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The Spin-Statistics Relation
and
Noncommutative Quantum Field Theory

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Contents

Chapter 1

Historical introduction

"The world is made of atoms"[*]

Democritos

Without a doubt the greatest breakthrough in 19th century chemistry was the construction of the table of elements by Dmitri Mendeleev in 1869. It is nothing less than perfect of an example of what a scientific theory should be like. In short, Mendeleev found definite patterns in the characteristics of different known fundamental elements, classified them into groups accordingly and predicted many new elements to be discovered along with their properties to astonishing accuracy. This achievement stands as a cornerstone of all chemistry done to date and it is no exaggeration to say it is one of the most important pieces of scientific knowledge.

The explanation for this particular structure remained a mystery until the development of quantum theory in the early 20th century. Finally it was Pauli who formulated his famous exclusion principle in 1925 [1]

> "There can never be two or more equivalent electrons in an atom. These are defined to be electrons for which - in strong magnetic fields - the value of all quantum numbers n, l, j, m_j is the same. If, in an atom, one electron occurs which has quantum numbers (in an external field) with these specific values, then this state is occupied."

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[*]Richard Feynman, when asked to name the most important of all scientific facts.

It took Pauli another 15 years to make the implications of his principle more precise. In two fundamental papers [2, 3] he formulated the famous spin-statistics theorem further to be discussed in chapter 3, accompanied by the latest developments on the subject in the framework of noncommutative quantum field theories in chapter 5.

In addition to the present chapter and chapters 3 and 5 already mentioned, chapter 2 deals with the general notion of spin and in chapter 4 a brief introduction to noncommutative quantum field theories is presented.

1.1   Experimental evidence

The Stern-Gerlach experiment

In 1922 Otto Stern and Walther Gerlach performed an experiment in order to determine whether particles have any so-called *intrinsic* angular momentum. As a classical example the earth has both the angular momentum from orbiting the sun and in addition the angular momentum from spinning around its axis, its *spin*. Shortly after the idea of the spin of the electron was presented, Pauli showed that for it to be produced by actual spinning, velocities exceeding the speed of light were needed. Thus spin in the quantum mechanical regime must be something completely different than its classical analogue. In the quantum mechanical case the term "intrinsic angular momentum" is often used instead of spin to avoid confusion with classical examples such as the earth.

In the Stern-Gerlach experiment a beam of silver atoms is directed through an inhomogeneous magnetic field and the distribution of final state particles is detected on a screen. Assuming that the particles have some intrinsic form of angular momentum they would behave like tiny magnets and be deflected either up or down depending on the spin orientation. The classical prediction is that there should be a completely *random* distribution of spin directions and thus a continuous pattern would appear on the screen. The marvelous thing of course is that this does not happen. At the end only two spots of dots of silver atoms are observed and nothing else.

This simple experiment thus proves that the silver atom carries a magnetic moment with only two possible orientations. In other words the spin
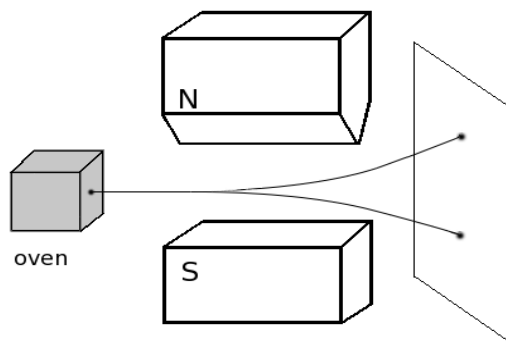
Figure 1.1: Layout of the Stern-Gerlach experiment

of the particle is quantized and has two values. It was known that for an angular momentum quantum number l there are $2l + 1$ states with different orientations of the angular momentum and that for orbital angular momentum l was restricted to integer values due to the uniqueness of the angular part of the wave function. This would then only allow a splitting into an *odd* number of states in contrast to the two spots observed.

To explain the experimental results it was necessary for the theoreticians to introduce half-integral angular momentum. This was first done by the hypothesis of the electron spin by Uhlenbeck and Gouldsmith in 1925. It stated that each electron carries an intrinsic angular momentum, or *spin*, of magnitude $\frac{1}{2}$ with the associated magnetic moment

$$\boldsymbol{\mu}_s = \frac{g_s e}{2mc}\, \mathbf{s}\,, \tag{1.1}$$

where g_s is the gyromagnetic ratio for the electron. To fit theory with experiment it seemed that a value $g_s = 2$ was needed which was exactly twice what nonrelativistic theory predicted. It was a great triumph for Dirac two years later when his equation, to be discussed in the next section, predicted exactly $g_s = 2$.

In group theoretical language the appearance of half-integral angular momentum means that spin is to be described by the group SU(2). SU(2) has the representations corresponding to $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...$, in contrast to SO(3) where l takes on only integer values. Classifications according to spin

are studied further in chapter 2.

Statistical significance

If the temperature of an ideal gas is sufficiently low at a given density, quantum effects become important and we have to think of the ensemble of particles in terms of a wave function.

The multi-particle wave functions correspond to representations of SU(2) and turn out to be either completely symmetric or antisymmetric with respect to an interchange of two identical particles. The antisymmetric ones represent systems of half-integral spin particles that respect the Pauli principle and obey *Fermi-Dirac* statistics. The symmetrical wave functions on the other hand are composed of integral spin particles that do not care about Pauli's principle and obey *Bose-Einstein* statistics. For a thorough treatment of the Fermi and Bose distributions, see [4].

The fundamental difference between the two different statistics is most apparent from the energy state occupation number distributions for the different systems which are written as

$$n_i^{BE} = \frac{g_i}{e^{(\epsilon_i - \mu)/k_b T} - 1} \tag{1.2}$$

$$n_i^{FD} = \frac{g_i}{e^{(\epsilon_i - \mu)/k_b T} + 1} \,, \tag{1.3}$$

where g_i is the degeneracy of the state, T the temperature, ϵ_i the state energy and μ the chemical potential. From these it is clear that at low energies in the boson case the values are not bounded from above, whereas for fermion systems one can at most have one particle per (nondegenerate) state.

For bosons this allows interesting properties in the low energy regime, for example Bose-Einstein condensation that earned a recent Nobel prize in 2001 [5]. For fermions this statistical behaviour is crucial, for it accounts for the stability of matter [6].

The stability of matter

The stability of matter is obviously an issue of fundamental importance. In [7, 8] Dyson and Lenard present their stability theorem of the N-electron

Hamiltonian, essentially setting a lower limit on the binding energy of an atom $H_N \geq CN$. Lieb and Thirring further advanced this model in [6,9] by using Thomas-Fermi theory to find a realistic value for the constant C.

The important requirement is that an assembly of N electrons has a binding energy proportional to N. The form $E \propto N$ is obtained by using Fermi-Dirac statistics, in contrast to Bose-Einstein statistics, which yields $E \propto N^{7/2}$. It is shown that for $E \propto N^{\alpha}$, $\alpha > 1$ there exists no minimum for the energy of a relativistic system and thus the system collapses. This is then the faith of any system obeying Bose-Einstein statistics whereas fermion systems, such as neutron stars, can remain stable due to the degeneracy pressure exerted by the identical fermions obeying the Pauli principle.

Because of its nature, even the smallest violation in the spin-statistics relation would be an extraordinary finding and there are continuos efforts to find any such deviations. The most recent test on the limits of any such violation is presented in [10]. A limit on the probability P of a two-electron system to have a symmetrical part in its wave function is found to be $P < 4.5 \times 10^{-28}$ with a 99.7% confidence level. The precision is hoped to increase in the ongoing two-year experiment by three orders of magnitude into the $10^{-30} - 10^{-31}$ region.

1.2 Relativistic theory[†]

The first step in high energy physics was taken in 1905 when Einstein published his special relativity theory. Among other things, it relates the total energy of an object to the relative velocity with respect to the observer[‡]

$$E = \frac{m}{\sqrt{1 - v^2}} \, . \tag{1.4}$$

Since now the classical value for the energy $E = E_{kin} + V(x)$, $E_{kin} = \frac{p^2}{2m}$ is replaced by its relativistic counterpart (1.4) the implication in quantum

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[†]For an introduction to relativistic quantum theory, see e.g. [11].

[‡]Throughout the thesis I will be using natural units, where $\hbar = c = 1$.

theory is to look for a relativistic alternative to the Schrödinger equation

$$\hat{E}\psi = \left[\frac{\hat{p}^2}{2m} + V(x)\right]\psi$$

$$i\partial_t\psi = \left[-\frac{\nabla^2}{2m} + V(x)\right]\psi. \tag{1.5}$$

Squaring both sides of (1.4) yields the relativistic dispersion relation $(p = Ev)$:

$$E^2 = E^2v^2 + m^2 = p^2 + m^2, \tag{1.6}$$

It did not take O. Klein too long after Schrödinger's original paper to publish the first wave-equation that satisfies (1.6). Later, W. Gordon coupled it with the electromagnetic field. It is interesting to note that Schrödinger himself first came up with the Klein-Gordon equation, but since the equation does not accommodate spin he failed to describe the energy levels of the hydrogen atom correctly and published his famous nonrelativistic equation instead. Also V. Fock came up with the equation independently.

