \right)^{1/p} < \infty,$$

where m is the Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$ normalized so that $m(\mathbb{T}) = 1$. For later use we recall that H^p can be viewed as the space of Poisson extensions of L^p functions on \mathbb{T} whose negative Fourier coefficients are zero. Indeed, if $1 \leq p < \infty$, then for each $g \in L^p(\mathbb{T})$ satisfying $\hat{g}_n := \int_{\mathbb{T}} \bar{\zeta}^n g(\zeta) dm(\zeta) = 0$ for $n < 0$, the *Poisson integral*

$$(1) \quad \mathcal{P}[g](z) = \int_{\mathbb{T}} g P_z dm \quad (z \in \mathbb{D}),$$

where $P_z(\zeta) = (1 - |z|^2)/|\zeta - z|^2$, defines an analytic function on \mathbb{D} . The map $g \mapsto \mathcal{P}[g]$ establishes a linear isometry from the space

$$(2) \quad L_a^p(\mathbb{T}) = \{g \in L^p(\mathbb{T}) : \hat{g}_n = 0 \text{ for } n < 0\}$$

onto H^p . By Fatou's theorem the boundary function $g \in L_a^p(\mathbb{T})$ can be recovered from $f = \mathcal{P}[g] \in H^p$ by taking the radial limit, which exists almost everywhere on \mathbb{T} . More precisely, $g(\zeta) = \lim_{r \rightarrow 1} f(r\zeta)$ for a.e. $\zeta \in \mathbb{T}$.

For every analytic map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and $1 \leq p < \infty$, the composition operator C_φ is bounded on the space H^p . This follows from the classical *Littlewood subordination principle*, which states that if $\varphi(0) = 0$, then

$$(3) \quad \|f \circ \varphi\|_{H^p} \leq \|f\|_{H^p},$$

for $f \in H^p$; see [19, 23, 43]. A natural follow-up question concerns the compactness of the operator C_φ . Recall that a linear operator T on a Banach space E is *compact* if $\overline{T(B_E)}$ is a compact subset of E , where $B_E = \{x \in E : \|x\|_E \leq 1\}$ is the closed unit ball of E . Compactness properties of composition operators have been quite intensively studied in connection with various function spaces. Among the prominent results we mention the exact formula due to J. Shapiro [42] for the *essential norm* $\|\cdot\|_e$ (i.e., the distance to all compact operators) of a composition operator on H^2 in terms of the *Nevanlinna counting function*

$$(4) \quad N(\varphi, z) = \sum_{w \in \varphi^{-1}(\{z\})} \log \frac{1}{|w|} \quad (z \in \mathbb{D} \setminus \{\varphi(0)\}).$$

Theorem 1 ([42]). *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic map. Then*

$$\|C_\varphi\|_e = \limsup_{|z| \rightarrow 1} \left(\frac{N(\varphi, z)}{-\log |z|} \right)^{1/2}.$$

In particular, C_φ is compact on H^2 if and only if

$$(5) \quad \lim_{|z| \rightarrow 1} \frac{N(\varphi, z)}{-\log |z|} = 0.$$

It was known, by an earlier result of Shapiro and P. Taylor, that if C_φ is compact on H^p for some $1 \leq p < \infty$, then it is compact on H^p for all $1 \leq p < \infty$. Consequently, condition (5) characterizes the compactness of C_φ on H^p for all $1 \leq p < \infty$.

Recall further that a linear operator $T: E \rightarrow E$ is *weakly compact* if $\overline{T(B_E)}$ is a weakly compact subset of E . For $1 < p < \infty$, the spaces H^p are reflexive so the identity operator of H^p , and hence all bounded linear operators $T: H^p \rightarrow H^p$, are weakly compact. In the interesting case $p = 1$ the weak compactness of composition operators was characterized by D. Sarason [41].

Theorem 2 ([41]). *Weakly compact composition operators on H^1 are compact.*

The operators C_φ have also been studied on many classical function spaces other than H^p spaces. For example, various characterizations have been obtained for the compactness and, in many cases, for the weak compactness of composition operators on Bergman and Bloch spaces, the spaces BMOA, VMOA, h^p , and CT. Some of these results will be discussed in more detail in the forthcoming sections.

Composition operators on spaces of vector-valued analytic functions

In 1998 P. Liu, E. Saksman, and H.-O. Tylli [34] initiated the study of compactness properties of composition operators on spaces of vector-valued analytic functions. For example, they considered the weak compactness of C_φ on X -valued Hardy, Bergman, and Bloch spaces, where X is a complex Banach space. These studies were further extended by J. Bonet, P. Domański, and M. Lindström [8] to the contexts of vector-valued weighted Bergman spaces and certain weak spaces $wE(X)$ of vector-valued analytic functions. We refer e.g. to [9, 31, 45] for other relevant vector-valued results. As an example we will next describe some results related to composition operators on the Hardy spaces $H^p(X)$.

Let $X = (X, \|\cdot\|_X)$ be any complex Banach space. Recall that a function $f: \mathbb{D} \rightarrow X$ is *analytic* if it is weakly analytic, i.e., if $x^* \circ f \in \mathcal{H}(\mathbb{D})$ for all functionals $x^* \in X^*$, where X^* is the dual space of X . Let $\mathcal{H}(\mathbb{D}, X)$ be the space of all analytic functions $f: \mathbb{D} \rightarrow X$ and let $1 \leq p < \infty$. Then a function $f \in \mathcal{H}(\mathbb{D}, X)$ belongs to the *vector-valued Hardy space* $H^p(X)$ if

$$\|f\|_{H^p(X)} = \sup_{0 < r < 1} \left(\int_{\mathbb{T}} \|f(r\zeta)\|_X^p dm(\zeta) \right)^{1/p} < \infty.$$

In particular, in the scalar-valued case where $X = \mathbb{C}$ the spaces $H^p(\mathbb{C})$ are just the classical Hardy spaces H^p .

For general Banach spaces X the Hardy spaces $H^p(X)$ have many properties analogous to those of H^p . However, certain aspects are much more delicate in the vector-valued setting. For example, recall that every function in H^p ($1 \leq p < \infty$) can be realized as a Poisson extension of some integrable function on \mathbb{T} , and $H^p = \mathcal{P}L_a^p := \{\mathcal{P}[g]: g \in L_a^p(\mathbb{T})\}$. If X is an arbitrary Banach space, then for a Bochner p -integrable function $g \in L^p(\mathbb{T}, X)$ satisfying $\hat{g}_n = 0$ for $n < 0$, the Poisson integral (1) defines an X -valued analytic function on \mathbb{D} , and $\mathcal{P}[g] \in H^p(X)$. Moreover, the space $\mathcal{P}L_a^p(X) = \{\mathcal{P}[g]: g \in L_a^p(\mathbb{T}, X)\}$ is a closed subspace of $H^p(X)$. Here

$$(6) \quad L_a^p(\mathbb{T}, X) = \{g \in L^p(\mathbb{T}, X): \hat{g}_n = 0 \text{ for } n < 0\},$$

and the Fourier coefficients \hat{g}_n are defined as in the scalar-valued case. In fact, $\mathcal{P}L_a^p(X)$ consists of the functions in $H^p(X)$ which admit the radial limit at almost every $\zeta \in \mathbb{T}$. However, $\mathcal{P}L_a^p(X) = H^p(X)$ only if X has the *analytic Radon–Nikodým property* (ARNP).

Recall that all reflexive Banach spaces X have the ARNP. On the other hand, for example, the sequence space c_0 does not have this property. To see this consider the bounded c_0 -valued analytic function $f(z) = (z^n)_{n=1}^\infty$ for which the radial limit $\lim_{r \rightarrow 1} f(r\zeta)$ does not exist for any $\zeta \in \mathbb{T}$. The possible absence of the radial limits has to be taken into account when studying composition operators on vector-valued Hardy spaces. We refer to e.g. [3, 28, 29] for further properties of vector-valued Hardy spaces, and to p. 723 of [B] for the relevant discussion of the ARNP.

For any analytic map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D}, X)$, the composed function

$$C_\varphi f = f \circ \varphi: \mathbb{D} \rightarrow X$$

is analytic, so that the composition operator C_φ is well-defined on $\mathcal{H}(\mathbb{D}, X)$. In analogy with the scalar-valued case one can show that every operator C_φ is bounded on each $H^p(X)$ for arbitrary X [31, 34]. Therefore it is reasonable to ask when C_φ is compact on $H^p(X)$.

Since composition operators fix the constant functions, it is easy to see that, in the interesting case where X is infinite dimensional, they are never compact. This observation suggests the study of weaker compactness properties, such as the weak compactness of C_φ on $H^1(X)$ spaces. A simple factorization argument shows that if C_φ is weakly compact on $H^1(X)$, then it is weakly compact on H^1 and X is reflexive. Further, by Theorems 1 and 2, the operator C_φ is compact on H^1 and (5) holds. In [34] the converse of this observation was established by extending some methods of Shapiro and C. Sundberg [42, 44] to the X -valued setting.

Theorem 3 ([34]). *The composition operator C_φ is weakly compact on $H^1(X)$ if and only if (5) holds and X is reflexive.*

Let us next consider another rather general approach introduced by Bonet, Domański, and Lindström [8] for studying C_φ on the vector-valued Bloch spaces and certain weighted Bergman spaces.

Let E be any Banach space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that E contains the constants and the closed unit ball B_E is compact in the topology of uniform convergence on compact subsets of \mathbb{D} . Then the *weak vector-valued space* $wE(X)$ associated with E consists of the functions $f \in \mathcal{H}(\mathbb{D}, X)$ such that

$$(7) \quad \|f\|_{wE(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_E < \infty.$$

It follows from the assumptions on E and the Dixmier–Ng theorem [37] that E is isometrically isomorphic to the dual of a Banach space V . Hence a linearization argument due to J. Mujica [36] shows that $wE(X)$ is isometric to $L(V, X)$, the space of bounded linear operators $V \rightarrow X$. In particular, $wE(X)$ is a Banach space.

It turns out that a composition operator is bounded $wE(X) \rightarrow wE(X)$ if and only if it is bounded $E \rightarrow E$; see also p. 736 of [B]. In [8] general sufficient and necessary conditions were given for the weak compactness of C_φ on weak vector-valued spaces $wE(X)$.

Theorem 4 ([8]). *Let X be a complex Banach space.*

- (i) *If C_φ is weakly compact on $wE(X)$, then X is reflexive and C_φ is weakly compact on E .*
- (ii) *If X is reflexive and C_φ is compact on E , then C_φ is weakly compact on $wE(X)$.*

Various properties of the weak spaces $wE(X)$ have earlier been considered in the special cases of $E = H^p$ and $E = h^p$ [3, 24, 25, 36]; see Section 3 for the definition of $wh^p(X)$ spaces.

Theorem 4 suggests the study of composition operators on the Hardy spaces $wH^p(X)$. In fact, by combining Theorems 2 and 4, one obtains the following counterpart of Theorem 3 for $wH^1(X)$.