The Klein-Gordon equation

Taking the usual Minkowski space representations for the energy and momentum $E = i\partial_t, p = -i\nabla$ and substituting into (1.6) we get:

$$-\partial_t^2\phi + \boldsymbol{\nabla}^2\phi = m^2\phi$$

$$\left[\partial_\mu\partial^\mu + m^2\right]\phi = 0 \quad \text{The Klein-Gordon equation.} \tag{1.7}$$

There seems to be a disaster, however, when looking at the energy eigenvalues of free particle solutions. Since $E = \pm\sqrt{p^2 + m^2}$ we have, in addition to the positive energy eigenvalues, also negative energy solutions. Further, since the probability density turns out to be proportional to the energy, we end up with negative probabilities. Something has to be wrong here, either the original equation or our interpretation of the result. This puzzled many leading scientists at the time but it was Dirac who found the solution in 1927. It turns out that the negative energy solutions have to be interpreted as *antiparticles* and that with this interpretation the Klein-Gordon equation describes spinless particles correctly.

The Dirac equation

Dirac's idea that won him the joint Nobel prize in 1933 with Schrödinger was to seek a form of the Klein-Gordon equation linear in the time derivative. He saw that the negative energy problems stem from the second order derivative with respect to time. After this insight everything else follows naturally. To have a Lorentz-covariant theory also the space part has to be first order. In addition, the Hamiltonian has to be hermitian $H = H^\dagger$ and of course (1.6) has to be satisfied. This led Dirac to the ansatz:

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$$
$$\partial_t \psi(x) = -i\alpha^k \partial_k \psi(x) + \beta m \psi(x) \,. \tag{1.8}$$

From the requirement that (1.6) be satisfied for plane waves it is easy to deduce the properties $(\alpha^k)^2 = \beta^2 = 1$ and $\{\alpha^i, \alpha^j\} = \{\alpha^i, \beta\} = 0$. These anticommutation relations determine the representations of the operators $\boldsymbol{\alpha}, \beta$. They have to be hermitian, traceless and of even dimensionality $N, N \geq 4$. The simplest way to satisfy these requirements is for $\boldsymbol{\alpha}, \beta$ to be 4×4 constant *matrices*. Probably the most used representation for these matrices is the so-called Dirac representation (This is not a unique choice since any set of similarly related matrices will do):

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{1.9}$$

where $\boldsymbol{\sigma}$ are the Pauli sigma matrices and $\mathbb{1}$ is a 2×2 unit matrix:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.10}$$

Further, by defining $\gamma^0 = \beta, \boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$ we arrive at a Lorentz-covariant notation where now the Dirac γ-matrices in the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{1.11}$$

satisfy the Dirac algebra $\{\gamma^\mu, \gamma^\mu\} = 2\eta^{\mu\nu}$. Here $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ is the metric of the space-time. With our new matrices we can turn the original ansatz (1.8) into the form

$$[i\gamma^\mu \partial_\mu - m]\psi(x) = [i\slashed{\partial} - m]\psi(x) = 0 \,, \tag{1.12}$$

which is the Dirac equation for a free particle. Here the Dirac slash notation $\gamma^\mu a_\mu = \slashed{a}$ has been used.

Dirac succeeded in his original goal of overcoming the problem of negative probability densities[§], which however required him to interpret the negative energy solutions as antiparticles. There was also an unexpected bonus easily derived from the equation - the fact that it describes spin $\frac{1}{2}$ particles.

The spin of the electron   Perhaps the most beautiful aspect of Dirac's equation (1.12) is that it directly implies a new form of angular momentum in addition to the quantum counterpart of the classical $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

Since the Dirac equation is relativistically covariant, the system it describes has Lorentz symmetry and, in particular, rotational symmetry. For such a system angular momentum is conserved, i.e. its commutator with the Hamiltonian vanishes. For $\mathbf{L}$ to be a constant of motion we thus require $[\mathrm{H}, \mathbf{L}] = 0$, but as it turns out (using $[x_i, p_j] = i\delta_{ij}$)

$$
\begin{aligned}
[\mathbf{L}, \mathrm{H}] &= [\mathbf{r} \times \mathbf{p}, \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m] = [\mathbf{r}, \boldsymbol{\alpha} \cdot \mathbf{p}] \times \mathbf{p} \\
&= \varepsilon_{ijk}[x_i, \alpha_l p_l]p_j\hat{\epsilon}_k = \varepsilon_{ijk}\alpha_l[x_i, p_l]p_j\hat{\epsilon}_k \\
&= \varepsilon_{ijk}\alpha_l i\delta_{il}p_j\hat{\epsilon}_k = i\varepsilon_{ijk}\alpha_i p_j\hat{\epsilon}_k = i\boldsymbol{\alpha} \times \mathbf{p} \neq 0 \,.
\end{aligned}
\tag{1.13}
$$

It is then obvious that to the orbital angular momentum $\mathbf{L}$ one has to add a second, similar, operator whose commutator with the Hamiltonian exactly cancels (1.13). Now we can make an educated guess and define an operator $\boldsymbol{\Sigma} = \left(\begin{smallmatrix} \sigma & 0 \\ 0 & \sigma \end{smallmatrix}\right)$ and calculate $[\mathrm{H}, \mathbf{S}]$ (using $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$). Here $\mathbf{S} = s \cdot \boldsymbol{\Sigma}$, s being a constant indicating the spin value.

$$
\begin{aligned}
[\mathbf{S}, \mathrm{H}] &= s[\boldsymbol{\Sigma}, \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m] = s[\boldsymbol{\Sigma}, \boldsymbol{\alpha} \cdot \mathbf{p}] = s[\sigma_i, \sigma_j p_j]\hat{\epsilon}_l\delta_{il} \\
&= 2si\varepsilon_{ijk}\sigma_k p_j\hat{\epsilon}_i = -2si\boldsymbol{\alpha} \times \mathbf{p}
\end{aligned}
\tag{1.14}
$$

Now in order for the total angular momentum $\mathbf{J} = \mathbf{S} + \mathbf{L}$ to be a constant of motion we see that the spin value has to be $s = \frac{1}{2}$ and hence the Dirac equation describes spin $\frac{1}{2}$ particles, i.e. fermions. The components of $\mathbf{S}$ satisfy $[S_i, S_j] = i\epsilon_{ijk}S_k$ and also $\mathbf{S}^2 = \frac{1}{4}(1 + 1 + 1) = \frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{4}$, which is what we require for an angular momentum operator.

[§]For the actual form and derivation of these probability densities, see e.g. [12].

Chapter 2

The Poincaré group and spin

The most important group in relativistic physics is the Poincaré group $\mathcal{P}$, the set of Lorentz transformations (Λ) and space-time translations (a) such that for a four-vector x^μ

$$
\begin{aligned}
x^\mu \to x'^\mu &= \Lambda^\mu{}_\nu x^\nu + a^\mu \\
x'^\mu \to x''^\mu &= \Lambda'^\mu{}_\rho (\Lambda^\rho{}_\nu x^\nu + a^\mu) + a'^\mu \,.
\end{aligned}
\tag{2.1}
$$

Here the second line displays the group multiplication properties, namely that the Lorentz transformations act on the translations. This implies a semidirect product between the two groups

$$
\mathcal{P} = \text{SO(1,3)} \ltimes \mathcal{T}_4 \,,
\tag{2.2}
$$

where SO(1,3) represents Lorentz transformations and $\mathcal{T}_4$ translations. The generators of translations and Lorentz transformations are P_μ and $M_{\mu\nu}$($J_l = \frac{1}{2}\epsilon_{ljk}M_{jk}$) respectively, with the corresponding group elements (unitary transformations)

$$
\begin{aligned}
U(a) &= e^{ia^\mu P_\mu} \\
U(\alpha) &= e^{i\alpha^{\mu\nu} M_{\mu\nu}} \,,
\end{aligned}
\tag{2.3}
$$

and the Minkowski space realizations

$$
\begin{aligned}
P_\mu f(x) &= i\partial_\mu f(x) \\
M_{\mu\nu} f(x) &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) f(x) \,.
\end{aligned}
\tag{2.4}
$$

These generators satisfy the following commutation relations

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu) \\
[M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha} M_{\nu\beta} - \eta_{\mu\beta} M_{\nu\alpha} - \eta_{\nu\alpha} M_{\mu\beta} + \eta_{\nu\beta} M_{\mu\alpha}) \, .
\end{aligned}
\tag{2.5}
$$

Wigner has presented a classification of all the irreducible representation of the Poincaré group in [13]. These give all the elementary states that obey the respective symmetry. We will now take a closer look at these representations and look for the operators whose eigenvalues label them.

2.1   Representations of the Poincaré group

The *Casimir operators* of a group are polynomials in the generators which commute with all the generators. The most familiar example is the angular momentum operator J^2 from ordinary quantum mechanics, the only Casimir operator for the rotation group SU(2). It clearly commutes with all the rotation generators J_i and the eigenvalues $j(j+1)$ label the states of particles with angular momentum j. The group SU(2) is characterized by the Lie algebra of its generators $[J_i, J_j] = i\epsilon_{ijk} J_k$ and is a fundamental part of the Poincaré group as is easily seen from the generators of the latter.

Let us relabel the components of $M_{\mu\nu}$ as $M_{jk} = J_i$ for the rotation generators and $M_{i0} = K_i$ for the boosts. These satisfy the Lie algebra

$$
[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k, \quad [K_i, J_j] = i\epsilon_{ijk} K_k \, .
$$

Now by further defining the linear combinations $N_i^+ = \frac{1}{2}(J_i + iK_j)$ and $N_i^- = \frac{1}{2}(J_i - iK_j)$ we get

$$
\begin{aligned}
[N_i^+, N_j^+] &= \frac{i}{4} \epsilon_{ijk} (J_k + i[J_i, K_j] + i[K_i, J_j] + J_k) \\
&= \frac{i}{2} \epsilon_{ijk} (J_k + iK_k) = i\epsilon_{ijk} N_k^+ \\
[N_i^-, N_j^-] &= i\epsilon_{ijk} N_k^- \\
[N_i^+, N_j^-] &= 0 \, .
\end{aligned}
\tag{2.6}
$$

We see that the SO(1,3) group is broken into two independent SU(2) groups. More precisely the particular complexification of the Lorentz algebra SO$(1,3)$

is equal to the direct product $SU(2) \times SU(2)$ and thus the representation content of $SU(2)$ is inherited into the Lorentz group and further into the Poincaré group. For this reason the representations of $SU(2)$ are of great importance.