Corollary 5. *The operator C_φ is weakly compact on $wH^1(X)$ if and only if it is weakly compact on H^1 and X is reflexive.*

The similarity of Theorem 3 and Corollary 5 raises the question whether the two different types of Hardy spaces, the “strong” spaces $H^p(X)$ and their weak counterparts $wH^p(X)$, are in fact the same. For $z \in \mathbb{D}$ and $1 \leq p < \infty$, we have $\sup_{x^* \in B_{X^*}} |x^*(f(z))|^p = \|f(z)\|_X^p$, so that

$$(8) \quad \sup_{x^* \in B_{X^*}} \sup_{0 < r < 1} \int_{\mathbb{T}} |x^*(f(r\zeta))|^p dm(\zeta) \leq \sup_{0 < r < 1} \int_{\mathbb{T}} \|f(r\zeta)\|_X^p dm(\zeta),$$

and $H^p(X)$ embeds continuously into $wH^p(X)$. However, it is known that the norms of $H^p(X)$ and $wH^p(X)$, when restricted to $H^p(X)$, are never equivalent for infinite-dimensional X ; see [24, 25, A, B]. Concrete examples

which explain this phenomenon will be discussed below in the contexts of vector-valued BMOA and harmonic Hardy spaces.

2. Composition operators and vector-valued BMOA

In Articles [A] and [C] the weak compactness of composition operators is studied on vector-valued versions of BMOA, the space of analytic functions of bounded mean oscillation. Before discussing these results in more detail let us recall some properties and characterizations of the scalar-valued BMOA space. This space consists of the analytic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$(9) \quad \|f\|_{*,1} = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^1} < \infty,$$

where the Möbius transformations $\sigma_a: \mathbb{D} \rightarrow \mathbb{D}$ are given by $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ for $a \in \mathbb{D}$. The quantity $\|f\|_{*,1}$ is a seminorm and $\|f\|_{\text{BMOA},1} = |f(0)| + \|f\|_{*,1}$ defines a complete norm on BMOA.

The space BMOA plays an important role in various questions related to complex analysis and operator theory. The importance of BMOA stems partly from the famous result of C. Fefferman that BMOA can be identified with the dual of the Hardy space H^1 . We refer to [2, 26, 27, 51] for this and other basic facts about BMOA. We next consider some well-known alternative characterizations of BMOA which will be useful in the sequel.

The name BMOA is related to the concept of *bounded mean oscillation* (BMO) introduced by F. John and L. Nirenberg [32]. Recall that an integrable function $g \in L^1(\mathbb{T})$ has bounded mean oscillation on \mathbb{T} , denoted by $g \in \text{BMO}(\mathbb{T})$, if the seminorm

$$\|g\|_{**} = \sup \frac{1}{m(I)} \int_I |g - g_I| dm$$

is finite, where $g_I = (\int_I g dm)/m(I)$ and the supremum is taken over all subintervals I of \mathbb{T} . It is well known that an equivalent seminorm on $\text{BMO}(\mathbb{T})$ is obtained if the averages over intervals I in this supremum are replaced by Poisson integrals (1); see [2, 26, 27]. More precisely,

$$\|g\|_{**} \sim \|g\|_{**, \mathcal{P}} := \sup_{a \in \mathbb{D}} \mathcal{P}[|g - \mathcal{P}[g](a)|](a),$$

where $a \sim b$ means that there exists an absolute constant $c > 0$ such that $c^{-1}a \leq b \leq ca$. Let us for a moment denote by $\text{BMO}_a(\mathbb{T})$ the Banach space of functions $g \in L_a^1(\mathbb{T})$ endowed with the norm $|\int_{\mathbb{T}} g dm| + \|g\|_{**, \mathcal{P}}$, where $L_a^1(\mathbb{T})$ is defined as in (2). Then, by using the properties of the Poisson integral and the fact that $g \mapsto \mathcal{P}[g]$ is an isometry $L_a^1(\mathbb{T}) \rightarrow H^1$, it is not difficult to check that $g \mapsto \mathcal{P}[g]$ defines a linear isometry from $\text{BMO}_a(\mathbb{T})$ onto BMOA. In other words, BMOA consists of the analytic Poisson extensions of BMO functions on the unit circle.

The main result of [32], known as the John–Nirenberg theorem, and the resulting *reverse Hölder inequality* imply the useful fact that the H^1 type

seminorm in (9) can be replaced by the corresponding H^p type seminorm for any $1 \leq p < \infty$; see [2, 26]. More precisely,

$$(10) \quad \|\cdot\|_{*,1} \sim \|\cdot\|_{*,p},$$

where $\|f\|_{*,p} = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^p}$ for $1 \leq p < \infty$.

We finally recall an important alternative characterization of BMOA which is related to Carleson measures. There are various ways to define these measures; we only recall here that a finite positive Borel measure μ on \mathbb{D} is a *Carleson measure* if and only if $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_a(z)| dA(z) < \infty$, where A is the two-dimensional Lebesgue measure on the plane. By some well-known estimates for the H^2 norm (see [26] or pp. 3–4 of [C]) one gets that

$$\|f\|_{*,2} \sim \|f\|_c := \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) |\sigma'_a(z)| \frac{dA(z)}{\pi} \right)^{1/2}.$$

In particular, $f \in \text{BMOA}$ if and only if the measure μ_f , given by $d\mu_f(z) = |f'(z)|^2 (1 - |z|^2) dA(z)$, is a Carleson measure.

It is well known that every composition operator is bounded on BMOA; see e.g. [11, 46]. Compactness and weak compactness of composition operators on BMOA have been studied by several authors [11, 15, 35, 46, 49, 50]. We recall here the following characterization due to W. Smith [46] of the compactness of C_φ on BMOA in terms of the Nevanlinna function (4).

Theorem 6 ([46]). *The operator C_φ is compact on BMOA if and only if*

$$(11) \quad \lim_{r \rightarrow 1} \sup_{\{a: |\varphi(a)| > r\}} \sup_{0 < |w| < 1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) = 0,$$

and, for all $0 < R < 1$,

$$(12) \quad \lim_{t \rightarrow 1} \sup_{\{a: |\varphi(a)| \leq R\}} m(\{\zeta \in \mathbb{T}: |(\varphi \circ \sigma_a)(\zeta)| > t\}) = 0.$$

In the literature there exist various necessary and sufficient conditions for a composition operator to be weakly compact on BMOA; see [15, 35]. However, it seems that the weak compactness of C_φ on BMOA has not yet been completely characterized.

Composition operators on $\text{BMOA}(X)$ and $w\text{BMOA}(X)$

Vector-valued bounded mean oscillation has been actively studied since the 1980s, when J. Bourgain [12] and O. Blasco [4, 5] established vector-valued versions of the the H^1 -BMO duality. More recently different versions of vector-valued BMOA spaces on the unit disk have been investigated in the setting of vector-valued multipliers [6, 7].

Let X be an arbitrary complex Banach space. We define $\text{BMOA}(X)$ as the space of analytic functions $f \in \mathcal{H}(\mathbb{D}, X)$ such that

$$(13) \quad \|f\|_{*,1,X} = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^1(X)} < \infty.$$

We equip $\text{BMOA}(X)$ with the complete norm $\|f\|_{\text{BMOA}(X)} = \|f(0)\|_X + \|f\|_{*,1,X}$. Clearly, $\text{BMOA}(\mathbb{C}) = \text{BMOA}$ in the special case where $X = \mathbb{C}$.

As in the scalar-valued case, $\text{BMOA}(X)$ is related to an X -valued BMO condition on the unit circle. We will next recall what this means precisely. We refer to [33] for the details which closely follow the corresponding scalar-valued argument.

Following [4, 5], we say that a Bochner integrable function $g \in L^1(\mathbb{T}, X)$ has *bounded mean oscillation*, denoted by $g \in \text{BMO}(\mathbb{T}, X)$, if the seminorm

$$\sup \frac{1}{m(I)} \int_I \|g - g_I\|_X dm$$

is finite. Here g_I is defined as in the scalar-valued case. It is again convenient to replace this seminorm by an equivalent quantity

$$\|g\|_{**, \mathcal{P}, X} = \sup_{a \in \mathbb{D}} \mathcal{P} [\|g - \mathcal{P}[g](a)\|_X](a),$$

and, if we equip the space $\text{BMO}_a(\mathbb{T}, X) = \text{BMO}(\mathbb{T}, X) \cap L_a^1(\mathbb{T}, X)$ with the complete norm $\|\int_{\mathbb{T}} g dm\|_X + \|g\|_{**, \mathcal{P}, X}$, where $L_a^1(\mathbb{T}, X)$ is defined as in (6), then the map $g \mapsto \mathcal{P}[g]$ defines an isometric embedding $\text{BMO}_a(\mathbb{T}, X) \rightarrow \text{BMOA}(X)$; see [33]. In particular, $\text{BMOA}(X)$ contains the analytic Poisson extensions of the X -valued BMO functions on \mathbb{T} . However, all of $\text{BMOA}(X)$ is obtained this way only if X has the analytic Radon–Nikodým property. In particular, $\text{BMOA}(X)$ is in general a bigger space than the image $\mathcal{P}[\text{BMO}_a(\mathbb{T}, X)]$.

The main objective of [A] is to study the operators C_φ on $\text{BMOA}(X)$ spaces. It is first observed, by applying an argument of Smith [46], that every operator C_φ is bounded on $\text{BMOA}(X)$ for arbitrary Banach spaces X . As in the context of $H^p(X)$ spaces, it turns out that for infinite-dimensional X the composition operator is never compact on $\text{BMOA}(X)$. On the other hand, if C_φ is weakly compact, then it is weakly compact on BMOA and X is reflexive. The main result of [A] is the following partial converse of this observation.

Theorem 7 (Theorem 7 of [A]). *Let X be reflexive and suppose that (11) and (12) hold. Then C_φ is weakly compact on $\text{BMOA}(X)$.*

The argument is essentially a vector-valued modification of the proof of Theorem 6. It also involves some techniques used in [34] such as the de la Vallée–Poussin operators and a general change of variables formula due to C. Stanton [47]. Further, it involves the following vector-valued version of the reverse Hölder inequality (10): For $1 \leq p < \infty$, it holds that

$$(14) \quad \|\cdot\|_{*,1,X} \sim \|\cdot\|_{*,p,X},$$

where $\|f\|_{*,p,X} = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^p(X)}$. A proof for general Banach spaces X can be found in [13] in the context of functions defined the unit ball of \mathbb{C}^n . This result generalizes a corollary of the analytic John–Nirenberg theorem due to A. Baernstein [2]. If X has the ARNP, then the equivalence

(14) is easily obtained by modifying the corresponding classical proof for functions defined on \mathbb{T} ; see [4, 33].

Since the problem of weak compactness of C_φ on the scalar-valued BMOA is still open, a full characterization is not available in the vector-valued setting. However, by combining Theorem 7 with suitable results from [35] or [15], some partial characterizations are obtained. For example, the following result is contained in Corollary 13 of [A]:

Corollary 8. *If φ is univalent, then C_φ is weakly compact on $\text{BMOA}(X)$ if and only if (11) and (12) hold and X is reflexive.*

Another natural vector-valued generalization of BMOA is given by the weak vector-valued space $w\text{BMOA}(X)$ as defined in (7). Recall that the scalar-valued BMOA space contains the constants and its unit ball is compact in the topology of uniform convergence on compact subsets of \mathbb{D} . Hence $w\text{BMOA}(X)$ is a Banach space and, by Theorem 4, one obtains a counterpart of Theorem 7 for composition operators on $w\text{BMOA}(X)$.

The final section of [A] exhibits concrete examples which indicate fundamental differences between $\text{BMOA}(X)$ and $w\text{BMOA}(X)$. In fact, although $\text{BMOA}(X) \subset w\text{BMOA}(X)$ as a continuous embedding (this can be deduced from (8) and (13)), the norms of $\text{BMOA}(X)$ and $w\text{BMOA}(X)$, when restricted to $\text{BMOA}(X)$, are never equivalent for infinite-dimensional X .