There are 1,2,3,4,5,... -dimensional representations of $SU(2)$ corresponding to $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2....$ The 2-dimensional, *fundamental*, representation of $SU(2)$ is given just by the σ-matrices that describe spin $\frac{1}{2}$ particles, the constituents of all known matter fields. The only standard model particle expected to carry zero spin is the Higgs boson, the particle responsible for giving all other fundamental particles a mass through spontaneous symmetry breaking. Spin 1 particles are the force mediators, the photon, the gluons and the Z and W bosons of weak interactions. The higher dimensional representations correspond to more exotic particles. Spin $\frac{3}{2}$ is expected from some of the superpartners of standard model bosons in theories with supersymmety and spin 2 seems to be preserved for the graviton, the hypothetical mediator of gravity.

For the Poincaré group there are two Casimir operators: $P_\mu P^\mu$ and $W_\mu W^\mu$, where the Pauli-Lubanski vector W^μ is defined by

$$W^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta\nu} M_{\alpha\beta} P_\nu \,. \tag{2.7}$$

For the eigenvalues it is useful to go to the rest frame[*] where $P^\mu = (m, 0, 0, 0)$. The eigenvalue for P^2 is m^2, but the eigenvalues of W^2 are less obvious. $W^\mu = \frac{1}{2}m\epsilon^{\mu\alpha\beta 0}M_{\alpha\beta} = \frac{1}{2}m\epsilon^{\mu jk0}M_{jk}$, which vanishes for $\mu = 0$ and $W^l = -\frac{1}{2}m\epsilon^{ljk}M_{jk} = -mJ^l$ and so $W^2 = m^2 J^2$. Thus the eigenvalues are $m^2 s(s+1)$, where s is the total angular momentum in the rest frame of the particle, its *spin*. To summarize

- P^2 has eigenvalue m^2.
- W^2 has eigenvalue $m^2 s(s+1)$.

Thus all massive quantum mechanical states, corresponding to a representation of the Poincaré group, can be described by their mass and spin[†]. The inherited representation content of $SU(2)$ can be seen as the reason *why* spin is quantized.

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[*]This suffices since P^2 and W^2 are Lorentz invariant.
[†]For an approachable treatment on these classifications, see e.g. [14].

Chapter 3

The spin-statistics theorem

In his original work [3] Pauli proved that the relation between spin and statistics holds provided that the following three requirements are met:*

1. The vacuum is the state of lowest energy. So long as no interaction between particles is considered the energy difference between this state of lowest energy and the state where a finite number of particles is present is finite.
2. Physical quantities (observables) commute with each other in two space-time points with a space-like distance. (Indeed due to the impossibility of signal velocities greater than that of light, measurements at two such points cannot disturb each other.)
3. The metric in the Hilbert-space of the quantum mechanical states is positive definite. This guarantees the positive sign of the values of physical probabilities.

Pauli had earlier shown that Bose(Dirac)-statistics used with half-integer(integer) -spin particles leads to the violation of at least one of these requirements† and thus did not lead to a valid theory according to present day consensus. Pauli's later proof in [3] is rather similar to, if not as clear as, the reasoning presented in section 3.2 and deals with negative energies and

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*Direct quotation from the work of 1950.
†For half-integer spins the 1st requirement is violated and for integer spins the 2nd one.

probabilities greater than one in the framework of relativistic quantum field theory.

The theorem is said to be known by many but fully understood by few. These early proofs on the theorem have been considered unsatisfactory for two reasons. They require relativistic formalism and leave the theorem difficult to understand. The more recent proof presented in the next section only requires rotational invariance, is thus nonrelativistic and so does away with at least one of the problems.

3.1 Rotational invariance and the spin-statistics relation

The connection between spin and statistics is used in many many fields that deal with non-relativistic phenomena, Bose-Einstein condensation, Cooper pairs and phonons to name a few. There has therefore been quite a bit of work done to prove the theorem without resorting to relativistic quantum field theory (section 3.2) in addition to the constant effort of making it easily understandable.

One of the most prominent treatments on the subject has been the work of E.C.G. Sudarshan who has clarified his original proof [15] recently in [16–18]. Sudarshan's proof is close to an earlier proof by Swinger [19] which however makes use of CPT invariance and is thus fully relativistic in nature. Today Schwinger's proof is best considered a proof of the CPT theorem starting from the requirement of the proper spin-statistics relation. Let us now go through Sudarshan's proof in detail.

There are four conditions for the kinematic part of the Lagrangian:

1. The Lagrangian is invariant under SU(2) and corresponds to a theory for fields, ξ, which are each a finite dimensional irreducible representation of SU(2).
2. The Lagrangian is composed of Hermitian fields $\xi = \xi^\dagger$.
3. It is at most linear in the time derivatives of the fields and these are only present in the kinematic part.
4. The kinematic part is bilinear in the field ξ.

The Dirac Lagrangian is in this form and as for bosons we can transform the Klein-Gordon Lagrangian into the Duffin-Kermer form $\mathcal{L} = \bar{\psi}(i\beta_\mu\partial^\mu - m)\psi$, which satisfies the requirements 1-4. The definitions of ψ and β are presented in [18]. We concentrate on the kinematic part since we expect the spin-statistics relation to hold irrespective of the processes present.

The heart of the proof is in the fact that for SU(2) the representations belonging to integral spin have a bilinear scalar product symmetric in the indices of its factors. Take the product of two real vectors

$$(V_1, V_2) = \sum_{j,k=1,2,3} V_{1j} V_{2k} \delta_{jk} \, . \tag{3.1}$$

In contrast, the similar product between half-integral representations is antisymmetric in the indices

$$(\psi_1, \psi_2) = \sum_{r,s=1}^{4} \psi_{1r} \psi_{2s} i\beta_{rs}, \tag{3.2}$$

where $\beta_{rs} = \left(\begin{smallmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{smallmatrix}\right)$, an imaginary *antisymmetric* matrix. Now let us have a look at the kinematic part of the Lagrangian which can generally be written as

$$\mathcal{L}_{kin} = \sum_{r,s} \frac{1}{2} K^0_{rs}(\xi_r \dot{\xi}_s - \dot{\xi}_r \xi_s) = \frac{1}{2} \sum_{r,s} \xi_r \Lambda_{rs} \xi_s, \tag{3.3}$$

where $\Lambda_{rs} = K^0_{rs}(\partial_t^{(r)} - \partial_t^{(s)})$. The requirement that the Lagrangian be SU(2) symmetric requires that the scalar fields appear as scalar products symmetrical in the indices r and s (3.1) and so Λ_{rs} too has to be symmetric in its indices. In contrast, for the spinor fields antisymmetry in r and s is needed due to the antisymmetric products (3.2). Because of the time derivative terms in the definition of Λ_{rs} we see that K^0_{rs} has to have the opposite symmetry.

We now want to go on to show independently that K^0_{rs} has to be antisymmetric for *commuting* fields and symmetric for *anticommuting* fields. This will give the proper spin-statistics relation.

Let us follow Schwinger's approach where it is assumed that the infinitesimal transformation generators are obtained by the variation of the action

$$I = \int_{\sigma_2}^{\sigma_1} d^4x \, \mathcal{L}[x], \tag{3.4}$$

where the σ_i denote the three-dimensional constant time slices. We only consider the variations in the kinematic part (3.3) and vary the field and its derivative

$$
\begin{aligned}
\xi_k &\rightarrow \xi_k + \delta\xi_k \\
\dot{\xi}_k &\rightarrow \dot{\xi}_k + \frac{d(\delta\xi_k)}{dt}.
\end{aligned}
\tag{3.5}
$$

Then the variation will be given by

$$
\begin{aligned}
\delta I &= \frac{1}{2} \int dt\, d^3x \left(\frac{\partial\mathcal{L}}{\partial\xi_r}\delta\xi_r + \frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\delta\dot{\xi}_r \right) \\
&= \frac{1}{2} \int dt\, d^3x \left\{ \frac{\partial\mathcal{L}}{\partial\xi_r} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\right) \right\} \delta\xi_r + \frac{1}{2} \int dt\, d^3x\, \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\delta\xi_r \right) \\
&= \frac{1}{2} \int dt\, d^3x \left\{ \frac{\partial\mathcal{L}}{\partial\xi_r} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\right) \right\} \delta\xi_r + \frac{1}{2} \int_\sigma d^3x\, \frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\delta\xi_r,
\end{aligned}
\tag{3.6}
$$

where on the second line we have used the identity

$$
\frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\delta\dot{\xi}_r = \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_r}\delta\xi_r \right) - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_r} \right)\delta\xi_r.
$$

Setting the first integral to zero in (3.6) will yield the Euler-Lagrange equations for the fields ξ. The generators of infinitesimal transformations are given by the surface integral term which, being integrated over a constant time slice, is *nonrelativistic*. Further, since δI is the generator of infinitesimal transformations of the field quantities ξ attached to the surface σ, we require

$$
[\,\xi_n\,,\,\delta I\,] = i\delta\xi_n\,.
\tag{3.7}
$$

We can now go on to plug the Lagrangian (3.3) into the last term of (3.6) and the variation still further into (3.7) to get

$$
\frac{1}{2} \int_\sigma d^3x \left[\xi_n, \sum_{r,s} K^0_{rs}(\xi_r\delta\xi_s - \delta\xi_r\xi_s) \right] = i\delta\xi_n.
\tag{3.8}
$$

By expanding the commutator we obtain

$$
\frac{1}{2} \int_\sigma d^3x \sum_{r,s} K^0_{rs}(\xi_n\xi_r\delta\xi_s - \xi_n\delta\xi_r\xi_s - \xi_r\delta\xi_s\xi_n + \delta\xi_r\xi_s\xi_n) = i\delta\xi_n.
\tag{3.9}
$$

Now we want to investigate what restrictions commuting and anticommuting fields have on the matrix K^0_{rs}.