Example 9 (Example 15 of [A]). *For any infinite-dimensional Banach space X and $n \in \mathbb{N}$, there exists a polynomial $f_n \in \mathcal{H}(\mathbb{D}, X)$ of degree n such that*

$$\|f_n\|_{w\text{BMOA}(X)} \leq c \quad \text{and} \quad \|f_n\|_{\text{BMOA}(X)} \geq \sqrt{\log n}.$$

Here $c > 0$ is an absolute constant.

The argument of Example 9 also implies the known fact that the norms of $wH^p(X)$ and $H^p(X)$ are not equivalent for $1 \leq p < \infty$ and any infinite-dimensional X . The argument is based on Dvoretzky's ℓ_n^2 -theorem (see [22]) and an (ℓ^2, BMOA) -multiplier result due to D. Girela [27].

Composition operators on $\text{BMOA}_C(X)$

In [C] composition operators are studied on the space $\text{BMOA}_C(X)$ introduced by Blasco [7]. This space consists of the functions $f \in \mathcal{H}(\mathbb{D}, X)$ such that

$$\|f\|_{C,X} = \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} \|f'(z)\|_X^2 (1 - |z|^2) |\sigma'_a(z)| \frac{dA(z)}{\pi} \right)^{1/2} < \infty.$$

We equip $\text{BMOA}_C(X)$ with the complete norm $\|f\|_{\text{BMOA}_C(X)} = \|f(0)\| + \|f\|_{C,X}$. In the special case $X = \mathbb{C}$ we have $\|f\|_{C,\mathbb{C}} = \|f\|_C$ so that $\text{BMOA}_C(\mathbb{C})$ and BMOA coincide with equivalent norms. Moreover, $f \in \text{BMOA}_C(X)$ if and only if $d\mu_f(z) = \|f'(z)\|_X^2 (1 - |z|^2) dA(z)$ is a Carleson measure.

For general Banach spaces X , the relationship of the spaces $\text{BMOA}(X)$ and $\text{BMOA}_c(X)$ is more complicated. In fact, it follows from a result in [7] that $\text{BMOA}(X)$ and $\text{BMOA}_c(X)$ coincide, and the respective norms are equivalent, if and only if X is isomorphic to a Hilbert space. More precisely, if $\text{BMOA}_c(X)$ embeds continuously into $\text{BMOA}(X)$ then X has *type 2*, and if $\text{BMOA}(X)$ embeds continuously into $\text{BMOA}_c(X)$ then X has *cotype 2*. Recall here that a Banach space is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2; see e.g. [7] or [22] for the definitions of type p and cotype q of a Banach space.

In [C] it is observed that $\text{BMOA}_c(X) = w\text{BMOA}(X)$, and the respective norms are equivalent, if and only if X is finite dimensional. In particular, there are reflexive Banach spaces X such that all of the above versions of X -valued BMOA are different. This observation motivates the study of composition operators on $\text{BMOA}_c(X)$ for general Banach spaces X .

In the light of Theorem 7 a particularly interesting question concerns sufficient conditions for C_φ to be weakly compact on $\text{BMOA}_c(X)$. The following main result of [C] provides such a condition.

Theorem 10 (Theorem 4.1 of [C]). *Let X be reflexive and suppose that C_φ is compact on BMOA. Then C_φ is weakly compact on $\text{BMOA}_c(X)$.*

The outline of the proof resembles that of Theorem 7. However, new difficulties arise in the setting of $\text{BMOA}_c(X)$. For example, no John–Nirenberg type theorem or reverse Hölder inequality are known to be available here. In the proof of Theorem 10 these difficulties are overcome by replacing condition (12) by the requirement that

$$(15) \quad \lim_{|w| \rightarrow 1} \sup_{\{a \in \mathbb{D} : |\varphi(a)| \leq R\}} \frac{N(\varphi \circ \sigma_a, w)}{-\log |w|} = 0,$$

for every $0 < R < 1$. In fact, it follows from the above argument that *the operator C_φ is compact on BMOA if and only if φ satisfies the conditions (11) and (15)*; see Theorem 2.1 and Corollary 4.5 of [C]. The proof of this result is based on some methods and results from [11], [42], and [50].

3. Composition operators on vector-valued harmonic functions and Cauchy transforms

Given a complex Banach space X and $1 \leq p < \infty$, the *harmonic Hardy space* $h^p(X)$ consists of the harmonic functions $f: \mathbb{D} \rightarrow X$ such that

$$\|f\|_{h^p(X)} = \sup_{0 < r < 1} \left(\int_{\mathbb{T}} \|f(r\zeta)\|_X^p dm(\zeta) \right)^{1/p} < \infty.$$

Recall here that a function $f: \mathbb{D} \rightarrow X$ is *harmonic* if it is weakly harmonic, i.e., if $x^* \circ f: \mathbb{D} \rightarrow \mathbb{C}$ is harmonic for all $x^* \in X^*$. Since the X -valued analytic functions are harmonic, the analytic Hardy space $H^p(X)$ is a closed subspace of $h^p(X)$.

By the classical Herglotz theorem the space $h^1 = h^1(\mathbb{C})$ can be viewed as the space of Poisson extensions of complex measures on the unit circle. Indeed, if $M(\mathbb{T})$ denotes the space of complex Borel measures on \mathbb{T} equipped with the total variation norm, then the Poisson integral $\mu \mapsto \mathcal{P}[\mu]$, where $\mathcal{P}[\mu](z) = \int_{\mathbb{T}} P_z d\mu$ (see (1)) defines an isometric isomorphism from $M(\mathbb{T})$ onto h^1 . The space $\mathcal{P}L^1 = \{\mathcal{P}[g] : g \in L^1(\mathbb{T})\}$ is a closed subspace of h^1 .