Commuting ξ

Let us first consider bosonic fields where the fields ξ commute and thus the variation $\delta\xi$ commutes with everything in (3.9). Then the left-hand side can be written as

$$\frac{1}{2}\int_\sigma d^3x \sum_{r,s} K^0_{rs}\{(\xi_n\xi_r\delta\xi_s - \xi_r\xi_n\delta\xi_s) - (\xi_n\xi_s\delta\xi_r - \xi_s\xi_n\delta\xi_r)\}. \qquad (3.10)$$

Inserting explicit space coordinates this can be written as

$$\frac{1}{2}\int_\sigma d^3x \sum_{r,s} K^0_{rs}\{[\xi_n(\mathbf{y}), \xi_r(\mathbf{x})]\delta\xi_s(\mathbf{x}) - [\xi_n(\mathbf{y}), \xi_s(\mathbf{x})]\delta\xi_r(\mathbf{x})\}. \qquad (3.11)$$

By switching the indices r and s in the second term we can write (3.11) as

$$[\xi_n, \delta I_{\delta\xi}] = \int_\sigma d^3x \sum_s \delta\xi_s(\mathbf{x}) \left[\xi_n(\mathbf{y}), \frac{1}{2}\sum_r (K^0_{rs} - K^0_{sr})\xi_r(\mathbf{x})\right]. \qquad (3.12)$$

Now this is a result as beautiful as they get. We see explicitly that for the action to be a generator of transformations K^0_{rs} cannot be symmetric and thus has to be *antisymmetric*. The possibility of it being anything else is ruled out by the requirement of SU(2) symmetry. This being consistent with the earlier derivation using scalar products we can conclude that *fields with integral spin have to be commuting, i.e. bosonic, fields.*

Anticommuting ξ

If we now take anticommutation rather than commutation relations for $\delta\xi$ in (3.9) we get analogously to (3.10)

$$\frac{1}{2}\int_\sigma d^3x \sum_{r,s} K^0_{rs}\{(\xi_n\xi_r\delta\xi_s + \xi_r\xi_n\delta\xi_s) + (\xi_n\xi_s\delta\xi_r + \xi_s\xi_n\delta\xi_r)\}, \qquad (3.13)$$

which is then simplified to give

$$[\xi_n, \delta I_{\delta\xi}] = \int_\sigma d^3x \sum_s \delta\xi_s(\mathbf{x}) \left\{\xi_n(\mathbf{y}), \frac{1}{2}\sum_r (K^0_{rs} + K^0_{sr})\xi_r(\mathbf{x})\right\}. \qquad (3.14)$$

The result is almost identical. We see that for this to be consistent with (3.7) K^0_{rs} cannot be antisymmetric and is thus *symmetric*. From this we conclude that *fields with half-integral spin have to be anticommuting, i.e. fermionic, fields.* This completes the proof of the spin-statistics theorem.

3.2 Commuting and anticommuting fields

To highlight the importance of the spin-statistics relation it is useful to have a look at the quantization of scalar and Dirac fields in relativistic quantum field theory[‡]. After all, Pauli himself has said that the requirement of positive energies is the best *a priori* argument in favor of his exclusion principle and here the connection to statistics is most obvious.

Quantization of the scalar field

We start from the Lagrangian density for the classical free scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x), \tag{3.15}$$

with the conjugated momentum field $\pi(t, \mathbf{x}) = \frac{\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi(t, \mathbf{x})$ and the mode expansion:

$$\phi(x) = \int d\mu(\mathbf{p})\Big[a(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{ip\cdot x}\Big]. \tag{3.16}$$

where $d\mu(\mathbf{p}) = \frac{d\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}$ and $p_0 = E_p = \sqrt{\mathbf{p}^2 + m^2}$. Inspired by the well known commutation relations of quantum mechanics

$$[\hat{x}_i, \hat{x}_j] = 0 \qquad [\hat{p}_i, \hat{p}_j] = 0 \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \tag{3.17}$$

we postulate[§] the *equal-time* commutation relations for the conjugate fields

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{3.18}$$
$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0.$$

Using the expressions for the ϕ and π fields and the commutation relations (3.18) one can derive for the creation and annihilation operators

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p} - \mathbf{p}') \tag{3.19}$$
$$[a(\mathbf{p}), a(\mathbf{p}')] = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0.$$

[‡]For an introduction see, for example, the excellent books by Peskin and Schroeder [20] and Lahiri and Pal [21].

[§]This is clearly justified by the previous section but let us consider this as an independent treatment.

We then calculate the total *normal ordered* Hamiltonian

$$
H = \int d^3\mathbf{x} : \mathcal{H} := \int d^3\mathbf{x} : [\pi\partial_t\phi - \mathcal{L}] :
$$
$$
= \int d^3\mathbf{x} : \frac{1}{2}[\pi^2 + \partial_i\phi\partial_i\phi + \phi^2] : \tag{3.20}
$$

The Hamiltonian can be further put in the form

$$
H = \int d^3\mathbf{p} \, E_p \frac{1}{2} : [a^\dagger(\mathbf{p})a(\mathbf{p}) + a(\mathbf{p})a^\dagger(\mathbf{p})] :
$$
$$
= \int d^3\mathbf{p} \, E_p \, a^\dagger(\mathbf{p})a(\mathbf{p}) = \int d^3\mathbf{p} \, E_p \, n(\mathbf{p}) \,. \tag{3.21}
$$

On the last line the Hamiltonian is expressed in terms of the number density operator $n(\mathbf{p})$ with the number operator defined as

$$
\hat{N} = \int d^3\mathbf{p} \, n(\mathbf{p}) = \int d^3\mathbf{p} \, a^\dagger(\mathbf{p})a(\mathbf{p}) \tag{3.22}
$$

The number operator, when acting on the multi-particle states

$$
|n\rangle = a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)\cdots a^\dagger(\mathbf{p}_n)|0\rangle, \tag{3.23}
$$

has the desired properties

$$
\hat{N}|0\rangle = 0
$$
$$
\hat{N}|n\rangle = n|n\rangle \tag{3.24}
$$

and we see clearly that the eigenvalues of $\hat{N}$ are not restricted.

Quantization of the Dirac field

Starting from the Dirac Lagrangian

$$
\mathcal{L} = \bar{\psi}(x)(i\slashed{\partial} - m)\psi(x) \,, \tag{3.25}
$$

we get the Hamiltonian density

$$
\mathcal{H} = \bar{\psi}(x)[-i\boldsymbol{\nabla} \cdot \boldsymbol{\gamma} + m]\psi(x) \,, \tag{3.26}
$$

where ψ is the Dirac field with the mode expansion:

$$\psi(x) = \int d\mu(\mathbf{p}) \sum_s \left[c_{\mathbf{p},s} u(\mathbf{p},\mathbf{s}) e^{-ip\cdot x} + d^\dagger_{\mathbf{p},s} v(\mathbf{p},\mathbf{s}) e^{ip\cdot x} \right]. \qquad (3.27)$$

Here the Dirac *spinors* u and v satisfy the equations

$$\displaylines{ \slashed{p} u(\mathbf{p},\mathbf{s}) = (\gamma^0 p_0 - \boldsymbol{\gamma}\mathbf{p}) u(\mathbf{p},\mathbf{s}) = m u(\mathbf{p},\mathbf{s}), \qquad (3.28) \\ \slashed{p} v(\mathbf{p},\mathbf{s}) = (\gamma^0 p_0 - \boldsymbol{\gamma}\mathbf{p}) v(\mathbf{p},\mathbf{s}) = -m v(\mathbf{p},\mathbf{s}). \qquad (3.29) }$$

i.e. the Dirac equation in momentum space. We first note how $[-\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m]$ acts on the spinors by using (3.28)

$$\begin{aligned} [-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m] u(\mathbf{p},\mathbf{s}) e^{-ip\cdot x} &= [\boldsymbol{\gamma}\cdot\mathbf{p} + m] u(\mathbf{p},\mathbf{s}) e^{-iE_p t + i\mathbf{p}\cdot\mathbf{x}} \\ &= \gamma^0 E_p u(\mathbf{p},\mathbf{s}) e^{-ip\cdot x}, \\ [-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m] v(\mathbf{p},\mathbf{s}) e^{ip\cdot x} &= [-\boldsymbol{\gamma}\cdot\mathbf{p} + m] v(\mathbf{p},\mathbf{s}) e^{iE_p t - i\mathbf{p}\cdot\mathbf{x}} \\ &= -\gamma^0 E_p v(\mathbf{p},\mathbf{s}) e^{ip\cdot x}. \qquad (3.30) \end{aligned}$$

We then use these results to calculate the normal-ordered total Hamiltonian

$$H = \int :\mathcal{H}: d^3\mathbf{x} \qquad (3.31)$$

Noting that the Dirac spinors are normalized to

$$\begin{aligned} u^\dagger(\mathbf{p},\mathbf{s}) u(\mathbf{p},\mathbf{s}') &= v^\dagger(\mathbf{p},\mathbf{s}) v(\mathbf{p},\mathbf{s}') = 2E_p \delta_{s,s'} \\ v^\dagger(\mathbf{p},\mathbf{s}) u(\mathbf{p},\mathbf{s}') &= u^\dagger(\mathbf{p},\mathbf{s}) v(\mathbf{p},\mathbf{s}') = 0, \end{aligned}$$

we get:

$$H = \int d^3\mathbf{x} \int d\mu(\mathbf{p}) d\mu(\mathbf{q}) \sum_{ss'} : \left[c^\dagger_{\mathbf{p},s} c_{\mathbf{q},s'} u^\dagger(\mathbf{p},\mathbf{s}) u(\mathbf{q},\mathbf{s}') E_q e^{ix(p-q)} + \right.$$
$$\left. d_{\mathbf{p},s} d^\dagger_{\mathbf{q},s'} v^\dagger(\mathbf{p},\mathbf{s}) v(\mathbf{q},\mathbf{s}') E_q e^{ix(q-p)} \right] : \quad (3.32)$$

By remembering the very useful form for the Dirac delta-function: $\delta(\mathbf{x}-\mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}$ and integrating over $\mathbf{x}$ we get

$$H = \int d^3\mathbf{p}\, E_p \sum_s : [c^\dagger_{\mathbf{p},s} c_{\mathbf{p},s} - d_{\mathbf{p},s} d^\dagger_{\mathbf{p},s}] : \qquad (3.33)$$

The next step is to do the normal-ordering and interpret the result. Now this is where we encounter something interesting and deeply physical. The

first guess here would be to use commutation relations of the creation and annihilation operators as with the scalar field to get

$$H = \int d^3\mathbf{p}\, E_p \sum_s [c^\dagger_{\mathbf{p},s} c_{\mathbf{p},s} - d^\dagger_{\mathbf{p},s} d_{\mathbf{p},s}]. \tag{3.34}$$

But now with the second term we can create as many particles as we like and thus lower the energy of the system indefinitely. There is however another way of quantizing the theory. If at the very beginning, instead of using commutation relations for the fields, were we to postulate *anticommutation* relations

$$\{\psi_\alpha(\mathbf{x}), \pi_\beta(\mathbf{y})\} = i\delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y})\,, \tag{3.35}$$
$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \{\pi_\alpha(\mathbf{x}), \pi_\beta(\mathbf{y})\} = 0,$$

we would get (analogously to the scalar case)

$$\{c_s(t, \mathbf{p}), c^\dagger_{s'}(t, \mathbf{p}')\} = \{d_s(t, \mathbf{p}), d^\dagger_{s'}(t, \mathbf{p}')\} = \delta_{ss'}\delta(\mathbf{p} - \mathbf{p}') \tag{3.36}$$

with all other anticommutators zero. Now carrying out the normal-ordering we get an appealing result

$$H = \int d^3\mathbf{p}\, E_p \sum_s [c^\dagger_{\mathbf{p},s} c_{\mathbf{p},s} + d^\dagger_{\mathbf{p},s} d_{\mathbf{p},s}]. \tag{3.37}$$

In terms of the number operators defined analogously to the scalar case $\hat{N}_c(s) = \int d^3\mathbf{p}\, n_{c,s}(\mathbf{p}) = \int d^3\mathbf{p}\, c^\dagger_{\mathbf{p},s} c_{\mathbf{p},s}$

$$H = \int d^3\mathbf{p}\, E_p \sum_s \left(n_{c,s}(\mathbf{p}) + n_{d,s}(\mathbf{p})\right). \tag{3.38}$$

To see that $\hat{N}_c(s)$ really is a number operator we note that

$$[\hat{N}_c(s), c_{\mathbf{p},s}] = -c_{\mathbf{p},s}\,, \qquad [\hat{N}^\dagger_c(s), c_{\mathbf{p},s}] = c^\dagger_{\mathbf{p},s}\,,$$

which gives directly the desired properties for creation and annihilation respectively:

$$\hat{N}_c(s) c^\dagger_{\mathbf{p},s}|n_k\rangle = (n_k + 1)c^\dagger_{\mathbf{p},s}|n_k\rangle\,,$$
$$\hat{N}_c(s) c_{\mathbf{p},s}|n_k\rangle = (n_k - 1)c_{\mathbf{p},s}|n_k\rangle\,.$$

Our new number operator has very different properties compared to the scalar equivalent (3.22). Operating on the ground state obviously still gives zero but from the anticommutation relations (3.36) we see directly that $c_s(\mathbf{p})c_s(\mathbf{p}) = d_s(\mathbf{p})d_s(\mathbf{p}) = 0$ and thus

$$\hat{N}|\mathbf{p}, s; \mathbf{p}, s\rangle = \int d^3\mathbf{p}' c^\dagger_{\mathbf{p}',s} c_{\mathbf{p}',s} \underbrace{c^\dagger_{\mathbf{p},s} c^\dagger_{\mathbf{p},s}}_{0} |0\rangle = 0|\mathbf{p}, s; \mathbf{p}, s\rangle \qquad (3.39)$$

and similarly for any state with a higher number of indistinguishable particles sharing the same quantum numbers.

Since the corresponding eigenvalues of the number operator are 0 and 1, we conclude that the states satisfy the Pauli exclusion principle. In other words, the system obeys Fermi-Dirac statistics (1.3). In the scalar case the eigenvalues of (3.22) are not restricted. The commutation relations used thus imply Bose-Einstein statistics (1.2).

Chapter 4

Noncommutative quantum field theory and twists

Perhaps the most famous appearance of noncommutativity in physical theories is the relation between position and momentum in ordinary quantum mechanics (3.17). This traditional picture is based on the assumption that phase space can be represented as a smooth manifold, which is of course true at relatively large scales or low energies.

On distances of the order of Planck scale ($\lambda_P \approx 1.6 \times 10^{-33} cm$) this picture is expected to break down due to the very high energy uncertainty involved [22]. This conclusion is intuitively clear when one considers the Heisenberg uncertainty relation $\Delta x \Delta p \geq \frac{1}{2}$ together with the general relativity result of black hole formation. When considering small enough length scales (corresponding to high energy) the energy uncertainty enables black hole formation and the smooth manifold structure is lost. As a result the notion of a point becomes meaningless and the simple commutation relation between space-time points is no longer expected to hold. This was first suggested long ago by Snyder (1947) [23] and Heisenberg (1954) [24]. They hoped that the UV-behaviour of quantum field theories could be improved by making $[x_i, x_j] \neq 0$.

4.1 Noncommutativity of space-time[*]

The form of noncommutativity of coordinates studied in this work can be written in the Heisenberg-like form:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{4.1}$$

where $\theta_{\mu\nu}$ is a *constant* real antisymmetric matrix. This particular form is motivated by string theory [27] where it was shown to appear as a low energy limit in string theories with a constant background B-field. This gives a justification for this particular type of noncommutativity and has led to intense study in this field (for reviews, see [28, 29]). From the relation (4.1) we can already extract two key problems to be faced with noncommuting coordinates: nonlocality and the violation of Lorentz invariance.

Due to (4.1) the coordinate operators cannot be simultaneously diagonalized, i.e. are no longer *independent*. Because of the induced uncertainty relation $\Delta x_i \Delta x_j \geq \frac{1}{2}|\theta_{ij}|$ we are forced to replace our notion of a point with that of a cell of dimension given by $|\theta_{ij}|$. This leads to the appearance of a UV-cutoff and seems to have been Snyder's [27] original motivation. Nonlocality induces serious practical and conceptual problems that are not very well understood and will not be further discussed in this work.

The Lorentz noninvariance is obvious since, in a Lorentz transformation, the left-hand side of (4.1) transforms as a tensor while the right-hand side stays constant. Among other things this poses a threat to causality. Since the normal light cone structure is no longer valid, the *microcausality condition* (the 2nd condition in Pauli's theorem) seems to be in doubt. There is a twist to this story however as it turns out that our noncommutative space-time follows another, hidden, symmetry. The twisted Poincaré symmetry introduced in section 4.3 repairs a large part of the damage induced by the loss of Lorentz invariance. The important topic of microcausality will be further discussed in chapter 5.

Let us have a closer look at the properties of our new space-time. In four dimensions there always exists a frame where the θ-matrix can be put in a

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[*]For an exhaustive treatment of noncommutative spaces, see [25, 26].

block-diagonal form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & -\theta' & 0 \end{pmatrix} \tag{4.2}$$

Here, again, we run into a major difficulty. Since the original Lorentz group is broken into $SO(1,1) \times SO(2)$, both of which are Abelian groups, we are left with only one-dimensional irreducible representations. Thus there is a serious problem in dealing with spinors, vectors and other higher dimensional representations and this of course contradicts much of what was said in previous chapters. The solution to this apparent problem is given by the twisted Poincaré symmetry discussed in section 4.3.

From the form (4.2) it is useful to classify different types of noncommutativity to clarify the causal structure:

- Space-space $\theta^2 = 0$
- Lightlike $\theta^{\mu\nu}\theta_{\mu\nu} = \theta^2 - \theta'^2 = 0$
- Time-space $\theta'^2 = 0$

4.2 Weyl-Moyal correspondence and the $\star$-product

The treatment of noncommutative (NC) space-times in quantum field theories (QFTs) is largely based on the method devised by Weyl [30] where each quantum operator is associated with a classical function of phase space variables.