Composition operators C_φ on harmonic Hardy spaces were probably first considered by Sarason [40], who showed that C_φ is compact on h^1 if and only if

$$(16) \quad \int_{\mathbb{T}} \frac{1 - |\varphi^*(\xi)|^2}{|\zeta - \varphi^*(\xi)|^2} dm(\xi) = \operatorname{Re} \left(\frac{\zeta + \varphi(0)}{\zeta - \varphi(0)} \right) \quad (\zeta \in \mathbb{T}).$$

In particular, condition (16) implies the compactness of C_φ on the subspaces H^1 and $\mathcal{P}L^1$ of h^1 . On the other hand, a subsequent result of Shapiro and Sundberg [44] shows that Shapiro's condition (5) implies the compactness of C_φ on h^1 . Therefore (16) and (5) are equivalent and, by Theorem 2, they characterize the weak compactness of composition operators C_φ on h^1 , $\mathcal{P}L^1$, and H^1 . Moreover, by an observation of Sarason, condition (16) is further equivalent to the requirement that $C_\varphi(h^1) \subset \mathcal{P}L^1$. The following theorem collects some of these results.

Theorem 11. *The following conditions are equivalent.*

- (i) C_φ is weakly compact on h^1 .
- (ii) C_φ is weakly compact on $\mathcal{P}L^1$.
- (iii) C_φ is weakly compact on H^1 .
- (iv) $C_\varphi(h^1) \subset \mathcal{P}L^1$.
- (v) φ satisfies condition (16).

In [B] this result is extended to composition operators on $h^1(X)$ spaces. Below $\mathcal{P}L^1(X) = \{\mathcal{P}[g] : g \in L^1(\mathbb{T}, X)\}$ is a closed subspace of $h^1(X)$.

Theorem 12 (Theorem 3.2 of [B]). *Let X be a reflexive Banach space. Then the following conditions are equivalent.*

- (i) C_φ is weakly compact on $h^1(X)$.
- (ii) C_φ is weakly compact on $\mathcal{P}L^1(X)$.
- (iii) C_φ is weakly compact on $H^1(X)$.
- (iv) $C_\varphi(h^1(X)) \subset \mathcal{P}L^1(X)$.
- (v) φ satisfies condition (16).

It is again easy to check that each of conditions (i) to (iii) implies the reflexivity of X . The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear, and by Theorem 11, each of conditions (i) to (iv) implies (v). The idea of employing a classical result of Dunford and Pettis for the more difficult implications (v) \Rightarrow (i) and (v) \Rightarrow (iv) is from [40].

For a measure $\mu \in M(\mathbb{T})$, the *Cauchy transform*

$$C[\mu](z) = \int_{\mathbb{T}} \frac{1}{1 - \bar{\zeta}z} d\mu(\zeta) \quad (z \in \mathbb{D})$$

defines an analytic function $\mathbb{D} \rightarrow \mathbb{C}$. It is well known that for $1 < p < \infty$ the map $f \mapsto \mathcal{C}[f dm]$, which essentially is the classical Riesz projection, is a bounded linear map from $L^p(\mathbb{T})$ onto H^p ; see [16, 23]. However, it is not bounded from $M(\mathbb{T})$ into $h^1(\mathbb{T})$, and, as a set, H^1 is strictly included in $\text{CT} = \{\mathcal{C}[\mu]: \mu \in M(\mathbb{T})\}$.

We identify the linear space CT with a quotient space of $M(\mathbb{T})$ by equipping it with the complete norm

$$\|\mathcal{C}[\mu]\|_{\text{CT}} = \inf\{\|\lambda\|_{M(\mathbb{T})}: \mathcal{C}[\lambda] = \mathcal{C}[\mu]\}.$$

We refer to the recent monograph [16] for the basic properties of the space CT . Note that in the literature the space CT is often denoted by K .

Let us next consider Cauchy transforms of countably additive Borel vector measures $\mu: \Sigma(\mathbb{T}) \rightarrow X$ for an arbitrary complex Banach space X , where $\Sigma(\mathbb{T})$ is the Borel σ -algebra on \mathbb{T} . Let $M(\mathbb{T}, X)$ denote the Banach space of such measures equipped with the total variation norm. For $\mu \in M(\mathbb{T}, X)$, the Cauchy transform $\mathcal{C}[\mu](z) = \int_{\mathbb{T}} (1 - \bar{\zeta}z)^{-1} d\mu(\zeta)$ again defines an analytic function $\mathbb{D} \rightarrow X$. Here the integral of a continuous map $g: \mathbb{T} \rightarrow X$ against a vector measure is defined via approximation by simple functions; see pp. 5–6 of [21]. We define $\text{CT}(X)$ as the linear space $\{\mathcal{C}[\mu]: \mu \in M(\mathbb{T}, X)\}$ equipped with the complete norm

$$\|\mathcal{C}[\mu]\|_{\text{CT}(X)} = \inf\{\|\lambda\|_{M(\mathbb{T}, X)}: \mathcal{C}[\lambda] = \mathcal{C}[\mu]\},$$

so that $\text{CT}(X)$ is isometrically isomorphic to a quotient space of $M(\mathbb{T}, X)$. The Herglotz theorem extends to the vector-valued setting, that is, the Poisson integral $\mu \mapsto \mathcal{P}[\mu]$ establishes an isometry from $M(\mathbb{T}, X)$ onto $h^1(X)$; see p. 722 of [B] for the references. Therefore $\text{CT}(X)$ can also be viewed as a quotient space of $h^1(X)$.

Composition operators on CT were first studied by P. Bourdon and J. Cima [10], who observed that every operator C_φ is bounded on CT . Later Cima and A. Matheson [14] showed that C_φ is (weakly) compact on CT if and only if (16) holds.

In Section 4 of [B] boundedness and weak compactness of composition operators are studied on $\text{CT}(X)$. Here it is crucial that C_φ essentially commutes with the quotient map $\pi: h^1(X) \rightarrow \text{CT}(X)$; see Lemma 4.2 of [B]. In the scalar-valued case this kind of factorization was established in [14]. The proof of the vector-valued version involves Singer's representation theorem and an argument of W. Hensgen [28]. The fact that any composition operator C_φ is bounded on $\text{CT}(X)$ follows easily from this factorization and the boundedness of C_φ on $h^1(X)$. Together with Theorem 12 this factorization also yields a characterization of the weak compactness of C_φ on $\text{CT}(X)$. Below $\text{CT}_r(X) = \{\mathcal{C}[f dm]: f \in L^1(\mathbb{T}, X)\}$ is a closed subspace of $\text{CT}(X)$.

Theorem 13 (Theorem 4.3 of [B]). *Let X be a reflexive Banach space. Then the following conditions are equivalent.*

- (i) C_φ is weakly compact on $\text{CT}(X)$.
- (ii) $C_\varphi(\text{CT}(X)) \subset \text{CT}_r(X)$.

(iii) φ satisfies condition (16).

In Section 5 of [B] composition operators are studied on weak spaces of vector-valued harmonic functions by extending the analytic approach of [8]. Let E be a Banach space of harmonic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that E contains the constant functions and B_E is compact in the topology of uniform convergence on compact subsets of \mathbb{D} . As in the analytic case, it follows that the weak vector-valued space

$$wE(X) = \{f: \mathbb{D} \rightarrow X \mid f \text{ harmonic, } \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_E < \infty\}$$

is a Banach space; see Lemma 5.1 of [B]. The following result extends Theorem 4 to the harmonic setting.

Theorem 14 (Theorem 5.2 of [B]). *Let X be a complex Banach space.*

- (i) *If C_φ is bounded on E , then C_φ is bounded on $wE(X)$.*
- (ii) *If C_φ is weakly compact on $wE(X)$, then X is reflexive and C_φ is weakly compact on E .*
- (iii) *If X is reflexive and C_φ is compact on E , then C_φ is weakly compact on $wE(X)$.*

The argument for (iii) closely follows that of Theorem 4 (ii), where the crucial trick is to transfer the composition operator on $wE(X)$ to an operator composition map on $L(V, X)$, and then apply a result of Saksman and Tylli [39].

In the special case of harmonic Hardy spaces one obtains a counterpart of Theorem 12 for the weak spaces $wh^1(X)$. It is known that the harmonic Hardy spaces $h^p(X)$ and $wh^p(X)$ differ from each other for $1 \leq p < \infty$ and any infinite-dimensional X ; see [3, 24, 25] and Example 9. The final Section 6 of [B] illustrates these differences by concrete examples. The following example is based on Dvoretzky's theorem and known estimates for lacunary polynomials.

Example 15 (Example 6.1 of [B]). *For any infinite-dimensional Banach space X , $n \in \mathbb{N}$, and $1 \leq p < \infty$, there exists a lacunary polynomial $f_n \in \mathcal{H}(\mathbb{D}, X)$ of degree 2^n such that*

$$\|f_n\|_{wh^p(X)} \leq c_p \quad \text{and} \quad \|f_n\|_{h^p(X)} \geq \sqrt{n}.$$

Here $c_p > 0$ depends only on p .

By a more careful development of these ideas an analytic function $f \in \mathcal{H}(\mathbb{D}, X)$ is constructed such that $f \in wH^p(X) \setminus H^p(X)$ (and $f \in wh^p(X) \setminus h^p(X)$). The difference of the spaces $\text{CT}(X)$ and $w\text{CT}(X)$ is also studied but in this case the results are less complete.

4. Weighted composition operators on BMOA

Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and $\psi: \mathbb{D} \rightarrow \mathbb{C}$ be analytic maps. Then the *weighted composition operator* $W_{\psi,\varphi}$ is the linear map defined for all $f \in \mathcal{H}(\mathbb{D})$ by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)) \quad (z \in \mathbb{D}).$$

Hence the operator $W_{\psi,\varphi}$ is a simultaneous generalization of both the composition operator C_φ and the operator $M_\psi: f \mapsto \psi \cdot f$ of pointwise multiplication.

Such weighted composition operators appear in various contexts. For example, recall that the isometries of the H^p spaces are in fact certain weighted composition operators; see [30]. These operators also appear in connection with many other classical operators on analytic function spaces. Boundedness and compactness of weighted composition operators have earlier been studied on various spaces of analytic functions, such as the Hardy, Bergman, and Bloch spaces; see [17,20,38] and the further references in [D].

In Article [D] weighted composition operators are studied on the space BMOA. Recall from Section 2 that this space consists of the functions $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{*,2} < \infty$ where the seminorm $\|\cdot\|_{*,2}$ is defined as in (10). In [D] we set $\|f\|_* = \|f\|_{*,2}$ and equip BMOA with the complete norm $\|f\|_{\text{BMOA}} = |f(0)| + \|f\|_*$. The space VMOA, i.e., the space of analytic functions of *vanishing mean oscillation*, is the closed subspace of BMOA consisting of the functions $f \in \text{BMOA}$ such that

$$\lim_{|a| \rightarrow 1} \|f \circ \sigma_a - f(a)\|_{H^2} = 0.$$

Alternatively, VMOA equals the closure in BMOA of the complex polynomials, see [27].

We refer to Section 2 for results concerning boundedness and compactness of composition operators on BMOA, and for the related references. Boundedness of the pointwise multipliers M_ψ on BMOA was first characterized by D. Stegenga [48]. In fact, M_ψ is bounded if and only if ψ is bounded and has logarithmic mean oscillation. It is not difficult to check that M_ψ can be compact on BMOA only if $\psi(z) = 0$ on \mathbb{D} .