To implement Weyl quantization we assume that each function can be represented by its Fourier transform

$$\tilde{f}(k) = \int d^D x e^{-ik_\mu x^\mu} f(x), \tag{4.3}$$

where $\tilde{f}(-k) = \tilde{f}(k)^*$ when $f(x)$ is real valued. We can now introduce the noncommutativity of space-time by replacing the coordinates x^μ by the operators $\hat{x}^\mu$ that satisfy the commutation relation (4.1). Given a function

$f(x)$ and its Fourier coefficients (4.3) we define its *Weyl symbol* by

$$\widehat{W}[f] = \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k) e^{ik_\mu \widehat{x}^\mu}, \tag{4.4}$$

with the symmetric Weyl operator ordering prescription chosen, i.e. $\widehat{W}[e^{ik_\mu x^\mu}] = e^{ik_\mu \widehat{x}^\mu}$. The operator $\widehat{W}[f]$ is hermitian when its *Weyl symbol* $f(x)$ is real valued. By inserting (4.3) we can rewrite (4.3) as

$$\widehat{W}[f] = \int d^D x f(x) \widehat{\Delta}(x), \tag{4.5}$$

$$\widehat{\Delta}(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu \widehat{x}^\mu} e^{-ik_\nu x^\nu}. \tag{4.6}$$

Using the Baker-Campbell-Hausdorff formula

$$e^{ik_\mu \widehat{x}^\mu} e^{ik'_\nu \widehat{x}^\nu} = e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{i(k+k')_\mu \widehat{x}^\mu}, \tag{4.7}$$

together with relation (4.5), the products of the operators $\widehat{\Delta}(x)$ can be calculated as

$$\begin{aligned} \widehat{\Delta}(x)\widehat{\Delta}(y) &= \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} e^{i(k+k')_\mu \widehat{x}^\mu} e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{-ik_\mu x^\mu - ik'_\nu x^\nu} \\ &= \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \int d^D z\, e^{i(k+k')_\mu \widehat{z}^\mu} \widehat{\Delta}(z) e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{-ik_\mu x^\mu - ik'_\nu x^\nu}, \end{aligned} \tag{4.8}$$

where $\widehat{W}[e^{i(k+k')_\mu z^\mu}] = e^{i(k+k')_\mu \widehat{z}^\mu}$ has been inserted.

If θ is invertible, the integrations in (4.9) over the momenta k and k' can be worked out to get

$$\widehat{\Delta}(x)\widehat{\Delta}(y) = \frac{1}{\pi^D |\det \theta|} \int d^D z\, \widehat{\Delta}(z) e^{-2i(\theta^{-1})_{\mu\nu}(x-z)^\mu (y-z)^\nu}. \tag{4.9}$$

The ⋆-product

The product of two Weyl symbols is defined as

$$\widehat{W}[f]\widehat{W}[g] = \widehat{W}[f \star g], \tag{4.10}$$

where we have introduced the $\star$-product

$$
\begin{aligned}
(f \star g)(x) &= \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \tilde{f}(k)\tilde{g}(k'-k) e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu} e^{ik'_\sigma x^\sigma} \\
&= f(x) e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g(x) \qquad\qquad\qquad (4.11) \\
&= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} f(x) \partial_{\nu_1} \cdots \partial_{\nu_n} g(x).
\end{aligned}
$$

This *Moyal-product*, or $\star$-product, (4.12) is associative and noncommutative and is only defined in the exponential form for a constant θ. It clearly reduces to the ordinary product of functions in the limit $\theta^{\mu\nu} \to 0$, which is what we expect. The commutator of functions is now naturally defined by the *Moyal bracket*

$$
[f(x), g(y)]_\star = f(x) \star g(y) - g(x) \star f(y) \qquad\qquad (4.12)
$$

and with this the commutator (4.1) is written as

$$
[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}. \qquad\qquad (4.13)
$$

4.3   Twisted Poincaré symmetry

The earliest works done on quantum field theories on noncommutative space-times disregarded the problem of the inadequate representation content apparent from (4.2). Instead all discussion on fundamental issues such as unitarity [31] and causality [32] was done with the usual representation content of the Poincaré algebra.

The breakthrough on this sector came when Chaichian et al. [33, 34] used a quantum group theoretical approach to tackle the problem[†]. They introduce a twist deformation on the universal enveloping $\mathcal{U}(\mathcal{P})$ of the usual Poincaré algebra $\mathcal{P}$ such that the noncommutative theory respects this new twisted symmetry.

The twist element $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes U(\mathcal{P})$ does not alter the multiplication properties in $\mathcal{U}(\mathcal{P})$ and so preserves the commutation relations among the generators P_μ and $M_{\mu\nu}$ (2.5). The essential implication of this is that *the representation content of the new theory is identical to that of the usual Poincaré*

[†]An introduction to the theory of quantum groups is given in [35–37].

algebra and is thus the justification needed for all the previous work done in the field. The price we need to pay is a change in the action of the Poincaré generators in the tensor product of representations, the *coproduct*, given in the standard case by

$$
\begin{aligned}
&\Delta_0 : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \\
&\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y, \qquad \forall Y \in \mathcal{P}.
\end{aligned}
\tag{4.14}
$$

The twist element changes this coproduct into the *twisted* coproduct

$$
\Delta_0(Y) \longmapsto \Delta_t(Y) = \mathcal{F} \Delta_0(Y) \mathcal{F}^{-1},
\tag{4.15}
$$

where the twist element $\mathcal{F}$ has to satisfy the twist equation

$$
(\mathcal{F} \otimes 1)(\Delta_0 \otimes \mathrm{id})\mathcal{F} = (1 \otimes \mathcal{F})(\mathrm{id} \otimes \Delta_0)\mathcal{F}.
\tag{4.16}
$$

Equation (4.16) is clearly satisfied if we take an *abelian* twist element written as

$$
\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu}.
\tag{4.17}
$$

Let us now consider the *commutative* algebra $\mathcal{A}$ (consistent with the coproduct Δ_0) of functions, $f(x)$, $g(x)$,..., in Minkowski space. The Poincaré algebra acts on the coordinates x_μ with commutative multiplication:

$$
m(f(x) \otimes g(x)) := f(x)g(x).
\tag{4.18}
$$

In $\mathcal{A}$ we have the representations of $\mathcal{U}(\mathcal{P})$ generated by the standard representations of the Poincaré algebra (2.4), acting on coordinates as follows:

$$
\begin{aligned}
&P_\mu x_\rho = i\partial_\mu x_\rho = i\eta_{\mu\rho} \\
&M_{\mu\nu} x_\rho = i(x_\mu \partial_\nu - x_\nu \partial_\mu) x_\rho = i(x_\mu \eta_{\nu\rho} - x_\nu \eta_{\mu\rho}).
\end{aligned}
\tag{4.19}
$$

When twisting $\mathcal{U}(\mathcal{P})$, one has to redefine the multiplication while retaining the action of the generators of the Poincaré algebra on the coordinates as in (4.19):

$$
\begin{aligned}
m_t(f(x) \otimes g(x)) = f(x) \star g(x) &= m \circ \left(e^{-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} f(x) \otimes g(x) \right) \\
&= m \circ \left(e^{\frac{i}{2}\theta^{\mu\nu} \partial_\mu \partial_\nu} f(x) \otimes g(x) \right).
\end{aligned}
\tag{4.20}
$$

Specifically for the coordinates we get

$$m_t(x_\mu \otimes x_\nu) = x_\mu \star x_\nu = m \circ e^{-\frac{i}{2}\theta^{\alpha\beta}P_\alpha \otimes P_\beta}(x_\mu \otimes x_\nu)$$

$$= m \circ [x_\mu \otimes x_\nu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\mu} \otimes \eta_{\beta\nu}]$$

$$= x_\mu x_\nu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\mu}\eta_{\beta\nu} ,$$

$$m_t(x_\nu \otimes x_\mu) = x_\nu \star x_\mu = x_\nu x_\mu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\nu}\eta_{\beta\mu} .$$

and so

$$[x_\mu, x_\nu]_\star = \frac{i}{2}\theta^{\alpha\beta}(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}) = i\theta_{\mu\nu} , \tag{4.21}$$

which is the normal Moyal bracket (4.13).

The equivalence of (4.13) and (4.21) implies directly that deforming the multiplication of functions (as presented here and in [33]) is equivalent to the noncommutative field theory constructed by the Weyl-Moyal correspondence discussed in the previous section. The obvious advantage to us here is the representation content which is identical to the usual Poincaré algebra by construction. Thus the standard way of classifying particles according to their mass and spin familiar from chapter 2 is carried on to noncommutative theories of the form (4.1).

In addition this approach uncovers a hidden symmetry of the space-time, a new form of relativistic invariance. In commutative theories relativistic invariance means symmetry under Poincaré transformations whereas in the noncommutative case symmetry under the twisted Poincaré transformations is needed. The discovery of this new symmetry has led to intense study in the field and many of its properties still remain mysteries.

Chapter 5

Spin-statistics in NC QFT

The latest chapter in the success story of Pauli's theorem has been written in the noncommutative regime. With noncommuting coordinates it is not obvious that the microcausality condition is satisfied, i.e. that at spacelike separations the commutators of any two observables vanish and so the preservation of the spin-statistics relation is held to question. This problem was first addressed in the Lagrangian formalism [38] and further studied in [39].

In the Lagrangian approach we need to pay attention to which kind of noncommutativity we are considering. Since time-space noncommutativity ($\theta_{0i} \neq 0$) induces violation of unitarity [31] and causality [32] and therefore leads to ill-behaving theories, they are not considered here. It further turns out that this approach leaves the question open in the case of lightlike noncommutativity, $\theta^{\mu\nu}\theta_{\mu\nu} = 0$. This problem has been a standing one until very recently [40] and is dealt with in section 5.2.

5.1   Microcausality

The spin-statistics relation in theories with $\theta_{0i} = 0$ was first studied in [38]*. Indeed the low-energy limit of string theory can only be found for these theories. In [38] it is shown that expectation values of equal-time commutation

---

*For an axiomatic approach, see [41].

relations with respect to two-particle states will vanish:

$$\langle 0 | [: \phi(x) \star \phi(x) : , \ : \phi(y) \star \phi(y) :] \Big|_{x_0 = y_0} | p, p' \rangle$$

$$= -\frac{2i}{(2\pi)^{2d}} \frac{1}{\sqrt{\omega_p \omega_{p'}}} (e^{-ip'x - ipy} + e^{-ipx - ip'y}) \times$$

$$\int \frac{d^3 k}{\omega_k} \sin\left(\vec{k}(\vec{x} - \vec{y})\right) \cos\left(\frac{1}{2} \theta^{\mu\nu} k_\mu p_\nu\right) \cos\left(\frac{1}{2} \theta^{\mu\nu} k_\mu p'_\nu\right), \qquad (5.1)$$

where the right-hand side of (5.1) is nonzero only when $\theta_{0i} \neq 0$. The conclusion is that in theories with space-space noncommutativity the spin-statistics relation is preserved. I only quote their result here since there has been some more recent work done [42] that seems to challenge the results of this simple approach.

In effect, the results of [42] reproduce the results of the initial work [38] but the interpretation differs: if in [38] spacelike separations consistent with the light wedge (explained below) were considered sufficient to guarantee microcausality, in [42] it was considered that only by taking all usual (in the light cone sense) spacelike separations could the violation of microcausality be avoided. Since the results of both papers correctly indicate a light wedge microcausality condition, the conclusion of [42] was that microcausality is violated in NC QFTs. However, this analysis missed the obvious fact of Lorentz noninvariance and indeed the light wedge microcausality condition should be regarded as a light wedge *locality* condition. The causality is not fundamentally affected by the nonlocality, i.e. the cause does not happen after the effect, as was shown already long ago in [32]. The question of microcausality is an important one and at the time of writing remains open and deserves further study.

The light wedge and twisted Poincaré symmetry

Let us take a closer look at the causal structure of the theory. In theories with commutative coordinates causality is implemented by demanding that the commutators of observables vanish outside their light cones, i.e. for $(x^0 + y^0)^2 - (\mathbf{x} + \mathbf{y})^2 < 0$. Due to the breaking of Lorentz invariance, the light cone structure of the O(1,3) group is replaced by that of the O(1,1) group, namely a *light wedge*, i.e. $(x^0 + y^0)^2 - (x^1 + y^1)^2 < 0$ (where we have considered $\theta_{23} = -\theta_{32} = \theta$, with all other components being zero). The idea was presented in [43] and further studied in [44].
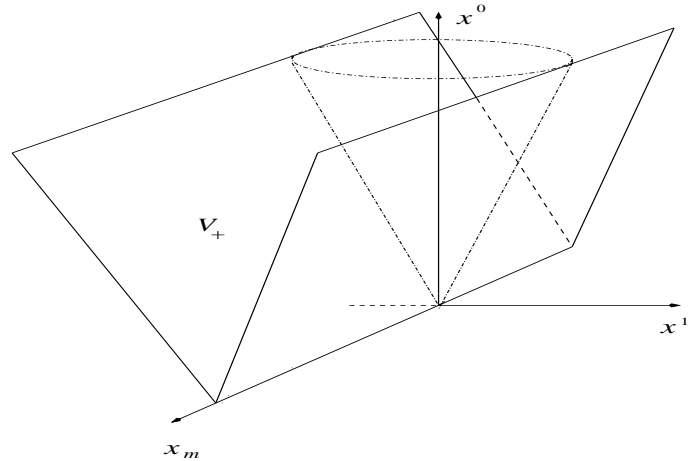
Figure 5.1: The light wedge. Adapted from [43].

This picture clarifies our problem. In the traditional light cone structure it can be argued that no spacelike vector can be Lorentz transformed into a timelike one and thus causality is preserved. Now the question is how to correctly combine twisted Poincaré transformations with the light wedge picture since in noncommutative theories Lorentz invariance is broken and only transformations respecting twisted Poincaré symmetry have any meaning of invariance. It is an interesting question whether it is possible to transform a vector outside the light wedge inside it in this framework. If this can be done the microcausality condition would seem to break down and the study of noncommutative theories would take an unexpected turn.

Once it is settled that NC QFTs with space-space noncommutativity, although nonlocal, do not violate causality, it still remains to show that the light wedge locality (microcausality) condition is compatible with the twisted Poincaré symmetry. It is well-known, as shown in [33], that under the action of *infinitesimal* twisted Poincaré transformations it is the usual space-time interval $x^2 = x_\mu x^\mu$ which is invariant and not the interval in the commutative directions, $\tilde{x}^2 = x_0 x^0 - x_1 x^1$, as the light wedge locality condition would require. However, the question here is related to *finite* twisted Poincaré transformations, which are not related in the usual, trivial way (by simple addition) to the infinitesimal ones.

The behavior of finite twisted Poincaré transformations has been studied in [45] and more recently in [46]. The commutation relations for finite

transformations have interesting properties compared to traditional theories where the parameters of transformations commute. By requiring that in a Poincaré transformation $x_\mu \to x'_\mu = \Lambda^\nu_{\ \mu} x_\nu + a_\mu$ the commutator $[x'_\mu, x'_\nu] = i\Lambda^\rho_{\ \mu} \Lambda^\delta_{\ \nu} \theta_{\rho\delta} + [a_\mu, a_\nu] = i\theta_{\mu\nu}$ remains invariant we get

$$[a_\mu, a_\nu] = i\theta_{\mu\nu} - i\Lambda^\rho_{\ \mu} \Lambda^\delta_{\ \nu} \theta_{\rho\delta} \,, \tag{5.2}$$
$$[\Lambda^\rho_{\ \mu}, \Lambda^\delta_{\ \nu}] = [\Lambda^\rho_{\ \mu}, a_\rho] = 0 \,.$$

Let us now have a look at the transformation properties of an interval with $(x^0 + y^0)^2 - (x^1 + y^1)^2 < 0$. It is easy to verify from (5.2) that only for a rotation in the noncommuting coordinates does the commutator $[a_2, a_3]$ remain zero whereas for a boost or a rotation between a noncommutative direction and the commutative one give $[a_2, a_3] = i\theta_{23}$ ($[a_0, a_i] = [a_1, a_i] = 0$ always). This suggests that these Lorentz transformations *require* translations, a peculiar fact further to be studied in [47]. Remark that these translations are not imposed from the outside, as in the case of usual Poincaré transformations, but they appear naturally when performing a finite twisted Lorentz transformation. Thus the actual value of these accompanying translations is arbitrary, the only thing known about them being their commutation relations. If we now look e.g. at the a boost Λ in the commutative direction

$$x_\mu \to x'_\mu = \Lambda^\nu_{\ \mu} x_\nu + a_\mu \,,$$
$$y_\mu \to y'_\mu = \Lambda^\nu_{\ \mu} y_\nu + b_\mu \,, \tag{5.3}$$

with $[a_2, a_3] = [b_2, b_3] = i\theta_{23}$. The noncommutative coordinates are transformed as $(x_2 - y_2) \to (x'_2 - y'_2) = (x_2 - y_2) + (a_2 - b_2)$ and with $(a_2 - b_2)$ being arbitrary we can set it to $(a_2 - b_2) = -(x_2 - y_2)$ thus doing away with the noncommuting coordinates and arriving precisely at the light wedge locality condition.

We have thus sketched the proof of the fact that it is indeed the light wedge locality condition which is compatible with finite twisted Poincaré transformations. The failure to appreciate this aspect has led to many erroneous statements such as the spin-statistics relation violation [48], the removal of UV/-IR mixing [49], or even the pure equivalence of commutative and noncommutative QFTs [50].

Although the requirement of microcausality is an important one it is in no way vital in the frame of this work. Indeed the nonrelativistic proof in section 3.1 did not require microcausality and Pauli himself has considered it as superfluous [51] for his proof in his later years. In the next section an

example is presented that makes more elegant use of the $\star$-product formalism and is enough to decide the faith of the spin-statistics relation in NC QFTs.

5.2 Quantum groups and the spin-statistics relation

It seems the final word in the defence for the spin-statistics relation in NC QFTs has come in the form of braided groups [40]. In contrast to the Lagrangian formulation we need not specify the form of noncommutativity and thus there is nothing special with the lightlike case.

The twisted Poincaré algebra is a *quantum group*. Quantum groups are not strongly correlated to any kind of statistics, i.e bosonic and fermionic commutation rules. However, they stem from the Yang-Baxter equation that also gives rise to the so-called *braided groups*. This leads to the fact that every quantum group which has a universal $\mathcal{R}$-*matrix*, such as the twisted Poincaré algebra, has a braided group analog. Braided groups in general have a deformed permutation rule, i.e. deformed statistics. Let us now see what sort of braiding, if any, the twisted Poincaré algebra possesses.

The $\mathcal{R}$-matrix relates, by a similarity transformation, the coproduct Δ_t to its opposite $\Delta_t^{op} = \sigma \circ \Delta_t$, where σ is the usual permutation operator of factors in the tensor product:

$$\mathcal{R}\Delta_t = \Delta_t^{op}\mathcal{R}, \quad \mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{H} \otimes \mathcal{H}. \tag{5.4}$$

The Hopf algebra $\mathcal{H}$ in which Δ_t and Δ_t^{op} are related by such an invertible $\mathcal{R}$-matrix is called a quasi-triangular Hopf algebra.