Clearly, the boundedness of both M_ψ and C_φ implies the boundedness of $W_{\psi,\varphi}$, and the compactness of either M_ψ or C_φ implies the compactness of $W_{\psi,\varphi}$. However, neither of these sufficient conditions is necessary, as is demonstrated by examples in Section 6 of [D].

The main purpose of [D] is to provide conditions which characterize the boundedness and compactness of $W_{\psi,\varphi}$ on BMOA and VMOA. These conditions will be formulated mainly in terms of the quantities

$$\alpha(\psi, \varphi, a) = |\psi(a)| \cdot \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{H^2}$$

and

$$\beta(\psi, \varphi, a) = \left(\log \frac{2}{1 - |\varphi(a)|^2} \right) \|\psi \circ \sigma_a - \psi(a)\|_{H^2},$$

where $a \in \mathbb{D}$.

Theorem 16 (Theorem 3.1 of [D]). *The operator $W_{\psi,\varphi}$ is bounded on BMOA if and only if*

$$\sup_{a \in \mathbb{D}} \alpha(\psi, \varphi, a) < \infty \quad \text{and} \quad \sup_{a \in \mathbb{D}} \beta(\psi, \varphi, a) < \infty.$$

The argument in the proof of Theorem 16 is a combination of Stegenga's multiplier result and the following modification of the subordination principle (3): There is a constant $c > 0$ such that

$$\|f \circ \varphi\|_{H^2} \leq c \|f\|_{H^2} \|\varphi\|_{H^2},$$

for all $f \in H^2$ and analytic maps $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ satisfying $f(0) = \varphi(0) = 0$; see Proposition 3.2 of [D]. The proof of this estimate is based mostly on properties of the Nevanlinna counting function and a related lemma due to Smith [46].

The following result characterizes the compactness of $W_{\psi,\varphi}$ on BMOA.

Theorem 17 (Theorem 4.1 of [D]). *The operator $W_{\psi,\varphi}$ is compact on BMOA if and only if*

$$\lim_{r \rightarrow 1} \sup_{\{a: |\varphi(a)| > r\}} \alpha(\psi, \varphi, a) = 0, \quad \lim_{r \rightarrow 1} \sup_{\{a: |\varphi(a)| > r\}} \beta(\psi, \varphi, a) = 0,$$

and, for all $0 < R < 1$,

$$\lim_{t \rightarrow 1} \sup_{\{a: |\varphi(a)| \leq R\}} \int_{\{\zeta \in \mathbb{T}: |(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a)(\zeta)| > t\}} |(\psi \circ \sigma_a)(\zeta)|^2 dm(\zeta) = 0.$$

The proof of Theorem 17 is partly analogous to that of Theorem 6. The first two conditions are obtained by modifying the proof of Theorem 16. The third condition is a weighted counterpart of (12).

In Section 5 of [D] the operators $W_{\psi,\varphi}$ are studied on the space VMOA. It is first shown that $W_{\psi,\varphi}$ is bounded on VMOA if and only if it is bounded on BMOA, $\psi \in \text{VMOA}$, and

$$\lim_{|a| \rightarrow 1} |\psi(a)| \cdot \|\varphi \circ \sigma_a - \varphi(a)\|_{H^2} = 0.$$

This generalizes a result of J. Arazy, S. Fisher, and J. Peetre [1] for composition operators. The following main result of that section provides estimates for the essential norm

$$\|W_{\psi,\varphi}\|_e = \inf\{\|W_{\psi,\varphi} - K\|: K \text{ is a compact operator on VMOA}\}$$

of the weighted composition operator $W_{\psi,\varphi}$ on VMOA.

Theorem 18 (Theorem 5.3 of [D]). *Assume that $W_{\psi,\varphi}$ is bounded on VMOA. Then*

$$\|W_{\psi,\varphi}\|_e \sim \limsup_{|a| \rightarrow 1} (\alpha(\psi, \varphi, a) + \beta(\psi, \varphi, a)).$$

In particular, $W_{\psi,\varphi}$ is compact on VMOA if and only if

$$\lim_{|a| \rightarrow 1} \alpha(\psi, \varphi, a) = 0 \quad \text{and} \quad \lim_{|a| \rightarrow 1} \beta(\psi, \varphi, a) = 0.$$

This result is partly based on a Carleson measure argument due to M. Tjani [49].

The above results yield as special cases some previously known characterizations of the boundedness and compactness of the operators M_ψ and C_φ on BMOA and VMOA. For example, in the special case where $\varphi(z) = z$ for $z \in \mathbb{D}$, so that $W_{\psi,\varphi} = M_\psi$, the conditions in Theorem 16 are equivalent to the boundedness criterion of Stegenga. In the special case where $\psi(z) = 1$ for $z \in \mathbb{D}$, so that $W_{\psi,\varphi} = C_\varphi$, the conditions in Theorem 17 are equivalent to (11) and (12); see Remark 4.4 of [D].

However, the estimate for the essential norm of $W_{\psi,\varphi}$ on VMOA appears to be new also in both of these special cases.

In the final section of [D] the main results of the article are compared with the corresponding results of S. Ohno and R. Zhao [38] for the Bloch spaces \mathcal{B} and \mathcal{B}_0 . In particular, a simple argument is given which shows that the boundedness (respectively compactness) of $W_{\psi,\varphi}$ on BMOA implies its boundedness (respectively compactness) on \mathcal{B} .

Errata

Article [A] contains the following errors known to the author:

- (i) Page 736, line 7: “(1) and (2)” should be “(2) and (3)”.
- (ii) Page 736, line 8: “(3)” should be “(1)”.
- (iii) Page 740, line 11: “ $S_n f \circ \varphi$ ” should be “ $(S_n f) \circ \varphi$ ”.

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