The braided permutation, or *braiding* of V and W, two (co)representation spaces of the quasi-triangular Hopf algebra $\mathcal{H}$ is given by

$$\Psi_{V,W}(v \otimes w) = P(\mathcal{R} \triangleright (v \otimes w)), \tag{5.5}$$

where $\triangleright$ is the action of $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$, with its first factor acting on V and the second factor acting on W, followed by the usual vector-space permutation P. This form of braiding is achieved by requiring that $\Psi_{V,W}$ be an intertwiner of representations. If we consider the action of an element of a quasi-triangular Hopf algebra $h \in \mathcal{H}$

$$h \bullet \Psi(v \otimes w) : \quad = \quad \Delta(h) \triangleright P(\mathcal{R} \triangleright (v \otimes w)) = P(\Delta^{op}(h)\mathcal{R} \triangleright (v \otimes w))$$

$$= P(\mathcal{R}\Delta(h) \triangleright (v \otimes w) = \Psi(h \bullet (v \otimes w)) \,. \qquad (5.6)$$

In general for a quasi-triangular Hopf algebra, $\mathcal{R}_{21} \neq \mathcal{R}^{-1}$ and consequently $\Psi \neq \Psi^{-1}$, i.e. the braiding is asymmetric. By finding the braiding Ψ one can see how nontrivial statistics emerges.

In the case of the twisted Poincaré algebra the $\mathcal{R}$-matrix can be presented as (using (4.15) and (5.4)):

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = e^{-i\theta^{\mu\nu}P_\mu \otimes P_\nu} \,, \qquad (5.7)$$

which satisfies $\mathcal{R}_{12}\mathcal{R}_{21} = 1$. The fact that $\mathcal{R}_{21} = \mathcal{R}^{-1}$ directly implies $\Psi = \Psi^{-1}$. As a result, the twisted Poincaré algebra is a so-called triangular Hopf algebra, Ψ is symmetric and not braided. It follows that *NC QFTs with twisted Poincaré symmetry do not experience nontrivial statistics*, although the notion of permutation is deformed using the $\mathcal{R}$-matrix.

This argument, although being beyond doubt, is not easily understood and can be clarified by a concrete example. Let us consider the free quantum scalar field with the Lagrangian (3.15). Since it is quadratic in the field and under the integration over the whole space-time one $\star$-product is known to vanish in each term we can use the familiar expansion (3.16):

$$\phi(x) = \int d\mu(\mathbf{p}) \left[a(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{ip\cdot x}\right] \,.$$

The products of such quantum fields have to be deformed by using the inverse of the abelian twist element (4.17). Here we have to choose a realization for the momentum operator.

The first option is to choose the ordinary Minkowski space realization $P_\mu = -i\partial_\mu$ in which case we will have a $\star$-product between exponentials

$$e^{ik_\mu x^\mu} \star e^{ip_\mu x^\mu} = e^{ik_\mu x^\mu} e^{ip_\mu x^\mu} e^{-\frac{i}{2}k_\mu \theta^{\mu\nu} p_\nu}$$

and the *usual commutation relations* between creation and annihilation operators. Now everything defined with these operators, such as the multi-particle states (3.23), remain as they were in the commutative space-time. This directly implies that the spin-statistics relation is preserved.

There is a second option available however[†]. Since the creation and annihilation operators are themselves representations of the momentum operator,

$$\mathcal{P}_\mu a(\mathbf{k}) = [P_\mu, a(\mathbf{k})] = -k_\mu a(\mathbf{k}), \quad \mathcal{P}_\mu a^\dagger = [P_\mu, a^\dagger(\mathbf{k})] = k_\mu a^\dagger(\mathbf{k}) \,,$$

[†]The third option is a combination of these two and was shown to give the same results in [39]. This route will not be further discussed here.

we can take $P_\mu = \int d^3\mathbf{k}\, k_\mu a^\dagger(\mathbf{k}) a(\mathbf{k})$ in which case we will have the $\star$-product between the creation and annihilation operators [52]

$$a^\dagger(\mathbf{k}) \star a^\dagger(\mathbf{p}) = m \circ \left(e^{-\frac{i}{2}\theta^{\mu\nu}\mathcal{P}_\mu \otimes \mathcal{P}_\nu} \right) \left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p}) \right) = a^\dagger(\mathbf{k})a^\dagger(\mathbf{p}) e^{-\frac{i}{2}k_\mu \theta^{\mu\nu} p_\nu} ,$$

but now with the usual multiplication between exponentials.

To move on with this latter choice we have to express the commutation relations of the operators using the $\star$-product as was previously done e.g. in (4.13)

$$a^\dagger(\mathbf{k}) \star a^\dagger(\mathbf{p}) = a^\dagger(\mathbf{p}) \star a^\dagger(\mathbf{k}) \; e^{-ik_\mu \theta^{\mu\nu} p_\nu} \tag{5.8}$$

$$a(\mathbf{k}) \star a^\dagger(\mathbf{p}) - e^{ik_\mu \theta^{\mu\nu} p_\nu} a^\dagger(\mathbf{p}) \star a(\mathbf{k}) = \delta(\mathbf{k} - \mathbf{p}) . \tag{5.9}$$

Now the multi-particle states become nontrivial as they are also naturally defined using the $\star$-product as

$$|n\rangle_\star = a^\dagger(\mathbf{p}_1) \star a^\dagger(\mathbf{p}_2) \star \cdots \star a^\dagger(\mathbf{p}_n)|0\rangle. \tag{5.10}$$

From here we want to show that no deformation of statistics comes about. To do this it is easiest to consider a two particle state and see whether or not it remains symmetric under the new concept of permutation given by the braiding (5.5). A two particle state is symmetric if for the braiding we have

$$m \circ \mathcal{F}^{-1} \left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p}) \right) = m \circ \mathcal{F}^{-1}\Psi \left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p}) \right) . \tag{5.11}$$

Using the definitions (5.5) and (5.7) and by noting that, since $\theta^{\mu\nu}$ is antisymmetric, we obviously have, for example, $\mathcal{F}_{12} = \mathcal{F}_{21}^{-1}$ we get

$$\begin{aligned}
m \circ \mathcal{F}^{-1}\Psi \left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p}) \right) &= m \circ \mathcal{F}^{-1}P \left(\mathcal{R} \triangleright (a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p})) \right) \\
&= m \circ \mathcal{F}^{-1}P \left(\mathcal{F}_{21}\mathcal{F}^{-1}(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p})) \right) = m \circ \mathcal{F}_{12} \left(a^\dagger(\mathbf{p}) \otimes a^\dagger(\mathbf{k}) \right) \\
&= m \circ \mathcal{F}_{21}^{-1} \left(a^\dagger(\mathbf{p}) \otimes a^\dagger(\mathbf{k}) \right) = m \circ \mathcal{F}^{-1} \left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p}) \right) .
\end{aligned} \tag{5.12}$$

By a similar treatment we can go on to show that (5.11) is equivalent to (5.8). The equivalence of the left-hand sides is obvious and as for the right-hand sides we have

$$\begin{aligned}
m \quad \circ \quad & \mathcal{F}^{-1}\Psi \left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p}) \right) = m \circ \mathcal{F}^{-1}P \left(\mathcal{R} \triangleright (a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p})) \right) \\
&= \quad m \circ \mathcal{F}^{-1}P \left(e^{-i\theta^{\mu\nu}P_\mu \otimes P_\nu} \triangleright (a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p})) \right) \\
&= \quad m \circ \mathcal{F}^{-1} e^{-ik_\mu \theta^{\mu\nu} p_\nu} \left(a^\dagger(\mathbf{p}) \otimes a^\dagger(\mathbf{k}) \right) \\
&= \quad a^\dagger(\mathbf{p}) \star a^\dagger(\mathbf{k}) \; e^{-ik_\mu \theta^{\mu\nu} p_\nu}
\end{aligned} \tag{5.13}$$

This means that the use of $\star$-commutation relations (5.8) with the braided permutation (5.5) preserves the symmetry of the multi-particle states of the scalar field. These braided permutations are considered more fully in [53]. The preservation of symmetry is interpreted as a different choice of representation for the wave functions in the same Hilbert space, i.e. a phase shift. In other words the $\star$-commutation relations induce a phase shift in the new multi-particle states $|n\rangle_\star$ with respect to the old ones $|n\rangle$. Due to this we end up with the same representation content of the permutation group and no deformation of statistics[‡]. In conclusion this second approach as well as the first lead to the preservation of the spin-statistics relation.

What is interesting when comparing the earlier results of the Lagrangian formalism to the results of quantum group theory is the unambiguous situation of lightlike noncommutativity. In the latter treatment no room is left for the violation of the spin-statistics relation and we can conclude that it holds in all noncommutative space-times defined by (4.1).

[‡]For a direct verification, see [54].

Chapter 6

Conclusions

In this work I have reviewed the status of one of the most important connections in physics, the spin-statistics relation. After a thorough historical introduction, a simple and clear proof on the theorem was presented starting from the requirement of rotational invariance followed by discussion based on relativistic quantum field theory. Next the notion of noncommutativity was introduced and some of its dramatic consequences studied. The controversial issue of microcausality in noncommutative quantum field theory was settled by showing for the first time that the light wedge microcausality condition is compatible with the twisted Poincaré symmetry. Finally it was proven that Pauli's age-old theorem stands even this test so dramatic for the whole structure of space-time. The quantum group theoretical proof presented originally in [40] and the proofs based on the Lagrangian formalism [38,39] show that the connection remains valid also in the noncommutative regime. This is in contrast to some earlier discussion on the subject [48] where the opposite conclusion was reached.

In the field of noncommutative quantum field theories there still remain many unanswered questions. There is work to be done in constructing gauge theories in the noncommutative setting as well as many fundamental theories requiring verification. The faith of microcausality along with proper understanding of other implications of nonlocality and Lorentz-noninvariance will require more study and a better understanding of the new relativistic symmetry present, the twisted Poincaré invariance.

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